Excluding a small minor

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Abstract

There are sixteen 3-connected graphs on eleven or fewer edges. For each of these graphs H we discuss the structure of graphs that do not contain a minor isomorphic to H.

Key words: Graph minor, splitter theorem, graph structure.

1 Introduction

Let G and H be graphs. In this paper, G is called *H-free* if no minor of G is isomorphic to H. We consider the problem of characterizing all H-free graphs, for certain fixed H.

In graph theory, many important problems are about H-free graphs. For instance, Hadwiger's Conjecture [7], made in 1943, states that every K_n-free graph is $n-1$ colorable. Today, this conjecture remains "one of the deepest unsolved problems in graph theory" [1]. Another long standing problem of this kind is Tutte's 4-flow conjecture [19], which asserts that every bridgeless Petersen-free graph admits a 4-flow. It is generally believed that knowing the structures of K_n-free graphs and Petersen-free graphs, respectively, would lead to a solution to the corresponding conjecture.

In their *Graph-Minors* project, Robertson and Seymour [16] obtained, for every graph H, an approximate structure for H-free graphs. This powerful result has many important consequences, yet it is not strong enough to handle the two conjectures mentioned above. An interesting contrast can be made for K_6-free graphs. By extending techniques developed in the Graph-Minors project, Kawarabayashi et. al. [10] proved that a sufficiently large 6-connected graph is K_6-free if and only if it is an *apex* graph, i.e. it has a vertex whose deletion results in a planar graph. However, no complete characterization for K_6-free graphs is known, not even when only 6-connected graphs are considered (in this special case, Jørgensen conjectured in [9] that they are all apex graphs).

Note that both K_6 and Petersen graph have fifteen edges. Currently, there is no connected graph H with that many edges for which H-free graphs are completely characterized. As an attempt to better understand these graphs, we try to exclude a graph with fewer than fifteen edges. We will focus on 3-connected graphs H since they provide the most insights on graph structures. By gradually increasing the size of H we hope eventually we will be able to characterize H-free graphs for some 15-edge graph H, including K_6 and Petersen. So this paper is the beginning of this project.

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The rest of the paper is arranged as follows. The next section includes preliminaries in this study. Then, in Section 3, we survey results on excluding a fixed graph H. In particular, we will see that the smallest 3-connected graphs H for which H-free graphs are not yet characterized are six graphs with eleven edges. In Section 4, we completely determine H-free graphs for each of these six graphs.

2 Preliminaries

In this paper all graphs are simple unless otherwise stated. We begin with a few definitions. A *wheel* on $n+1$ vertices ($n \geq 3$), denoted by W_n, is obtained from a cycle on n vertices by adding a new vertex and making this vertex adjacent to all vertices on the cycle. Notice that the smallest wheel W_3 is K_4. Let G be a graph. If u, v are nonadjacent vertices of G, then $G+uv$ is obtained from G by adding a new edge uv. If v has degree at least four, then by *splitting* v we mean the operation of first deleting v from G, then adding two new adjacent vertices v', v'' and joining each neighbor of v to exactly one of v', v'' such that each of v', v'' has degree at least three in the new graph. The operations of adding an edge and splitting a vertex are also known as *undeletion* and *uncontraction*, respectively. The next is a classical result of Tutte [18], which explains how 3-connected graphs are generated.

Theorem 2.1 (*Tutte's wheel theorem.*) *A graph is 3-connected if and only if it is obtained from a wheel by repeatedly adding edges and splitting vertices.*

The next is a useful theorem of Seymour [17] which we will use repeatedly in this paper.

Theorem 2.2 (*Seymour's splitter theorem.*) *Suppose a 3-connected graph $H \neq W_3$ is a proper minor of a 3-connected graph $G \neq W_n$. Then G has a minor J, which is obtained from H by either adding an edge or splitter a vertex.*

If a 3-connected graph H is a minor of a non-3-connected graph G, then H has to be a minor of a "3-connected component" of G. To make this fact more clear we need some definitions. Let G_1, G_2 be disjoint graphs. The 0-*sum* of G_1, G_2 is the disjoint union of these two graphs; a 1-*sum* of G_1, G_2 is obtained by identifying one vertex of G_1 with one vertex of G_2; a 2-*sum* of G_1, G_2 is obtained by identifying one edge of G_1 with one edge of G_2, and the common edge could be deleted after the identification. Notice that, if G is a k-sum ($k = 0, 1, 2$) of G_1, G_2, then both G_1 and G_2 are minors of G. The following is a well known fact, so we omit its proof, which is easy. Let us write $H \preceq G$ if H is a minor of G.

