Online Appendix to

Ranking Tournaments with No Errors II: Minimax Relation

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The purpose of this online appendix is to present proofs of Lemma 4.4 and Lemma 5.6 in the submitted version of this paper.

4 Basic Reductions

Lemma 4.5. *If $T_2 = F_4$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_3v_1, v_2v_3v_4v_2, v_1v_5v_3v_1, v_3v_4v_5v_3, v_1v_2v_3v_4v_1, v_1v_5v_2v_3v_1, v_1v_5v_3v_4v_1, v_2v_3v_4v_5v_2, v_1v_5v_2v_3v_4v_1\}$ and
- $\mathcal{F}_2 = \{\{v_2v_3, v_5v_3\}, \{v_3v_1, v_3v_4\}, \{v_1v_2, v_1v_5, v_3v_4\}, \{v_1v_5, v_2v_3, v_3v_4\}, \{v_1v_5, v_2v_3, v_4v_5\}, \{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}, \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}, \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}, \{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}, \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}, \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 9$ and $|\mathcal{F}_2| = 11$. Recall that $(b_2, a_2) = (v_5, v_6)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_5v_2v_3v_1) + y(v_1v_5v_3v_4v_1)$ is minimized;

(4) subject to (1)-(3), $y(v_2v_3v_4v_5v_2)$ is minimized;

(5) subject to (1)-(4), $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$ is minimized; and

(6) subject to (1)-(5), $y(v_1v_5v_3v_1)$ is minimized.

Let us make some simple observations about $\boldsymbol{y}$.

(7) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

(8) If $y(v_1v_5v_2v_3v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Furthermore, $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_5v_2v_3v_4v_1$. So the first half follows instantly from Lemma 3.5(v). Let $\uplus$ stand for the multiset sum. Then $v_1v_5v_2v_3v_4v_1 \uplus v_1v_2v_3v_1 = v_1v_5v_2v_3v_1 \uplus v_1v_2v_3v_4v_1$, $v_1v_5v_2v_3v_4v_1 \uplus v_1v_5v_3v_1 = v_1v_5v_2v_3v_1 \uplus v_1v_5v_3v_4v_1$, and $v_1v_5v_2v_3v_4v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_4v_1 \uplus v_2v_3v_4v_5v_2$. Suppose on the contrary that $y(v_1v_2v_3v_1) > 0$. Let $\theta = \min\{y(v_1v_5v_2v_3v_4v_1), y(v_1v_2v_3v_1)\}$ and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_2v_3v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_2v_3v_4v_1)$ with $y(v_1v_5v_2v_3v_4v_1) - \theta$, $y(v_1v_2v_3v_1) - \theta$, $y(v_1v_5v_2v_3v_1) + \theta$, and $y(v_1v_2v_3v_4v_1) + \theta$. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_5v_2v_3v_4v_1) < y(v_1v_5v_2v_3v_4v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (2) on $\boldsymbol{y}$. So $y(v_1v_2v_3v_1) = 0$. Similarly, $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

(9) If $y(v_1v_5v_2v_3v_1) > 0$, then v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_5 provided $y(v_1v_2v_3v_4v_1) > 0$. Furthermore, $y(v_3v_4v_5v_3) = 0$.

To justify this, note that both v_1v_2 and v_5v_3 are chords of the cycle $v_1v_5v_2v_3v_1$, so they are saturated by $\boldsymbol{y}$ in F_4 by Lemma 3.5(v). Since $v_1v_5v_2v_3v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_1 \uplus v_2v_3v_4v_5v_2$, from (3) we deduce that $y(v_3v_4v_5v_3) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_2v_3v_4v_1) > 0$. If v_4v_5 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc v_4v_5 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_2v_3v_4v_5v_2$; if v_4v_5 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of $v_1v_5v_2v_3v_1$, $v_1v_2v_3v_4v_1$, and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_2v_3v_4v_5v_2$, and $C' = C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

(10) If $y(v_1v_5v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_2 provided $y(v_1v_5v_2v_3v_1) > 0$, and so is v_1v_2 provided $y(v_2v_3v_4v_5v_2) > 0$. Furthermore, $y(v_1v_2v_3v_1) = 0$.

To justify this, note that both v_3v_1 and v_4v_5 are chords of the cycle $v_1v_5v_3v_4v_1$, so they are saturated by $\boldsymbol{y}$ in F_4 by Lemma 3.5(v). Since $v_1v_5v_3v_4v_1 \uplus v_1v_2v_3v_1 = v_1v_5v_3v_1 \uplus v_1v_2v_3v_4v_1$, from (3) we deduce that $y(v_1v_2v_3v_1) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_5v_2v_3v_1) > 0$. If v_4v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_3v_4v_1$, and the arc v_4v_2 contains arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$; if v_4v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C_1 \in \mathcal{C}_0^y$, then the multiset sum of C_1, $v_1v_5v_2v_3v_1$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_2$, and $C_1' = C_1[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

Next, consider the case when $y(v_2v_3v_4v_5v_2) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_4v_1$, $v_2v_3v_4v_5v_2$, and the arc v_1v_2 contains arc-disjoint cycles $v_3v_4v_5v_3$ and $v_1v_2v_3v_4v_1$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C_2 \in \mathcal{C}_0^y$, then the multiset sum of C_2, $v_2v_3v_4v_5v_2$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_4v_1$, and $C_2' = C_2[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_1v_2 is saturated by $\boldsymbol{y}$ in F_4.

(11) If $y(v_1v_2v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_5 provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 3.5(v). Suppose $y(v_1v_5v_3v_1) > 0$. If v_4v_5 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc v_4v_5 contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$; if v_4v_5 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of $v_1v_2v_3v_4v_1$, $v_1v_5v_3v_1$, and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_3v_4v_5v_3$, and $C' = C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

(12) If $y(v_2v_3v_4v_5v_2) > 0$, then both v_4v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4; so is v_1v_2 provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 3.5(v). Suppose $y(v_1v_5v_3v_1) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_5v_2$, and the arc v_1v_2 contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of C, $v_2v_3v_4v_5v_2$, and $v_1v_5v_3v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_1$, and $C' = C[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_1v_2 is saturated by $\boldsymbol{y}$ in F_4.

Claim 1. $y(\mathcal{C}_2) = \tau_w(F_4 \backslash v_6)$.

To justify this, observe that v_2v_3 is a special arc of T and v_2 is a near-sink. By Lemma 3.4(iv), we may assume that v_2v_3 is saturated by $\boldsymbol{y}$ in T. Depending on whether v_2v_3 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_3 is contained in some cycle in $\mathcal{C}_0^y$.

Choose $C \in \mathcal{C}_0^y$ that contains v_2v_3 and, subject to this, has the maximum number of arcs in $F_4 \backslash v_6$. We proceed by considering three subcases.

• C contains v_1v_2. In this subcase, C contains the path $P = v_1v_2v_3v_4v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Since no arc on C (and hence on P) is saturated by $\boldsymbol{y}$ in F_4, we have $y(v_1v_5v_2v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$ by (8) – (10). Since the multiset sum of $v_1v_5v_3v_1$ and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_3v_4v_5v_3$, and $C' = C[v_5, v_1] \cup \{v_1v_5\}$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_1v_5v_3v_1) = 0$. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

• C contains v_4v_2. In this subcase, C contains the path $P = v_4v_2v_3v_1v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Since no arc on C (and hence on P) is saturated by $\boldsymbol{y}$ in F_4, $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0 by (8) and (10)-(12). Since the multiset sum of $v_3v_4v_5v_3$ and C contains three arc-disjoint cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_2$, and $C' = C[v_5, v_4] \cup \{v_4v_5\}$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_3v_4v_5v_3) = 0$. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

• C contains neither v_1v_2 nor v_4v_2. In this subcase, we may assume that both v_1v_2 and v_4v_2 are outside $\mathcal{C}_0^y$, for otherwise, each cycle containing v_1v_2 or v_4v_2 passes through v_2v_3, and thus one of the preceding subcases occurs. Clearly, C contains v_3v_4 or v_3v_1.

Assume first that C contains v_3v_4. If C contains v_4v_1, then it also contains v_1v_5. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$. If C does not contain v_4v_1, then C contains v_4v_5. By Lemma 3.5(ii) and (iv), each arc in the set $\{v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. If v_1v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where K is as defined above. Again, K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$. So we assume that v_1v_2 is not saturated by $\boldsymbol{y}$ in T. Since v_1v_2 is outside $\mathcal{C}_0^y$, so are v_4v_1 and v_3v_1. By Lemma 3.5(iii), both v_4v_1 and v_3v_1 are saturated by $\boldsymbol{y}$ in T and hence in F_4. Moreover, by (8)-(10), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_5v_3v_4v_1)$ are all 0. Since the multiset sum of the cycles $v_1v_5v_3v_1$, C, and the unsaturated arc v_1v_2 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$. By Lemma 3.5(vi), we have $y(v_1v_5v_3v_1) = 0$. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

Assume next that C contains v_3v_1. Then C contains v_1v_5. By Lemma 3.5(ii) and (iv), each arc in the set $\{v_1v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. If v_4v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$. So we assume that v_4v_2 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T (recall that v_4v_2 is outside $\mathcal{C}_0^y$). By Lemma 3.5(iv), v_3v_4 is outside $\mathcal{C}_0^y$. By Lemma 3.5(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T and hence in F_4. By (8) and (10)-(12), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0. Since the multiset sum of the cycles $v_3v_4v_5v_3$, C, and the unsaturated arc v_4v_2 contains two arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$, we have $y(v_3v_4v_5v_3) = 0$ by Lemma 3.5(vi). So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

Case 1.2. v_2v_3 is outside $\mathcal{C}_0^y$.

By the previous observation, v_2v_3 is saturated by $\boldsymbol{y}$ in F_4 now. Note also that v_5v_3 is outside $\mathcal{C}_0$. If v_5v_3 is saturated by $\boldsymbol{y}$ in T, so is it in F_4, and hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_3, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$. So we assume that v_5v_3 is unsaturated. By (8), (9), and (12), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0. Observe that both v_3v_1 and v_3v_4 are outside $\mathcal{C}_0^y$, for otherwise, since each cycle passing through v_3v_1 or v_3v_4 contains v_1v_5 or v_4v_5, from Lemma 3.5(iv) we deduce that v_5v_3 is saturated, a contradiction. If both v_3v_1 and v_3v_4 are saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_3v_4\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$. So we assume that

(13) at most one of v_3v_1 and v_3v_4 is saturated by $\boldsymbol{y}$ in F_4.

Since $\mathcal{C}_0^y \neq \emptyset$, there is a cycle $C \in \mathcal{C}_0^y$ passing through v_4v_1, or v_1v_5, or v_4v_5; subject to this, let C be chosen to have the maximum number of arcs in $F_4 \backslash v_6$. We proceed by considering three subcases.

• C contains both v_4v_1 and v_1v_5. In this subcase, since v_5v_3 is unsaturated, by Lemma 3.5(iii), v_3v_1 and v_3v_4 are both saturated by $\boldsymbol{y}$ in F_4, a contradiction.

• C contains v_1v_5 but not v_4v_1. In this subcase, from the choice of C, we see that v_4v_1 is outside $\mathcal{C}_0^y$, because every cycle containing v_4v_1 passes through v_1v_5. Since v_5v_3 is unsaturated, Lemma 3.5(iii) implies that v_3v_1 is saturated by $\boldsymbol{y}$ in F_4, and thus v_3v_4 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T by (13). Once again, by Lemma 3.5(iii), v_4v_1 is saturated by $\boldsymbol{y}$ in F_4, and v_4v_5 is outside $\mathcal{C}_0^y$. Since both v_5v_3 and v_3v_4 are unsaturated, it follows from Lemma 3.5(i) that v_4v_5 is saturated by $\boldsymbol{y}$ in F_4. If v_4v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4 \backslash v_6)$. If v_4v_2 is not saturated by $\boldsymbol{y}$ in F_4, then $y(v_1v_2v_3v_4v_1) = 0$ by (11). Moreover, since the multiset sum of the cycles $v_1v_2v_3v_1$, C, and the unsaturated arcs v_5v_3, v_3v_4, and v_4v_2 contains two arc-disjoint cycles $v_2v_3v_4v_2$ and $v_1v_5v_3v_1$, we have $y(v_1v_2v_3v_1) = 0$ by Lemma 3.5(vi). Therefore, $y(\mathcal{C}_2) = w(J)$, where $J = \{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

• C contains v_4v_5. In this subcase, we may assume that both v_4v_1 and v_1v_5 are outside $\mathcal{C}_0^y$, otherwise one of the preceding subcases occurs. By Lemma 3.5(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T and hence in F_4, which together with (13) implies that v_3v_1 is not saturated by $\boldsymbol{y}$ in F_4. Using (10) and (11), we deduce that $y(v_1v_5v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$. Using Lemma 3.5(iii) and the triangle $v_1v_5v_3v_1$, we see that v_1v_5 is outside $\mathcal{C}_0^y$. Using Lemma 3.5(i) and the triangle $v_1v_5v_3v_1$, we also deduce that v_1v_5 is saturated by $\boldsymbol{y}$ in T and hence in F_4. If v_1v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_1v_5, v_3v_4\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4 \backslash v_6)$. So we assume that v_1v_2 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T, because v_1v_2 is outside $\mathcal{C}_0^y$, by the hypothesis of the present case. Since the multiset sum of the cycles C, $v_2v_3v_4v_2$, and unsaturated arcs v_5v_3, v_3v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$, we have $y(v_2v_3v_4v_2) = 0$ by Lemma 3.5(vi). It follows that $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_5, v_2v_3, v_3v_4\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4 \backslash v_6)$. This completes the proof of Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, let $\mathcal{G}_2 = \mathcal{F}_2 \backslash \{\{v_1v_5, v_2v_3, v_4v_5\}, \{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}\}$. From the proof of Claim 1, we see that $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{G}_2$. Observe that if $y(\mathcal{C}_2) = w(J)$ for $J = \{v_1v_5, v_2v_3, v_4v_5\}$ or $\{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}$, then both v_1v_5 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4, so $\mathcal{C}_0^y = \emptyset$ in this case, which has been excluded by Lemma 3.2(ii).

Let us make some further observations about $\boldsymbol{y}$.

(14) $y(v_1v_5v_2v_3v_4v_1) = 0$.

Suppose on the contrary that $y(v_1v_5v_2v_3v_4v_1) > 0$. By (8), we have $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$, and each arc in the set $\{v_1v_2, v_3v_1, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2(v_1v_2)) = w(v_1v_2)$, $y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$, $y(\mathcal{C}_2(v_4v_2)) = w(v_4v_2)$, $y(\mathcal{C}_2(v_4v_5)) = w(v_4v_5)$, and $y(\mathcal{C}_2(v_5v_3)) = w(v_5v_3)$. It follows that $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_1v_5v_2v_3v_1) = w(v_3v_1)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, and $y(v_1v_5v_3v_4v_1) = w(v_5v_3)$. From Claim 1 we deduce that $y(v_1v_5v_2v_3v_4v_1)$ is also integral, and hence $\nu_w^*(T)$ is an integer by Lemma 3.2(iii).

(15) $y(v_1v_5v_2v_3v_1)$ or $y(v_1v_5v_3v_4v_1)$ is 0.

Assume the contrary: both $y(v_1v_5v_2v_3v_1)$ and $y(v_1v_5v_3v_4v_1)$ are positive. By (9) and (10), we have $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = 0$, and each arc in the set $\{v_1v_2, v_5v_3, v_3v_1, v_4v_2, v_4v_5\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2(v_1v_2)) = w(v_1v_2)$, $y(\mathcal{C}_2(v_5v_3)) = w(v_5v_3)$, $y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$, $y(\mathcal{C}_2(v_4v_2)) = w(v_4v_2)$, and $y(\mathcal{C}_2(v_4v_5)) = w(v_4v_5)$. It follows that $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, $y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1) = w(v_3v_1)$, and $y(v_1v_5v_3v_1) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$. Given the above equations and (14), to prove that $y(C)$ is integral for all $C \in \mathcal{C}_2$, it suffices to show that one of $y(v_1v_5v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_5v_3v_1)$ is integral.

By Lemma 3.1 and Claim 1, each arc $e \in K$ satisfies $w(e) = z(e) = y(\mathcal{C}_2(e))$. Let us proceed by considering four subcases.

If $v_2v_3 \in K$, then $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2)$, which implies that $y(v_1v_5v_2v_3v_1)$ is integral.

If $v_3v_4 \in K$, then $w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_3v_4v_1)$, which implies that $y(v_1v_5v_3v_4v_1)$ is integral.

If $v_4v_1 \in K$, then $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1)$, which implies that $y(v_1v_5v_3v_4v_1)$ is integral.

If $v_5v_2 \in K$, then $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2)$, which implies that $y(v_1v_5v_2v_3v_1)$ is integral.

Since each $K \in \mathcal{G}_2$ contains at least one arc in the set $\{v_2v_3, v_3v_4, v_4v_1, v_5v_2\}$, it follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So $y(v_1v_5v_2v_3v_1)$ is a positive integer, and hence $\nu_w^*(T)$ is an integer by Lemma 3.2(iii). Therefore we may assume that (15) holds.

Depending on what $K \in \mathcal{G}_2$ is, we distinguish among nine cases.

Case 2.1. $K = \{v_1v_5, v_2v_3, v_3v_4\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_2v_3v_4v_2) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$ and $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields $w(v_1v_5) = y(\mathcal{C}_2(v_1v_5)) = y(v_1v_5v_3v_1)$, $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_1v_2v_3v_1)$, and $w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.2. $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_5v_3v_4v_1) = y(v_3v_4v_5v_3) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = 0$, which together with (14) yields $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1)$,

$w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2)$, $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.3. $K = \{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = 0$, which together with (14) yields $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_2v_3v_4v_2)$, $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_5v_3v_1)$, $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_5v_3v_4v_1)$, and $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.4. $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_5v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$, which together with (14) yields $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1)$, $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1)$, $w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$, $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_2v_3v_4v_5v_2)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.5. $K = \{v_1v_2, v_1v_5, v_3v_4\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$ and $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following three equations:

$w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1)$;

$w(v_1v_5) = y(\mathcal{C}_2(v_1v_5)) = y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$; and

$w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$.

Depending on the value of $y(v_1v_5v_2v_3v_1)$, we consider two subcases.

- $y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_1v_5v_3v_1) = w(v_1v_5)$. If $y(v_2v_3v_4v_5v_2) > 0$, then $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$ and $w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$ by (12). Thus both $y(v_3v_4v_5v_3)$ and $y(v_2v_3v_4v_5v_2)$ are integral, and hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3)$. If $y(v_2v_3v_4v_2)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_2v_3v_4v_2)$ is not integral. Thus $[y(v_2v_3v_4v_2)] + [y(v_3v_4v_5v_3)] = 1$. Since each arc in K is saturated by $\boldsymbol{y}$ in F_4, both v_2v_3 and v_4v_2 are outside $\mathcal{C}_0^y$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_2v_3v_4v_2)$ and $y(v_3v_4v_5v_3)$ with $y(v_2v_3v_4v_2) + [y(v_3v_4v_5v_3)]$ and $\lfloor y(v_3v_4v_5v_3) \rfloor$ respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_3v_4v_5v_3) < y(v_3v_4v_5v_3)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (5) on $\boldsymbol{y}$.
- $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, $y(v_3v_4v_5v_3) = 0$ and v_5v_3 is saturated by $\boldsymbol{y}$ in F_4 by (9). So $w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_2v_3v_4v_5v_2)$ and $w(v_5v_3) = y(v_1v_5v_3v_1)$. It follows that $y(v_1v_5v_2v_3v_1) = w(v_1v_5) - w(v_5v_3)$. If $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) = w(v_3v_4)$; otherwise, by (12), both v_1v_5 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4. Thus $y(v_2v_3v_4) = w(v_4v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_3v_4) - w(v_4v_2)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.6. $K = \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$;

$w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1)$;

$w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$; and

$w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$.

Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

- $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, by (10) and (15), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$. So $y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_2v_3v_4v_5v_2) > 0$, then both v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$

in F_4 by (10) and (12). So $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_4v_1)$ and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$. Since $y(v_1v_5v_3v_4v_1) = w(v_4v_1) - y(v_1v_2v_3v_4v_1)$ and $y(v_2v_3v_4v_5v_2) = w(v_4v_5) - y(v_3v_4v_5v_3)$, it follows that $y(v_1v_5v_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_2v_3v_4v_5v_2)$ are all integral. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. Since each arc in K is saturated by $\boldsymbol{y}$ in F_4, both v_1v_2 and v_2v_3 are outside $\mathcal{C}_0^y$. By Lemma 3.2(i), we may assume that $w(e) = \lceil z(e) \rceil$ for all arcs e in T. Thus, from (3) we deduce that $y(v_1v_2v_3v_4v_1) = \min\{w(v_1v_2), w(v_2v_3) - w(v_4v_2)\}$ and $y(v_1v_5v_3v_4v_1) = w(v_4v_1) - y(v_1v_2v_3v_4v_1)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, from (9) and (15), we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$, and that both v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$ $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1)$. Thus $y(v_1v_2v_3v_1) = w(v_1v_2) - w(v_4v_1)$ is integral, so is $y(v_1v_5v_2v_3v_1)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

• $y(v_1v_5v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$. Suppose $y(v_2v_3v_4v_5v_2) > 0$. Then v_5v_3 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$, It follows that $y(v_1v_2v_3v_1)$ and hence $y(C)$ is integral for any $C \in \mathcal{C}_2$. If $y(v_1v_5v_3v_1) = 0$, then $y(v_1v_2v_3v_1) = w(v_3v_1)$, which implies that $y(C)$ is integral for any $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. Observe that $y(v_1v_2v_3v_1)$ is integral, for otherwise, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively. Since v_1v_2 and v_2v_3 are outside $\mathcal{C}_0^y$, we see $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (5) on $\boldsymbol{y}$. From the above observation, it is easy to see that $y(C)$ is integral for any $C \in \mathcal{C}_2$.

Case 2.7. $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$.

In this case, by Lemma 3.1(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$;

$w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$;

$w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2)$; and

$w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$.

Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, by (10) and (15), $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = w(v_5v_2)$, and $y(v_1v_5v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$. Thus $y(v_3v_4v_5v_3)$ and $y(v_1v_5v_3v_4v_1)$ are also integral.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, by (9) and (15), we have $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$. So $y(v_1v_5v_3v_1) = w(v_5v_3)$. If $y(v_1v_2v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). So $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$ and $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_2v_3v_4v_5v_2)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1)$, $y(v_2v_3v_4v_2)$, and $y(v_1v_5v_3v_1)$ are integral, and $y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_5v_2)$. If $y(v_2v_3v_4v_5v_2)$ is an integer, then $y(C)$ is integral for any $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2)$ is not integral. We propose to show that

(16) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. By Lemma 3.2(iii), we may assume that $w(v_1v_2) = w(v_4v_2) = w(v_5v_3) = 0$. Thus $y(C) = 0$ for all $C \in \mathcal{C}_2 \backslash \{v_1v_5v_2v_3v_1, v_2v_3v_4v_5v_2\}$. Observe that v_3v_4 is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ that contains v_3v_4. It is then easy to see that an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ can be obtained from $\boldsymbol{y}$ by modifying $y(D)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_2v_3v_4v_5v_2)$ and by possibly rerouting D, so that $y'(v_1v_5v_2v_3v_1) < y(v_1v_5v_2v_3v_1)$, contradicting (3). Since $y(v_2v_3v_4v_5v_2) < w(v_3v_4)$, we have $x(v_3v_4) = 0$ by Lemma 3.1(ii). Since both $y(v_1v_5v_2v_3v_1)$ and $y(v_2v_3v_4v_5v_2)$ are positive, $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$ by Lemma 3.1(i). So $x(v_4v_5) = x(v_3v_1) + x(v_1v_5)$.

Let us show that if $w(v_4v_1) > 0$, then $x(v_4v_1) = x(v_3v_1)$. For this purpose, note that both v_4v_1 and v_4v_5 are contained in some cycles in $\mathcal{C}_0^y$, for otherwise, we can obtain a new optimal solution $\boldsymbol{y}'$ from $\boldsymbol{y}$ satisfying (1) and (2), but $y'(v_1v_5v_2v_3v_1) = \lfloor y(v_1v_5v_2v_3v_1) \rfloor$ and $y'(v_2v_3v_4v_5v_2) = y(v_2v_3v_4v_5v_2) + [y(v_1v_5v_2v_3v_1)]$, which again contradicts (3). Thus $x(v_4v_5) = x(v_1v_5) + x(v_4v_1)$ by Lemma 3.1(iii). Combining it with the equality established in the preceding paragraph, we obtain the $x(v_4v_1) = x(v_3v_1)$. If $w(v_4v_1) = 0$, then we may assume that $x(v_4v_1) = x(v_3v_1)$ (replacing the smaller of these two with the larger if necessary).

Similarly, we can prove that $x(uv_3) = x(uv_4)$ for each $u \in V(T_1) \backslash \{b, a_1\}$, where b is the hub of the 1-sum. Let T' be the the digraph obtained from T by identifying v_3 and v_4; the resulting vertex is still denoted by v_4. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to $A(T')$ by replacing $w(uv_4)$ with $w(uv_3) + w(uv_4)$ for each $u \in V(T_1) \backslash \{b, a_1\}$. Note that T' is Möbius-free by Lemma 2.7, $\boldsymbol{x}$ corresponds to a feasible solution $\boldsymbol{x}'$ to $\mathbb{P}(T', \boldsymbol{w}')$, and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y}'$ to $\mathbb{P}(T', \boldsymbol{w}')$ with $y'(v_4v_5v_4) = y'(v_4v_2v_4) = 0$, both having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. So $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. By Lemma 3.3, the optimal value $\nu_w^*(T)$ of $\mathbb{P}(T', \boldsymbol{w}')$ is integral. So (16) is established.

- $y(v_1v_5v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_2v_3v_4v_2)$ and $y(v_2v_3v_4v_5v_2)$ are integral. Assume first that $y(v_1v_2v_3v_4v_1) > 0$. Then, by (11), the arc v_3v_1 is saturated by $\boldsymbol{y}$ in F_4. So $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1)$. If $y(v_1v_5v_3v_1) = 0$, then $y(v_3v_4v_5v_3) = w(v_5v_3)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$. If $y(v_1v_5v_3v_1) > 0$, then v_4v_5 is is saturated by $\boldsymbol{y}$ in F_4 by (11). Thus $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$, which is integral. It follows that $y(v_3v_4v_5v_3) = w(v_4v_5) - w(v_5v_2)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$. Assume next that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1)$ is integral and $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) = w(v_5v_3)$. Clearly, we may assume that neither $y(v_1v_5v_3v_1)$ nor $y(v_3v_4v_5v_3)$ is integral, otherwise we are done. Similar to (16), we can show that

(17) $\nu_w^*(T)$ is an integer.

The proof goes along the same line as that of (16). In fact, we only need to replace $y(v_1v_5v_2v_3v_1)$ and $y(v_2v_3v_4v_5v_2)$ with $y(v_1v_5v_3)$ and $y(v_3v_4v_5v_3)$, respectively. So we omit the details here.

Case 2.8. $K = \{v_2v_3, v_5v_3\}$.

In this case, by Lemma 3.1(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following two equations:

$w(v_2v_3) = y(v_1v_2v_3v_1) + y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_2v_3v_1)$; and

$w(v_5v_3) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$.