Lemma 2.3 *Let H be 3-connected and let G be a k-sum of G_1, G_2, where $k = 0, 1, 2$. Then $G \succeq H$ if and only if $G_1 \succeq H$ or $G_2 \succeq H$.*

Let H be a 3-connected graph. We use $\mathcal{F}(H)$ to denote the class of 3-connected H-free graphs. Since every non-3-connected graph is a k-sum ($k = 0, 1, 2$) of two smaller graphs, we deduce the following from the last lemma immediately.

Lemma 2.4 *Let H be a 3-connected graph. Then a graph is H-free if and only if it is constructed by repeatedly taking 0-, 1-, and 2-sums, starting from graphs in $\{K_1, K_2, K_3\} \cup \mathcal{F}(H)$.*

Because of this lemma, in order to characterize H-free graphs, we only need to determine $\mathcal{F}(H)$, which is exactly what we will do in this paper.

Finally, we state a technical lemma. For any graph $G=(V,E)$, let $\rho(G)=|E|-|V|$. If G is connected and H is a minor of G, it is not difficult to verify that H can be obtained from G by deleting and contracting edges, and without using the operation of deleting vertices. Thus the following lemma is obvious. This result is also apparent to those who are familiar with matroids since ρ is basically the corank function.

Lemma 2.5 *Suppose H is a minor of a connected graph G. Then $\rho(H)\le\rho(G)$. Moreover, if $\rho(H)=\rho(G)$ then $H=G/X$, for some $X\subseteq E(G)$ with $|X|=|V(G)|-|V(H)|$.*

3 Known results

In this section we survey known results on excluding a single 3-connected graph. Most of these results are easy to prove, thanks to Theorem 2.2. However, we will not formally prove any of them. In stead, we will simply point out the main idea of these proofs, whenever it is possible. For these results, since the proof technique is exactly what we are going to use in the next section, their proofs can be constructed easily by mimicking the proofs given in the next section. In our survey below, we order the results according to the number of edges of the graph to be excluded. By Theorem 2.1, $K_4=W_3$ is the smallest 3-connected graph, which has six edges. Moreover, every 3-connected graph contains a wheel, and thus W_3, as a minor, which implies the following result [5] immediately.

Theorem 3.1 (Dirac 1952) *$\mathcal{F}(K_4)=\emptyset$.*

Equivalently, K_4-free graphs are precisely the 0-, 1-, 2-sums of K_1, K_2, and K_3. This class is better known as *series-parallel* graphs since 2-summing a graph with K_3 is a series-parallel extension.

Since K_4 is cubic, none of its vertices can be split. On the other hand, since K_4 is complete, no edge can be added either. Therefore, by Theorem 2.1, all other 3-connected graphs contain W_4, and so the following holds.

Theorem 3.2 *$\mathcal{F}(W_4)=\{K_4\}$.*

As we have seen, K_4 and W_4 are the only 3-connected graphs with eight or fewer edges. Next, we consider 3-connected graphs with nine edges. By Theorem 2.1, these graphs are constructed from W_4. In fact, it is easy to check that there are three such graphs: Prism, $K_5\backslash e$, and $K_{3,3}$. We make an interesting observation on these graphs, which follows immediately from Theorem 2.1.

Proposition 3.1 *Every 3-connected non-wheel graph contains Prism, $K_5\backslash e$, or $K_{3,3}$ as a minor*

Notice that both Prism and $K_{3,3}$ are cubic, so none of their vertices can be split. Since adding any edge to any of them creates a $K_5\backslash e$ minor, we deduce from Proposition 3.1 and Theorem 2.2 the following result of [21]. Let $\mathcal{W}=\{W_n: n\ge 3\}$.

Theorem 3.3 (Wagner 1960) *$\mathcal{F}(K_5\backslash e)=\{K_{3,3},Prism\}\cup\mathcal{W}$.*

Prism-free graphs are characterized in [6] and [12]. Let $\mathcal{K}$ be the class of 3-connected graphs G for which there exists a set X of three vertices such that $G-X$ is edgeless. Equivalently, such a graph G is obtained from $K_{3,n}$ ($n\ge 1$) by adding edges to its color class of size three. The following result can also be proved using Proposition 3.1 and Theorem 2.2 by considering how to add an edge and how to split a vertex in a non-wheel graph $G\in\mathcal{K}\cup\{K_5\}$.

Theorem 3.4 (Dirac 1963, Lovasz 1965) *$\mathcal{F}(Prism) = \{K_5\} \cup \mathcal{W} \cup \mathcal{K}$.*

Hall [8] characterized $K_{3,3}$-free graphs using Kuratowski Theorem [11], which states that *a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.* Notice that no edge can be added to K_5, and splitting any vertex of K_5 creates a $K_{3,3}$ minor, so the next result follows from Theorem 2.2 immediately. Let $\mathcal{P}$ denote the class of 3-connected planar graphs.

Theorem 3.5 (Hall 1943) *$\mathcal{F}(K_{3,3}) = \{K_5\} \cup \mathcal{P}$.*

Next, we consider 3-connected graphs on ten edges. By Theorem 2.1, these graphs (other than W_5) are constructed from Prism, $K_5 \backslash e$, and $K_{3,3}$ by adding an edge or splitting a vertex. It is routine to verify that there are exactly four such graphs: W_5, $Prism + e$, $K_{3,3} + e$, and K_5. During this verification we used the observation that Prism and $K_{3,3}$ are cubic and so none of their vertices can be split. Together with Theorem 2.2, this observation also implies the following two results immediately.

Theorem 3.6 *$\mathcal{F}(Prism + e) = \{Prism\} \cup \mathcal{F}(Prism) = \{Prism, K_5\} \cup \mathcal{W} \cup \mathcal{K}$.*

Theorem 3.7 *$\mathcal{F}(K_{3,3} + e) = \{K_{3,3}\} \cup \mathcal{F}(K_{3,3}) = \{K_{3,3}, K_5\} \cup \mathcal{P}$.*

Using Theorem 2.2, Oxley [15] characterized W_5-free graphs.

Theorem 3.8 (Oxley 1989) *$\mathcal{F}(W_5)$ consists of $\mathcal{K}$ and 3-connected minors of graphs in $\{$Cube, Octahedron, Pyramid, $K_5^{\perp}\}$.*

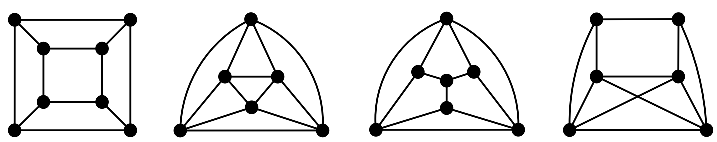

Figure 3.1: Cube, Octahedron, Pyramid, and $K_5^{\perp}$

Wagner [20] characterized K_5-free graphs. A 3-*sum* of two 3-connected graphs G_1, G_2 is obtained by identifying a triangle of G_1 with a triangle of G_2. Some common edges could be deleted after the identification, as long as no degree-two vertices are created. It is not difficult to verify that the resulting graph is always 3-connected.

Theorem 3.9 (Wagner 1937) *$\mathcal{F}(K_5) = \{V_8\} \cup \{$3-sums of 3-connected planar graphs$\}$.*

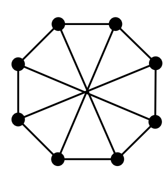

Figure 3.2: Wagner graph V_8

There is only one result on graphs with eleven edges. In a recent paper [3], the two authors of this paper characterized $Cube/e$-free graphs, which we state below. An *augmentation* of a graph is obtained by replacing a $K_{3,n}$- or a fan-subgraph with a larger one. That is, if two cubic vertices have the same set of neighbors, then we can add a new cubic vertex of the same set of neighbors; if two cubic vertices x, y are in a triangle xyz, then we can replace edge xy with a new vertex v and three edges vx, vy, vz.

Theorem 3.10 (Liu and Ding 2011) *$\mathcal{F}(Cube/e)$ consists of augmentations of 3-connected minors of graphs in Figure 3.3.*