Since v_2v_3 is saturated by $\boldsymbol{y}$ in F_4, we have $w(uv_2) = z(uv_2) = 0$ for any $u \in V(T_1) \backslash \{b, a_1\}$ in this case. Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, from (10) and (15) we deduce that $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and that both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2) = w(v_4v_5)$ and $y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_2v_3v_4v_5v_2) > 0$, then both v_1v_2 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4 by (10) and (12). Thus $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$. It follows that $y(v_3v_4v_5v_3)$, $y(v_2v_3v_4v_5v_2)$, and $y(v_1v_5v_3v_4v_1)$ are all integral. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$, and $y(v_1v_5v_3v_4v_1) = w(v_5v_3) - w(v_3v_1) - w(v_4v_5)$. Moreover, $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_4v_1) = w(v_2v_3) - w(v_4v_2)$ if $y(v_1v_2v_3v_4v_1) > 0$, and $y(v_2v_3v_4v_2) = w(v_2v_3)$ otherwise. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether if $y(v_2v_3v_4v_5v_2) > 0$.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, by (9) and (15) we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$ and that v_1v_2 is saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2)$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_3v_1, v_4v_2, and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, and $y(v_2v_3v_4v_2) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1) = w(v_3v_1)$. It follows that $y(v_1v_2v_3v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_1v_5v_2v_3v_1)$ are all integral. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) = w(v_1v_2)$. If $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) + y(v_1v_5v_2v_3v_1) = w(v_2v_3) - w(v_1v_2)$. Since $y(v_1v_5v_2v_3v_1) > 0$, we see that v_3v_4 is outside $\mathcal{C}_0^y$, for otherwise, we can obtain an optimal solution $\boldsymbol{y'}$ to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_5v_2v_3v_1) < y(v_1v_5v_2v_3v_1)$, contradicting (3). It follows that $y(v_2v_3v_4v_2) = \min\{w(v_4v_2), w(v_3v_4)\}$ and $y(v_1v_5v_2v_3v_1) = w(v_2v_3) - w(v_1v_2) - y(v_2v_3v_4v_2)$. If $y(v_2v_3v_4v_5v_2) > 0$, then $y(v_2v_3v_4v_2) = w(v_4v_2)$ by (12) and $y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) - w(v_4v_2)$. Thus we always have $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. Since v_2 is a near-sink, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.4(i).

• $y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$. In this subcase, depending on whether $y(v_2v_3v_4v_5v_2) > 0$, we distinguish between two subsubcases.

(a) We first assume that $y(v_2v_3v_4v_5v_2) > 0$. Now, in view of (12), v_4v_2 is saturated by $\boldsymbol{y}$ in F_4, which yields $w(v_4v_2) = y(v_2v_3v_4v_2)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) - w(v_4v_2)$. Thus $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. By Lemma 3.4(i), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_5v_3v_1) = 0$. If $y(v_1v_2v_3v_4v_1) = 0$, then $y(v_1v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_4v_2)$. Since $\boldsymbol{y}$ satisfies (1), we have $y(v_1v_2v_3v_1) = \min\{w(v_1v_2), w(v_3v_1)\}$ and $y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_4v_2) - y(v_1v_2v_3v_1)$. If $y(v_1v_2v_3v_4v_1) > 0$, then $y(v_1v_2v_3v_1) = w(v_3v_1)$ by (11) and $y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_3v_1) - w(v_4v_2)$. Assume $y(v_1v_2v_3v_4v_1)$ is not integral. Then $[y(v_1v_2v_3v_4v_1)] + [y(v_2v_3v_4v_5v_2)] = 1$. We propose to show that

(18) v_4v_1 is saturated by $\boldsymbol{y}$ in F_4.

Suppose the contrary. If v_4v_1 is not saturated by $\boldsymbol{y}$ in T, we set $\theta = \min\{w(v_4v_1) - z(v_4v_1), [y(v_2v_3v_4v_5v_2)]\}$, and let $\boldsymbol{y'}$ arise from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_4v_1)$ and $y(v_2v_3v_4v_5v_2)$ with $y(v_1v_2v_3v_4v_1) + \theta$ and $y(v_2v_3v_4v_5v_2) - \theta$, respectively. Since v_1v_2 is outside $\mathcal{C}_0^y$, $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (4). If v_4v_1 is saturated by $\boldsymbol{y}$ in T but contained in a cycle $C \in \mathcal{C}_0^y$, let $C' = C[v_5, v_4] \cup \{v_4v_5\}$ and $\sigma = \min\{y(C), [y(v_2v_3v_4v_5v_2)]\}$, and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_4v_1)$, $y(v_2v_3v_4v_5v_2)$, $y(C)$, and $y(C')$ with $y(v_1v_2v_3v_4v_1) + \sigma$, $y(v_2v_3v_4v_5v_2) - \sigma$, $y(C) - \sigma$, and $y(C') + \sigma$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (4) again. So (18) is established.

By (18), we have $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

(b) We next assume that $y(v_2v_3v_4v_5v_2) = 0$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_1)+y(v_1v_2v_3v_4v_1) = w(v_2v_3)-w(v_4v_2)$. Thus $w(v_iv_2) = \lceil z(v_iv_2)\rceil = z(v_iv_2)$ for $i = 1,4,5$. By Lemma 3.4(i), $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) + y(v_2v_3v_4v_2) = w(v_2v_3)$ and $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) = w(v_5v_3)$. If $y(v_1v_2v_3v_1)$ is integral, then $w(v_iv_2) = \lceil z(v_iv_2)\rceil = z(v_iv_2)$ for $i = 1,4,5$. Hence, by Lemma 3.4(i), $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_2v_3v_1)$ is not integral. We propose to show that

(19) $\nu^*_w(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T,\boldsymbol{w})$. Since $0 < y(v_1v_2v_3v_1) < w(v_1v_2)$ and $0 < y(v_2v_3v_4v_2) < w(v_4v_2)$, by Lemma 3.1(i) and (ii), we have $x(v_1v_2) = x(v_4v_2) = 0$ and $x(v_3v_1) = x(v_3v_4)$.

Let us show that $x(v_1v_5) = x(v_4v_5)$. If both $y(v_1v_5v_3v_1)$ and $y(v_3v_4v_5v_3)$ are positive, then, by Lemma 3.1(i), we have $x(v_1v_5v_3v_1) = x(v_3v_4v_5v_3) = 1$, which implies $x(v_1v_5) = x(v_4v_5)$, as desired. If one of $y(v_1v_5v_3v_1)$ and $y(v_3v_4v_5v_3)$ is zero, then the other equals $w(v_5v_3)$. By Lemma 3.2(iii), we may assume that $w(v_5v_3) = 0$. Since v_2v_3 is saturated by $\boldsymbol{y}$ in F_4, both v_1v_2 and v_4v_2 are outside $\mathcal{C}^y_0$. If v_3v_4 is also outside $\mathcal{C}^y_0$, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_5v_3)$ and $y(v_1v_5v_3v_1)$ with $y(v_3v_4v_5v_3) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1)\rfloor$, respectively, then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$. Since $y'(v_3v_4v_5v_3)$ is a positive integer, $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution by Lemma 3.2(iii). So we may assume that v_3v_4 is contained in some cycle in $\mathcal{C}^y_0$; the same holds for v_3v_1. Let C_1 and C_2 be two cycles in $\mathcal{C}^y_0$ passing through v_3v_1 and v_3v_4, respectively. By Lemma 3.1(iii), we have $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$. Thus $x(v_1v_5) = x(v_4v_5)$ also holds.

Similarly, we can prove that $x(uv_1) = x(uv_4)$ for each vertex $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Let $T' = (V', A')$ be the digraph obtained from T by identifying v_1 and v_4; the resulting vertex is still denoted by v_1. Let $\boldsymbol{w'}$ be the restriction of $\boldsymbol{w}$ to A'. Then $\boldsymbol{x}$ corresponds to a feasible solution $\boldsymbol{x'}$ to $\mathbb{P}(T',\boldsymbol{w'})$ with $x'(v_1v_5) = x(v_4v_1) + x(v_1v_5) = x(v_4v_5)$ by Lemma 3.1(iii), and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y'}$ to $\mathbb{D}(T',\boldsymbol{w'})$; both having the same objective value $\nu^*_w(T)$ as $\mathbb{P}(T,\boldsymbol{w})$ and $\mathbb{D}(T,\boldsymbol{w})$. By the LP-duality theorem, $\boldsymbol{x'}$ and $\boldsymbol{y'}$ are optimal solutions to $\mathbb{P}(T',\boldsymbol{w'})$ and $\mathbb{D}(T',\boldsymbol{w'})$, respectively. By Lemma 3.3, $\mathbb{D}(T',\boldsymbol{w'})$ has an integral optimal solution. So $\nu^*_w(T)$ is an integer. This proves (19).

Case 2.9. $K = \{v_3v_1, v_3v_4\}$.

In this case, by Lemma 3.1(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following two equations:

$w(v_3v_1) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$; and

$w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_3v_4v_1)$.

Since each $e \in K$ is saturated by $\boldsymbol{y}$ in F_4, we have $w(uv_i) = z(uv_i) = 0$ for $i = 2,3$ and all $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, from (9) and (15) we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$ and that v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4. So $w(v_1v_2) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$ and $w(v_5v_3) = y(v_1v_5v_3v_1)$. If $y(v_1v_2v_3v_4v_1) > 0$, then both v_4v_2 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). Thus $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. If

$y(v_2v_3v_4v_5v_2) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12), which implies that $y(v_2v_3v_4v_2) = w(v_4v_2)$; if $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) = w(v_3v_4)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$, regardless of the value of $y(v_2v_3v_4v_5v_2)$.

- $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, from (10) and (15) we deduce that $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and that v_4v_5 is saturated by $\boldsymbol{y}$ in F_4. So $w(v_3v_1) = y(v_1v_5v_3v_1)$ and $w(v_4v_5) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$. If $y(v_2v_3v_4v_5v_2) > 0$, then v_1v_2, v_4v_2, and v_5v_3 are all saturated by $\boldsymbol{y}$ in F_4 by (10) and (12). So $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, and $y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3) - y(v_1v_5v_3v_1)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$ and hence $y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1) = w(v_3v_4) - w(v_4v_5) - w(v_4v_2)$; if $y(v_1v_2v_3v_4v_1) = 0$, then $y(v_2v_3v_4v_2) + y(v_1v_5v_3v_4v_1) = w(v_3v_4) - w(v_4v_5)$. Since all arcs in $F_4 \backslash v_6$ except $\{v_1v_5, v_4v_1, v_4v_5\}$ are outside $\mathcal{C}_0^y$ and $y(v_1v_5v_3v_4v_1) > 0$, by (ii) we have $y(v_1v_2v_3v_4v_1) = \min\{w(v_1v_2), w(v_2v_3) - w(v_4v_2)\}$ if $y(v_1v_2v_3v_4v_1) > 0$ and $y(v_2v_3v_4v_2) = \min\{w(v_4v_2), w(v_2v_3)\}$ otherwise. So $y(v_1v_5v_3v_4v_1)$ is integral, and hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, regardless of the value of $y(v_1v_2v_3v_4v_1)$.
- $y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$. In this subcase, depending on whether $y(v_2v_3v_4v_5v_2) > 0$, we distinguish between two subsubcases.

(a) We first assume that $y(v_2v_3v_4v_5v_2) > 0$. By (12), both v_4v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4, which implies $w(v_4v_2) = y(v_2v_3v_4v_2)$ and $w(v_5v_3) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_1v_2) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$. Moreover, if $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_5 is saturated by $\boldsymbol{y}$ in F_4 by (11), which yields one more equation $w(v_4v_5) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_3v_4v_1) = 0$. So we assume that $y(v_1v_5v_3v_1) = 0$. Then $y(v_1v_2v_3v_1) = w(v_3v_1)$, $y(v_3v_4v_5v_3) = w(v_5v_3)$ and $y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) = w(v_3v_4) - w(v_4v_2) - w(v_5v_3)$. If $y(v_1v_2v_3v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1)$ is integral. Similar to (18), we can prove that v_4v_1 is saturated by $\boldsymbol{y}$ in F_4. Then $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$, a contradiction.

(b) We next assume that $y(v_2v_3v_4v_5v_2) = 0$. Suppose $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) = w(v_3v_1)$ and $y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) = w(v_3v_4)$. If neither $y(v_1v_5v_3v_1)$ nor $y(v_3v_4v_5v_3)$ is integral, then neither $y(v_1v_2v_3v_1)$ nor $y(v_2v_3v_4v_2)$ is integral. Similar to (19), we can show that $\nu_w^*(T)$ is an integer. So we may assume that $y(v_1v_5v_3v_1)$ or $y(v_3v_4v_5v_3)$ is integral. Observe that both of them are integral, for otherwise, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively. Since v_1v_2, v_2v_3, and v_4v_2 are all outside $\mathcal{C}_0^y$, $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, with $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, contradicting (5).

Suppose $y(v_1v_2v_3v_4v_1) > 0$. Then $y(v_2v_3v_4v_2) = w(v_4v_2)$. If $y(v_1v_5v_3v_1) > 0$, then v_4v_5 is saturated by $\boldsymbol{y}$ in F_4 by (11), which implies $y(v_3v_4v_5v_3) = w(v_4v_5)$, $y(v_1v_2v_3v_4v_1) = w(v_3v_4) - w(v_4v_2) - w(v_4v_5)$, and $y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_1v_5v_3v_1)$ is not integral, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively. Since both v_1v_2 and v_2v_3 are outside $\mathcal{C}_0$, $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, with $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, contradicting (5). So $y(v_1v_5v_3v_1)$ is integral and hence is zero by Lemma 3.2(iii). It follows that $y(v_1v_2v_3v_1) = w(v_3v_1)$ and $y(v_1v_2v_3v_4v_1) + y(v_3v_4v_5v_3) = w(v_3v_4) - w(v_4v_2)$. If $y(v_3v_4v_5v_3)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_3v_4v_5v_3)$ is not integral. Let us show that

(20) $\nu_w^*(T)$ is an integer.

By Lemma 3.2(iii), we may assume that $w(v_3v_1) = w(v_4v_2) = 0$. Recall that $w(v_5v_2) = z(v_5v_2) = 0$ and $w(uv_i) = z(uv_i) = 0$ for $i = 2, 3$ and all $u \in V(T_1)\backslash\{b, a_1\}$. So we may assume that $x(uv_2) = x(uv_3)$. Let $T' = (V', A')$ be the digraph obtained from T by identifying v_2 and v_3; the resulting vertex is still denoted by v_3, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A'. Then $\boldsymbol{x}$ corresponds to a feasible solution $\boldsymbol{x}'$ to $\mathbb{P}(T', \boldsymbol{w}')$, and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y}'$ to $\mathbb{D}(T', \boldsymbol{w}')$; both having the same objective value $\nu_w^*(T)$ as $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$. By the LP-duality theorem, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively. By Lemma 3.3, $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution. So $\nu_w^*(T)$ is an integer. This proves (20) and hence Claim 2.

Since $\tau_{\boldsymbol{w}}(F_4\backslash v_6) > 0$, from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 4.4. ∎

Lemma 4.6. *If $T_2 = G_2$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_4v_1, v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_2v_4v_1, v_1v_6v_3v_4v_1, v_1v_6v_3v_2v_4v_1\}$ and
- $\mathcal{F}_2 = \{\{v_1v_6, v_1v_2\}, \{v_1v_6, v_2v_4\}, \{v_1v_6, v_4v_1\}, \{v_3v_1, v_4v_1\}, \{v_4v_1, v_6v_3\}, \{v_2v_4, v_6v_3, v_6v_4\}, \{v_2v_4, v_3v_1, v_3v_4, v_6v_4\}, \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}, \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 6$ and $|\mathcal{F}_2| = 9$. Recall that $(b_2, a_2) = (v_4, v_5)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(4) subject to (1)-(3), $y(v_1v_6v_4v_1)$ is minimized;

Let us make some simple observations about $\boldsymbol{y}$.

(5) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The three statements below follow instantly from Lemma 3.5(v).

(6) If $y(v_1v_6v_3v_2v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$ is saturated by $\boldsymbol{y}$ in G_2.

(7) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2.

(8) If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2.

Claim 1. $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$.

To justify this, observe that if both v_1v_2 and v_1v_6 are saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_1v_6\}$; if both v_3v_1 and v_4v_1 are saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$ in either case. So we assume that

(9) at most one of v_1v_2 and v_1v_6 is saturated by $\boldsymbol{y}$ in G_2. The same holds for v_3v_1 and v_4v_1.

As v_2v_4 is a special arc of T and v_2 is a near-sink, by Lemma 3.4(iv), we may assume that v_2v_4 is saturated by $\boldsymbol{y}$ in T. Depending on whether v_2v_4 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_4 is contained by some cycle in $\mathcal{C}_0^y$.

In this case, we proceed by considering two subcases.

• v_3v_1 is saturated by $\boldsymbol{y}$ in G_2. In this subcase, by (9), v_4v_1 is not saturated by $\boldsymbol{y}$ in G_2 and hence in T, because v_4v_1 is outside $\mathcal{C}_0$. By the hypothesis of the present case and Lemma 3.5(iii), v_1v_2 is saturated by $\boldsymbol{y}$ in T. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, a cycle $C \in \mathcal{C}_0^y$ containing v_1v_2 must pass through v_2v_4. Thus, by Lemma 3.5(iv), v_4v_1 is saturated by $\boldsymbol{y}$ in G_2, a contradiction. It follows that v_1v_2 is saturated by $\boldsymbol{y}$ in G_2. So, by (9), v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2. If v_1v_6 is contained in some cycle $C \in \mathcal{C}_0^y$, applying Lemma 3.5(iv) to the cycle $C[v_1, v_4] \cup \{v_4v_1\}$ in $\mathcal{C}_2$, we see that v_4v_1 is saturated by $\boldsymbol{y}$ in T, a contradiction. So v_1v_6 is outside $C \in \mathcal{C}_0^y$. By Lemma 3.5(iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_2 and v_6v_4 is outside $\mathcal{C}_0^y$. Using Lemma 3.5(i), we further deduce that v_6v_4 is saturated by $\boldsymbol{y}$ in G_2. If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$. By (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$. If v_6v_3 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, applying Lemma 3.5(iii) to the cycle $C[v_6, v_4] \cup \{v_4v_1, v_1v_6\} \in \mathcal{C}_2$, we see that v_4v_1 or v_1v_6 is saturated, a contradiction. If v_6v_3 is not saturated by $\boldsymbol{y}$ in T then, by Lemma 3.5(iii), v_3v_2 is saturated by $\boldsymbol{y}$ in G_2 and and v_3v_4 is outside $\mathcal{C}_0^y$. Using Lemma 3.5(i), we further deduce that v_3v_4 is saturated by $\boldsymbol{y}$ in G_2. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}$. By (5), J is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$.

• v_3v_1 is not saturated by $\boldsymbol{y}$ in G_2. In this subcase, we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ by (6) and (7). Assume first that v_1v_2 is saturated by $\boldsymbol{y}$ in G_2. Then v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2 by (9). Thus v_6v_3 is saturated by $\boldsymbol{y}$ in G_2 by Lemma 3.5(iii) and (iv). If v_4v_1 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$; otherwise, both v_6v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2 by Lemma 3.5(iii) and (iv). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$. By (5), K is an MFAS in either subsubcase, and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$.

Assume next that v_1v_2 is not saturated by $\boldsymbol{y}$ in G_2. By (8), we have $y(v_1v_6v_2v_4v_1) = 0$. By the hypothesis of the present case and by Lemma 3.5(iii) and (iv), v_4v_1 is saturated by $\boldsymbol{y}$ in G_2. If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. By (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in G_2. Thus v_1v_6 is saturated by $\boldsymbol{y}$ in G_2 by Lemma 3.5(iii) and (iv). We propose to show that

(10) $y(v_1v_6v_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_4v_1) > 0$. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2. Then the multiset sum of $v_1v_6v_4v_1$ and C contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$. Set $\theta = \min\{y(v_1v_6v_4v_1), y(C)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$, $y(v_1v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (4) on $\boldsymbol{y}$. It follows that v_3v_1 is also outside $\mathcal{C}_0^y$, because every cycle containing v_3v_1 in $\mathcal{C}_0^y$ must pass through v_1v_2. So neither v_1v_2 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T.

Let us show that v_6v_3 is outside $\mathcal{C}_0^y$, for otherwise, let $C \in \mathcal{C}_0^y$ contain v_6v_3. Then the multiset sum of $v_1v_6v_4v_1$, C, and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6v_4\}$. Set $\theta = \min\{y(v_1v_6v_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (4) on $\boldsymbol{y}$. Let $D \in \mathcal{C}_0^y$ be a cycle containing v_2v_4. Then the multiset sum of D, $v_1v_6v_4v_1$, and the unsaturated arcs v_6v_3, v_3v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_1v_6v_3v_1$. Thus, by Lemma 3.5(vi), we obtain $y(v_1v_6v_4v_1) = 0$; this contradiction proves (10).

From (10), we deduce that $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1\}$. So, by (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$.

Case 1.2. v_2v_4 is outside $\mathcal{C}_0^y$.

In this case, v_2v_4 is saturated by $\boldsymbol{y}$ in G_2. So v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$. Assume first that v_1v_6 is saturated by $\boldsymbol{y}$ in G_2. Then v_1v_2 is not saturated by $\boldsymbol{y}$ by (9). By (6) and (8), we have $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$ and hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_2v_4\}$. It follows from (5) that K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$. Assume next that v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2. If v_4v_1 is not saturated by $\boldsymbol{y}$ in T, then v_6v_4 is outside $\mathcal{C}_0^y$ by Lemma 3.5(iii). So v_3v_4 is contained in some cycle in $\mathcal{C}_0^y$ because $\mathcal{C}_0^y \neq \emptyset$. Using Lemma 3.5(iii), we deduce that both v_6v_3 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. Using (6), we obtain $y(v_1v_6v_3v_2v_4v_1) = 0$. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_4, v_6v_3, v_6v_4\}$. If v_4v_1 is saturated by $\boldsymbol{y}$ in T, then so is it in G_2 because v_4v_1 is outside $\mathcal{C}_0^y$. By (9), v_3v_1 is not saturated by $\boldsymbol{y}$ in G_2. By Lemma 3.5(iii), v_6v_3 is saturated by $\boldsymbol{y}$ in G_2. By (6) and (7), we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. Hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. In either subsubcase, K is an MFAS by (5) and thus $y(\mathcal{C}_2) = \tau_w(G_2 \backslash v_5)$. This proves Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we may assume that

(11) $y(v_1v_6v_3v_2v_4v_1) = 0$.

Otherwise, by (6), we have $w(e) = y(\mathcal{C}_2(e))$ for each e in the set $\{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer, so is $y(v_1v_6v_3v_2v_4v_1)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{F}_2$. Depending on what K is, we distinguish among nine cases.

Case 2.1. $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$. It follows instantly that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.2. $K = \{v_1v_6, v_4v_1\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_4v_1) = y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. By Lemma 3.1 (iii), we further obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$. It follows that $y(v_1v_2v_4v_1) = w(v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.3. $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, $y(v_1v_6v_4v_1) = w(v_6v_4)$, and $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$. Note that if $y(v_1v_6v_3v_4v_1) > 0$, we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (7). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_3v_4v_1) = 0$.

Case 2.4. $K = \{v_2v_4, v_6v_3, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Note that if $y(v_1v_6v_2v_4v_1) > 0$, we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$ by (8); if $y(v_1v_6v_3v_4v_1) > 0$, we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (7). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subcase.

Case 2.5. $K = \{v_2v_4, v_3v_1, v_3v_4, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Note that if $y(v_1v_6v_2v_4v_1) > 0$, we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$ by (8). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$.

Case 2.6. $K = \{v_1v_6, v_2v_4\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) = w(v_2v_4)$ and $y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. Moreover, in this case v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$, and $w(uv_2) = z(uv_2) = 0$ for any $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Examining the cycles in $\mathcal{C}_2$, we see that $z(v_3v_2) = z(v_6v_2) = 0$ and so $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 3, 6$. Thus $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.4(i).

Case 2.7. $K = \{v_4v_1, v_6v_3\}$.

In this case, by Lemma 3.1 (i) and (iii), we have $y(v_1v_6v_3v_4v_1) = 0$, $y(v_1v_6v_3v_1) = w(v_6v_3)$, and $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_4v_1)$. Lemma 3.2(iii) allows us to assume that $w(v_6v_3) = 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2 by (8). So $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$; the same holds if $y(v_1v_6v_2v_4v_1) = 0$ and $y(v_1v_2v_4v_1)$ is integral. So we assume that $y(v_1v_6v_2v_4v_1) = 0$ and $y(v_1v_2v_4v_1)$ is not integral. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2, let $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$, and set $\theta = \min\{y(C), y(v_1v_6v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$, $y(v_1v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (4) on $\boldsymbol{y}$. Similarly, we can prove that v_6v_2 is outside $\mathcal{C}_0^y$. Examining cycles in $\mathcal{C}_2$, we see that $w(v_6v_2) = z(v_6v_2) = 0$. Now we propose to show that

(12) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_2v_4v_1)$ and $y(v_1v_6v_4v_1)$ are positive, we have $x(v_1v_2) + x(v_2v_4) = x(v_1v_6) + x(v_6v_4)$ by Lemma 3.1(i). Since $y(v_1v_2v_4v_1) < w(v_1v_2)$, we have $x(v_1v_2) = 0$ by Lemma 3.1(ii). So $x(v_2v_4) = x(v_1v_6) + x(v_6v_4)$. If each of v_3v_1 and v_3v_2 is contained in some cycle in $\mathcal{C}_0^y$, then $x(v_3v_1) = x(v_3v_2)$ by Lemma 3.1(iv). If one of v_3v_1 and v_3v_2 is outside $\mathcal{C}_0^y$, say v_3v_1, then we may assume that $w(v_3v_1) = 0$ and $x(v_3v_1) = x(v_3v_2)$. Similarly, we can prove that $x(uv_1) = x(uv_2)$ for each $u \in V(T_1)\backslash\{a_1, b\}$.

Let $T' = (V', A')$ be obtained from T by deleting vertex v_2, let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by defining $w'(uv_1) = w(uv_1) + w(uv_2)$ for $u = v_3$ or $u \in V(T_1)\backslash\{b, a_1\}$ and $w'(v_iv_j) = w(v_iv_j) + w(v_2v_4)$ for $(i, j) = (1, 6)$ or $(6, 4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through the path uv_2v_4 with $u \in (V(T_1)\backslash\{a_1, b\}) \cup \{v_3\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_1v_6v_4$, and set $y'(C') = y(C) + y(C')$ and $y'(v_1v_6v_4v_1) = y(v_1v_6v_4v_1) + y(v_1v_2v_4v_1)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$ respectively, both having the same value $\nu^*_w(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu^*_w(T)$ is an integer by the hypothesis of Theorem 1.5.

Case 2.8. $K = \{v_1v_6, v_1v_2\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. Moreover, v_3v_1 is outside $\mathcal{C}^y_0$. Depending on whether $y(v_1v_6v_3v_4v_1) = 0$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, we first assume that $y(v_1v_6v_2v_4v_1) > 0$. Then $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (8). Thus $y(v_1v_6v_3v_1) + y(v_1v_6v_2v_4v_2) = w(v_1v_6) - w(v_6v_4)$. Let us show that $y(v_1v_6v_3v_1)$ is integral. Suppose not. If v_6v_3 is outside $\mathcal{C}^y_0$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6v_2v_4v_1)$ with $y(v_1v_6v_3v_1) + [y(v_1v_6v_2v_4v_1)]$ and $\lfloor y(v_1v_6v_2v_4v_1)\rfloor$, respectively; if v_6v_3 is contained in some cycle C in $\mathcal{C}^y_0$, set $\theta = \min\{y(C), [y(v_1v_6v_2v_4v_1)]\}$ and $C' = C[v_4, v_6] \cup \{v_6v_2, v_2v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1)+\theta$, $y(v_1v_6v_2v_4v_1)-\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. In both subsubcases, $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_2v_4v_1) < y(v_1v_6v_2v_4v_1)$, contradicting (2). We next assume that $y(v_1v_6v_2v_4v_1) = 0$. The proof of this subsubcase is similar to that in the preceding one (with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6v_2v_4v_1)$). Thus we reach a contradiction to (4).

• $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, by (7), both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_4v_1) = w(v_6v_4)$, and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6) - w(v_3v_1) - w(v_6v_4)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_6v_2v_4v_1)$ is not integral. Then $[y(v_1v_6v_2v_4v_1)]+[y(v_1v_6v_3v_4v_1)] = 1$. Observe that v_6v_2 is outside $\mathcal{C}^y_0$, for otherwise, let C be a cycle in $\mathcal{C}^y_0$ containing v_6v_2, let $C' = C[v_4, v_6]\cup\{v_6v_3, v_3v_4\}$, let $\theta = \min\{y(C), [y(v_1v_6v_3v_4v_1]\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, $y(C')$ with $y(v_1v_6v_3v_4v_1) - \theta$, $y(v_1v_6v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (2). Similarly, we can show that v_3v_2 is also outside $\mathcal{C}^y_0$. Thus $w(v_3v_2) = z(v_3v_2) = 0$. By Lemma 3.2(iii), we may assume that $w(v_1v_2)$, $w(v_3v_1)$, and $w(v_6v_4)$ are all 0. We propose to show that

(13) $\nu^*_w(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $y(v_1v_6v_2v_4v_1) > 0$ and $y(v_1v_6v_3v_4v_1) > 0$, from Lemma 3.1(i) we deduce that $x(v_6v_2) + x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_6v_2v_4v_1) < w(v_6v_2)$, we have $x(v_6v_2) = 0$ by Lemma 3.1(ii). It follows that $x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $w(v_6v_4) = 0$ and v_6v_2 is outside $\mathcal{C}^y_0$, $x(uv_6) = x(uv_2)$ for each $u \in V(T_1)\backslash\{b, a_1\}$. Let $T' = (V', A')$ be the tournament obtained from T by deleting vertex v_2, let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_6)$ with $w(uv_6)+w(uv_2)$ for each $u \in V(T_1)\backslash\{b, a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j) + w(v_2v_4)$ for $(i, j) = (6, 3)$ or $(3, 4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u \in V(T_1)\backslash\{b, a_1\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_6v_3v_4$, and set $y'(C') = y(C')+y(C)$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1)+y(v_1v_6v_2v_4v_1)$. From the LP-duality theorem, we deduce that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, both having the same value $\nu^*_w(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu^*_w(T)$ is an integer by the hypothesis of Theorem 1.5.

Case 2.9. $K = \{v_3v_1, v_4v_1\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$. Assume first that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_1v_6v_3v_4v_1) > 0$, then v_6v_4 is saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_6v_4v_1) = w(v_6v_4)$ and hence $y(v_1v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_6v_4)$; if $y(v_1v_6v_3v_4v_1) = 0$, then $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. If $y(v_1v_2v_4v_1)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_4v_1)$ is not integral. Then we can prove that both v_6v_2 and v_1v_2 are outside $\mathcal{C}_0^y$ and that $\nu_w^*(T)$ is an integer. The proof is the same as that of (12) (with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6v_3v_4v_1)$ when $y(v_1v_6v_3v_4v_1) > 0$), so we omit the details here .

Assume next that $y(v_1v_6v_2v_4v_1) > 0$. Then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_4v_1) = w(v_6v_4)$, and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_1v_2) - w(v_6v_4)$. If $y(v_1v_6v_3v_4v_1)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_6v_3v_4v_1)$ is not integral. Then we can prove that both v_6v_2 and v_3v_2 are outside $\mathcal{C}_0^y$ and that $\nu_w^*(T)$ is an integer. The proof is the same as that of (13), so we omit the details here. Thus Claim 2 is established.

Since $\tau_{\boldsymbol{w}}(F_4 \backslash v_6) > 0$, from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 4.5. ∎

Lemma 4.7. *If $T_2 = G_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_4v_1, v_1v_6v_3v_1, v_2v_4v_6v_2, v_3v_4v_6v_3, v_1v_6v_2v_4v_1, v_1v_6v_3v_4v_1, v_2v_4v_6v_3v_2, v_1v_6v_3v_2v_4v_1, v_1v_2v_4v_6v_3v_1\}$ and
- $\mathcal{F}_2 = \{\{v_2v_4, v_6v_3\}, \{v_1v_2, v_1v_6, v_4v_6\}, \{v_1v_2, v_6v_2, v_6v_3\}, \{v_1v_6, v_2v_4, v_3v_4\}, \{v_1v_6, v_2v_4, v_4v_6\}, \{v_1v_6, v_4v_1, v_4v_6\}, \{v_2v_4, v_3v_1, v_3v_4\}, \{v_3v_1, v_4v_1, v_4v_6\}, \{v_4v_1, v_4v_6, v_6v_3\}, \{v_4v_1, v_6v_2, v_6v_3\}, \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}, \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}, \{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 9$ and $|\mathcal{F}_2| = 13$. Recall that $(b_2, a_2) = (v_4, v_5)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(4) subject to (1)-(3), $y(v_1v_2v_4v_1) + y(v_3v_4v_6v_3)$ is minimized;

Let us make some simple observations about $\boldsymbol{y}$.

(5) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

(6) If $y(v_1v_2v_4v_6v_3v_1) > 0$, then each arc in the set $\{v_1v_6, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. Furthermore, $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_2v_4v_6v_3v_1$. So the first half follows instantly from Lemma 3.5(v). Once again let $\uplus$ stand for the multiset sum. Then $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_2v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_2v_4v_6v_2$, $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_3v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3v_4v_6v_3$, and $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_3v_2v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_2v_4v_6v_3v_2$. It follows from the optimality of $\boldsymbol{y}$ that $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

(7) If $y(v_1v_6v_3v_2v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_4v_6, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. Furthermore, $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_2v_4v_6v_3v_1$. So the first half follows instantly from Lemma 3.5(v). Observe that $v_1v_6v_3v_2v_4v_1 \uplus v_3v_4v_6v_3 = v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_3v_2$ and $v_1v_6v_3v_2v_4v_1 \uplus v_2v_4v_6v_2 = v_1v_6v_2v_4v_1 \uplus v_2v_4v_6v_3v_2$. Since $\boldsymbol{y}$ satisfies (2), it is clear that $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$.

(8) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3; so is v_1v_2 if $y(v_2v_4v_6v_3v_2) > 0$. Furthermore, $y(v_2v_4v_6v_2) = 0$.

To justify this, note that both v_3v_1 and v_4v_6 are chords of the cycle $v_1v_2v_4v_6v_3v_1$, so they are saturated by $\boldsymbol{y}$ in G_3 by Lemma 3.5(v). Suppose $y(v_2v_4v_6v_3v_2) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then $v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_3v_2 \uplus \{v_1v_2\} = v_1v_2v_4v_1 \uplus v_3v_4v_6v_3$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of C, $v_1v_6v_3v_4v_1$, and $v_2v_4v_6v_3v_2$ contains arc-disjoint cycles $v_1v_2v_4v_1$, $v_3v_4v_6v_3$, and $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_2, v_2v_4\}$. Thus we can obtain an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that contradicts the assumption (3) on $\boldsymbol{y}$. Moreover, since $v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_2 = v_3v_4v_6v_3 \uplus v_1v_6v_2v_4v_1$, it follows from (3) that $y(v_2v_4v_6v_2) = 0$.

(9) If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3; so is v_3v_1 if $y(v_3v_4v_6v_3) > 0$ or $y(v_2v_4v_6v_3v_2) > 0$.

The first half follows instantly from Lemma 3.5(v). To prove the second half, assume the contrary. If v_3v_1 is not saturated by $\boldsymbol{y}$ in T, then $v_3v_4v_6v_3 \uplus v_1v_6v_2v_4v_1 \uplus \{v_3v_1\} = v_2v_4v_6v_2 \uplus v_1v_6v_3v_1$, and $v_2v_4v_6v_3v_2 \uplus v_1v_6v_2v_4v_1 \uplus \{v_3v_1\} = v_2v_4v_6v_2 \uplus v_1v_6v_3v_1$; if v_3v_1 is saturated by $\boldsymbol{y}$ in T but contained in some cycle C in $\mathcal{C}_0^y$, then the multiset sum of C, $v_1v_6v_2v_4v_1$, and $v_3v_4v_6v_3$ (resp. $v_2v_4v_6v_3v_2$) contains arc-disjoint cycles $v_2v_4v_6v_2$, $v_1v_6v_3v_1$, and $C' = C[v_4, v_3] \cup \{v_3v_4\}$ (resp. $C' = C[v_4, v_3] \cup \{v_3v_2, v_2v_4\}$). Since $\boldsymbol{y}$ satisfies (2), we have $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$, a contradiction.

(10) If $y(v_2v_4v_6v_3v_2) > 0$, then both v_3v_4 and v_6v_2 are saturated by $\boldsymbol{y}$ in G_3 by Lemma 3.5(v).

(11) If v_1v_6 is contained in a cycle in $\mathcal{C}_0^y$, then both v_4v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3.

Since both $C[v_1, v_4] \cup \{v_4v_1\}$ and $C[v_6, v_4] \cup \{v_4v_6\}$ are cycles in $\mathcal{C}_2$, the statement follows instantly from Lemma 3.5(iv).

(12) If v_6v_3 is contained in a cycle in $\mathcal{C}_0^y$, then v_4v_6 is saturated by $\boldsymbol{y}$ in G_3; so is v_1v_6 or v_4v_1.

The first half follows instantly from Lemma 3.5(iv). To prove the second half, we may assume, by (11), that v_1v_6 is outside $\mathcal{C}_0^y$. Let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_3. Then both $C[v_6, v_4] \cup \{v_4v_6\}$ and $C[v_6, v_4] \cup \{v_4v_1, v_1v_6\}$ are cycles in $\mathcal{C}_2$. Thus, by Lemma 3.5(iv), v_4v_6 and at least one of v_1v_6 and v_4v_1 are saturated by $\boldsymbol{y}$ in G_3.

Claim 1. $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

To justify this, observe that v_2v_4 is a special arc of T and v_2 is a near-sink. By Lemma 3.4(iv), we may assume that v_2v_4 is saturated by $\boldsymbol{y}$ in T. Let $\mathcal{G}_2 = \{\{v_1v_2, v_1v_6, v_4v_6\}, \{v_1v_2, v_6v_2, v_6v_3\}, \{v_2v_4, v_3v_1, v_3v_4\}, \{v_3v_1, v_4v_1, v_4v_6\}\}$. Then $\mathcal{G}_2 \subset \mathcal{F}_2$. Observe that

(13) if $y(v_1v_2v_4v_6v_3v_1) = 0$, then for each $K \in \mathcal{G}_2$, not all arcs in K are saturated by $\boldsymbol{y}$ in G_3.

Suppose the contrary: all arcs in K are saturated by $\boldsymbol{y}$ in G_3. Examining cycles in $\mathcal{C}_2$, we see that $y(\mathcal{C}_2) = w(K)$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$. So we may assume that (13) holds.

Depending on whether v_2v_4 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_4 is contained in some cycle in $\mathcal{C}_0^y$.

We proceed by considering four subcases.

• Neither v_4v_1 nor v_4v_6 is saturated by $\boldsymbol{y}$ in G_3. In this subcase, by Lemma 3.5(iii) and (iv), both v_1v_2 and v_6v_2 are saturated by $\boldsymbol{y}$ in G_3. By (6)-(9), $y(v_1v_2v_4v_6v_3v_1)$, $y(v_1v_6v_3v_2v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6v_2v_4v_1)$ are all zero. By (12) and (13), v_6v_3 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. By Lemma 3.5(iii), both v_3v_2 and v_3v_4 are saturated by $\boldsymbol{y}$ in G_3. By Lemma 3.5(i) and (iii), at least one of v_1v_6 and v_3v_1 is saturated by $\boldsymbol{y}$ in G_3. Thus $y(\mathcal{C}_2) = w(K)$, where K is $\{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$ or $\{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3\backslash v_5)$.

• v_4v_6 is saturated by $\boldsymbol{y}$ in G_3 while v_4v_1 is not. In this subcase, by Lemma 3.5(iii), v_1v_2 is saturated by $\boldsymbol{y}$ in G_3. By (6), we have $y(v_1v_2v_4v_6v_3v_1) = 0$. By (11) and (13), v_1v_6 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. By Lemma 3.5(i) and (iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. So, by (12) and (13), v_6v_3 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. It follows from Lemma 3.5(i) and (iii) that v_3v_1, v_3v_2, and v_3v_4 are all saturated by $\boldsymbol{y}$ in G_3. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3\backslash v_5)$.

• v_4v_1 is saturated by $\boldsymbol{y}$ in G_3 while v_4v_6 is not. In this subcase, by Lemma 3.5(iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. By (7)-(9), $y(v_1v_6v_3v_2v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6v_2v_4v_1)$ are all zero. By (12), v_6v_3 is outside $\mathcal{C}_0^y$. Furthermore, we may assume that v_6v_3 is not saturated by $\boldsymbol{y}$, for otherwise $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_2, v_6v_3\}$. Then, by Lemma 3.5(iii) and (iv), both v_3v_2 and v_3v_4 are saturated by $\boldsymbol{y}$ in G_3. If v_3v_1 is also saturated by $\boldsymbol{y}$ in G_3, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$; otherwise, by Lemma 3.5(i) and (iii), both v_1v_2 and v_1v_6 are saturated by $\boldsymbol{y}$ in G_3. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$.

• Both v_4v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3. In this subcase, if $y(v_1v_2v_4v_6v_3v_1) > 0$, then v_1v_6 is saturated by $\boldsymbol{y}$ in G_3 and $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ by (6). Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. So we assume that $y(v_1v_2v_4v_6v_3v_1) = 0$. Then v_3v_1 is not saturated by $\boldsymbol{y}$ in G_3 by (13). Thus $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$ by (7) and (8). If $y(v_1v_6v_2v_4v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in G_3 and $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$ by (9). By (13), v_1v_6 is not saturated by $\boldsymbol{y}$ in G_3. Hence, by Lemma 3.5(iii), v_6v_3 is saturated by $\boldsymbol{y}$ in G_3. Therefore, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. So we may assume that $y(v_1v_6v_2v_4v_1) = 0$ and that v_1v_6 is not saturated by $\boldsymbol{y}$ in G_3, for otherwise $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. Thus, by Lemma 3.5(iii) and (iv), v_6v_3 is saturated by $\boldsymbol{y}$ in G_3. We may further assume that v_6v_2 is not saturated by $\boldsymbol{y}$ in G_3, for otherwise, $y(\mathcal{C}_2) = w(J)$, where $J = \{v_4v_1, v_6v_2, v_6v_3\}$. Then $y(v_2v_4v_6v_3v_2) = 0$ by (10). We propose to show that

(14) $y(v_3v_4v_6v_3) = 0$.

Assume the contrary: $y(v_3v_4v_6v_3) > 0$. Since neither v_1v_6 nor v_3v_1 is saturated by $\boldsymbol{y}$ in G_3, we distinguish among four subsubcases.

(a) Neither v_1v_6 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T. In this subsubcase, set $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3v_4v_6v_3)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$ and $y(v_1v_6v_3v_1)$ with $y(v_3v_4v_6v_3) - \theta$ and $y(v_1v_6v_3v_1) + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4).

(b) v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_1v_6 is contained in some cycle $C_1 \in \mathcal{C}_0^y$. In this subsubcase, since v_6v_3 is saturated by $\boldsymbol{y}$ in G_3, cycle C_1 contains the path $v_6v_2v_4$. Thus the multiset sum of C_1, $v_3v_4v_6v_3$, and v_3v_1 contains two arc-disjoint cycles $v_2v_4v_6v_2$ and $v_1v_6v_3v_1$. By Lemma 3.5(iv), we have $y(v_3v_4v_6v_3) = 0$, a contradiction.

(c) v_1v_6 is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $C_2 \in \mathcal{C}_0^y$. In this subsubcase, it is clear that C_2 contains the path $v_1v_2v_4$. Observe that the multiset sum of C_2, $v_3v_4v_6v_3$, and the unsaturated v_1v_6 contains two arc-disjoint cycles $v_1v_6v_3v_1$ and $C_2' = C_2[v_4, v_3] \cup \{v_3v_4\}$. Set $\theta = \min\{y(C_2), y(v_3v_4v_6v_3), w(v_1v_6) - z(v_1v_6)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_2)$, $y(v_3v_4v_6v_3)$, $y(v_1v_6v_3v_1)$, and $y(C_2')$ with $y(C_2) - \theta$, $y(v_3v_4v_6v_3) - \theta$, $y(v_1v_6v_3v_1) + \theta$, and $y(C_2') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4).

(d) v_1v_6 and v_3v_1 are contained in some cycles C_1 and C_2 in $\mathcal{C}_0^y$, respectively. In this subsubcase, if v_3v_1 is also on C_1, then the multiset sum of C_1 and $v_3v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_2v_4v_6v_2$, and $C_1' = C_1[v_4, v_3] \cup \{v_3v_4\}$. From the optimality of $\boldsymbol{y}$, we deduce that $y(v_3v_4v_6v_3) = 0$. If v_3v_1 is outside C_1, then the multiset sum of C_1, C_2, and $v_3v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_2v_4v_6v_2$, $C_1' = C_1[v_4, v_1] \cup \{v_1v_2, v_2v_4\}$, and $C_2' = C_2[v_4, v_3] \cup \{v_3v_4\}$. From the optimality of $\boldsymbol{y}$, we again deduce that $y(v_3v_4v_6v_3) = 0$.

By (14), we have $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. So K is an MFAS by (5) and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

Case 1.2. v_2v_4 is outside $\mathcal{C}_0^y$.

In this case, v_2v_4 is saturated by $\boldsymbol{y}$ in G_3, and v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$. Since $\mathcal{C}_0^y \neq \emptyset$, there exists a cycle $C \in \mathcal{C}_0^y$ containing v_3v_4. From (6), (7), and (10), we see that $y(v_1v_2v_4v_6v_3v_1)$, $y(v_1v_6v_3v_2v_4v_1)$, and $y(v_2v_4v_6v_3v_2)$ are all zero. If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_3, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_4, v_6v_3\}$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in G_3. By Lemma 3.5(iii) and (iv), v_4v_6 is saturated by $\boldsymbol{y}$ in G_3.

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in G_3. Then, by Lemma 3.5(iii) and (iv), v_1v_6 is saturated by $\boldsymbol{y}$ in G_3. By (13), v_1v_2 is not saturated by $\boldsymbol{y}$ in G_3 and hence in T. By (9), $y(v_1v_6v_2v_4v_1) = 0$. If v_6v_3 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of C, $v_2v_4v_6v_2$, and the unsaturated arcs v_6v_3, v_4v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_3v_4v_6v_3$; if v_6v_3 is saturated by $\boldsymbol{y}$ in T but contained in some cycle C in $\mathcal{C}_0^y$, then the multiset sum of C, $v_2v_4v_6v_2$, and the unsaturated arcs v_4v_1 and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_3v_4v_6v_3$. By Lemma 3.5(vi), we have $y(v_2v_4v_6v_2) = 0$ in either subcase. So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_2v_4\}$.

Assume next that v_4v_1 is saturated by $\boldsymbol{y}$ in G_3. Then, by (13), v_3v_1 and at least one of v_1v_2 and v_1v_6 are not saturated by $\boldsymbol{y}$ in G_3. By Lemma 3.5(iii) and (iv), both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$; using Lemma 3.5(i) and (iii), we further deduce that v_1v_6 is saturated by $\boldsymbol{y}$ in G_3. Thus, by (13), v_1v_2 is not saturated by $\boldsymbol{y}$ in G_3. It follows from (8) and (9) that $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$. Therefore $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1, v_4v_6\}$. So K is an MFAS by (5) and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$. This proves Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we may assume that

(15) $y(v_1v_2v_4v_6v_3v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

Assume the contrary: $y(v_1v_2v_4v_6v_3v_1) = 0$. Then, from (6) we deduce that $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ and that each arc in the set $\{v_1v_6, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. So $y(v_1v_2v_4v_1) = w(v_4v_1)$, $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, $y(v_2v_4v_6v_2) = w(v_6v_2)$, and $y(v_2v_4v_6v_3v_1) = w(v_3v_2)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer; so is $y(v_1v_2v_4v_6v_3v_1)$. Thus Lemma 3.2(iii) allows us to assume that $y(v_1v_2v_4v_6v_3v_1) = 0$.

If $y(v_1v_6v_3v_2v_4v_1) > 0$, then from (7) we deduce that $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$ and that each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_4v_6, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer; so is $y(v_1v_6v_3v_2v_4v_1)$. Thus Lemma 3.2(iii) allows us to further assume that $y(v_1v_6v_3v_2v_4v_1) = 0$.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{F}_2$. Depending on what K is, we distinguish among 13 cases.

Case 2.1. $K = \{v_1v_6, v_2v_4, v_4v_6\}$.

In this case, by Lemma 3.1 (i), we have $y(v_2v_4v_6v_2) = y(v_1v_6v_2v_4v_1) = y(v_2v_4v_6v_3v_2) = y(v_1v_6v_3v_2v_4v_1) = y(v_1v_2v_4v_6v_3v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_1v_6v_3v_4v_1) > 0$, then by (8) we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_3v_4v_1) = 0$.

Case 2.2. $K = \{v_4v_1, v_4v_6, v_6v_3\}$.

In this case, by Lemma 3.1 (i), we have $y(v_3v_4v_6v_3) = y(v_1v_6v_3v_4v_1) = y(v_2v_4v_6v_3v_2) = y(v_1v_6v_3v_2v_4v_1) = y(v_1v_2v_4v_6v_3v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_4v_1)$, $y(v_2v_4v_6v_2) = w(v_4v_6)$, and $y(v_1v_6v_3v_1) = w(v_6v_3)$. If $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$.

Case 2.3. $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3)+y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_2v_4v_6v_2)+y(v_1v_6v_2v_4v_1) = w(v_6v_2)$. Observe that if $y(v_1v_6v_3v_4v_1) > 0$, then by (8) we have $y(v_3v_4v_6v_3) = w(v_4v_6) - w(v_3v_2)$ and $y(v_2v_4v_6v_2) = 0$; if $y(v_1v_6v_3v_4v_1) = 0$ and $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have $y(v_2v_4v_6v_2) = w(v_4v_6) - w(v_3v_2) - w(v_3v_4)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1)$ or $y(v_1v_6v_3v_4v_1)$ is zero.

Case 2.4. $K = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, and $y(v_2v_4v_6v_2) = w(v_6v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.5. $K = \{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, $y(v_1v_2v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$. Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have $y(v_2v_4v_6v_2) = w(v_4v_6) - w(v_3v_2) - w(v_3v_4)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1)$ is zero.

Case 2.6. $K = \{v_1v_6, v_2v_4, v_3v_4\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3v_2) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_2v_4v_6v_2) = w(v_6v_2)$ by (10). Since v_4v_1 and v_1v_2 are outside $\mathcal{C}_0^y$ and $\boldsymbol{y}$ satisfies (2), it is easy to see that $y(v_1v_2v_4v_1) = \min\{w(v_1v_2), w(v_4v_1)\}$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$. Thus we may assume that $y(v_2v_4v_6v_3v_2) = 0$. Since both v_4v_6 and v_6v_2 are outside $\mathcal{C}_0^y$, by (4) we have $y(v_2v_4v_6v_2) = \min\{w(v_6v_2), w(v_4v_6) - w(v_3v_4)\}$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.7. $K = \{v_2v_4, v_3v_1, v_3v_4\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_2v_4)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_3v_4)$.

Assume first that $y(v_1v_6v_3v_4v_1) > 0$. Then, by (8), we have $y(v_2v_4v_6v_2) = 0$ and $y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (8) and (10), we obtain $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$; if $y(v_2v_4v_6v_3v_2) = 0$ and $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we get $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_2v_4) - w(v_1v_2)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$; if $y(v_1v_6v_2v_4v_1) = y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_2v_4v_1) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subcase.

Assume next that $y(v_1v_6v_3v_4v_1) = 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we have $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3v_2) = w(v_4v_6) - y(v_3v_4v_6v_3) = w(v_4v_6) - w(v_3v_4)$, and so $y(v_1v_6v_2v_4v_1) = w(v_2v_4) + w(v_3v_4) - w(v_1v_2) - w(v_4v_6)$. Observe that if $y(v_2v_4v_6v_3v_2) > 0$, then we have one more equation $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ by (10). Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_2v_4v_6v_3v_2) = 0$. So we assume that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_2v_4v_6v_2) = w(v_6v_2)$ and $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_2v_4) - w(v_6v_2)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) = w(v_2v_4)$. Since $\boldsymbol{y}$ satisfies (2) and (4) and since v_4v_1, v_4v_6, v_1v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$, if $y(v_1v_2v_4v_1) > 0$, then $y(v_1v_2v_4v_1) = \min\{w(v_4v_1), w(v_1v_2)\}$ or $y(v_2v_4v_6v_2) = \min\{w(v_4v_6) - y(v_3v_4v_6v_3), w(v_6v_2)\}$, regardless of the value of $y(v_2v_4v_6v_3v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.8. $K = \{v_1v_2, v_6v_2, v_6v_3\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

- $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, by (8), we have $y(v_2v_4v_6v_2) = 0$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. So $y(v_1v_6v_3v_4v_1) = w(v_6v_3) - w(v_3v_1) - w(v_4v_6)$. Observe that if $y(v_2v_4v_6v_3v_2) > 0$, then we have one more equation $y(v_3v_4v_6v_3) = w(v_3v_4) - y(v_1v_6v_3v_4v_1)$ by (10). So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_2v_4v_6v_3v_2) = 0$.
- $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, assume first that $y(v_1v_6v_2v_4v_1) > 0$. If $y(v_3v_4v_6v_3) > 0$ or $y(v_2v_4v_6v_3v_2) > 0$, then, by (9), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_1)$, and $y(v_2v_4v_6v_2) = w(v_4v_6) + w(v_3v_1) - w(v_6v_3)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10). Thus $y(v_2v_4v_6v_3v_2)$ and $y(v_1v_6v_2v_4v_1)$ are integral. If $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$, then $y(v_2v_4v_6v_2) = w(v_4v_6)$ and $y(v_1v_6v_2v_4v_1) = w(v_6v_2) -$

$w(v_4v_6)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subsubcase. Assume next that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10) and $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_4)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) = w(v_6v_3)$. Note that both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$. As $\boldsymbol{y}$ satisfies (2) and (4), we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_1v_6), w(v_3v_1)\}$, no matter whether $y(v_2v_4v_6v_3v_2) > 0$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.9. $K = \{v_4v_1, v_6v_2, v_6v_3\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_4v_1)$, $y(v_2v_4v_6v_2) = w(v_6v_2)$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10), so $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_4)$; if $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. Clearly, v_1v_6 is outside $\mathcal{C}_0^y$. We propose to show that

(16) $y(v_1v_6v_3v_1)$ is integral.

Suppose on the contrary that $y(v_1v_6v_3v_1)$ is not integral. If v_3v_1 is outside $\mathcal{C}_0^y$, then from (2) and (4) we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$, a contradiction. So we assume that v_3v_1 is contained in some cycle C in $\mathcal{C}_0^y$. Then C contains the path $v_1v_2v_4$. Set $C' = C[v_4, v_3] \cup \{v_3v_2, v_2v_4\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $C' = C[v_4, v_3] \cup \{v_3v_4\}$ otherwise, and set $\theta = \min\{[y(v_2v_4v_6v_3v_2)], y(C)\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $\theta = \min\{[y(v_3v_4v_6v_3)], y(C)\}$ otherwise. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_2v_4v_6v_3v_2)$ (resp. $y(v_3v_4v_6v_3)$), $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_2v_4v_6v_3v_2)-\theta$ (resp. $y(v_3v_4v_6v_3)-\theta$), $y(v_1v_6v_3v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $y'(v_2v_4v_6v_3v_2) < y(v_2v_4v_6v_3v_2)$ or $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (2) or (4). So (16) is established.

From (16) it follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.10. $K = \{v_2v_4, v_6v_3\}$.

In this case, by Lemma 3.1 (i), we have $y(v_2v_4v_6v_3v_2) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$ and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$. It follows that all arcs in $G_3\backslash v_5$ are outside $\mathcal{C}_0^y$ except possibly v_3v_4. If $y(v_1v_6v_3v_4v_1) > 0$, then, by (8), we have $y(v_2v_4v_6v_2) = 0$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$. Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$. So we assume that $y(v_1v_6v_3v_4v_1) = 0$.

If $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we obtain $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) = w(v_4v_6)$. Furthermore, $y(v_1v_6v_3v_1) = w(v_3v_1)$ if $y(v_3v_4v_6v_3) > 0$ and $y(v_1v_6v_3v_1) = w(v_6v_3)$ otherwise. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_3v_4v_6v_3) = 0$. So we may assume that $y(v_1v_6v_2v_4v_1) = 0$.