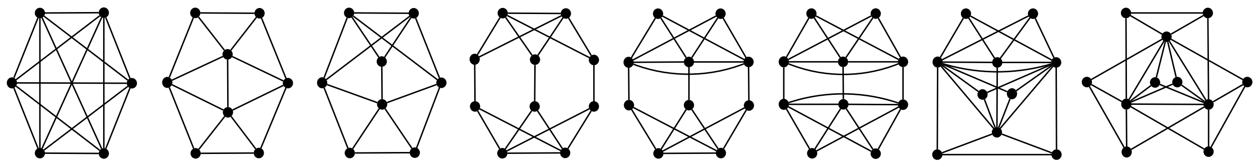

Figure 3.3: Maximal $Cube/e$-free graphs

Beyond the ten graphs listed above, there are only three other 3-connected graphs H, all happen to have twelve edges, for which H-free graphs are completely characterized. Robertson characterized V_8-free graphs, Maharry [14] characterized Cube-free graphs, and Ding [2] characterized Octahedron-free graphs, which extends a partial characterization of Maharry [13]. Robertson's result is not published, but it can be found in many papers, for instance, in [2]. This result is often stated as a characterization of internally 4-connected V_8-free graphs, yet it can be easily turned into a complete characterization of all V_8-free graphs. We will not get into the detail of these three results, but we do point out that, in all three cases, graphs in $\mathcal{F}(H)$ can be further "decomposed" into graphs that belong to a few well defined classes (like what happened in Theorem 3.9).

4 Excluding a 3-connected graph on eleven edges

By Theorem 2.1, 3-connected graphs on eleven edges are constructed from those on ten edges: W_5, $Prism+e$, $K_{3,3}+e$, and K_5. It is not difficult to verify that there are seven such graphs: $K_5^{\perp}$ (from Figure 3.1) and the six graphs shown below. Notice that $K_5^{\perp}$ is the unique graph obtained from K_5 by splitting a vertex. Moreover, the first two graphs in Figure 4.1 are simple modifications of $K_{3,3}$; the middle two are planar dual to each other, and so are the last two. Since $Cube/e$-free graphs are characterized, we characterize H-free graphs in this section for the remaining six graphs.

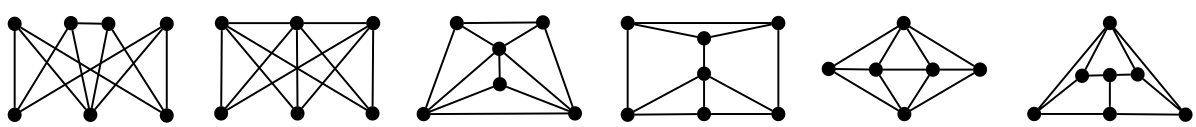

Figure 4.1: $K_{3,3}^{\nabla}$, $K_{3,3}^{\ddagger}$, W_5+e, $(W_5+e)^{*}$, $Octahedron\backslash e$, $Cube/e$

A typical $\mathcal{F}(H)$ consists of a few isolated graphs and a few well defined infinite families. Theorem 3.3 is a good example of such a result. In fact, its proof also illustrate how our other proofs go. The main tool we use is Theorem 2.2. To capture the isolated graphs, we repeatedly perform edge additions and vertex splittings, starting from some small graphs, which are usually small wheels. In the proofs of the first few results, we are going to include as much detail as possible, to help the reader to understand the process. Since the isolated graphs are getting bigger in the last few results, we will skip some of the details. In fact, in the last result, some extensions are performed by computer. We also use Theorem 2.2 to handle the infinite families. We prove that, for each graph in the family, all its H-free edge-additions and vertex-splittings still belong to the family. Since graphs in these families are not defined by abstract properties, but by special constructions, it is understandable that there have to be a lot of case checking. In fact, the main work in this part is to find ways to efficiently organize the cases.

4.1 Excluding $K_5^{\perp}$, $K_{3,3}^{\nabla}$, and $K_{3,3}^{\ddagger}$

In this subsection we consider the three nonplanar graphs. The first result is known to many people. We include a proof for completeness.

Theorem 4.1 $\mathcal{F}(K_5^{\perp}) = \{K_5\} \cup \mathcal{F}(K_5) = \{K_5, V_8\} \cup \{$*3-sums of 3-connected planar graphs*$\}$.

Proof. The second equation follows from Theorem 3.9, so we only need to prove the first. Since K_5 and K_5-free graphs are $K_5^{\perp}$-free, it follows that $\mathcal{F}(K_5^{\perp}) \supseteq \{K_5\} \cup \mathcal{F}(K_5)$. To prove $\mathcal{F}(K_5^{\perp}) \subseteq \{K_5\} \cup \mathcal{F}(K_5)$, let $G \in \mathcal{F}(K_5^{\perp})$. We need to show that $G = K_5$ or $G \in \mathcal{F}(K_5)$. If $G \in \mathcal{F}(K_5)$ then we are done, so we assume that $G \notin \mathcal{F}(K_5)$, meaning that $G \succeq K_5$. If $G \neq K_5$, by Theorem 2.2, G has a minor J, which is obtained from K_5 by adding an edge or splitting a vertex. Since K_5 is complete, no edge can be added, so J is obtained by splitting a vertex of K_5, which means $J = K_5^{\perp}$, contradicting the assumption that $G \in \mathcal{F}(K_5^{\perp})$. Therefore, $G = K_5$, and thus the theorem is proved. ∎

Theorem 4.2 $\mathcal{F}(K_{3,3}^{\nabla}) = \mathcal{K} \cup \mathcal{P} \cup \{$*3-connected graphs on* $\leq$ *6 vertices*$\}$.