If $y(v_3v_4v_6v_3) = 0$, then $y(v_1v_6v_3v_1) = w(v_6v_3)$. Recall that both v_4v_6 and v_6v_2 are outside $\mathcal{C}_0^y$. If $y(v_1v_2v_4v_1) > 0$, then from (4) we deduce that $y(v_2v_4v_6v_2) = \min\{w(v_4v_6), w(v_6v_2)\}$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_4v_1) > 0$. It remains to consider the subcase when $y(v_3v_4v_6v_3) > 0$. Since both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$, from (4) we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$. If $y(v_1v_2v_4v_1) = 0$, then $y(v_2v_4v_6v_2) = w(v_2v_4)$; otherwise, by (4), at least one of v_4v_6 and v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. It follows that $y(v_2v_4v_6v_2) = \min\{w(v_6v_2), w(v_4v_6) - y(v_3v_4v_6v_3)\}$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_4v_1) = 0$.

Case 2.11. $K = \{v_3v_1, v_4v_1, v_4v_6\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, $y(v_2v_4v_6v_2) = 0$ by (8). If $y(v_2v_4v_6v_3v_2) > 0$, then, by (8) and (10), we have $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_3v_4v_6v_3) = w(v_4v_6)$. Depending on the value of $y(v_1v_6v_2v_4v_1)$, we distinguish between two subsubcases.

(a) $y(v_1v_6v_2v_4v_1) > 0$. By (9), $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_1v_2)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_6v_2v_4v_1)$ is not integral. By Lemma 3.2(iii), we may assume that $w(v_3v_1)$, $w(v_1v_2)$, and $w(v_4v_6)$ are all zero. Observe that v_6v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_2. Then C passes through v_2v_4. Let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$, let $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_4v_1)-\theta$, $y(v_1v_6v_2v_4v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (3). Similarly, we can prove that v_3v_2 is outside $\mathcal{C}_0^y$. Thus $w(v_3v_2) = z(v_3v_2) = 0$. We propose to show that

(17) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $y(v_1v_6v_2v_4v_1) > 0$ and $y(v_1v_6v_3v_4v_1) > 0$, by Lemma 3.1(i) we have $x(v_6v_2) + x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_6v_2v_4v_1) < w(v_6v_2)$, by Lemma 3.1(ii) we obtain $x(v_6v_2) = 0$, which implies $x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since v_6v_2 is outside $\mathcal{C}_0^y$, for each vertex u in $V(T_1)\backslash\{b, a_1\}$, we obtain $x(uv_6) = x(uv_2)$. Let $T' = (V', A')$ be obtained from T by deleting vertex v_2, let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_6)$ with $w(uv_6) + w(uv_2)$ for each u in $V(T_1)\backslash\{b, a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j) + w(v_2v_4)$ for $(i, j) = (6, 3)$ or $(3, 4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be defined from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u \in V(T_1)\backslash\{b, a_1\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_6v_3v_4$, and set $y'(C') = y(C)+y(C')$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1)+y(v_1v_6v_2v_4v_1)$. Then $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 1.5. So (17) follows.

(b) $y(v_1v_6v_2v_4v_1) = 0$. Then $y(v_1v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$. If $y(v_1v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_4v_1)$ is not integral. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2. Since the multiset sum of C and $v_1v_6v_3v_4v_1$ contains arc-disjoint cycles $v_1v_2v_4v_1$ and $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_4\}$. By Lemma 3.5(vi), we have $y(C) = 0$, a contradiction. Similarly, we can prove that v_6v_2 and v_3v_2 are outside $\mathcal{C}_0^y$ as well. Thus $w(v_iv_2) = z(v_iv_2) = 0$ for $i = 3, 6$. We propose to show that

(18) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_2v_4v_1)$ and

$y(v_1v_6v_3v_4v_1)$ are positive, by Lemma 3.1(i) we have $x(v_1v_2) + x(v_2v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_2v_4v_1) < w(v_1v_2)$, by Lemma 3.1(ii) we obtain $x(v_1v_2) = 0$, which implies that $x(v_2v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Since v_1v_2 is outside $\mathcal{C}_0^y$, for each vertex $u \in V(T_1)\backslash\{b, a_1\}$, we obtain $x(uv_1) = x(uv_2)$. Let $T' = (V', A')$ be obtained from T by deleting vertex v_2, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_1)$ with $w(uv_1)+w(uv_2)$ for each $u \in V(T_1)\backslash\{b, a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j)+w(v_2v_4)$ for $(i, j) = (1, 6)$, $(6, 3)$, and $(3, 4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be defined from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u \in V(T_1)\backslash\{b, a_1\}$, let C' be obtained from C by replacing uv_2v_4 with $uv_1v_6v_3v_4$, and set $y'(C') = y(C)+y(C')$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1)+y(v_1v_2v_4v_1)$. Then $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 1.5. This proves (18).

- $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, $y(v_1v_2v_4v_1) = w(v_1v_2)$. By (9), if $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_2v_4v_1) = w(v_4v_1)-w(v_1v_2)$; otherwise, $y(v_1v_2v_4v_1) = w(v_4v_1)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (10), we have $y(v_3v_4v_6v_3) = w(v_3v_4)$, $y(v_2v_4v_6v_2) = w(v_6v_2) - y(v_1v_6v_2v_4v_2)$, and $y(v_2v_4v_6v_3v_2) = w(v_4v_6) - w(v_3v_4) - y(v_2v_4v_6v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2) = 0$. Thus $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_2)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_2v_4v_6v_2)$ is not integral. By Lemma 3.2(iii), we may assume that $w(v_3v_1) = w(v_4v_1) = 0$. Observe that v_6v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_2. Then C passes through v_2v_4. Let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$, let $\theta = \min\{y(C), y(v_3v_4v_6v_3)\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_2v_4v_6v_2)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3) - \theta$, $y(v_2v_4v_6v_2) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4). Similarly, we can show that v_3v_2 is outside $\mathcal{C}_0^y$. So $w(v_3v_2) = z(v_3v_2) = 0$. Moreover, $\nu_w^*(T)$ is an integer; the proof is the same as that of (17) (with $y(v_2v_4v_6v_2)$ and $y(v_3v_4v_6v_3)$ in place of $y(v_1v_6v_2v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$, respectively), so we omit the details here.

Case 2.12. $K = \{v_1v_6, v_4v_1, v_4v_6\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (10), we have $y(v_2v_4v_6v_2) = w(v_6v_2)$ and $y(v_3v_4v_6v_3) = w(v_3v_4)$, so $y(v_2v_4v_6v_3v_2) = w(v_4v_6) - w(v_6v_2) - w(v_3v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, It remains to assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_2)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_2v_4v_6v_2)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

Case 2.13. $K = \{v_1v_2, v_1v_6, v_4v_6\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. Clearly, v_3v_1 is outside $\mathcal{C}_0^y$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

- $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, $y(v_2v_4v_6v_2) = 0$ and $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (8). If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_3v_4)$

by (10). Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_3v_4v_6v_3) = w_{46}$ and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6) - w(v_3v_1)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_1v_6v_2v_4v_1)$ is not integral. Then we can prove that $\nu^*_w(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

• $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, $y(v_1v_6v_3v_1)+y(v_1v_6v_2v_4v_1) = w(v_1v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ and $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ by (10). Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (9). So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$. Thus we may assume that $y(v_2v_4v_6v_3v_2) = 0$. We proceed by considering two subsubcases.

(a) Assume first that $y(v_3v_4v_6v_3) = 0$. Then $y(v_2v_4v_6v_2) = w(v_4v_6)$. If $y(v_1v_6v_3v_1)$ is integral, then so is $y(C)$ for all $C \in \mathcal{C}_2$. Thus we assume that $y(v_1v_6v_3v_1)$ is not integral. If v_6v_3 is outside $\mathcal{C}^y_0$, then it follows from (4) that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_6v_3)\}$; this contradiction implies that v_6v_3 is contained in a cycle C in $\mathcal{C}^y_0$. Let $C' = C[v_4, v_6] \cup \{v_6v_2, v_2v_4\}$, let $\theta = \min\{[y(v_1v_6v_2v_4v_1)], y(C)\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_2v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_2v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_2v_4v_1) < y(v_1v_6v_2v_4v_1)$, contradicting (2).

(b) Assume next that $y(v_3v_4v_6v_3) > 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_6v_2v_4v_1) = w(v_1v_6) - w(v_3v_1)$ by (9); otherwise, $y(v_1v_6v_3v_1) = w(v_1v_6)$. If $y(v_3v_4v_6v_3)$ is integral, then so is $y(C)$ for all $C \in \mathcal{C}_2$. Thus we assume that $y(v_3v_4v_6v_3)$ is not integral. Let us prove that

(19) $\nu^*_w(T)$ is an integer.

By Lemma 3.2(iii), we may assume that $w(v_1v_2) = w(v_1v_6) = 0$. Let $T' = (V', A')$ be obtained from T by deleting v_1, and let $\boldsymbol{w}$ be the restriction of $\boldsymbol{w}$ to A'. It is routine to check that $\mathbb{D}(T', \boldsymbol{w}')$ has the same optimal value $\nu^*_w(T)$ as $\mathbb{D}(T, \boldsymbol{w})$. Hence $\nu^*_w(T)$ is an integer by the hypothesis of Theorem 1.5. This proves (19) and hence Claim 2.

Since $\tau_{\boldsymbol{w}}(G_3\backslash v_5) > 0$, from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 4.6. ∎

5 Composite Reductions

Lemma 5.7. *If $T_2/S = F_4$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_6)$, $s^* = v_2$, and $v_0 = v_3$. To establish the statement, by Lemma 3.4(ii), it suffices to prove that

(1) the optimal value $\nu^*_w(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of F_4, there are at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. Then (1) follows immediately from Lemma 4.4; the argument can be found in that of (3) in the proof of Lemma 5.5.

Lemma 5.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1,2,3\}$.

Let t be the subscript in $\{1,2,3\}$ with $v_5 \in \varphi(s_t)$, if any. By (2), t is well defined. In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically; and

(7) subject to (5) and (6), $y(v_1v_5s_tv_3v_1) + y(v_1v_5v_3v_4v_1)$ is minimized.

Let us make a few observations about $\boldsymbol{y}$ before proceeding.

(8) If $y(v_1v_5s_iv_3v_4v_1) > 0$ for some $i \in \{1,2,3\}$, then each arc in the set $\{v_1s_i, v_3v_1, v_4s_i, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in T_2. Furthermore, $y(v_1s_jv_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$ for any $j \in \{1,2,3\} \backslash \{i\}$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_5s_iv_3v_4v_1$. So the first half follows instantly from Lemma 3.5(v). Once again let $\uplus$ stand for the multiset sum. Then $v_1v_5s_iv_3v_4v_1 \uplus v_1s_jv_3v_1 = v_1v_5s_iv_3v_1 \uplus v_1s_jv_3v_4v_1$, $v_1v_5s_iv_3v_4v_1 \uplus v_1v_5v_3v_1 = v_1v_5s_iv_3v_1 \uplus v_1v_5v_3v_4v_1$, and $v_1v_5s_iv_3v_4v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_4v_1 \uplus v_5s_iv_3v_4v_5$. Since $\boldsymbol{y}$ satisfies (6), we deduce that $y(v_1s_jv_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

(9) If $y(v_1v_5s_iv_3v_1) > 0$ for some $i \in \{1,2,3\}$, then both v_1s_i and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2; so are v_4s_i and v_4v_5 if $y(v_1s_jv_3v_4v_1) > 0$. Furthermore, $y(v_3v_4v_5v_3) = 0$.

Since both v_1s_i and v_5v_3 are chords of the cycle $v_1v_5s_iv_3v_1$, the first half follows instantly from Lemma 3.5(v). To establish the second half, observe that $v_1v_5s_iv_3v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_1 \uplus v_5s_iv_3v_4v_5$. Hence $y(v_3v_4v_5v_3) = 0$ by (7). Suppose $y(v_1s_jv_3v_4v_1) > 0$. Since the multiset sum of the cycles $v_1v_5s_iv_3v_1$, $v_1s_jv_3v_4v_1$, and the arc v_4v_5 (resp. v_4s_i) contains arc-disjoint cycles $v_1s_jv_3v_1$ and $v_5s_iv_3v_4v_5$ (resp. $v_4s_iv_3v_4$), from (7) we deduce that both v_4s_i and v_4v_5 are are saturated by $\boldsymbol{y}$ in T_2.

(10) If $y(v_1v_5v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, $y(v_1s_iv_3v_1) = 0$ for any $i \in \{1,2,3\}$.

Since both v_3v_1 and v_4v_5 are chords of the cycle $v_1v_5v_3v_4v_1$, the first half follows instantly from Lemma 3.5(v). To establish the second half, observe that $v_1v_5v_3v_4v_1 \uplus v_1s_iv_3v_1 = v_1v_5v_3v_1 \uplus v_1s_iv_3v_4v_1$. Since $\boldsymbol{y}$ satisfies (7), we have $y(v_1s_iv_3v_1) = 0$.

The following two statements can be seen from Lemma 3.5(v).

(11) If $y(v_1s_iv_3v_4v_1) > 0$, then both v_3v_1 and v_4s_i are saturated by $\boldsymbol{y}$ in T_2, for $i \in \{1,2,3\}$.

(12) If $y(v_5s_iv_3v_4v_5) > 0$, then both v_4s_i and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2, for $i \in \{1,2,3\}$.

We proceed by considering two cases, depending on whether $\varphi(s_k) = \{v_4\}$ for some $k \in \{1,2,3\}$ (see (4)).

Case 1. $\varphi(s_k) = \{v_4\}$ for some $k \in \{1,2,3\}$.

By Lemma 5.2(i), we may assume that $k = 1$; that is, $\varphi(s_1) = \{v_4\}$. Let i and j be the subscripts in $\{2,3\}$, if any (possibly $i = j$), such that $v_5 \in \varphi(s_i)$ and $v_1 \in \varphi(s_j)$. Then

(13) $\mathcal{C}_2^y \subseteq \{v_4s_1v_3v_4, v_1s_jv_3v_1, v_1s_jv_3v_4v_1, v_1v_5s_iv_3v_1, v_5s_iv_3v_4v_5, v_1v_5s_iv_3v_4v_1, v_1v_5v_3v_1, v_3v_4v_5v_3, v_1v_5v_3v_4v_1\}$.

We propose to show that

(14) if $w(v_3v_4) > 0$, then $y(v_4s_1v_3v_4)$ is a positive integer.

For this purpose, note that $z(s_1v_3) = w(s_1v_3) > 0$ by Lemma 5.2(iii). If s_1v_3 is outside $\mathcal{C}_0^y$, then $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$. So we assume that s_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. If C contains v_4s_1, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). Moreover, the multiset sum of C and each cycle in the set $\{v_1s_jv_3v_4v_1, v_5s_iv_3v_4v_5, v_1v_5s_iv_3v_4v_1, v_3v_4v_5v_3, v_1v_5v_3v_4v_1\}$ contains the cycle $v_4s_1v_3v_4$, a cycle in $\{v_1s_jv_3v_1, v_1v_5s_iv_3v_1, v_1v_5v_3v_1\}$, and a cycle $C' \in \mathcal{C}_0$ that are arc-disjoint, where $C' = C[v_5, v_4] \cup \{v_4v_5\}$ or $C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. From the optimality of $\boldsymbol{y}$, we thus deduce that $y(v_1s_jv_3v_4v_1)$, $y(v_5s_iv_3v_4v_5)$, $y(v_1v_5s_iv_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_1v_5v_3v_4v_1)$ are all zero. Hence $y(v_4s_1v_3v_4) = w(v_3v_4) > 0$. So we assume that C does not contain v_4s_1. Furthermore, v_4s_1 is outside $\mathcal{C}_0^y$, because every cycle using v_4s_1 passes through s_1v_3. Note that v_4s_1 is not saturated by $\boldsymbol{y}$ in T, for otherwise $y(v_4s_1v_3v_4) = w(v_4s_1) > 0$, as desired. By Lemma 3.5(vii), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 and C contains v_3v_1. It follows from (8), (10) and (11) that $y(v_1v_5s_iv_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$ and $y(v_1s_jv_3v_4v_1)$ are all zero. As the multiset sum of C, each of $v_5s_iv_3v_4v_5$ and $v_3v_4v_5v_3$, and the unsaturated arc v_4s_1 contains arc-disjoint cycles $v_4s_1v_3v_4$ and one of $v_1v_5s_iv_3v_1$ and $v_1v_5v_3v_1$, both $y(v_5s_iv_3v_4v_5)$ and $y(v_3v_4v_5v_3)$ are zero by Lemma 3.5(vi). So $y(v_4s_1v_3v_4) = w(v_3v_4) > 0$. This proves (14).

By (14) and Lemma 3.2(iii), we may assume that $w(v_3v_4) = 0$. It follows that $w(v_3v_1) \geq z(v_3v_1) > 0$, for otherwise, $\tau_w(T_2 \backslash a_2) = w(v_3v_1) + w(v_3v_4) = 0$, contradicting (α). Since $z(v_4s_1) > 0$ and $w(v_3v_4) = 0$, the arc v_4s_1 is contained in some cycle in $\mathcal{C}_0^y$. From the proof of (14) we see that

(15) $y(v_1s_jv_3v_4v_1)$, $y(v_5s_iv_3v_4v_5)$, $y(v_1v_5s_iv_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_1v_5v_3v_4v_1)$ are all zero.

(16) If $w(v_1s_j) \geq z(v_1s_j) > 0$, then $y(v_1s_jv_3v_1)$ is a positive integer.

To justify this, note that $z(s_jv_3) = w(s_jv_3) > 0$ by Lemma 5.2(iii). Assume first that s_jv_3 is outside $\mathcal{C}_0^y$. If $i \neq j$, then $y(v_1s_jv_3v_1) = w(s_jv_3) > 0$. So we assume that $i = j$. Then $y(v_1s_iv_3v_1) + y(v_1v_5s_iv_3v_1) = w(s_iv_3)$. If $y(v_1v_5s_iv_3v_1) > 0$, then v_1s_i is saturated by $\boldsymbol{y}$ in T_2 by (9). Thus $y(v_1s_iv_3v_1) = w(v_1s_i)$. Next assume that s_jv_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Since $w(v_3v_4) = 0$, cycle C contains v_3v_1. It follows that v_1s_j is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1s_jv_3v_1) = w(v_1s_j) > 0$ and hence (16) is established.

By (16) and Lemma 3.2(iii), we may assume that $w(v_1s_j) = 0$. By (3), we have $z(v_5s_i) > 0$ and $\varphi(s_i) = \{v_5\}$. By (13)-(16), we obtain

(17) $\mathcal{C}_2^y \subseteq \{v_1v_5s_iv_3v_1, v_1v_5v_3v_1\}$.

(18) $y(v_1v_5s_iv_3v_1)$ is a positive integer.

To justify this, note that $z(s_iv_3) = w(s_iv_3) > 0$ by Lemma 5.2(iii). If s_iv_3 is outside $\mathcal{C}_0^y$, then $y(v_1v_5s_iv_3v_1) = w(s_iv_3) > 0$ by (17), as desired. So we assume that s_iv_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Applying Lemma 3.5(iii) to the cycle $v_1v_5s_iv_3v_1$, we deduce that (v_5, s_i) is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1v_5s_iv_3v_1) = w(v_5s_i) > 0$ and hence (18) holds.

By (18) and Lemma 3.2(iii), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution, which implies (1).

Case 2. $\varphi(s_k) \neq \{v_4\}$ for any $k \in \{1, 2, 3\}$.

By (3), the hypothesis of the present case, and Lemma 5.2(i), we may assume that $v_1 \in \varphi(s_1)$ and $v_5 \in \varphi(s_2)$. Then

(19) $\mathcal{C}_2^y \subseteq \{v_1s_1v_3v_1, v_1s_1v_3v_4v_1, v_1v_5s_2v_3v_1, v_5s_2v_3v_4v_5, v_1v_5s_2v_3v_4v_1, v_1v_5v_3v_1, v_3v_4v_5v_3,$
$v_1v_5v_3v_4v_1, v_4s_1v_3v_4, v_4s_2v_3v_4\}$.

By Lemma 5.2(vi), we have

(20) if $v_4 \in \varphi(s_i)$, then $z(v_4s_{3-i}) = 0$ and $y(v_4s_{3-i}v_3v_4) = 0$ for $i = 1, 2$.

Claim 1. $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

To justify this, observe that

(21) if K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

In view of Lemma 5.2(iii), we distinguish among three subcases, depending on whether s_iv_3 is contained in a cycle in $\mathcal{C}_0^y$.

Subcase 1.1. Both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$. In this subcase, s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$. If v_5v_3 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_5v_3, s_1v_3, s_2v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_5v_3 is not saturated by $\boldsymbol{y}$ in T_2.

(22) Both v_3v_1 and v_3v_4 are outside $\mathcal{C}_0^y$. Furthermore, at least one of them is not saturated by $\boldsymbol{y}$ in T_2.

Indeed, the first half follows directly from Lemma 3.5(iii). To justify the second half, assume the contrary. Then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_3v_4\}$. Thus K is an MFAS of $T_2\backslash a_2$ by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

By (22), (8), (9), and (12), we have

(23) $y(v_1v_5s_2v_3v_1)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_4v_1)$ are all zero.

Since $\mathcal{C}_0^y \neq \emptyset$, some cycle $C \in \mathcal{C}_0^y$ contains v_1v_5 or v_4v_5. Thus there are two possibilities to consider.

- C contains v_1v_5. Now by (22) and Lemma 3.5(iii), v_3v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence v_3v_4 is not saturated by $\boldsymbol{y}$ in T_2. It follows from Lemma 3.5(i) and (iii) that both v_4v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. If $z(v_4s_i) = w(v_4s_i)$ for $i = 1, 2$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, v_4s_2\}$. Thus K is an MFAS of $T_2\backslash a_2$ by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that $0 < z(v_4s_i) < w(v_4s_i)$ for $i = 1$ or 2. Then $z(v_4s_{3-i}) = w(v_4s_{3-i}) = 0$ by (2). If $i = 2$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, s_2v_3\}$, and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. If $i = 1$, then $y(v_1s_1v_3v_4v_1) = 0$ by (11). Since the multiset sum of the cycles $v_1s_1v_3v_1$, C, and the unsaturated arcs $\{v_4s_1, v_5v_3, v_3v_4\}$ contains arc-disjoint cycles $v_4s_1v_3v_4$ and $v_1v_5v_3v_1$, we have $y(v_1s_1v_3v_1) = 0$ by Lemma 3.5(vi). Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, s_1v_3, v_4s_2\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.
- C contains v_4v_5. Now by (22) and Lemma 3.5(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 and hence v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2. It follows from Lemma 3.5(i) and (iii) that v_1v_5 is saturated by $\boldsymbol{y}$ in T_2. By (10) and (11), we have $y(v_1v_5v_3v_4v_1) = y(v_1s_1v_3v_4v_1) = 0$. If v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_5, v_3v_4, v_1s_1\}$. Thus $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_1s_1 is not saturated by $\boldsymbol{y}$ in T_2 and hence not in T by (22). Since the multiset sum of the cycles C, $v_4s_1v_3v_4$, and the unsaturated arcs $\{v_3v_1, v_5v_3, v_1s_1\}$ contains arc-disjoint cycles $v_1s_1v_3v_1$ and $v_3v_4v_5v_3$, we have $y(v_4s_1v_3v_4) = 0$ by Lemma 3.5(vi). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_5, v_3v_4, s_1v_3\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

Subcase 1.2. s_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, we choose C so that it contains as many edges in $T_2\backslash a_2$ as possible.

Assume first that C contains v_1s_1. Then C contains the path $v_1s_1v_3v_4v_5$. By Lemma 3.5(iii), each arc in the set $\{v_3v_1, v_4v_1, v_4s_1, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in T_2. By (2), (8) and (10), we have $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. Since the multiset sum of C and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$ contains arc-disjoint cycles $v_3v_4v_5v_3$, $C' = C[v_5, v_1] \cup \{v_1v_5\}$, and one of $v_1s_1v_3v_1$ and $v_5s_2v_3v_4v_5$, from the optimality of $\boldsymbol{y}$ we deduce that $y(v_1v_5v_3v_1) = y(v_1v_5s_2v_3v_1) = 0$. If

s_2v_3 is outside $\mathcal{C}_0^y$, then s_2v_3 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 5.2(iii). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4s_1, s_2v_3, v_5v_3\}$. Hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$. So we assume that s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$. Since v_3v_1 is saturated by $\boldsymbol{y}$ in T_2, every cycle in $\mathcal{C}_0^y$ containing s_2v_3 passes through v_3v_4. By Lemma 3.5(iii), both v_4s_2 and v_5s_2 are saturated by $\boldsymbol{y}$ in T_2. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$.

Assume next that v_1s_1 is not on C. Then we may further assume that v_1s_1 is outside $\mathcal{C}_0^y$. We proceed by considering three subsubcases.

• C contains v_3v_1. Now v_1s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). Hence $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = y(v_1s_1v_3v_4v_1) = 0$ by (8), (10) and (11). If v_4s_1 is not saturated by $\boldsymbol{y}$ in T_2, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). Moreover, for each $D \in \{v_3v_4v_5v_3, v_5s_2v_3v_4v_5\}$, if v_4s_1 is on C, then the multiset sum of C and D contains arc-disjoint cycles $v_4s_1v_3v_4$, $C' = C[v_5, v_4] \cup \{v_4v_5\}$, and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$; if v_4s_1 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of C, D and the arc v_4s_1 contains $v_4s_1v_3v_4$ and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$ that are arc-disjoint. It follows from the optimality of $\boldsymbol{y}$ or Lemma 3.5(iv) that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. So $y(\mathcal{C}_2) = w(K)$ if s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$ and $y(\mathcal{C}_2) = w(J)$ otherwise, where $K = \{v_1s_1, v_3v_4, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$. So we assume that v_4s_1 is saturated by $\boldsymbol{y}$ in T_2. If s_2v_3 is outside $\mathcal{C}_0^y$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$, which implies that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$. So we further assume that s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$. By Lemma 3.5(iii), v_5s_2 is saturated by $\boldsymbol{y}$ in T_2. If v_4s_2 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, v_3v_4 is saturated by $\boldsymbol{y}$ in T_2, and $w(v_4s_1) = z(v_4s_1) = 0$. Similar to the case when v_4s_1 is not saturated by $\boldsymbol{y}$ in T_2, we can show that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

• C contains both v_3v_4 and v_4v_1. Now v_1s_1, v_4s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). If s_2v_3 is outside $\mathcal{C}_0^y$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$; otherwise, v_5s_2 and v_4s_2 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

• C contains both v_3v_4 and v_4v_5. Now v_4s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii) and $y(v_1v_5v_3v_4v_1) = y(v_1v_5s_2v_3v_4v_1) = 0$ by (8) and (10). If v_1s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$ or $w(J)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, both v_3v_1 and v_4v_1 are saturated by $\boldsymbol{y}$ in T_2, and every cycle in $\mathcal{C}_0^y$ containing s_2v_3 traverses $v_3v_4v_5$. Since the multiset sum of C, each of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$, and the unsaturated arc v_1s_1 contains $v_1s_1v_3v_1$ and one of $v_3v_4v_5v_3$ and $v_5s_2v_3v_4v_5$ that are arc-disjoint, we have $y(v_1v_5v_3v_1) = y(v_1v_5s_2v_3v_1) = 0$ by Lemma 3.5(iv). So $y(\mathcal{C}_2) = w(K)$ if s_2v_3 is outside $\mathcal{C}_0^y$ and $y(\mathcal{C}_2) = w(J)$ otherwise, where $K = \{v_3v_1, v_4v_1, v_4s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_4s_2, v_5s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

Subcase 1.3. s_2v_3 is contained in some cycle $C \in \mathcal{C}_0^y$ and s_1v_3 is saturated by $\boldsymbol{y}$ in T_2. In this subcase, both v_5s_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). If v_4s_2 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_3, v_5v_3, v_4s_2, v_5s_2\}$; otherwise, $z(v_4s_2) > 0$ and $w(v_4s_1) = z(v_4s_1) = 0$ by Lemma 5.2(vii). In this case C contains v_3v_1, so v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). By (8) and (10)-(12), we have $y(v_1v_5s_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1s_1v_3v_4v_1)$, and $y(v_5s_2v_3v_4v_5)$ are all zero. Since the multiset sum of the cycles C, $v_3v_4v_5v_3$, and the unsaturated arc v_4s_2 contains arc-disjoint cycles $v_4s_2v_3v_4$ and

$v_1v_5v_3v_1$, by Lemma 3.5(iv), we have $y(v_3v_4v_5v_3) = 0$. It follows that $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_3, v_5v_3, v_3v_4, v_5s_2\}$.

Combining the above three subcases, we see that the equality $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$ holds. So Claim 1 is established.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, note that $y(\mathcal{C}_2) = w(K)$ for some MFAS K of $T_2\backslash a_2$ by Claim 1. Depending on what K is, we distinguish among eight cases.

Subcase 2.1. K is one of $\{v_1v_5, v_3v_4, v_1s_1\}$, $\{v_1s_1, v_3v_4, s_2v_3, v_5v_3\}$, $\{v_1s_1, v_3v_4, v_5s_2, v_5v_3\}$, $\{v_1v_5, v_3v_4, s_1v_3\}$, and $\{s_1v_3, v_3v_4, v_5s_2, v_5v_3\}$.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_1s_1)$ or $w(s_1v_3)$, each of them is positive by Lemma 5.2(iii) and the assumption that $v_1 \in \varphi(s_1)$.

Subcase 2.2. $K = \{v_3v_1, v_4v_1, v_4s_1, s_2v_3, v_5v_3\}$.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, $y(v_4s_2v_3v_4) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$, $y(v_3v_4v_5v_3) = w(v_5v_3)$. If $y(v_5s_2v_3v_4v_5) = 0$, then $y(v_4s_2v_3v_4) = w(s_2v_3) > 0$ by Lemma 5.2(iii). If $y(v_5s_2v_3v_4v_5) > 0$, then v_4s_2 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). So $w(v_4s_2) = y(\mathcal{C}_2(v_4s_2))$. It follows that $y(v_4s_2v_3v_4) = w(v_4s_2)$, and hence $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.3. $K = \{v_3v_1, v_4v_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_5s_2)$, and $y(v_3v_4v_5) = w(v_5v_3)$. Since $v_5 \in \varphi(s_2)$, we have $w(v_5s_2) > 0$. So $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.4. $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, s_2v_3\}$ or $\{v_3v_1, v_4v_1, v_4v_5, s_1v_3, v_4s_2\}$.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_2v_3v_4) = w(s_2v_3) > 0$ or $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$ by Lemma 5.2(iii).

Subcase 2.5. $K = \{v_1s_1, v_4s_1, s_2v_3, v_5v_3\}$ or $\{v_1s_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$.

We only consider the subcase when $K = \{v_1s_1, v_4s_1, s_2v_3, v_5v_3\}$, as the other subcase can be justified likewise.

By Lemma 3.1(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4v_1) = w(v_1s_1)$, $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, and $y(v_4s_2v_3v_4) + y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$. We may assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$, for otherwise, by (8) or (10), we have $y(v_1s_1v_3v_1) = 0$ and hence $y(v_1s_1v_3v_4v_1) = w(v_1s_1) > 0$.

If $y(v_1v_5s_2v_3v_1) = 0$, then $y(v_5s_2v_3v_4v_5)+y(v_4s_2v_3v_4) = w(s_2v_3)$. Observe that $y(v_4s_2v_3v_4) > 0$, for otherwise, $y(v_5s_2v_3v_4v_5) = w(s_2v_3) > 0$. By (6), we obtain $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $w(v_4s_2)$, which is a positive integer. So we assume that $y(v_1v_5s_2v_3v_1) > 0$. Then $y(v_3v_4v_5v_3) = 0$ by (9). Note that $y(v_1s_1v_3v_4v_1) > 0$, for otherwise, $y(v_1s_1v_3v_1) = w(v_1s_1) > 0$. Thus, by

(9), both v_4s_2 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. It follows that $y(v_4s_2v_3v_4) = w(v_4s_2)$ and $y(v_5s_2v_3v_4v_5) = w(v_4v_5)$. So $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer.

Subcase 2.6. $K = \{s_1v_3, v_4s_2, v_5s_2, v_5v_3\}$ or $\{s_1v_3, s_2v_3, v_5v_3\}$.

We only consider the subcase when $K = \{s_1v_3, s_2v_3, v_5v_3\}$, as the other subcase can be justified likewise.

By Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4v_1) = w(s_1v_3)$, $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$, and $y(v_4s_2v_3v_4) + y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$.

We may assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$, for otherwise, by (8) or (10), we have $y(v_1s_1v_3v_1) = 0$ and hence $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_4v_1) = w(v_1s_1) > 0$, which together with (6) implies that $y(v_4s_1v_3v_4) = w(s_1v_3)$ or $w(v_4s_1)$, so $y(v_1s_1v_3v_4v_1) = w(v_1s_1) - y(v_4s_1v_3v_4)$. Since $w(s_1v_3) > 0$, at least one of $y(v_4s_1v_3v_4)$ and $y(v_1s_1v_3v_4v_1)$ is a positive integer.

If $y(v_1v_5s_2v_3v_1) = 0$, then $y(v_5s_2v_3v_4v_5) + y(v_4s_2v_3v_4) = w(s_2v_3)$, which together with (6) implies that $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $w(v_4s_2)$, so $y(v_5s_2v_3v_4v_5) = w(s_2v_3) - y(v_4s_2v_3v_4)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$ and $y(v_5s_2v_3v_4v_5)$ is a positive integer. So we assume that $y(v_1v_5s_2v_3v_1) > 0$. Thus, by (9), we have $y(v_1v_5v_3v_1) = w(v_5v_3)$. If $y(v_1s_1v_3v_4v_1) > 0$, then $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_4v_5)$, and $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer. So we further assume that $y(v_1s_1v_3v_4v_1) = 0$. Then $y(v_1s_1v_3v_1) + y(v_4s_1v_3v_4) = w(s_1v_3)$. If $y(v_4s_1v_3v_4) = 0$, then $y(v_1s_1v_3v_1) = w(s_1v_3) > 0$. So we assume that $y(v_4s_1v_3v_4) > 0$. By Lemma 5.2(vii), we have $y(v_4s_2v_3v_4) = 0$, so $y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$. Observe that if $y(v_1s_1v_3v_1)$ or $y(v_1v_5s_2v_3v_1)$ is an integer, then accordingly $y(v_4s_1v_3v_4)$ or $y(v_5s_2v_3v_4v_5)$ is an integer. Since $w(s_iv_3) > 0$ for $i = 1, 2$ by Lemma 5.2(iii), at least one of $y(v_1s_1v_3v_1)$, $y(v_4s_1v_3v_4)$, $y(v_1v_5s_2v_3v_1)$, and $y(v_5s_2v_3v_4v_5)$ is a positive integer, as claimed.

It remains to consider the subcase when neither $y(v_1s_1v_3v_1)$ nor $y(v_1v_5s_2v_3v_1)$ is an integer. We propose to show that

(24) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $0 < y(v_1s_1v_3v_1) < w(v_1s_1)$ and $0 < y(v_4s_1v_3v_4) < w(v_4s_1)$, by Lemma 3.1(i) and (ii), we have $x(v_1s_1) = x(v_4s_1) = 0$ and $x(v_1s_1v_3v_1) = x(v_4s_1v_3v_4) = 1$, which implies $x(v_3v_1) = x(v_3v_4)$. Furthermore, since $y(v_1v_5s_2v_3v_1) > 0$ and $y(v_5s_2v_3v_4v_5) > 0$, we have $x(v_1v_5s_2v_3v_1) = x(v_5s_2v_3v_4v_5) = 1$, which implies $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$. Thus $x(v_1v_5) = x(v_4v_5)$. Similarly, for each vertex $u \in V \backslash (V(T_2) \backslash a_2)$, we deduce that $x(uv_1) = x(uv_4)$. Let $T' = (V', A')$ be obtained from T by identifying v_1 and v_4; the resulting vertex is still denoted by v_1. Let $\boldsymbol{w'}$ be obtained from the restriction of $\boldsymbol{w}$ by setting $w'(v_1v_5) = w(v_1v_5) + w(v_4v_5)$, $w'(v_3v_1) = w(v_3v_1) + w(v_3v_4)$, $w'(v_1s_i) = w(v_1s_i) + w(v_4s_i)$ for $1 \le i \le r$, and $w'(uv_1) = w(uv_1) + w(uv_4)$ for each $u \in V \backslash (V(T_2) \backslash a_2)$. By the LP-duality theorem, $\boldsymbol{x}$ and $\boldsymbol{y}$ naturally correspond to solutions to $\mathbb{P}(T', \boldsymbol{w'})$ and $\mathbb{D}(T', \boldsymbol{w'})$ respectively with the same optimal value $\nu_w^*(T)$. From the hypothesis of Theorem 1.5, we deduce that $\nu_w^*(T)$ is an integer. This proves (24).

Subcase 2.7. $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, v_4s_2\}$.

In this case, by Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_iv_3v_4) = w(v_4s_i)$ for $i = 1, 2$, $y(v_1s_1v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5s_2v_3v_1) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) + y(v_1v_5v_3v_4v_1) + y(v_1v_5s_2v_3v_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_5v_3) + y(v_5s_2v_3v_4v_5) = w(v_4v_5)$. We may assume that $w(v_4s_i) = 0$ for $i = 1, 2$, for otherwise, $y(v_4s_1v_3v_4)$ or $y(v_4s_2v_3v_4)$ is a positive integer. Note that both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$. So s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$, and hence $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4v_1) = w(s_1v_3)$ and $y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$. If $y(v_1v_5s_2v_3v_4v_1) > 0$ or $y(v_1v_5v_3v_4v_1) > 0$, then $y(v_1s_1v_3v_4v_1) = w(s_1v_3) > 0$ by (8) or (10). So we assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. Then $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1s_1v_3v_1) = w(s_1v_3) - y(v_1s_1v_3v_4v_1)$. Since $w(s_1v_3) > 0$, at least one of $y(v_1s_1v_3v_1)$ and $y(v_1s_1v_3v_4v_1)$ is a positive integer.

Subcase 2.8. $K = \{v_3v_1, v_3v_4\}$.

In this case, by Lemma 3.1(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5s_2v_3v_1) = w(v_3v_1)$, $y(v_4s_1v_3v_4) + y(v_4s_2v_3v_4) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) + y(v_1s_1v_3v_4v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(v_3v_4)$. Since both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$, we see that s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$. Hence $y(v_1s_1v_3v_1) + y(v_4s_1v_3v_4) + y(v_1s_1v_3v_4) = w(s_1v_3)$ and $y(v_4s_2v_3v_4) + y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$.

If $y(v_1v_5s_2v_3v_4v_1) > 0$ or $y(v_1v_5v_3v_4v_1) > 0$, then $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_4v_1) = w(s_1v_3)$ by (8) and (10). It follows from (6) that either $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$ or $y(v_4s_1v_3v_4) = w(v_4s_1)$ and $y(v_1s_1v_3v_4v_1) = w(s_1v_3) - y(v_4s_1v_3v_4)$. Since $w(s_1v_3) > 0$, at least one of $y(v_4s_1v_3v_4)$ and $y(v_1s_1v_3v_4)$ is a positive integer. So we assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. If $y(v_1v_5s_2v_3v_1) = 0$, then either $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $y(v_4s_2v_3v_4) = w(v_4s_2)$ by (12), so $y(v_5s_2v_3v_4v_5) = w(s_2v_3) - w(v_4s_2)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$ and $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Suppose $y(v_1v_5s_2v_3v_1) > 0$. Then $y(v_1v_5v_3v_1) = w(v_5v_3)$ by (9). If $y(v_1s_1v_3v_4v_1) > 0$, then $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_4v_5)$, and $y(v_4s_1v_3v_4) = w(v_4s_1)$ by (9) and (11). It follows that $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer. So we assume that $y(v_1s_1v_3v_4v_1) = 0$. If $y(v_5s_2v_3v_4v_5) = 0$, then $y(v_4s_1v_3v_4) + y(v_4s_2v_3v_4) = w(v_3v_4)$. By Lemma 5.2(vii), at most one of $w(v_4s_1)$ and $w(v_4s_2)$ is nonzero. Thus either $y(v_4s_1v_3v_4) = 0$ or $y(v_4s_2v_3v_4) = 0$, and hence either $y(v_1s_1v_3v_1) = w(s_1v_3) > 0$ or $y(v_1v_5s_2v_3v_1) = w(s_2v_3) > 0$. So we further assume that $y(v_5s_2v_3v_4v_5) > 0$. If $y(v_1s_1v_3v_1)$ or $y(v_1v_5s_2v_3v_1)$ is an integer, then accordingly $y(v_4s_1v_3v_4)$ or $y(v_5s_2v_3v_4v_5)$ is an integer. Since $w(s_iv_3) > 0$ for $i = 1, 2$, at least one of $y(v_1s_1v_3v_1)$, $y(v_4s_1v_3v_4)$, $y(v_1v_5s_2v_3v_1)$, and $y(v_5s_2v_3v_4v_5)$ is a positive integer, as claimed.

It remains to consider the subcase when neither $y(v_1s_1v_3v_1)$ nor $y(v_1v_5s_2v_3v_1)$ is an integer. Now we can prove that $\nu_w^*(T)$ is an integer. Since the proof is the same as that of (24), we omit the details here.

Combining the above subcases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 5.8. *If $T_2/S = G_2$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_4, v_5)$, $s^* = v_2$, and $v_0 = v_4$. To establish the statement, by Lemma 3.4(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2 (i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of G_2, there are at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. Then (1) follows immediately from Lemma 4.5; the argument can be found in that of (3) in the proof of Lemma 5.5.

Lemma 5.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(7) subject to (5) and (6), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(8) subject to (5)-(7), $y(v_1v_6v_4v_1)$ is minimized.

Let us make some observations about $\boldsymbol{y}$ before proceeding.

(9) If K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The statements below follow instantly from Lemma 3.5(v).

(10) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in T_2.

(11) If $y(v_1v_6s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both v_1s_i and v_6v_4 are saturated by $\boldsymbol{y}$ in T_2.

(12) If $y(v_1v_6v_3s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then each arc in the set $\{v_3v_1, v_3v_4, v_6v_4, v_1s_i, v_6s_i\}$ is saturated by $\boldsymbol{y}$ in T_2.

Claim 1. $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

To justify this, we may assume that

(13) at most one of v_3v_1 and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

We proceed by considering two cases, depending on whether $v_1 \in \varphi(s_i)$ for some i.

Case 1.1. $v_1 \notin \varphi(s_i)$ for any $i \in \{1, 2, 3\}$.

By (2), (3) and Lemma 5.2(i), we may assume that $\varphi(s_1) = \{v_6\}$ and $\varphi(s_2) = \{v_3\}$. Thus

(14) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_3v_4v_1, v_1v_6s_1v_4v_1, v_1v_6v_3s_2v_4v_1\}$.

By Lemma 5.2(iii), $z(s_iv_4) = w(z_iv_4) > 0$. If s_iv_4 is outside $\mathcal{C}_0^y$ for $i = 1$ or 2, then s_iv_4 is saturated by $\boldsymbol{y}$ in T_2. In view of (14), we have $y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(s_2v_4) > 0$, and hence (1) follows from Lemma 3.2(iii). Similarly, if v_6s_1 or v_3s_2 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(v_3s_2) > 0$, and hence (1) follows from Lemma 3.2(iii). So we assume that

(15) s_iv_4 is contained in some cycle in $\mathcal{C}_0^y$ for $i = 1$ and 2. Furthermore, neither v_6s_1 nor v_3s_2 is saturated by $\boldsymbol{y}$ in T_2.

By (15) and Lemma 3.5(iii), at least one of v_1v_6 and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(v_1v_6)$. By (9), $\{v_1v_6\}$ is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. If v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, then v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2

by (13). So, by Lemma 3.5(vi), v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 and, by (10) and (12), we have $y(v_1v_6v_3s_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

Case 1.2. $v_1 \in \varphi(s_i)$ for some $i \in \{1, 2, 3\}$.

By (2), (3) and Lemma 5.2(i), we may assume that $v_1 \in \varphi(s_1)$, $v_6 \in \varphi(s_i)$, and $v_3 \in \varphi(s_j)$, with $\{1\} \neq \{i, j\} \subseteq \{1, 2, 3\}$. Furthermore,

(16) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_3v_4v_1, v_1s_1v_4v_1, v_1v_6s_iv_4v_1, v_1v_6v_3s_jv_4v_1\}$.

We may further assume that s_1v_4 is contained in some cycle in $\mathcal{C}_0^y$ and v_1s_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1s_1v_4v_1) = w(s_1v_4) > 0$ or $y(v_1s_1v_4v_1) = w(v_1s_1) > 0$. Hence (1) follows instantly from Lemma 3.2(iii). It follows from Lemma 3.5(vii) that v_4v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (13), v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2. By (10) and (12), we obtain $y(v_1v_6v_3s_jv_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. If v_6v_3 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. Thus, by Lemma 3.5(vii), v_1v_6 is saturated by $\boldsymbol{y}$ in T_2. We propose to show that

(17) $y(v_1v_6v_4v_1) = y(v_1v_6s_iv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_4v_1) > 0$ or $y(v_1v_6s_iv_4v_1) > 0$. Then v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1s_1. Then the multiset sum of the cycles C and $v_1v_6v_4v_1$ (resp. $v_1v_6s_iv_4v_1$) contains arc-disjoint cycles $v_1s_1v_4v_1$ and $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$ (resp. $C' = C[v_4, v_1] \cup \{v_1v_6, v_6s_i, s_iv_4\}$). Set $\theta = \min\{y(v_1v_6v_4v_1), y(C)\}$ (resp. $\min\{y(v_1v_6s_iv_4v_1), y(C)\}$). Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$ (resp. $y(v_1v_6s_iv_4v_1)$), $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$ (resp. $y(v_1v_6s_iv_4v_1) - \theta$), $y(v_1v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. It is easy to see that $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$ or $y'(v_1v_6s_iv_4v_1) < y(v_1v_6s_iv_4v_1)$, contradicting (8) or (6). Since v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, every cycle in $\mathcal{C}_0^y$ containing v_3v_1 passes through v_1s_1. Thus v_3v_1 is outside $\mathcal{C}_0^y$, and neither v_1s_1 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T.

Observe that v_6v_3 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_3. Then the multiset sum of the cycles C, $v_1v_6v_4v_1$ (resp. $v_1v_6s_iv_4v_1$), and the unsaturated arc v_3v_1 contain arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6v_4\}$ (resp. $C' = C[v_4, v_6] \cup \{v_6s_i, s_iv_4\}$). Set $\theta = \min\{y(v_1v_6v_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$ (resp. $\theta = \min\{y(v_1v_6s_iv_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$). Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$ (resp. $y(v_1v_6s_iv_4v_1)$), $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$ (resp. $y(v_1v_6s_iv_4v_1) - \theta$), $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. It is easy to see that $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$ or $y'(v_1v_6s_iv_4v_1) < y(v_1v_6s_iv_4v_1)$, contradicting (8) or (6). Hence v_6v_3 is not saturated by $\boldsymbol{y}$ in T.

Let C be a cycle in $\mathcal{C}_0^y$ containing s_1v_4. Then the multiset sum of the cycles C, each of the cycles $v_1v_6v_4v_1$ and $v_1v_6s_iv_4v_1$, and the unsaturated arcs v_6v_3, v_3v_1, and v_1s_1 contains arc-disjoint cycles $v_1s_1v_4v_1$ and $v_1v_6v_3v_1$. So, by Lemma 3.5(vi), we have $y(v_1v_6v_4v_1) = y(v_1v_6s_iv_4v_1) = 0$; this contradiction establishes (17).

Using (17), we obtain $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. This proves Claim 1.

The above proof yields the following statement, which will be used later.

(18) If Case 1.1 occurs, then every MFAS comes from $\{\{v_3v_1, v_4v_1\}, \{v_1v_6\}, \{v_4v_1, v_6v_3\}\}$. If Case 1.2 occurs, then every MFAS comes from $\{\{v_3v_1, v_4v_1\}, \{v_1v_6, v_4v_1\}, \{v_4v_1, v_6v_3\}\}$.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we first show that

(19) if $v_3 \in \varphi(s_i)$ for $i \in \{1,2,3\}$, then $y(v_1v_6v_3s_iv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_3s_iv_4v_1) > 0$. Then $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (12). So Lemma 3.2(iii) allows us to assume that $w(v_3v_1) = w(v_3v_4) = w(v_6v_4) = 0$. Let j and k be subscripts in $\{1,2,3\}$, if any, such that $v_6 \in \varphi(s_j)$ and $v_1 \in \varphi(s_k)$. If both $y(v_1s_kv_4v_1)$ and $y(v_1v_6s_jv_4v_1)$ are integral, then, by Claim 1, $y(v_1v_6v_3s_iv_4v_1)$ is a positive integer, so Claim 2 holds. Thus we may assume that $y(v_1s_kv_4v_1)$ or $y(v_1v_6s_jv_4v_1)$ is not integral. Then, by (11) and Lemma 3.2(iii), we have $j, k \neq i$. Furthermore, both v_1s_k and v_6s_j are outside $\mathcal{C}_0^y$, for otherwise, we can construct an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3s_jv_4v_1) < y(v_1v_6v_3s_jv_4v_1)$, contradicting (6).

Consider first the case when $y(v_1v_6s_jv_4v_1)$ is not integral. If $j = k$ and $y(v_1s_kv_4v_1) > 0$, then $y(v_1s_kv_4v_1) = w(v_1s_k) > 0$ by (11), so Claim 2 holds. Thus we may assume that $j \neq k$ if $y(v_1s_kv_4v_1) > 0$. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_jv_4v_1)$ and $y(v_1v_6v_3s_iv_4v_1)$ are positive, $x(v_1v_6s_jv_4v_1) = x(v_1v_6v_3s_iv_4v_1) = 1$ by Lemma 3.1(i). By Lemma 5.2(vi), $x(v_6s_j) = x(v_3s_i) = 0$. It follows that $x(s_jv_4) = x(v_6v_3) + x(s_iv_4)$. If v_6v_3 is outside $\mathcal{C}_0^y$, then $x(v_6v_3) = 0$ by Lemma 3.1(ii), because $z(v_6v_3) = y(v_1v_6v_3s_iv_4v_1) < w(v_6v_3)$. Thus $x(s_iv_4) = x(s_jv_4)$, contradicting Lemma 5.2(iv). So we assume that v_6v_3 is contained in some cycle in $\mathcal{C}_0^y$. Since $w(v_3v_4) = w(v_6v_4) = 0$ and (v_6, s_j) is outside $\mathcal{C}_0^y$, for any $u \in V \backslash (V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3s_iv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_j, then it passes through s_jv_4. By Lemma 3.1(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3s_i) + x(s_iv_4) = x(us_j) + x(s_jv_4)$. Hence $x(uv_6) = x(us_j)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_j, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_jv_4)$ for each $e \in \{v_6v_3, v_3s_i, s_iv_4\}$ and replacing $w(uv_6)$ with $w(uv_6) + w(us_j)$ for each $u \in V \backslash (V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3s_iv_4v_1) = y(v_1v_6s_jv_4v_1) + y(v_1v_6v_3s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_jv_4 for any $u \in V \backslash (V(T_2)\backslash a_2)$, let C' be the cycle arising from C by replacing the path us_jv_4 with the path $uv_6v_3s_iv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

In the other case when $y(v_1v_6s_jv_4v_1) = 0$ and $y(v_1s_kv_4v_1)$ is not integral, the proof goes along the same line, so we omit the details here.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some FAS K of $T_2\backslash a_2$ as described in (18). Recall that

(20) in Case 1.1, we have $v_1 \notin \varphi(s_i)$ for any $i \in \{1,2,3\}$, $\varphi(s_1) = \{v_6\}$, and $\varphi(s_2) = \{v_3\}$; in Case 1.2, we have $v_1 \in \varphi(s_1)$, $v_6 \in \varphi(s_i)$, and $v_3 \in \varphi(s_j)$, with $\{1\} \neq \{i,j\} \subseteq \{1,2,3\}$.

Depending on what K is, we distinguish among four cases.

Case 2.1. $K = \{v_4v_1, v_6v_3\}$ in Case 1.1 or $K = \{v_1v_6, v_4v_1\}$ in Case 1.2.

Consider first the subcase when $K = \{v_4v_1, v_6v_3\}$ in Case 1.1. Now $y(v_1v_6v_3v_1) = w(v_6v_3)$ and $y(v_1v_6v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1)$ (see (20)). If $y(v_1v_6s_1v_4v_1) = 0$, then $y(v_1v_6v_4v_1) = w(v_4v_1)$. If $y(v_1v_6s_1v_4v_1) > 0$, then $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (11), and hence $y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_6v_4)$. By the hypothesis of the present section, $w(K) = \tau_w(T_2\backslash a_2) > 0$. So at least one of $y(v_1v_6v_3v_1)$, $y(v_1v_6v_4v_1)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Next consider the subcase when $K = \{v_1v_6, v_4v_1\}$ in Case 1.2. Now $y(v_1s_1v_4v_1) = w(v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. So at least one of $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_3v_1)$ is a positive integer.

Case 2.2. $K = \{v_1v_6\}$ or $\{v_3v_1, v_4v_1\}$ in Case 1.1.

We only consider the subcase when $K = \{v_1v_6\}$, as the proof in the other subcase goes along the same line. Now $y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6)$, and v_3v_1 is outside $\mathcal{C}_0^y$.

Observe that $y(v_1v_6v_3v_4v_1) > 0$, for otherwise, if $y(v_1v_6s_1v_4v_1) > 0$, then $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (11), and hence $y(v_1v_6v_3v_1)+y(v_1v_6s_1v_4v_1) = w(v_1v_6)-w(v_6v_4)$; if $y(v_1v_6s_1v_4v_1) = 0$, then $y(v_1v_6v_3v_1)+y(v_1v_6v_4v_1) = w(v_1v_6)$. Let us show that $y(v_1v_6v_3v_1)$ is integral. Assume first that $y(v_1v_6s_1v_4v_1) > 0$. If v_6v_3 is outside $\mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ with $y(v_1v_6v_3v_1)+[y(v_1v_6s_1v_4v_1)]$ and $\lfloor y(v_1v_6s_1v_4v_1)\rfloor$, respectively; if v_6v_3 is contained in a cycle $C \in \mathcal{C}_0^y$, set $\theta = \min\{y(C), [y(v_1v_6s_1v_4v_1)]\}$ and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_1) > y(v_1v_6v_3v_1)$ while $y'(v_1v_6s_1v_4) < y(v_1v_6s_1v_4v_1)$ in either situation, so $\boldsymbol{y}'$ is a better choice than $\boldsymbol{y}$ (see (6)), a contradiction. Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Imitating the above proof, with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6s_1v_4v_1)$, we can reach a contradiction to (8).

Since $y(v_1v_6v_3v_4v_1) > 0$, by (10), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_6v_4v_1) = w(v_6v_4)$; so Lemma 3.2(iii) allows us to assume that $w(v_3v_1) = w(v_6v_4) = 0$. Thus the previous equality concerning $w(v_1v_6)$ becomes $y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. So we may assume that neither $y(v_1v_6s_1v_4v_1)$ nor $y(v_1v_6v_3v_4v_1)$ is integral, for otherwise, at least one of them is a positive integer. Observe that v_6s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ that contains v_6s_1, let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$, and let $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6s_1v_4v_1)+\theta$, $y(v_1v_6v_3v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7).

We propose to show that $\nu_w^*(T)$ is an integer. For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, $x(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$ by Lemma 3.1(i). Since $y(v_1v_6s_1v_4v_1) < w(v_6s_1)$, we have $x(v_6s_1) = 0$ by Lemma 3.1(ii). Thus $x(s_1v_4) = x(v_6v_3) + x(v_3v_4)$. Since $w(v_6v_4) = 0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$ or $v_6s_1v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 3.1(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_4) = x(us_1) + x(s_1v_4)$ or $x(uv_6) + x(v_6s_1) + x(s_1v_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_6) = x(us_j)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e = v_6v_3$ and v_3v_4 and replacing $w(uv_6)$ with $w(uv_6) + w(us_1)$ for any $u \in V\backslash V(T_2)\backslash a_2$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3v_4v_1) = y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4 for any $u \in V\backslash(V(T_2)\backslash a_2)$, let C' be the cycle arising from C by replacing the path us_1v_4 with the path $uv_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$

is an integer.