Proof. Let $\mathcal{L} = \mathcal{K} \cup \mathcal{P} \cup \{$3-connected graphs on ≤ 6 vertices$\}$. We first verify that $\mathcal{L} \subseteq \mathcal{F}(K_{3,3}^{\nabla})$. Since $K_{3,3}^{\nabla}$ is nonplanar, every planar graph is $K_{3,3}^{\nabla}$-free. Since $K_{3,3}^{\nabla}$ has seven vertices, every graph on ≤ 6 vertices is $K_{3,3}^{\nabla}$-free. Finally, in every minor of any graph in $\mathcal{K}$, there are three or fewer vertices that meet all its edges. However, it requires four or more vertices to meet all edges of $K_{3,3}^{\nabla}$, which implies that all graphs in $\mathcal{K}$ are $K_{3,3}^{\nabla}$-free.

Next, for any $G \in \mathcal{F}(K_{3,3}^{\nabla})$, we prove that $G \in \mathcal{L}$. If G is $(K_{3,3}+e)$-free, then the result follows from Theorem 3.7. Thus we assume $G \succeq K_{3,3}+e$. Since $K_{3,3}+e \in \mathcal{L}$, we can choose $H \in \mathcal{L}$ such that $G \succeq H \succeq K_{3,3}+e$ and such that H has as many edges as possible. Note that H is not planar, so either $|V(H)| = 6$ or $H \in \mathcal{K}$, which allows us to make the following assumption.

(*) Let the vertices of H be x_1, x_2, x_3, y_1, y_2, ..., y_m such that every x_i is adjacent to every y_j. In addition, if $m > 3$ then no y_i is adjacent to any other y_j, and if $m = 3$ then some x_i has degree ≥ 4.

Suppose $G \neq H$. Then H is a proper minor of G. By Theorem 2.2, G has a minor J obtained from H by adding an edge or splitting a vertex. We prove that $J \succeq K_{3,3}^{\nabla}$, which implies $G \succeq K_{3,3}^{\nabla}$, contradicting the assumption $G \in \mathcal{F}(K_{3,3}^{\nabla})$. This contradiction will prove $G = H$ and that proves the theorem.

We first assume $J = H + e$. Then $m > 3$ since otherwise $|V(H+e)| = 6$, implying $H + e \in \mathcal{L}$ and contradicting the maximality of H. Also by the maximality of H, we deduce that $e = y_iy_j$, for some $i \neq j$. Thus J contains the first graph in Figure 4.2 as a subgraph, which implies $J \succeq K_{3,3}^{\nabla}$, as required.

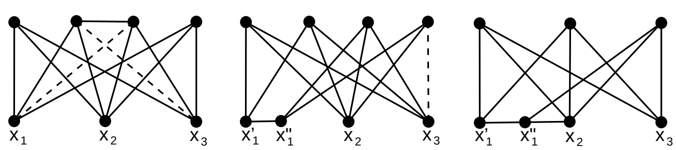

Figure 4.2: $J \succeq K_{3,3}^{\nabla}$, by deleting the dashed edges

Now we assume that J is obtained from H by splitting a vertex v. To simplify our analysis, we may further assume that v is some x_i. This is clear if $m > 3$ since $d_H(v) \geq 4$ while $d_H(y_j) = 3$ for every j. If $m = 3$ and v is some y_j, since $d_H(v) \geq 4$, we can interchange $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ without violating definition (*), which justifies the assumption that v is some x_i. Without loss of generality, let $i = 1$. Let

x_1', x_1'' be the two new vertices. Let $y_1, ..., y_k$ be adjacent to x_1', and $y_{k+1}, ..., y_m$ be adjacent to x_1''. By symmetry, let $k \geq m/2$. Then $k \geq 2$. Since $d_J(x_1'') \geq 3$, either $m - k \geq 2$ or x_1'' is adjacent to x_2 or x_3. We may assume $k < m$ because otherwise $J \in \mathcal{K}$, contradicting the maximality of H. Thus J contains the second or third graph in Figure 4.2 as a subgraph, which implies $J \succeq K_{3,3}^{\nabla}$, as required. ■