Case 2.3. $K = \{v_4v_1, v_6v_3\}$ in Case 1.2.

In this case, $y(v_1v_6v_3v_1) = w(v_6v_3)$ and $y(v_1s_1v_4v_1)+y(v_1v_6v_4v_1)+y(v_1v_6s_iv_4v_1) = w(v_4v_1)$ (see (20)). By Lemma 3.2(iii), we may assume that $w(v_6v_3) = 0$. Let us show that

(21) $y(v_1v_6s_iv_4v_1) = 0$.

Assume the contrary. Then, by (11), we have $y(v_1v_6v_4v_1) = w(v_6v_4)$, and v_1s_i is saturated by $\boldsymbol{y}$ in T_2. Lemma 3.2(iii) allows us to assume that $w(v_6v_4) = 0$ and that $y(v_1v_6s_iv_4v_1)$ is not integral. It follows from (6) and Lemma 3.5(v) that $i \neq 1$ and v_1s_1 is outside $\mathcal{C}_0^y$. We propose to prove that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6s_iv_4v_1)$ are positive, by Lemma 3.1(i), we have $x(v_1s_1v_4v_1) = x(v_1v_6s_iv_4v_1) = 1$. Since $y(v_1s_1v_4v_1) < w(v_1s_1)$, by Lemma 3.1(ii), we obtain $x(v_1s_1) = 0$, so $x(s_1v_4) = x(v_1v_6)+x(v_6s_i)+x(s_iv_4)$. If v_1v_6 is outside $\mathcal{C}_0^y$, then $x(v_1v_6) = 0$, because $z(v_1v_6) = y(v_1v_6s_iv_4v_1) < w(v_1v_6)$. By Lemma 5.2(vi), $x(v_1s_1) = x(v_6s_i) = 0$. Hence, $x(s_1v_4) = x(s_iv_4)$, contradicting Lemma 5.2(iv). So we assume that v_1v_6 is contained in some cycle in $\mathcal{C}_0^y$. Since $w(v_6v_3) = w(v_6v_4) = 0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_iv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 3.1(iv), we obtain $x(uv_1)+x(v_1v_6)+x(v_6s_i)+x(s_iv_4) = x(us_1)+x(s_1v_4)$. Hence $x(uv_1) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e)+w(s_1v_4)$ for $e \in \{v_1v_6, v_6s_i, s_iv_4\}$ and replacing $w(uv_1)$ with $w(uv_1)+w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$, Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6s_iv_4v_1) = y(v_1s_1v_4v_1) + y(v_1v_6s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4, let C' arise from C by replacing the path us_1v_4 with the path $uv_1v_6s_iv_4$, and set $y'(C') = y(C')+y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. So we may assume that (21) holds.

By (21), the equality concerning $w(v_4v_1)$ becomes $y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. As $w(v_4v_1) = w(K) = \tau_w(T_2\backslash a_2) > 0$, neither $y(v_1s_1v_4v_1)$ nor $y(v_1v_6v_4v_1)$ is integral. Observe that v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle containing v_1s_1 in $\mathcal{C}_0^y$, let $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$, and let $\theta = \min\{y(C), y(v_1v_6v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_1v_4v_1)$, $y(v_1v_6v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1s_1v_4v_1)+\theta$, $y(v_1v_6v_4v_1)-\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, contradicting (8). Moreover, $i \neq 1$, for otherwise, it can be shown similarly that v_6s_1 is outside $\mathcal{C}_0^y$, which implies $z(v_6s_1) = 0$, contradicting that $v_6 \in \varphi(s_1)$. Let us show that

(22) $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_4v_1)$ are positive, we have $x(v_1s_1v_4v_1) = x(v_1v_6v_4v_1) = 1$ by Lemma 3.1(i). By (16) and Lemma 3.2(iii), we have $y(v_1s_1v_4v_1) < w(v_1s_1)$ and hence $x(v_1s_1) = 0$. So $x(s_1v_4) = x(v_1v_6) + x(v_6v_4)$. Note that if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. For any $u \in V\backslash(V(T_2)\backslash a_2)$, if there exists a cycle $C \in \mathcal{C}_0^y$ containing uv_1 and passing through $v_1v_6v_4$, then by Lemma 3.1(iv), we obtain $x(uv_1) + x(v_1v_6) + x(v_6v_4) = x(us_1) + x(s_1v_4)$, and hence $x(uv_1) = x(us_1)$. Otherwise, since $w(v_6v_3) = 0$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_iv_4$. By Lemma 3.1(i) and (iv), we have $x(v_6v_4) \geq x(v_6s_i) + x(s_iv_4)$ and

$x(uv_1)+x(v_1v_6)+x(v_6s_i)+x(s_iv_4) = x(us_1)+x(s_1v_4)$. Since $x(v_1v_6v_4v_1) = 1$ and $x(v_1v_6s_iv_4v_1) \geq 1$, we see that $x(v_6v_4) \leq x(v_6s_i) + x(s_iv_4)$. Hence, $x(uv_1) = x(us_1)$ also holds. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e = v_1v_6$ and v_6v_4 and replacing $w(uv_1)$ with $w(uv_1) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_4v_1) = y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4 for any $u \in V\backslash(V(T_2)\backslash a_2)$, let C' arise from C by replacing the path us_1v_4 with the path $uv_1v_6v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value as $\boldsymbol{x}$ and $\boldsymbol{y}$. From the hypothesis of Theorem 1.5, (22) follows.

Case 2.4. $K = \{v_3v_1, v_4v_1\}$ in Case 1.2.

In this case, $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1s_1v_4v_1)+y(v_1v_6v_4v_1)+y(v_1v_6s_iv_4v_1)+y(v_1v_6v_3v_4v_1) = w(v_4v_1)$ (see (20)). By Lemma 3.2(iii), we may assume that $w(v_3v_1) = 0$.

If $y(v_1v_6s_iv_4v_1) = y(v_1v_6v_3v_4v_1) = 0$, then $y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. Since $w(v_4v_1) = w(K) = \tau_w(T_2\backslash a_2) > 0$, we see that $y(v_1s_1v_4v_1)$ is not integral. Imitating the proof of (22), it can be shown that $\nu_w^*(T)$ is an integer. So we assume that at least one of $y(v_1v_6v_3v_4v_1)$ and $y(v_1v_6s_iv_4v_1)$ is positive. By (10) or (11), v_6v_4 is saturated by $\boldsymbol{y}$ in T_2, and hence $y(v_1v_6v_4v_1) = w(v_6v_4)$. By Lemma 3.2(iii), we may assume that $w(v_6v_4) = 0$. If neither $y(v_1v_6s_iv_4v_1)$ nor $y(v_1v_6v_3v_4v_1)$ is integral then, imitating the proof in Case 2.2, it can be shown that $\nu_w^*(T)$ is an integer. It remains to consider the subcase when precisely one of them is positive. Now it can be shown that $\nu_w^*(T)$ is an integer. Since the proof is the same as that contained in the argument of (21), we omit the routine details here.

Combining the above four cases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 5.9. *If $T_2/S = G_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_4, v_5)$, $s^* = v_2$, and $v_0 = v_4$. To establish the statement, by Lemma 3.2(iii) and Lemma 3.4(ii), it suffices to prove that

(1) $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is an integer.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2 (i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. (The statement is exactly the same as (3) in the proof of Lemma 5.7.)

Lemma 5.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

Let t be the subscript in $\{1, 2, 3\}$ with $v_1 \in \varphi(s_t)$, if any. By (2), t is well defined. In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(7) subject to (5) and (6), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(8) subject to (5)-(7), $y(v_1s_tv_4v_1) + y(v_3v_4v_6v_3)$ is minimized.

Let us make some observations about $\boldsymbol{y}$ before proceeding.

(9) If K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The statements below follow instantly from Lemma 3.5(v) and the choice of $\boldsymbol{y}$.

(10) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, for any $i \in \{1, 2, 3\}$, we have $y(v_6s_iv_4v_6) = 0$; if $y(v_3s_iv_4v_6v_3) > 0$, then v_1s_i is saturated by $\boldsymbol{y}$ in T_2.

(11) If $y(v_1v_6s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both v_1s_i and v_4v_6 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, if $y(v_3v_4v_6v_3) > 0$, then v_3v_1 is saturated by $\boldsymbol{y}$ in T_2; for any $1 \le j \ne i \le 3$, if $y(v_3s_jv_4v_6v_3) > 0$, then both v_3v_1 and v_1s_j are saturated by $\boldsymbol{y}$ in T_2.

(12) If $y(v_3s_iv_4v_6v_3) > 0$ for some $i \in \{1, 2, 3\}$, then both v_3v_4 and v_6s_i are saturated by $\boldsymbol{y}$ in T_2.

(13) If $v_1 \in \varphi(s_i)$ for some $i \in \{1, 2, 3\}$, then $y(v_1s_iv_4v_6v_3v_1) = 0$.

Assume the contrary: $y(v_1s_iv_4v_6v_3v_1) > 0$. Then v_1v_6, v_3v_4, and v_4v_1 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(v). Let j and k be subscripts in $\{1, 2, 3\}$, if any, such that $v_3 \in \varphi(s_j)$ and $v_6 \in \varphi(s_k)$ (possibly $j = k$). As before, let $\uplus$ denote the multiset sum. Then $v_1s_iv_4v_6v_3v_1 \uplus v_1v_6v_3v_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3v_4v_6v_3$, $v_1s_iv_4v_6v_3v_1 \uplus v_1v_6s_kv_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_6s_kv_4v_6$, and $v_1s_iv_4v_6v_3v_1 \cup v_1v_6v_3s_jv_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3s_jv_4v_6v_3$. Thus, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6s_kv_4v_1)$, and $y(v_1v_6v_3s_jv_4v_1)$ are all zero. So $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1s_iv_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. Clearly, we may assume that $w(v_1v_6) = w(v_4v_1) = w(v_3v_4) = 0$, otherwise (1) holds. By (3), we have $\{j, k\} \ne \{i\}$. Let us show that one of $y(v_6s_kv_4v_6)$, $y(v_3s_jv_4v_6v_3)$, and $y(v_1s_iv_4v_6v_3v_1)$ is a positive integer or $\nu^*_w(T)$ is an integer. We proceed by considering two cases.

- k exists and $i \ne k$. In this case, observe first that v_6s_k is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ and hence (1) holds. Next, v_kv_4 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $k \ne j$, then $y(v_6s_kv_4v_6) = w(s_kv_4) > 0$; if $k = j$, then $y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6v_3) = w(s_kv_4) > 0$, and $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ by Lemma 3.5(v) provided $y(v_3s_kv_4v_6v_3) > 0$. So $y(v_6s_kv_4v_6)$ is a positive integer, and hence (1) also holds. Moreover, both v_6s_k and v_3s_j are outside $\mathcal{C}^y_0$, for otherwise, let C_1 (resp. C_2) be a cycle in $\mathcal{C}^y_0$ containing v_6s_k (resp. v_3s_j). Since $C_1 \uplus v_1s_iv_4v_6v_3v_1 = v_6s_kv_4v_6 \uplus C'_1$ and $C_2 \uplus v_1s_iv_4v_6v_3v_1 = v_3s_jv_4v_6v_3 \uplus C'_2$, where $C'_1 = C_1[v_4, v_6] \cup \{v_6v_3, v_3v_1, v_1s_i, s_iv_4\}$ and $C'_2 = C_2[v_4, v_3] \uplus \{v_3v_1, v_1s_i, s_iv_4\}$, by Lemma 3.5(viii), we have $y(C_i) = 0$ for $i = 1, 2$, a contradiction. It follows that v_6s_k is not saturated by $\boldsymbol{y}$ in T, and s_kv_4 is contained in some cycle in $\mathcal{C}^y_0$. By Lemma 3.5(vii), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, so $y(v_1s_iv_4v_6v_3v_1) + y(v_6s_kv_4v_6) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$. If $j = k$ and $y(v_3s_kv_4v_6v_3) > 0$, then v_6s_k is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(v), a contradiction. So either $j \ne k$ or $j = k$ and $y(v_3s_kv_4v_6v_3) = 0$. Since $w(v_6s_k) > 0$ and v_6s_k is outside $\mathcal{C}^y_0$, we have $y(v_6s_kv_4v_6) > 0$. Assume $y(v_6s_kv_4v_6)$ is not integral. Let us show that $\nu^*_w(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_6s_kv_4v_6)$ and $y(v_1s_iv_4v_6v_3v_1)$ are positive, by Lemma 3.1(i), we have $x(v_6s_kv_4v_6) = x(v_1s_iv_4v_6v_3v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_6s_k) = 0$. Hence $x(s_kv_4) = x(v_6v_3) + x(v_3v_1) + x(v_1s_i) + x(s_iv_4)$. Since $w(v_3v_4) = 0$ and v_6s_k is outside $\mathcal{C}^y_0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}^y_0$ contains uv_6, then it passes through $v_6v_3v_1s_iv_4$. Moreover, if a cycle in $\mathcal{C}^y_0$ contains us_k, then it passes through s_kv_4. By Lemma 3.1(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_1) + x(v_1s_i) + x(s_iv_4) = x(us_k)+x(s_kv_4)$. Hence $x(uv_6) = x(us_k)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_k, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(v_4 s_k)$ for $e \in \{v_6v_3, v_3v_1, v_1s_i, s_iv_4\}$ and replacing $w(uv_6)$ with $w(uv_6) + w(us_k)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1s_iv_4v_6v_3v_1) = y(v_1s_iv_4v_6v_3v_1) + y(v_6s_kv_4v_6)$; for each $C \in \mathcal{C}_0^y$ passing through us_kv_4, let C' arise from C by replacing the path us_kv_4 with the path $uv_6v_3v_1s_iv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

• Either k does not exist or $i = k$. In this case, by (3), we see that j exists; that is, $v_3 \in \varphi(s_j)$. Similar to the above case, we can show that either $y(v_3s_jv_4v_6v_3)$ is a positive integer or $\nu_w^*(T)$ is an integer. Since the proof goes along the same line (with v_3s_j and $y(v_3s_jv_4v_6v_3)$ in place of v_6s_k and $y(v_6s_kv_4v_6)$, respectively), we omit the details here. Hence we may assume that (13) holds.

(14) If $v_3 \in \varphi(s_j)$ for some $j \in \{1, 2, 3\}$, then $y(v_1v_6v_3s_jv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_3s_jv_4v_1) > 0$. Then v_3v_1, v_3v_4, and v_4v_6 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(v). Let i and k be subscripts in $\{1, 2, 3\}$, if any, such that $v_1 \in \varphi(s_i)$ and $v_6 \in \varphi(s_k)$ (possibly $i = k$). Since $v_1v_6v_3s_jv_4v_1 \uplus v_3v_4v_6v_3 = v_1v_6v_3v_4v_1 \uplus v_3s_jv_4v_6v_3$, and $v_1v_6v_3s_jv_4v_1 \cup v_6s_kv_4v_6 = v_1v_6s_kv_4v_1 \uplus v_3s_jv_4v_6v_3$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_3v_4v_6v_3) = y(v_6s_kv_4v_6) = 0$. So $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_3s_jv_4v_6v_3) = w(v_4v_6)$. Clearly, we may assume that $w(v_3v_1) = w(v_3v_4) = w(v_4v_6) = 0$, otherwise (1) holds. By (3), we have $\{i, k\} \neq \{j\}$. Let us show that one of $y(v_1s_iv_4v_1)$, $y(v_1v_6s_kv_4v_1)$, and $y(v_1v_6v_3s_jv_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. We proceed by considering two cases.

• i exists and $i \neq j$. In this case, observe first that v_1s_i is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1s_iv_4v_1) = w(v_1s_i) > 0$ and hence (1) holds. Next, s_iv_4 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $i \neq k$, then $y(v_1s_iv_4v_1) = w(s_iv_4) > 0$; if $i = k$, then $y(v_1s_iv_4v_1) + y(v_1v_6s_iv_4) = w(s_iv_4) > 0$, and $y(v_1s_iv_4v_1) = w(v_1s_i) > 0$ by Lemma 3.5(v) provided $y(v_1v_6s_iv_4v_1) > 0$. So $y(v_1s_iv_4v_1)$ is a positive integer, and hence (1) also holds. Moreover, both v_1s_i and v_6s_k are outside $\mathcal{C}_0^y$, for otherwise, let C_1 (resp. C_2) be a cycle in $\mathcal{C}_0^y$ containing v_1s_i (resp. v_6s_k). Since $C_1 \uplus v_1v_6v_3s_jv_4v_1 = v_1s_iv_4v_1 \cup C_1'$ and $C_2 \uplus v_1v_6v_3s_jv_4v_1 = v_1v_6s_kv_4v_1 \uplus C_2'$, where $C_1' = C_1[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3s_j, s_jv_4\}$ and $C_2' = C_2[v_4, v_6] \cup \{v_6v_3, v_3s_j, s_jv_4\}$, by Lemma 3.5(viii), we have $y(C_i) = 0$ for $i = 1, 2$, a contradiction. It follows that v_1s_i is not saturated by $\boldsymbol{y}$ in T and s_iv_4 is contained in some cycle in $\mathcal{C}_0^y$. By Lemma 3.5(vii), v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, so $y(v_1s_iv_4v_1) + y(v_1v_6s_kv_4v_1) + y(v_1v_6v_3s_jv_4v_1) = w(v_4v_1)$. If $i = k$ and $y(v_1v_6s_kv_4v_1) > 0$, then v_1s_i is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(v), a contradiction. So either $i \neq k$ or $i = k$ and $y(v_1v_6s_kv_4v_1) = 0$. Since $w(v_1s_i) > 0$ and v_1s_i is outside $\mathcal{C}_0^y$, we have $y(v_1s_iv_4v_1) > 0$. Assume $y(v_1s_iv_4v_1)$ is not integral. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_iv_4v_1)$ and $y(v_1v_6v_3s_jv_4v_1)$ are positive, by Lemma 3.1(i), we have $x(v_1s_iv_4v_1) = y(v_1v_6v_3s_jv_4v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_1s_i) = 0$. Hence $x(s_iv_4) = x(v_1v_6) + x(v_6v_3) + x(v_3s_j) + x(s_jv_4)$. Since $w(v_3v_1) = w(v_3v_4) = 0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_6v_3s_jv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_i, then it passes through s_iv_4. By Lemma 3.1(iv), we obtain $x(uv_1) + x(v_1v_6) + x(v_6v_3) + x(v_3s_j) + x(s_jv_4) = x(us_i) + x(s_iv_4)$. Hence $x(uv_1) = x(us_i)$. Clearly, we may assume that this equality holds in any other situation.

Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(v_4s_i)$ for $e \in \{v_1v_6, v_6v_3, v_3s_j, s_jv_4\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(us_i)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3s_jv_4v_1) = y(v_1v_6v_3s_jv_4v_1) + y(v_1s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_iv_4, let C' be obtained from C by replacing the path us_iv_4 with the path $uv_1v_6v_3s_jv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

• Either i does not exist or $i = j$. In this case, by (3), we see that k exists; that is, $v_6 \in \varphi(s_k)$. Similar to the above case, we can show that either $y(v_1v_6s_kv_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. Since the proof goes along the same line (with v_6s_k and $y(v_1v_6s_kv_4v_1)$ in place of v_1s_i and $y(v_1s_iv_4v_1)$, respectively), we omit the details here. Hence we may assume that (14) holds.

We proceed by considering two cases, depending on whether $\varphi(s_i) = \{v_1\}$ for some i.

Case 1. $\varphi(s_i) = \{v_1\}$ for some $i \in \{1, 2, 3\}$.

By Lemma 5.2(i), we may assume that $\varphi(s_1) = \{v_1\}$. Let j and k be subscripts in $\{1, 2, 3\}$, if any, such that $v_3 \in \varphi(s_j)$ and $v_6 \in \varphi(s_k)$ (possibly $j = k$). By (13) and (14), we have

(15) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_4v_1, v_1v_6s_kv_4v_1, v_3s_jv_4v_6v_3, v_1s_1v_4v_1, v_6s_kv_4v_6, v_1v_6v_3v_1, v_3v_4v_6v_3\}$.

Observe that neither s_1v_4 nor v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1s_1v_4v_1) = w(s_1v_4)$ or $w(v_1s_1)$; both of them are positive, so (1) holds. By Lemma 5.2(iii), $z(s_1v_4) = w(z_1v_4) > 0$. Thus there exists a cycle $C \in \mathcal{C}_0^y$ containing s_1v_4; subject to this, C is chosen to contain v_1s_1 if possible. If v_1s_1 is outside C, then v_1s_1 is not saturated by $\boldsymbol{y}$ in T. By Lemma 3.5(vii), v_4v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence $y(v_1s_1v_4v_1) + y(v_1v_6s_kv_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$.

(16) If $w(v_4v_1) > 0$, then either $y(v_1s_1v_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer.

To justify this, assume $y(v_1s_1v_4v_1)$ is not a positive integer. Then at least one of $y(v_1v_6s_kv_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ is positive. Observe that v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ containing v_1s_1. If $y(v_1v_6v_3v_4v_1) > 0$ then, using $D \uplus v_1v_6v_3v_4v_1 = v_1s_1v_4v_1 \uplus D'$, where $D' = D[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_4\}$, and applying Lemma 3.5(viii), we deduce that $y(D) = 0$, a contradiction. If $y(v_1v_6s_kv_4v_1) > 0$, then a contradiction can be reached similarly. Since $w(v_1s_1) > 0$, we obtain $y(v_1s_1v_4v_1) > 0$. As $y(v_1s_1v_4v_1)$ is not integral, at least one of $y(v_1v_6s_kv_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ is not integral. Let us show that $\nu_w^*(T)$ is an integer.

We only consider the case when $y(v_1v_6v_3v_4v_1)$ is not integral, as the proof in the other case when $y(v_1v_6v_3v_4v_1) = 0$ and $y(v_1v_6s_kv_4v_1) > 0$ goes along the same line.

Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, by Lemma 3.1(i), we have $x(v_1s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_1s_1) = 0$, because v_1s_1 is not saturated by $\boldsymbol{y}$. It follows that $x(s_1v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Observe that there is no cycle D in $\mathcal{C}_0^y$ that contains the path $v_1v_6s_kv_4$, for otherwise, let $\theta = \min\{y(D), y(v_1v_6v_3v_4v_1)\}$, let $D' = D[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(D)$, $y(D')$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6s_kv_4v_1)$ with $y(D) - \theta$, $y(D') + \theta$, $y(v_1v_6v_3v_4v_1) - \theta$, and $y(v_1v_6s_kv_4v_1) + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7). For any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 3.1(iv), we obtain $x(uv_1) +$

$x(v_1v_6) + x(v_6v_3) + x(v_3v_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_1) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e \in \{v_1v_6, v_6v_3, v_3v_4\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1s_1v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4, let C' be obtained from C by replacing the path us_1v_4 with the path $uv_1v_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. So (16) follows.

By (16) and Lemma 3.2(iii), we may assume that $w(v_4v_1) = 0$ hereafter.

(17) If k exists (so $v_6 \in \varphi(s_k)$) and $w(v_4v_6) > 0$, then either $y(v_6s_kv_4v_6)$ is a positive integer or $\nu_w^*(T)$ is an integer.

To justify this, observe first that v_6s_k is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$, so (17) holds. Next, s_kv_4 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $j \neq k$, then $y(v_6s_kv_4v_6) = w(s_kv_4) > 0$; if $j = k$, then $y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6v_3) = w(s_kv_4)$, and $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ by Lemma 3.5(v) provided $y(v_3s_kv_4v_6v_3) > 0$, so (17) also holds. By Lemma 5.2(iii), s_kv_4 is saturated by $\boldsymbol{y}$ in T, so s_kv_4 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_6s_k if possible. Clearly, if v_6s_k is not on C, then v_6s_k is not saturated by $\boldsymbol{y}$ in T. By Lemma 3.5(vii), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, and hence $y(v_6s_kv_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$.

Assume $y(v_6s_kv_4v_6)$ is not a positive integer. Then at least one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive, say the former. Note that v_6s_k is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ containing v_6s_k. Set $D' = D[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$ and $\theta = \min\{y(v_3v_4v_6v_3), y(C)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_6s_kv_4v_6)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3) - \theta$, $y(v_6s_kv_4v_6) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (8). Since $w(v_6s_k) > 0$, we have $y(v_6s_kv_4v_6) > 0$. As $y(v_6s_kv_4v_6)$ is not integral, $y(v_3v_4v_6v_3)$ or $y(v_3s_jv_4v_6v_3)$ is not integral. If $y(v_3s_jv_4v_6v_3) > 0$, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(v), so $y(v_3v_4v_6v_3) = w(v_3v_4)$. Hence we may assume that exactly one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive. Let us show that $\nu_w^*(T)$ is an integer.

We only consider the case when $y(v_3v_4v_6v_3)$ is not integral, because the proof in the other case when $y(v_3v_4v_6v_3) = 0$ and $y(v_3s_jv_4v_6v_3) > 0$ goes along the same line.

Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_6s_kv_4v_6)$ and $y(v_3v_4v_6v_3)$ are positive, we have $x(v_6s_kv_4v_6) = x(v_3v_4v_6v_3) = 1$ by Lemma 3.1(i). Since v_6s_k is not saturated by $\boldsymbol{y}$ in T, we obtain $x(v_6s_k) = 0$ by Lemma 3.1(ii). It follows that $x(s_kv_4) = x(v_6v_3) + x(v_3v_4)$. For any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_k, then it passes through s_kv_4. By Lemma 3.1(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_4) = x(us_k) + x(s_kv_4)$. Hence $x(uv_6) = x(us_k)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_k, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_kv_4)$ for $e = v_6v_3$ and v_3v_4 and replacing $w(uv_6)$ with $w(uv_6) + w(us_k)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_3v_4v_6v_3) = y(v_3v_4v_6v_3) + y(v_6s_kv_4v_6)$; for each $C \in \mathcal{C}_0^y$ passing through us_iv_4, let C' be the cycle arising from C by replacing the path us_kv_4 with the path $uv_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. So (17) holds.

By (17) and Lemma 3.2(iii), we may assume that if $w(v_4v_6) > 0$, then k does no exist, and hence j exists (so $v_3 \in \varphi(s_j)$) by (3).

(18) If $w(v_4v_6) > 0$, then at least one of $y(v_1v_6v_3v_1)$, $y(v_3v_4v_6v_3)$, and $y(v_3s_jv_4v_6v_3)$ is a positive integer.

To justify this, note that neither s_jv_4 nor v_3s_j is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_3s_jv_4v_6v_3) = w(s_jv_4)$ or $w(v_3s_j)$; both of them are positive, so (18) holds. By Lemma 5.2(iii), s_jv_4 is saturated by $\boldsymbol{y}$ in T, so s_jv_4 is contained in a cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_3s_j if possible. Clearly, if v_3s_j is not on C, then v_3s_j is not saturated by $\boldsymbol{y}$ in T. By Lemma 3.5(iii), at least one of v_4v_6 and v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. Furthermore, by Lemma 3.5(iv), if v_6v_3 is contained in some cycle in $\mathcal{C}_0^y$, then v_4v_6 is saturated by $\boldsymbol{y}$ in T_2. If v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_3v_4v_6v_3) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$ by Lemma 3.5(v) provided $y(v_3s_jv_4v_6v_3) > 0$. So at least one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is a positive integer, and hence (18) holds. Thus we may assume that v_4v_6 is not saturated by $\boldsymbol{y}$ in T_2, which implies that v_6v_3 saturated by $\boldsymbol{y}$ in T_2. It follows that $y(v_1v_6v_3v_1)+y(v_3v_4v_6v_3)+y(v_3s_jv_4v_6v_3) = w(v_6v_3)$. If $w(v_6v_3) = 0$, then $K = \{v_4v_1, v_6v_3, v_6s_j\}$ is an FAS of T with total weight zero, so $\tau_w(T_2\backslash a_2) = 0$, contradicting the hypothesis (α) of this section. Therefore $w(v_6v_3) > 0$. If $y(v_3s_jv_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (15) and Lemma 3.5(v). So we may further assume that exactly one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive, and thus $y(v_1v_6v_3v_1) > 0$.

Let us show that $y(v_1v_6v_3v_1)$ is an integer. Suppose not. Then $y(v_3v_4v_6v_3)$ or $y(v_3s_jv_4v_6v_3)$ is not integral, say the former (the proof in the other case goes along the same line). Since v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 and $w(v_6s_j) = 0$, the arc v_1v_6 is outside $\mathcal{C}_0^y$. If v_3v_1 is also outside $\mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$ and $y(v_1v_6v_3v_1)$ with $y(v_3v_4v_6v_3)-\theta$ and $y(v_1v_6v_3v_1)+\theta$, respectively, where $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3v_4v_6v_3)\}$; if v_3v_1 is contained in some cycle $C \in \mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3) - \sigma$, $y(v_1v_6v_3v_1) + \sigma$, $y(C) - \sigma$, $y(C') + \sigma$, respectively, where $C' = C[v_4, v_3] \cup \{v_3v_4\}$ and $\sigma = \min\{w(v_1v_6) - z(v_1v_6), y(C), y(v_3v_4v_6v_3)\}$. It is easy to see that in either situation $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (8). This proves (18).

By (16)-(18), we may assume that $w(v_4v_1) = w(v_4v_6) = 0$. Since each of $\{v_4v_1, v_4v_6, v_1v_6\}$, $\{v_4v_1, v_4v_6, v_6v_3\}$, and $\{v_4v_1, v_4v_6, v_3v_1\}$ is a minimal FAS of $T_2\backslash a_2$,

$$\epsilon = \min\{w(v_1v_6), w(v_6v_3), w(v_3v_1)\} > 0$$

by the hypothesis (α) of this section. By Lemma 3.5(vii), we obtain $y(v_1v_6v_3v_1) = \epsilon > 0$. Thus (1) is established in the present case.

Case 2. $\varphi(s_i) \neq \{v_1\}$ for any $i \in \{1, 2, 3\}$.

By the hypothesis of the present case, we may assume that $v_6 \in \varphi(s_1)$, $v_3 \in \varphi(s_2)$, and $v_1 \in \varphi(s_i)$ for $i = 1$ or 2. By (13) and (14), we have

(19) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_3v_4v_6v_3, v_1v_6v_3v_4v_1, v_6s_1v_4v_6, v_1v_6s_1v_4v_1, v_3s_2v_4v_6v_3, v_1s_1v_4v_1, v_1s_2v_4v_1\}$

and $y(v_1s_iv_4v_1) = 0$ for $i = 1$ or 2.

Claim 1. $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

To justify this, note that $z(s_iv_4) = w(s_iv_4) > 0$ for $i = 1$ and 2 by Lemma 5.2(iii). Depending on the saturation of s_1v_4 and s_2v_4, we distinguish among three subcases.

Subcase 1.1. s_1v_4 is contained in some cycle $C \in \mathcal{C}_0^y$. In this subcase, v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, v_4v_6 is not saturated by $\boldsymbol{y}$ in T, because it is outside $\mathcal{C}_0^y$. By Lemma 3.5(iii), v_6s_1 is saturated by $\boldsymbol{y}$ in T_2. By (11), we have $y(v_1v_6s_1v_4v_1) = 0$, which together with (19) implies $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$, so (1) holds. Clearly, v_4v_1 is outside $\mathcal{C}_0^y$. We proceed by considering two subsubcases.

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in T_2 (and hence in T). Then, by Lemma 3.5(iii), v_1s_1 and at least one of v_1s_2 and s_2v_4 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, v_1s_2 is outside $\mathcal{C}_0^y$. If v_1s_2 is not saturated by $\boldsymbol{y}$ in T, then $y(v_3s_2v_4v_6v_3) = 0$, for otherwise, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_2v_4v_1)$ and $y(v_3s_2v_4v_6v_3)$ with $y(v_1s_2v_4v_1) + \theta$ and $y(v_3s_2v_4v_6v_3) - \theta$, where $\theta = \min\{w(v_4v_1) - z(v_4v_1), w(v_1s_2) - z(v_1s_2), y(v_3s_2v_4v_6v_3)\} > 0$. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (6). It follows from (19) that $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$, so (1) holds. Thus we may assume that v_1s_2 is saturated by $\boldsymbol{y}$ in T_2. If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. By Lemma 3.5(iii), v_1v_6 is outside $\mathcal{C}_0^y$, for otherwise, v_4v_1 would be saturated by $\boldsymbol{y}$ in T_2, a contradiction. So we may assume that v_1v_6 is not saturated by $\boldsymbol{y}$ in T. By Lemma 3.5(iii), v_6s_1 is saturated by $\boldsymbol{y}$ in T_2. If v_6v_3 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_6v_3, v_6s_1, v_1s_1, v_1s_2\}$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 3.5(iii), v_6v_3 is outside $\mathcal{C}_0^y$. Furthermore, v_3v_1, v_3s_2, and v_3v_4 are all saturated by $\boldsymbol{y}$ in T_2. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_3v_4, v_6s_1, v_1s_1, v_1s_2, v_3s_2\}$. By (9), J is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

Next assume that v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. We may assume that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_6\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. Thus, by (10), we have $y(v_1v_6v_3v_4v_1) = 0$. If $y(v_1v_6s_1v_4v_1) = 0$ and v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1, v_4v_6\}$. So we may assume that $y(v_1v_6s_1v_4v_1) > 0$ or v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2. Consider the situation when $y(v_1v_6s_1v_4v_1) > 0$. Now, by (11), v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, and $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Moreover, at least one of v_1s_2 and s_2v_4 is saturated by $\boldsymbol{y}$ in T_2 (otherwise, $y(v_1s_2v_4v_1)$ can be made larger). If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$ or $\{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$; if v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2, then v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iiv). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. So we may assume that $y(v_1v_6s_1v_4v_1) = 0$ and v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 3.5(vii), v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. If v_6s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6s_1, v_6v_3\}$. So we further assume that v_6s_1 is not saturated by $\boldsymbol{y}$ in T_2. We propose to show that

(20) $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$.

We only prove that $y(v_3s_2v_4v_6v_3) = 0$, as the proof of the other equality $y(v_3v_4v_6v_3) = 0$ goes along the same line. Assume the contrary: $y(v_3s_2v_4v_6v_3) > 0$. Depending on the saturation of v_1v_6 and v_3v_1, we consider several possibilities.

• Both v_1v_6 and v_3v_1 are not saturated by $\boldsymbol{y}$ in T. Define $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3s_2v_4v_6v_3)\}$. Then $\theta > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3s_2v_4v_6v_3)$ and

$y(v_1v_6v_3v_1)$ with $y(v_3s_2v_4v_6v_3)-\theta$ and $y(v_1v_6v_3v_1)+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_3s_2v_4v_6v_3) < y(v_3s_2v_4v_6v_3)$, contradicting (6).

• v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_1v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. Since v_6v_3 is saturated by $\boldsymbol{y}$ in T_2, cycle C passes through $v_6s_1v_4$. Thus the multiset sum of the cycles C, $v_3s_2v_4v_6v_3$ and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_6s_1v_4v_6$ and $v_1v_6v_3v_1$. From Lemma 3.5(vi) we deduce that $y(v_3s_2v_4v_6v_3)=0$, a contradiction.

• v_1v_6 is is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $D \in \mathcal{C}_0^y$. It is clear that D passes through $v_1s_iv_4$ for $i=1$ or 2. Furthermore, the multiset sum of D, $v_3s_2v_4v_6v_3$, and the unsaturated arc v_1v_6 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $D' = D[v_4,v_3] \cup \{v_3s_2, s_2v_4\}$. Define $\theta = \min\{y(D), y(v_3s_2v_4v_6v_3), w(v_1v_6)-z(v_1v_6)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(D)$, $y(D')$, $y(v_3s_2v_4v_6v_3)$, and $y(v_1v_6v_3v_1)$ with $y(D)-\theta$, $y(D')+\theta$, $y(v_3s_2v_4v_6v_3)-\theta$, and $y(v_1v_6v_3v_1)+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_3s_2v_4v_6v_3) < y(v_3s_2v_4v_6v_3)$, contradicting (6).

• v_1v_6 and v_3v_1 are contained in some cycles C and D in $\mathcal{C}_0^y$, respectively. If v_3v_1 is on C, then the multiset sum of C and $v_3s_2v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_6s_1v_4v_6$, and $C' = C[v_4,v_3] \cup \{v_3s_2, s_2v_4\}$; if v_3v_1 is outside C, then the multiset sum of C, D, and $v_3s_2v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_6s_1v_4v_6$, $C' = C[v_4,v_1] \cup \{v_1s_i, s_iv_4\}$ for $i=1$ or 2, and $D' = D[v_4,v_3] \cup \{v_3s_2, s_2v_4\}$. In either situation from the optimality of $\boldsymbol{y}$ we deduce that $y(v_3s_2v_4v_6v_3)=0$.

Combining the above observations, we see that (20) holds. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

Subcase 1.2. s_1v_4 is saturated by $\boldsymbol{y}$ in T_2 and s_2v_4 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_3s_2 if possible. In this subcase, observe first that both v_1s_1 and v_6s_1 are outside $\mathcal{C}_0^y$. Next, v_3s_2 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_3s_2v_4v_6v_3) = w(v_3s_2) > 0$, so (1) holds. If both v_6v_3 and v_1s_2 are saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_4, v_1s_2, v_6v_3\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. We proceed by considering two subsubcases.

(a) v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. Now v_4v_6 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii).

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in T. Then both v_1v_6 and v_1s_2 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 3.5(iii). If v_1s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$; otherwise, v_1s_1 is not saturated by $\boldsymbol{y}$ in T. By (11), we have $y(v_1v_6s_1v_4v_1)=0$. Let us show that

(21) $y(v_6s_1v_4v_6)=0$.

Indeed, if v_6v_3 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles C, $v_6s_1v_4v_6$, and the unsaturated arcs v_4v_1, v_1s_1, and v_6v_3 (or v_3s_2 if it is outside C) contains arc-disjoint cycles $v_1s_1v_4v_1$ and $v_3s_2v_4v_6v_3$. Thus, by Lemma 3.5(vi), we have $y(v_6s_1v_4v_6)=0$. If v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$, then C contains v_3v_4 or v_3s_2. Thus the multiset sum of cycles C, $v_6s_1v_4v_6$, and the unsaturated arcs v_4v_1 and v_1s_1 contains arc-disjoint cycles $v_1s_1v_4v_1$ and one of $v_3v_4v_6v_3$ and $v_3s_2v_4v_6v_3$. Thus, by Lemma 3.5(vi), we have $y(v_6s_1v_4v_6)=0$. This proves (21).

It follows from (19) and (21) that $y(v_1s_1v_4v_1) = w(s_1v_4) > 0$, so (1) holds. Thus we may assume that v_4v_1 is saturated by $\boldsymbol{y}$ in T (and hence in T_2). Then we may further assume that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_3v_1\}$. Thus $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. By Lemma 3.5(vii), v_1v_6 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (10), we have $y(v_1v_6v_3v_4v_1) = 0$. Let us show that

(22) $y(v_1v_6s_1v_4v_1) = 0$.

To justify this, we consider four possibilities, depending on the saturation of v_6v_3 and v_3v_1.

• Both v_6v_3 and v_3v_1 are saturated by $\boldsymbol{y}$ in T. Now define $\theta = \min\{w(v_6v_3)-z(v_6v_3), w(v_3v_1)-z(v_3v_1), y(v_1v_6s_1v_4v_1)\}$. Then $\theta > 0$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ with $y(v_1v_6v_3v_1)+\theta$ and $y(v_1v_6s_1v_4v_1)-\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6).

• v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Now the multiset sum of the cycles C, $v_1v_6s_1v_4v_1$ and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$. Define $\theta = \min\{w(v_3v_1) - z(v_3v_1), y(C), y(v_1v_6s_1v_4v_1)\}$. Then $\theta > 0$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6s_1v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6).

• v_6v_3 is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $D \in \mathcal{C}_0^y$. Now D passes through $v_1s_2v_4$. Since the multiset sum of the cycles D, $v_1v_6s_1v_4v_1$, and the unsaturated arc v_6v_3 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $v_1s_2v_4v_1$, by Lemma 3.5(vi), we have $y(v_1v_6s_1v_4v_1) = 0$, a contradiction.

• v_6v_3 and v_3v_1 are contained in some cycles C and D in $\mathcal{C}_0^y$, respectively. Now if v_3v_1 is on C, then the multiset sum of the cycles C and $v_1v_6s_1v_4v_1$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_1s_2v_4v_1$, and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$; otherwise, the multiset sum of the cycles C, D, and $v_1v_6s_1v_4v_1$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_1s_2v_4v_1$, and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$, and $D' = D[v_4, v_3] \cup C[v_3, v_4]$. In each situation from the optimality of $\boldsymbol{y}$ we deduce that $y(v_1v_6s_1v_4v_1) = 0$.

Combining the above observations, we see that (22) holds. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. By (9), K is an MFAS of $T_2 \backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$.

(b) v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. Now v_1s_2 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 3.5(vii), v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. Since $z(v_1s_2) > 0$, by Lemma 5.2(vii), we have $z(v_1s_1) = 0$. Furthermore, we may assume that $y(v_1v_6v_3v_4v_1) = 0$, for otherwise, both v_3v_1 and v_4v_6 saturated by $\boldsymbol{y}$ in T_2 by (10). Hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_3v_1\}$. If $y(v_1v_6s_1v_4v_1) = 0$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3, s_1v_4\}$; if $y(v_1v_6s_1v_4v_1) > 0$ then, by (11), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, and either v_3v_1 is saturated by $\boldsymbol{y}$ in T_2 or $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_4v_1, v_4v_6, v_3v_1\}$ or $\{v_4v_1, v_4v_6, v_6v_3\}$. Therefore $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$.

Subcase 1.3. s_iv_4 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1$ and 2. In this subcase, since $\mathcal{C}_0^y \neq \emptyset$, v_3v_4 is contained in some cycle in $\mathcal{C}_0^y$. By (12), we have $y(v_3s_2v_4v_6v_3) = 0$. Thus $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$ and (1) holds. This completes the proof of Claim 1.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2^y$ or $\nu_w^*(T)$ is an integer.

To justify this, note that $y(\mathcal{C}_2) = w(K)$ for some MFAS K of $T_2 \backslash a_2$ by Claim 1. From the proof of Claim 1, we see that K has ten possibilities. So we proceed by considering them accordingly.

Subcase 2.1. K is one of $\{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$, $\{v_4v_1, v_6v_3, v_6s_1\}$, and $\{v_4v_1, v_6v_3, s_1v_4\}$.

In this subcase, by (15) and (19), we have $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$ if $K = \{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$, $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$ if $K = \{v_4v_1, v_6v_3, v_6s_1\}$, and $y(v_6s_1v_4v_6) = w(s_1v_4) > 0$ if $K = \{v_4v_1, v_6v_3, s_1v_4\}$, as desired.

Subcase 2.2. $K = \{v_3v_1, v_3v_4, v_6s_1, v_1s_1, v_1s_2, v_3s_2\}$.

In this subcase, by (15) and (19), we have $y(v_6s_1v_4v_6) + y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ and $y(v_3v_4v_6v_3)+y(v_1v_6v_3v_4v_1) = w(v_3v_4)$. So we may assume that $y(v_1v_6s_1v_4v_1) > 0$, for otherwise, $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$. It follows from Lemma 3.5(v) that v_4v_6 is saturated by $\boldsymbol{y}$ in T_2. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10), and hence $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$; if $y(v_1v_6v_3v_4v_1) = 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ and so $y(v_6s_1v_4v_6) = w(v_4v_6) - y(v_3v_4v_6v_3)$. Since $w(v_6s_1) > 0$, at least one of $y(v_6s_1v_4v_6)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Subcase 2.3. $K = \{v_6v_3, v_6s_1, v_1s_1, v_1s_2\}$ or $\{v_6v_3, s_1v_4, v_1s_2\}$.

In this subcase, we only consider the situation when $K = \{v_6v_3, s_1v_4, v_1s_2\}$, as the proof in the other situation goes along the same line.

Given the arcs in K, we have $y(v_1s_2v_4v_1) = w(v_1s_2)$, $y(v_1s_1v_4v_1)+y(v_6s_1v_4v_6)+y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3)$. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10). Thus $y(v_1s_1v_4v_1) + y(v_1v_6s_1v_4) = w(s_1v_4)$. If $y(v_1v_6s_1v_4v_1) > 0$, then one more equality $y(v_1s_1v_4v_1) = w(v_1s_1)$ holds by (11). Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer. So we assume that $y(v_1v_6v_3v_4v_1) = 0$ in the following discussion.

Assume first that $y(v_1v_6s_1v_4v_1) > 0$. Then $y(v_1s_1v_4v_1) = w(v_1s_1)$ and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$ by (11). If $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$, then $y(v_6s_1v_4v_6) = w(v_4v_6)$, and hence $y(v_1v_6s_1v_4v_1) = w(s_1v_4) - y(v_1s_1v_4v_1) - y(v_6s_1v_4v_6)$. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$, $y(v_6s_1v_4v_6)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer. So we assume that $y(v_3v_4v_6v_3)$ or $y(v_3s_2v_4v_6v_3)$ is positive. By (11), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$; by (12), one more equality $y(v_3v_4v_6v_3) = w(v_3v_4)$ holds if $y(v_3s_2v_4v_6v_3) > 0$. Thus $y(v_6s_1v_4v_6)$, $y(v_1v_6s_1v_4v_1)$, $y(v_3v_4v_6v_3)$, and $y(v_3s_2v_4v_6v_3)$ are all integers.

Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Then $y(v_1s_1v_4v_1) + y(v_6s_1v_4v_6) = w(s_1v_4)$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_6v_3v_4v_6) = w(v_3v_4)$ by (12), so $y(v_1v_6v_3v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3) - w(v_3v_4)$; if $y(v_3s_2v_4v_6v_3) = 0$, then $y(v_1v_6v_3v_1) + y(v_6v_3v_4v_6) = w(v_6v_3)$. Since both v_1v_6 and v_3v_1 are outside $\mathcal{C}_0^y$, from the choice of $\boldsymbol{y}$, we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$. This implies that in either situation $y(v_3s_2v_4v_6v_3)$ and $y(v_6v_3v_4v_6)$ are integers. On the other hand, since both v_4v_6 and v_6s_1 are outside $\mathcal{C}_0^y$, by (8), we obtain $y(v_6s_1v_4v_6) = \min\{w(v_6s_1), w(v_4v_6)-y(v_6v_3v_4v_6)-y(v_3s_2v_4v_6v_3)\}$, which is also an integer. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_6s_1v_4v_6)$ is a positive integer.

Subcase 2.4. $K = \{v_1v_6, v_4v_6, v_4v_1\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. By Lemma 3.2(iii) and Lemma 5.2(vi), we may assume that $w(v_1v_6) = w(v_4v_1) = 0$ and thus $w(v_4v_6) = w(K) > 0$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (12), and thus we may assume that $w(v_3v_4) = 0$. Hence $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) = w(v_4v_6)$ or $y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. If $y(v_6s_1v_4v_6)$ is an integer, then one of $y(v_3v_4v_6v_3)$, $y(v_6s_1v_4v_6)$, and $y(v_6s_2v_4v_6v_3)$ is a positive integer. So we assume that $y(v_6s_1v_4v_6)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the proof of (17) (with $y(v_6s_1v_4v_6)$ in place of $y(v_6s_iv_4v_6)$).

Subcase 2.5. $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$.

In this subcase, we have $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) = w(v_1s_2)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6)$, and $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) =$

$w(v_4v_6)$. By Lemma 3.2(iii), we may assume that $w(v_1s_1) = w(v_1s_2) = 0$.

Assume first that $y(v_1v_6v_3v_4v_1) > 0$. Then $y(v_6s_1v_4v_6) = 0$ and $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (10). So $y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. By (12), one more equality $y(v_3v_4v_6v_3) = w(v_3v_4)$ holds if $y(v_3s_2v_4v_6v_3) > 0$. So both $y(v_3v_4v_6v_3)$ and $y(v_3s_2v_4v_6v_3)$ are integers. By Lemma 3.2(iii), we may assume that $w(v_3v_1)$ and $w(v_4v_6)$ are both zero. Thus $y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6) > 0$. By Lemma 3.2(iii), we may assume that neither $y(v_1v_6v_3v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral. Observe that v_6s_1 is outside $\mathcal{C}_0^y$, for otherwise, let $C \in \mathcal{C}_0^y$ be a cycle containing v_6s_1. Then C contains s_1v_4. Let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$ and $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_4v_1) - \theta$, $y(v_1v_6s_1v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7). Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, we have $x(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$ by Lemma 3.1(i). So $x(v_6s_1) + x(s_1v_4) = x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_6s_1v_4v_1) < w(v_6s_1)$, by Lemma 3.1(ii), we have $x(v_6s_1) = 0$, which implies $x(s_1v_4) = x(v_6v_3) + x(v_3v_4)$. For any $u \in V \backslash (V(T_2) \backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 3.1(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_6) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by setting $w'(uv_6) = w(uv_6) + w(us_1)$ for any $u \in V \backslash (V(T_2) \backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through us_1v_4 with $u \in V \backslash (V(T_2) \backslash a_2)$, let C' arise from C by replacing the path us_1v_4 with $uv_6v_3v_4$, and set $y'(C') = y(C) + y(C')$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1)$. It is easy to see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

Assume next that $y(v_1v_6v_3v_4v_1) = 0$. Then both $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ are integers, for otherwise, neither of them is integral, because their sum is $w(v_1v_6)$. If $y(v_3v_4v_6v_3)$ or $y(v_3s_2v_4v_6v_3)$ is positive, then $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (11), a contradiction. So $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Since v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, the arc v_3v_1 is outside $\mathcal{C}_0^y$. If v_3v_1 is is saturated by $\boldsymbol{y}$ in T_2, then $y(v_1v_6v_3v_1) = w(v_3v_1)$; this contradiction implies that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2 (and hence in T). If v_6v_3 is outside $\mathcal{C}_0^y$, then from the choice of $\boldsymbol{y}$ we see that $y(v_1v_6v_3v_1) = \min\{w(v_6v_3), w(v_3v_1)\}$, a contradiction again. So we assume that v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Define $\theta = \min\{w(v_3v_1) - z(v_3v_1), y(C), y(v_1v_6s_1v_4v_1)\}$. Let $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6). By Lemma 3.2(iii), we may assume $w(v_1v_6) = 0$. Thus $z(v_4v_1) = w(v_4v_1) = 0$; the remainder of the proof is exactly the same as that in the preceding subcase.

Subcase 2.6. $K = \{v_4v_1, v_4v_6, v_6v_3\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_6v_3)$, $y(v_6s_1v_4v_6) = w(v_4v_6)$, and $y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1)$. Since $w(K) = \tau_w(T_2 \backslash a_2) > 0$, we have $w(v_4v_1) > 0$. By Lemma 5.2(vi), $y(v_1s_1v_4v_1)$ or $y(v_1s_2v_4v_1)$ is zero. By Lemma 3.2(iii), we may assume that $w(v_6v_3) = w(v_4v_6) = 0$ and $y(v_1v_6s_1v_4v_1) > 0$. So $y(v_1s_1v_4v_1) = w(v_1s_1)$ by (11). By

Lemma 3.2(iii), we may further assume that $w(v_1s_1)=0$. Thus $y(v_1s_2v_4v_1)+y(v_1v_6s_1v_4v_1)=w(v_4v_1)$, and hence neither $y(v_1s_2v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral. Observe that v_1s_2 is outside $\mathcal{C}_0^y$, for otherwise, let $C\in\mathcal{C}_0^y$ be a cycle containing v_1s_2. Then C contains s_2v_4. Let $C'=C[v_4,v_1]\cup\{v_1v_6,v_6s_1,s_1v_4\}$ and $\theta=\min\{y(C),y(v_1v_6s_1v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_2v_4v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1s_2v_4v_1)+\theta$, $y(v_1v_6s_1v_4v_1)-\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1)<y(v_1v_6s_1v_4v_1)$, contradicting (6). Furthermore, since $w(v_1s_1)=0$, the arc v_3v_1 is also outside $\mathcal{C}_0^y$. Thus $w(v_3v_1)=z(v_3v_1)=0$. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T,\boldsymbol{w})$. Since both $y(v_1s_2v_4v_1)$ and $y(v_1v_6s_1v_4v_1)$ are positive, we have $x(v_1s_2v_4v_1)=x(v_1v_6s_1v_4v_1)=1$ by Lemma 3.1(i). Since $y(v_1s_2v_4v_1)<w(v_1s_2)$, we have $x(v_1s_2)=0$ by Lemma 3.1(ii). It follows that $x(s_2v_4)=x(v_1v_6)+x(v_6s_1)+x(s_1v_4)$. Since $w(v_1s_1)=0$ and v_1s_2 is outside $\mathcal{C}_0^y$, for any $u\in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_1v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_2, then it passes through s_2v_4. By Lemma 3.1(iv), we obtain $x(uv_1)+x(v_1v_6)+x(v_6,s_1)+x(s_1v_4)=x(us_2)+x(s_2v_4)$. Hence $x(uv_1)=x(us_2)$. Clearly, we may assume that this equality holds in any other situation. Let $T'=(V',A')$ be obtained from T by deleting s_2, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e)+w(s_2v_4)$ for $e\in\{v_1v_6,v_6s_1,s_1v_4\}$, replacing $w(uv_1)$ with $w(uv_1)+w(us_2)$ for any $u\in V\backslash(V(T_2)\backslash a_2)$, and replacing $w(v_3v_1)$ with $w(v_3v_1)+w(v_3s_2)$. Let $\boldsymbol{x}'$ be obtained from $\boldsymbol{x}$ by setting $x(v_3v_1)=x(v_3s_2)$. Since $w(v_3v_1)=0$ and $w'(v_3v_1)=w(v_3s_2)$, we have $(\boldsymbol{w}')^T\boldsymbol{x}'=\boldsymbol{w}^T\boldsymbol{x}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6s_1v_4v_1)=y(v_1v_6s_1v_4v_1)+y(v_1s_2v_4v_1)$; for each $C\in\mathcal{C}_0^y$ passing through us_2v_4, let C' arise from C by replacing the path us_2v_4 with the path $uv_1v_6s_1v_4$, and set $y'(C')=y(C')+y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T',\boldsymbol{w}')$ and $\mathbb{D}(T',\boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

Subcase 2.7. $K=\{v_4v_1,v_4v_6,v_3v_1\}$.

In this subcase, we have $y(v_1v_6v_3v_1)=w(v_3v_1)$, $y(v_1s_1v_4v_1)+y(v_1s_2v_4v_1)+y(v_1v_6s_1v_4v_1)+y(v_1v_6v_3v_4v_1)=w(v_4v_1)$, and $y(v_6s_1v_4v_6)+y(v_3v_4v_6v_3)+y(v_3s_2v_4v_6v_3)=w(v_4v_6)$. By Lemma 3.2(iii), we may assume that $w(v_3v_1)=0$.

Assume first that $y(v_1v_6v_3v_4v_1)>0$. Then $y(v_6s_1v_4v_6)=0$ by (10). If $y(v_3s_2v_4v_6v_3)>0$, then $y(v_3v_4v_6v_3)=w(v_3v_4)$ by (12); otherwise, $y(v_3v_4v_6v_3)=w(v_4v_6)$. So both $y(v_3v_4v_6v_3)$ and $y(v_3s_2v_4v_6v_3)$ are integers in either situation. Thus we may assume that $w(v_4v_6)=0$. The remainder of the proof is exactly the same as that of (16).

Assume next that $y(v_1v_6v_3v_4v_1)=0$. Consider first the subsubcase when $w(v_4v_1)=0$. Then $w(v_4v_6)=w(K)>0$. If $y(v_3s_2v_4v_6v_3)>0$, then $y(v_3v_4v_6v_3)=w(v_3v_4)$ by (12), so $y(v_6s_1v_4v_6)+y(v_3s_2v_4v_6v_3)=w(v_4v_6)-w(v_3v_4)$; if $y(v_3s_2v_4v_6v_3)=0$, then $y(v_6s_1v_4v_6)+y(v_3v_4v_6v_3)=w(v_4v_6)$. It can be shown that $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the proof of (17).