Theorem 4.3 $\mathcal{F}(K_{3,3}^{\ddagger})$ *consists of 3-connected planar graphs and 3-connected minors of the three graphs in Figure 4.3.*

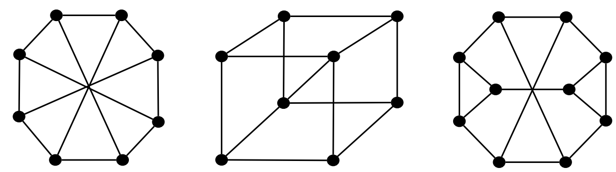

Figure 4.3: Maximal 3-connected nonplanar $K_{3,3}^{\ddagger}$-free graphs

Proof. The first graph in Figure 4.3 is V_8. Let us denote the other two by A_1 and A_2, respectively. Since planar graphs are clearly $K_{3,3}^{\ddagger}$-free, to prove the forward containment, we only need to show that V_8, A_1, A_2 are $K_{3,3}^{\ddagger}$-free. By Lemma 2.5, this is clear for V_8 since $\rho(K_{3,3}^{\ddagger}) = 5 > 4 = \rho(V_8)$. We also deduce from the same lemma that, if $K_{3,3}^{\ddagger}$ is a minor of A_1 or A_2, then the minor is obtained by only contracting edges. In A_1, contracting any edge incident with a degree-four vertex results in a planar graph, which is $K_{3,3}^{\ddagger}$-free. On the other hand, contracting any other edge results in three pairwise adjacent vertices of degree four, which do not appear in $K_{3,3}^{\ddagger}$. Thus A_1 is $K_{3,3}^{\ddagger}$-free. In A_2, let C be the 4-cycle formed by edges not incident with any of the two triangles. Note that the four edges of C are symmetric and contracting any of them results in a planar graph, which implies that no edge of C is contracted. Since no deletion is allowed, edges in a triangle cannot be contracted either. Therefore, since no two cubic vertices are adjacent in $K_{3,3}^{\ddagger}$, all edges not in C or the two triangles have to be contracted. But this is impossible because the result has only ten edges, which proves that A_2 is $K_{3,3}^{\ddagger}$-free. In summary, V_8, A_1, A_2 are $K_{3,3}^{\ddagger}$-free and thus all graphs described in the theorem belong to $\mathcal{F}(K_{3,3}^{\ddagger})$.

Next, for any graph $G \in \mathcal{F}(K_{3,3}^{\ddagger})$, we prove that either G is planar or G is a minor of V_8, A_1, or A_2. If G is $(K_{3,3}+e)$-free, by Theorem 3.7, either G is planar or $G = K_5$ or $K_{3,3}$. Since both K_5 and $K_{3,3}$ are minors of A_1, the theorem holds for $(K_{3,3}+e)$-free, and thus we may assume that G contains a $K_{3,3}+e$ minor. In the following, we generate all graphs, starting from $K_{3,3}+e$, by repeatedly adding edges and splitting vertices. We will only keep those that are $K_{3,3}^{\ddagger}$-free and we prove that the process terminates at V_8, A_1, A_2. Consequently, by Theorem 2.2, G is a minor of V_8, A_1, or A_2, which will prove the theorem.

Remark. We should warn the reader that the following analysis is tedious. We include the details because this is the first proof in this paper that involves nontrivial case analysis and we want to show how our method works. The task we are facing is clearly finite, so it is possible to solve the problem using computer, which is exactly what we did. We wrote a computer program with which we verified our case checking (and we did not miss any case!). Therefore, those who trust a computer on this type of computation can skip the following details.

In this proof, we will denote the generated graphs by Γ_k^a, Γ_k^b, and so on, where k is the number of edges of the graph. Let $K_{3,3}+e$ be labeled as in the figure below. By symmetry there is one addition Γ_{11}^a (obtained by adding 45) and one split Γ_{11}^b (obtained by splitting 1). For any generated graph Γ, let $F(\Gamma) = \{e : \Gamma + e \succeq K_{3,3}^{\ddagger}\}$, which is the set of *forbidden* edges. For instance, $F(K_{3,3}+e) = \{13, 23\}$. Since $F(\Gamma_{11}^a) \supseteq F(K_{3,3}+e)$, using symmetry we deduce that $F(\Gamma_{11}^a) \supseteq \{13, 23, 46, 56\}$. From the construction of

Γ_{11}^b and the fact $13 \in F(K_{3,3}+e) \subseteq F(\Gamma_{11}^b)$ we deduce that $1'3 \in F(\Gamma_{11}^b)$. Then by symmetry we obtain $F(\Gamma_{11}^b) \supseteq \{13, 23, 1'3, 12, 16\}$. For the purpose of reducing the amount of case checking, we will keep track of these sets using the same type of arguments, which we will not explicitly explain every time.

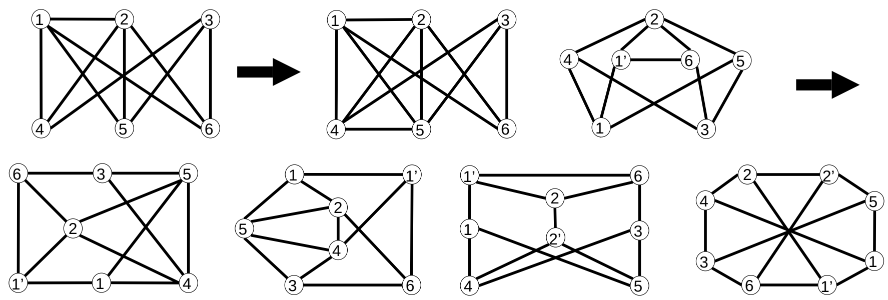

Figure 4.4: The first two steps: Γ_{11}^a, Γ_{11}^b, and Γ_{12}^a, Γ_{12}^b, Γ_{12}^c, Γ_{12}^d

Since $F(\Gamma_{11}^a) \supseteq \{13, 23, 46, 56\}$, no addition to Γ_{11}^a is $K_{3,3}^\ddagger$-free. Since all degree-four vertices of Γ_{11}^a are symmetric, we only need to split 1, which give rise to two graphs (up to isomorphism): Γ_{12}^a and Γ_{12}^b. As before, using the construction and symmetry we obtain $F(\Gamma_{12}^a) \supseteq \{13, 23, 46, 56, 1'3, 12, 1'5, 1'4, 16\}$ and $F(\Gamma_{12}^b) \supseteq \{13, 23, 46, 56, 1'3, 14, 1'2, 1'5, 16\}$.

From $F(\Gamma_{11}^b) \supseteq \{13, 23, 1'3, 12, 16\}$ we deduce that any addition to Γ_{11}^b has to be between two vertices in $\{1', 4, 5, 6\}$. By symmetry, we may add 45 or 46, which give rise to two graphs isomorphic to Γ_{12}^a and Γ_{12}^b, respectively. In Γ_{11}^b only vertex 2 can be split, which give rise to Γ_{12}^c and $\Gamma_{12}^d (= V_8)$. Since $F(\Gamma_{12}^d) \supseteq F(\Gamma_{11}^b) \supseteq \{13, 23\}$, we deduce by symmetry that no addition to Γ_{12}^d is $K_{3,3}^\ddagger$-free. On the other hand, Γ_{12}^d is cubic so no split is possible either. Therefore, the process terminates at Γ_{12}^d. In the following we assume that this situation does not occur any more. To be precise, we assume that:

(*) *if both vertices* 1 *and* 2 *are split in* $K_{3,3}+e$ *and the split at* 1 *is* $\{2,i\}$-$\{j,k\}$*, where set* $\{i,j,k\}$ *equals* $\{4,5,6\}$*, then the split at* 2 *is* $\{1,i\}$-$\{j,k\}$.

We further observe from $F(\Gamma_{12}^c) \supseteq F(\Gamma_{11}^b)$ that $F(\Gamma_{12}^c) \supseteq \{13, 23, 1'3, 12, 16, 2'3, 12'\}$.