Consider next the subsubcase when $w(v_4v_1)>0$. Observe that $y(v_1v_6s_1v_4v_1)>0$ and $y(v_3s_2v_4v_6v_3)=0$, for otherwise, since $w(v_1s_1)w(v_1s_2)=0$ by Lemma 5.2(vi), at most one of $y(v_1s_1v_4v_1)$ and $y(v_1s_2v_4v_1)$ is positive. Hence, if $y(v_1v_6s_1v_4v_1)=0$, then either $y(v_1s_1v_4v_1)=w(v_4v_1)$ or $y(v_1s_2v_4v_1)=w(v_4v_1)$; if $y(v_1v_6s_1v_4v_1)>0$ and $y(v_3s_2v_4v_6v_3)>0$, then, by (11), we have $y(v_1s_1v_4v_1)=w(v_1s_1)$, $y(v_1s_2v_4v_1)=w(v_1s_2)$. So $y(v_1v_6s_1v_4v_1)=w(v_4v_1)-w(v_1s_1)-w(v_1s_2)$. By Lemma 3.2(iii), we see that $\nu_w^*(T)$ is an integer. The preceding observation together with (11) implies that $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_1s_1)$, and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) = w(v_4v_6)$. Lemma 3.2(iii) allows us to assume that $w(v_1s_1) = 0$ and that neither $y(v_1s_2v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral.

It can then be shown that v_1s_2 is outside $\mathcal{C}_0^y$ and $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the preceding case.

Combining the above seven subcases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

To establish the corresponding lemmas for the cases when $T_2/S \in \{G_4, G_5, G_6\}$, we need some further preparations.

Lemma 5.10. *If* $T_2/S \in \{G_5, G_6\}$*, then we may assume that* $\min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\} = 0$.

Proof. Let $\theta = \min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\}$ and $C_0 = v_1v_3v_4v_1$. Assume the contrary: $\theta > 0$. Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized; and

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Let $\mathcal{C}_2' = \mathcal{C}_2 \backslash \{C_0\}$. Note that every cycle in $\mathcal{C}_2'$ passes through b. By Lemma 3.5(vii), at least one of v_1v_3, v_3v_4, and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, say v_1v_3 (by symmetry). Thus $w(v_1v_3) = \theta$. We propose to show that

(3) there is no cycle $C \in \mathcal{C}_2'$ with $y(C) > 0$ passing through v_1v_3.

Assume the contrary: v_1v_3 is contained in some cycle $C_1 \in \mathcal{C}_2'$ with $y(C_1) > 0$. Clearly, $|C_1| \geq 4$. If neither v_3v_4 nor v_4v_1 is saturated by $\boldsymbol{y}$ in T, then $\theta_1 = \min\{w(v_3v_4) - z(v_3v_4), w(v_4v_1) - z(v_4v_1)\} > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$ and $y(C_0)$ with $y(C_1) - \theta_1$ and $y(C_0) + \theta_1$, respectively. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(C_1) < y(C_1)$, contradicting (2). Thus at least one of v_3v_4 and v_4v_1 is saturated by $\boldsymbol{y}$ in T. We proceed by considering two cases.

• Both v_3v_4 and v_4v_1 are saturated by $\boldsymbol{y}$ in T. In this case, let $C_2 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ be a cycle containing v_3v_4 with $y(C_2) > 0$; subject to this, C_2 is chosen to contain v_4v_1, if possible. If v_4v_1 is on C_2, then the multiset sum of C_1 and C_2 contains three arc-disjoint cycles C_0, $C_1' = \{bv_1\} \cup C_2[v_1, b]$, and $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$. Define $\epsilon = \min\{y(C_1), y(C_2)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_i)$ and $y(C_i')$ with $y(C_i) - \epsilon$ and $y(C_i') + \epsilon$, respectively, for $i = 1, 2$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \epsilon$, a contradiction. If v_4v_1 is outside C_2, then there exists a cycle $C_3 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ containing v_4v_1 with $y(C_3) > 0$. Observe that the multiset sum of C_1, C_2, and C_3 contains four arc-disjoint cycles C_0, $C_1' = \{bv_1\} \cup C_3[v_1, b]$, $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$, and $C_3' = C_3[b, v_4] \cup C_2[v_4, b]$. Define $\epsilon = \min_{1 \leq i \leq 3} y(C_i)$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_i)$ and $y(C_i')$ with $y(C_i) - \epsilon$ and $y(C_i') + \epsilon$, respectively, for $1 \leq i \leq 3$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \epsilon$, a contradiction again.

• Exactly one of v_3v_4 and v_4v_1 is saturated by $\boldsymbol{y}$ in T. In this case, by symmetry, we may assume that v_3v_4 is saturated while v_4v_1 is not. Let $C_2 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ be a cycle containing v_3v_4 with $y(C_2) > 0$. Then the multiset sum of C_1, C_2, and the unsaturated arc v_4v_1 contains two arc-disjoint cycles C_0 and $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$. Clearly, $C_2' \in \mathcal{C}_2'$ if $C_2 \in \mathcal{C}_2'$. Define $\epsilon = \min\{y(C_1), y(C_2), w(v_4v_1) - z(v_4v_1)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0) +$

ϵ, replacing $y(C_1)$ with $y(C_1) - \epsilon$, and replacing $y(C_2)$ and $y(C_2')$ with $y(C_2) - \epsilon$ and $y(C_2') + \epsilon$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(C_1) < y(C_1)$, contradicting (2).

Combining the above two cases, we see that (3) holds. So $y(C_0) = \theta > 0$, and hence $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.2(iii). This proves the lemma. ∎

Let $Q = V(T_2)\backslash(S \cup \{b_2, a_2\})$. Then $Q = \{v_2, v_3\}$ if $T_2/S = G_4$, $Q = \{v_1, v_3, v_4\}$ if $T_2/S = G_5$, and $Q = \{v_1, v_2, v_3, v_4\}$ if $T_2/S = G_6$. Moreover, $v_1v_3v_4v_1$ is the unique cycle in $T[Q]$ when $T_2/S = G_5$ or G_6. Let $T' = T$ if $T_2/S = G_4$, and let T' be obtained from T be reversing precisely one arc e on $v_1v_3v_4v_1$ with $w(e) = 0$ (see Lemma 5.9) so that $T[Q]$ is acyclic if $T_2/S = G_5$ and G_6. From Lemma 2.3 we see that T' is also Möbius-free. Note that every integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ naturally corresponds to an integral optimal solution to $\mathbb{D}(T', \boldsymbol{w})$ with the same value, and vice versa. So we shall not make effort to distinguish between $\mathbb{D}(T, \boldsymbol{w})$ and $\mathbb{D}(T', \boldsymbol{w})$. Let us label the vertices in Q as $q_1, q_2, \ldots, q_t$ such that q_jq_i is an arc in T' for $1 \le i < j \le t$, where $t = |Q|$.

Lemma 5.11. *Suppose $T_2/S \in \{G_4, G_5, G_6\}$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Then we may assume that the following statements hold:*

(i) For each $q_i \in Q$, there exists exactly one $s_k \in S$ such that $z(q_is_k) > 0$;

(ii) $z(q_jq_i) = w(q_jq_i) = 0$ for $1 \le i < j \le t$, where $t = |Q|$;

(iii) If $z(q_is_k)z(q_js_k) > 0$ for some $1 \le i < j \le t$ and $s_k \in S$, then $x(q_is_k) \ne x(q_js_k)$.

Proof. As remarked above the lemma, we may simply treat T, $\mathbb{P}(T, \boldsymbol{w})$, and $\mathbb{D}(T, \boldsymbol{w})$ as T' and $\mathbb{P}(T', \boldsymbol{w})$, and $\mathbb{D}(T', \boldsymbol{w})$, respectively, in our proof.

(i) By Lemma 5.2(vi), for each vertex $q_i \in Q$, there exists at most one $s_k \in S$ with $z(q_is_k) > 0$. Assume on the contrary that $z(q_is_k) = 0$ for all $s_k \in S$. Then no cycle in $\mathcal{C}^y$ passes through q_i. Let $G = T\backslash q_i$ and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to the arcs of G. By the hypothesis of Theorem 1.5, $\mathbb{D}(G, \boldsymbol{w}')$ has an integral optimal solution, and so does $\mathbb{D}(T', \boldsymbol{w})$. Hence we assume that (i) holds.

(ii) Assume the contrary: $z(q_jq_i) > 0$; subject to this, $j + i$ is minimized. If there exists exactly one $s_k \in S$ such that $z(q_is_k)z(q_js_k) > 0$, then the proof is the same as that of Lemma 5.2(i) (with s_k, q_i, and q_j in place of v_0, s_i, and s_j, respectively), so we omit the details here. In view of Lemma 5.2(i), we may assume that $z(q_is_1)z(q_js_2) > 0$. We proceed by considering two cases.

Case 1. $x(q_jq_i) = 0$. In this case, we may assume that $x(uq_j) = x(uq_i)$ for any $u \in V\backslash(S \cup Q)$. Indeed, if $z(uq_j)z(uq_i) > 0$, then Lemma 3.1(iv) implies $x(uq_j) = x(uq_i)$; if $z(uq_j)z(uq_i) = 0$, then $w(us_i')w(us_j') = 0$ by Lemma 3.2(i). Thus we may modify $x(uq_j)$ and $x(uq_i)$ so that they become equal. Let $T' = (V', A')$ be obtained from T by identifying q_j with q_i; we still use q_i to denote the resulting vertex. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uq_i)$ with $w(uq_j) + w(uq_i)$ for any $u \in V\backslash(S \cup Q)$. Let $\boldsymbol{x}'$ and $\boldsymbol{y}'$ be the projections of $\boldsymbol{x}$ and $\boldsymbol{y}$ onto T', respectively. From the LP-duality theorem, it is easy to see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w}')$ and $\mathbb{D}(T, \boldsymbol{w}')$, respectively, with the same value as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Case 2. $x(q_jq_i) > 0$. In this case, $z(q_jq_i) = w(q_jq_i) > 0$ by Lemma 3.1(iii). Let C_1 and C_2 be two cycles in $\mathcal{C}^y$ that passes through q_jq_i and q_js_2, respectively. Clearly, both C_1 and C_2 pass through b. By Lemma 3.1(iv), we have $x(q_jq_i) + x(q_is_1) + x(s_1b) = x(q_js_2) + x(s_2b)$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(e_1)$ with $w(e_1) + w(q_jq_i)$ for $e_1 = q_js_2$ and s_2b and replacing $w(e_2)$ with $w(e_2) - w(q_jq_i)$ for $e_2 = q_jq_i$, q_is_1, and s_1b. Let $\boldsymbol{x}' = \boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle passing through q_jq_i, let C' be the cycle arising from C by replacing the path $q_jq_is_1b$ with q_js_2b. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w}')$ and $\mathbb{D}(T, \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $w'(A) < w(A)$, by the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Combining the above two cases, we may assume that $z(q_jq_i) = 0$.

(iii) Since the proof is the same as that of Lemma 5.2(iv) (with s_k, q_i, and q_j in place of v_0, s_i, and s_j, respectively), we omit the routine details here. ∎

Lemma 5.12. *If $T_2/S = G_4$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_1, v_5)$, $s^* = v_4$, and $Q = \{v_2, v_3\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0$ for $u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 5.10(i), we see that

(2) there exists at least one and at most two vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

Lemma 5.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2\}$.

By Lemma 5.10(ii), we obtain

(4) $w(v_2v_3) = z(v_2v_3) = 0$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in \mathcal{C}_2^y$.

To justify this, we distinguish between two cases.

Case 1. $\varphi(s_i) = \{v_2\}$ for $i = 1$ or 2.

In this case, by Lemma 5.2(i) and Lemma 5.10(i), we may assume that $\varphi(s_1) = \{v_2\}$ and $\varphi(s_2) = \{v_3\}$. By (4), we obtain

(7) $\mathcal{C}_2^y \subseteq \{v_1v_2s_1v_1, v_1v_3s_2v_1\}$.

From Lemma 3.5(vii), we deduce that $y(v_1v_2s_1v_1) = \min\{w(v_1v_2), w(v_2s_1), w(s_1v_1)\}$ and $y(v_1v_3s_2v_1) = \min\{w(v_1v_3), w(v_3s_2), w(s_2v_1)\}$. If both $y(v_1v_2s_1v_1)$ and $y(v_1v_3s_2v_1)$ are zero, then $\tau_w(T_2\backslash a_2) = \min\{w(v_1v_2), w(v_2s_1), w(s_1v_1)\} + \min\{w(v_1v_3), w(v_3s_2), w(s_2v_1)\} = 0$, contradicting (α). Therefore, $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_2v_1)$ is a positive integer.

Case 2. $\varphi(s_i) \neq \{v_2\}$.

In this case, Lemma 5.10(i), (2) and (3) allow us to assume that $\varphi(s_1) = \{v_2, v_3\}$. By (4), we have

(8) $\mathcal{C}_2^y \subseteq \{v_1v_2s_1v_1, v_1v_3s_1v_1\}$.

By Lemma 5.2(iii), we also obtain $z(s_1v_1) = w(s_1v_1) > 0$. Assume first that s_1v_1 is outside $\mathcal{C}_0^y$. Then both v_2s_1 and v_3s_1 are outside $\mathcal{C}_0^y$, and s_1v_1 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1v_2s_1v_1) + y(v_1v_3s_1v_1) = w(s_1v_1) > 0$. Observe that both $y(v_1v_2s_1v_1)$ and $y(v_1v_3s_1v_1)$ are integral, for otherwise, $0 < y(v_1v_is_1v_1) < w(v_is_1)$ for $i = 2, 3$, by Lemma 3.1(i) and (ii), we have $x(v_2s_1) = x(v_3s_1) = 0$, contradicting Lemma 5.9(iii). Hence $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_1v_1)$ is a positive integer.

Assume next that s_1v_1 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 3.5(vii), we see that $y(v_1v_is_1v_1) = \min\{w(v_1v_i), w(v_is_1)\}$ for $i = 2, 3$. If $y(v_1v_is_1v_1) = 0$ for $i = 2, 3$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{2} \min\{w(v_1v_i), w(v_is_1)\} = 0$, contradicting (α). Therefore $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_1v_1)$ is a positive integer. So the above Claim is established.

From the above Claim and Lemma 3.2(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 5.13. *If $T_2/S = G_5$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_2, v_6)$, $s^* = v_5$, and $Q = \{v_1, v_3, v_4\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 5.10(i), we see that

(2) there exists at least one and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

Lemma 5.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

By Lemma 5.10(ii), we obtain

(4) $w(e) = z(e) = 0$ for $e \in \{v_1v_3, v_3v_4, v_4v_1\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in \mathcal{C}_2^y$.

To justify this, we consider three possible cases (see the structure of G_5), depending on the size of $\varphi(s_i)$ for $1 \leq i \leq 3$.

Case 1. $|\varphi(s_i)| = 1$ for each $1 \leq i \leq 3$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_3\}$, and $\varphi(s_3) = \{v_4\}$. By (4), we obtain

(7) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_3v_2\}$.

From Lemma 3.5(vii), we deduce that $y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}$, $y(v_2v_3s_2v_2) = \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\}$, and $y(v_2v_4s_3v_2) = \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\}$. If $y(v_2v_1s_1v_2)$, $y(v_2v_3s_2v_2)$, and $y(v_2v_4s_3v_2)$ are all zero, then $\tau_w(T_2\backslash a_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} + \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\} + \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\} = 0$, contradicting (α). Therefore, at least one of $y(v_2v_1s_1v_2)$, $y(v_2v_3s_2v_2)$, and $y(v_2v_4s_3v_2)$ is a positive integer.

Case 2. $|\varphi(s_i)| = 1$ for exactly one $i \in \{1, 2, 3\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_3, v_4\}$. By (4), we have

(8) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_2v_2\}$.

From Lemma 3.5(vii), we see that $y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}$. If $y(v_2v_1s_1v_2) > 0$, we are done. So we assume that $y(v_2v_1s_1v_2) = 0$. Since $w(v_1s_1)w(s_1v_2) > 0$, we obtain $w(v_2v_1) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} = 0$. By Lemma 5.2(iii), we have $z(s_2v_2) = w(s_2v_2) > 0$.

Assume first that s_2v_2 is outside $\mathcal{C}_0^y$. Then both v_3s_2 and v_4s_2 are outside $\mathcal{C}_0^y$, and s_2v_2 is saturated by $\boldsymbol{y}$ in T_2. Hence $y(v_2v_3s_2v_2) + y(v_2v_4s_2v_2) = w(s_2v_2) > 0$. Observe that both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are integral, for otherwise, since $0 < y(v_2v_is_2v_2) < w(v_is_2)$ for $i = 3, 4$, by Lemma 3.1(i) and (ii), we have $x(v_3s_2) = x(v_4s_2) = 0$, contradicting Lemma 5.9(iii). Hence both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are positive integers.

Assume next that s_2v_2 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 3.5(vii), we see that $y(v_2v_is_2v_2) = \min\{w(v_2v_i), w(v_is_2)\}$ for $i = 3, 4$. If $y(v_2v_is_2v_2) = 0$ for $i = 3, 4$, then $\tau_w(T_2\backslash a_2) = w(v_2v_1) + \sum_{i=3}^{4} \min\{w(v_2v_i), w(v_is_2)\} = 0$, contradicting (α). Therefore $y(v_2v_3s_2v_2)$ or $y(v_2v_4s_2v_2)$ is a positive integer.

Case 3. $|\varphi(s_i)| \neq 1$ for any $i \in \{1, 2, 3\}$.

In this case, by Lemma 5.10(i), (2), and (3), we may assume that $\varphi(s_1) = \{v_1, v_3, v_4\}$ (see the structure of G_5). By (4), we obtain

(9) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_1v_2, v_2v_4s_1v_2\}$.

By Lemma 5.2(iii), we have $z(s_1v_2) = w(s_1v_2) > 0$.

Assume first that s_1v_2 is outside $\mathcal{C}_0^y$. Then v_is_1 is outside $\mathcal{C}_0^y$ for each $i \in \{1, 3, 4\}$, and s_1v_2 is saturated by $\boldsymbol{y}$ in T_2. So $\sum_{i\in\{1,3,4\}} y(v_2v_is_1v_2) = w(s_1v_2) > 0$. Observe that $y(v_2v_is_1v_2)$ is integral for each $i \in \{1, 3, 4\}$, for otherwise, symmetry allows us to assume that $y(v_2v_1s_1v_2)$ is not integral. Then $y(v_2v_3s_1v_2)$ or $y(v_2v_4s_1v_2)$ is not integral, say $y(v_2v_3s_1v_2)$. Since $0 < y(v_2v_is_1v_2) < w(v_is_1)$ for $i = 1, 3$, by Lemma 3.1(i) and (ii), we have $x(v_1s_1) = x(v_3s_1) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_2v_is_1v_2)$ is a positive integer for each $i \in \{1, 3, 4\}$.

Assume next that s_1v_2 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 3.5(vii), we deduce that $y(v_2v_is_1v_2) = \min\{w(v_2v_i), w(v_is_1)\}$ for $i \in \{1, 3, 4\}$. If $y(v_2v_is_1v_2) = 0$ for each $i \in \{1, 3, 4\}$, then $\tau_w(T_2\backslash a_2) = \sum_{i\in\{1,3,4\}} \min\{w(v_2v_i), w(v_is_1)\} = 0$, contradicting (α). Hence $y(v_2v_is_1v_2)$ is a positive integer for some $i \in \{1, 3, 4\}$. This proves the Claim.

From the Claim and Lemma 3.2(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 5.14. *If $T_2/S = G_6$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_6, v_7)$, $s^* = v_5$, and $Q = \{v_1, v_2, v_3, v_4\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 5.10(i), we see that

(2) there exists at least one and at most four vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

Lemma 5.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $1 \leq i \leq 4$.

By Lemma 5.10(ii), we obtain

(4) $w(e) = z(e) = 0$ for $e \in \{v_1v_3, v_3v_4, v_4v_1, v_1v_2, v_3v_2, v_4v_2\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in \mathcal{C}_2^y$.

To justify this, we consider five possible cases (see the structure of G_6), depending on the size of $\varphi(s_i)$ for $1 \leq i \leq 4$.

Case 1. $|\varphi(s_i)| = 1$ for each $1 \le i \le 4$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_i) = \{v_i\}$ for each $1 \le i \le 4$. By (4), we obtain

(7) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_3v_6, v_6v_4s_4v_6\}$.

From Lemma 3.5(vii), we deduce that $y(v_6v_is_iv_6) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\}$ for each $1 \le i \le 4$. If $y(v_6v_is_iv_6) = 0$ for $1 \le i \le 4$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^4 \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\} = 0$, contradicting (α). Hence $y(v_6v_is_iv_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$.

Case 2. $|\varphi(s_i)| = 1$ for exactly one $i \in \{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_2, v_3, v_4\}$. By (4), we have

(8) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_2v_6, v_6v_4s_2v_6\}$.

From Lemma 3.5(vii), we see that $y(v_6v_1s_1v_6) = \min\{w(v_6v_1), w(v_1s_1), w(s_1v_6)\}$. If $y(v_6v_1s_1v_6) > 0$, we are done. So we assume that $y(v_6v_1s_1v_6) > 0$. Since $w(v_1s_1)w(s_1v_6) > 0$, we obtain $w(v_6v_1) = \min\{w(v_6v_1), w(v_1s_1), w(s_1v_6)\} = 0$. By Lemma 5.2(iii), we have $z(s_2v_6) = w(s_2v_6) > 0$.

Assume first that s_2v_6 is outside $\mathcal{C}_0^y$. Then v_is_2 is outside $\mathcal{C}_0^y$ for $i \in \{2, 3, 4\}$, and s_2v_6 is saturated by $\boldsymbol{y}$ in T_2. So $\sum_{i=2}^4 y(v_6v_is_2v_6) = w(s_2v_6) > 0$. Observe that $y(v_6v_is_2v_6)$ is integral for each $i \in \{2, 3, 4\}$, for otherwise, symmetry allows us to assume that $y(v_6v_2s_2v_6)$ is not integral. Then one of $y(v_6v_3s_2v_6)$ and $y(v_6v_4s_2v_6)$ is not integral, say $y(v_6v_3s_2v_6)$. Since $0 < y(v_6v_is_2v_6) < w(v_is_2)$ for $i = 2, 3$, by Lemma 3.1(i) and (ii), we have $x(v_2s_2) = x(v_3s_2) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_6v_is_2v_6)$ is a positive integer for each $i \in \{2, 3, 4\}$.

Assume next that s_2v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. By Lemma 3.5(vii), we obtain $y(v_6v_is_2v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i \in \{2, 3, 4\}$. If $y(v_6v_is_2v_6) = 0$ for $i \in \{2, 3, 4\}$, then $\tau_w(T_2\backslash a_2) = w(v_6v_1) + \sum_{i=2}^4 \min\{w(v_6v_i), w(v_is_2)\} = 0$, contradicting (α). Hence $y(v_6v_is_2v_6)$ is a positive integer for some $i \in \{2, 3, 4\}$.

Case 3. $|\varphi(s_i)| = 1$ for exactly two i's in $\{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_i\}$ for $i = 1, 2$ and $\varphi(s_3) = \{v_3, v_4\}$. By (4), we obtain

(9) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_3v_6, v_6v_4s_3v_6\}$.

From Lemma 3.5(vii), we see that $y(v_6v_is_iv_6) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\}$ for $i = 1, 2$. If $y(v_6v_is_iv_6) > 0$, we are done. So we assume that $y(v_6v_is_iv_6) = 0$. Since $w(v_is_i)w(s_iv_6) > 0$, we obtain $w(v_6v_i) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\} = 0$ for $i = 1, 2$. By Lemma 5.2(iii), we have $z(s_3v_6) = w(s_3v_6) > 0$.

Assume first that s_3v_6 is outside $\mathcal{C}_0^y$. Then v_is_3 is outside $\mathcal{C}_0^y$ for $i = 3, 4$, and s_3v_6 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_6v_3s_3v_6) + y(v_6v_4s_3v_6) = w(s_3v_6) > 0$. Observe that both $y(v_6v_3s_3v_6)$ and $y(v_6v_4s_3v_6)$ are integral, for otherwise, since $0 < y(v_6v_is_3v_6) < w(v_is_3)$ for $i = 3, 4$, by Lemma 3.1(i) and (ii), we have $x(v_3s_3) = x(v_4s_3) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_6v_is_3v_6)$ is a positive integer for $i = 3, 4$.

Assume next that s_3v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. By Lemma 3.5(vii), we obtain $y(v_6v_is_3v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i = 3, 4$. If $y(v_6v_is_3v_6) = 0$ for $i = 3, 4$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^2 w(v_6v_i) + \sum_{i=3}^4 \min\{w(v_6v_i), w(v_is_3)\} = 0$, contradicting (α). Hence $y(v_6v_is_3v_6)$ is a positive integer for $i = 3$ or 4.

Case 4. $1 < |\varphi(s_i)| < 4$ if $\varphi(s_i) \neq \emptyset$, for $i \in \{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1, v_2\}$ and $\varphi(s_2) = \{v_3, v_4\}$. By (4), we obtain

(10) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_2v_6, v_6v_4s_2v_6\}$.

By Lemma 5.2(iii), we have $z(s_iv_6) = w(s_iv_6) > 0$ for $i = 1, 2$.

Assume first that s_1v_6 is outside $\mathcal{C}_0^y$. Then both v_1s_1 and v_2s_1 are outside $\mathcal{C}_0^y$, and s_1v_6 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_6v_1s_1v_6) + y(v_6v_2s_1v_6) = w(s_1v_6) > 0$. Observe that both $y(v_6v_1s_1v_6)$ and $y(v_6v_2s_1v_6)$ are integral, for otherwise, since $0 < y(v_6v_is_1v_6) < w(v_is_1)$ for $i = 1, 2$, by Lemma 3.1(i) and (ii), we have $x(v_1s_1) = x(v_2s_1) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_6v_is_1v_6)$ is a positive integer for $i = 1, 2$. Similarly, we can show that if s_2v_6 is outside $\mathcal{C}_0^y$, then $y(v_6v_is_2v_6)$ is a positive integer for $i = 3, 4$.

Assume next that s_iv_6 is contained in some cycle in $\mathcal{C}_0^y$ for $i = 1, 2$. By Lemma 3.5(vii), we have $y(v_6v_is_1v_6) = \min\{w(v_6v_i), w(v_is_1)\}$ for $i = 1, 2$, and $y(v_6v_is_2v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i = 3, 4$. If $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_2v_6)$ and $y(v_6v_4s_2v_6)$ are all zero, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{2} \min\{w(v_6v_i), w(v_is_1)\} + \sum_{i=3}^{4} \min\{w(v_6v_i), w(v_is_2)\} = 0$, contradicting (α). So at least one of $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_2v_6)$, and $y(v_6v_4s_2v_6)$ is a positive integer.

Case 5. $|\varphi(s_i)| > 2$ if $\varphi(s_i) \neq \emptyset$, for $i \in \{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1, v_2, v_3, v_4\}$. By (4), we obtain

(11) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_1v_6, v_6v_4s_1v_6\}$.

By Lemma 5.2(iii), we have $z(s_1v_6) = w(s_1v_6) > 0$.

Assume first that s_1v_6 is outside $\mathcal{C}_0^y$. Then $\sum_{i=1}^{4} y(v_6v_is_1v_6) = w(s_1v_6)$. If $y(v_6v_is_1v_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$, we are done. So we assume the contrary. Thus at least two of $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_1v_6)$, and $y(v_6v_4s_1v_6)$ are not integral, say $y(v_6v_1s_1v_6)$ and $y(v_6v_2s_1v_6)$. Since $0 < y(v_6v_is_1v_6) < w(v_is_1)$ for $i = 1, 2$, by Lemma 3.1 (i) and (ii), we have $x(v_1s_1) = x(v_2s_1) = 0$, contradicting Lemma 5.9(iii).

Assume next that s_1v_6 is contained in some cycle of $\mathcal{C}_0^y$. By Lemma 3.5(vii), we have $y(v_6v_is_1v_6) = \min\{w(v_6v_i), w(v_is_1)\}$ for $1 \leq i \leq 4$. If $y(v_6v_is_1v_6)$ is zero for $1 \leq i \leq 4$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{4} \min\{w(v_6v_i), w(v_is_1)\} = 0$, contradicting (α). So $y(v_6v_is_1v_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$. This proves the Claim.

From the above Claim and Lemma 3.2(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