From $F(\Gamma_{12}^a)$ we see that no addition to Γ_{12}^a is possible. By symmetry we will split 2 and 4. By (*) there is only one way to split 2, which results in Γ_{13}^a. By symmetry, splitting 4 results in Γ_{13}^b and Γ_{13}^c. Similarly, no edge can be added to $F(\Gamma_{12}^b)$ either. Splitting at 5 results in Γ_{13}^b, Γ_{13}^d, and V_8+24, which is not $K_{3,3}^\ddagger$-free. By (*), there is only one way to split 2, which is 15-46, and the result is isomorphic to Γ_{13}^c. Using the isomorphism $1'1526342' \to 11'2344'56$ and $F(\Gamma_{12}^b) \supseteq \{65, 64, 31', 32\}$ we also conclude that $F(\Gamma_{13}^c) \supseteq \{42, 45, 4'1, 4'3\}$. Finally, in Γ_{12}^c no splitting applies, and, by $F(\Gamma_{12}^c)$, any addition should involve neither 1 nor 3. From early analysis we have seen that adding edges to Γ_{11}^b between vertices in $\{1', 4, 5, 6\}$ would result in Γ_{12}^a or Γ_{12}^b, which have been analyzed. So we may assume that none of these are added to Γ_{12}^c. It follows that we only need to add edges incident with either 2 or $2'$. By symmetry, we may add either 24 or $1'2'$, which give rise to Γ_{13}^c or a graph that contains $K_{3,3}^\ddagger$ (by contracting $11'$ and 36), respectively.

Since no addition is possible to Γ_{12}^a, the only potential additions to Γ_{13}^a are between $22'$ and $1'456$. By symmetry, none of these is possible, so no addition to Γ_{13}^a is possible. Since $\Gamma_{13}^b, \Gamma_{13}^c, \Gamma_{13}^d$ are obtained similarly, the same argument (together with $F(\Gamma_{13}^c) \supseteq \{42, 45, 4'1, 4'3\}$) shows that no addition to any of these is possible either. So we only need to consider splits of these four graphs. By symmetry, Γ_{13}^a has only one split, obtained by splitting at 4, with respect to $12'$-35. The result is isomorphic to Γ_{14}^a, with isomorphism $11'22'344'56 \to 44'55'1'3621$. By symmetry and (*), Γ_{13}^b has only one split Γ_{14}^a, obtained by splitting at 5, with respect to 23-$14'$. Since additions to $\Gamma_{13}^a, \Gamma_{13}^b$ are impossible, we deduce that no addition is possible to Γ^a_{14}. In Γ^c_{13}, splitting 2 results in Γ^a_{14}, and splitting 5 results in Γ^b_{14} and other two graphs that contain $K^{\ddagger}_{3,3}$ (they properly contain V_8, by deleting $1'2$). Again, all potential additions to Γ^b_{14} are between $55'$ and $1234'$, and by symmetry, we deduce that no addition to Γ^b_{14} is possible. Finally, $F(\Gamma^d_{13}) = A_1$ and it has only one split, which contains $K^{\ddagger}_{3,3}$. Thus the process terminates at A_1, as required.

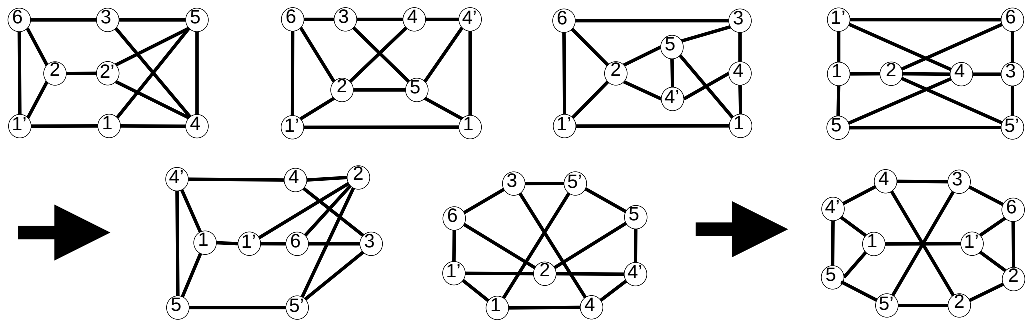

Figure 4.5: Γ^a_{13}, Γ^b_{13}, Γ^c_{13}, Γ^d_{13}, and Γ^a_{14}, Γ^b_{14}, Γ^a_{15}

Since no addition to $\Gamma^a_{14}, \Gamma^b_{14}$ is possible, we only need to consider splits. Note that the only non-cubic vertex is 2 in both cases, so by (*), there is only split in each graph. Splitting Γ^a_{14} results in Γ^a_{15} and splitting Γ^b_{14} results in an isomorphic copy of Γ^a_{15}. Since Γ^a_{15} is cubic, no split is possible. Moreover, using the same argument it is easy to see that no addition is possible either. Thus the process terminates at $\Gamma^a_{15} = A_2$, which proves the theorem. ∎

4.2 Excluding $W_5 + e$

Theorem 4.4 $\mathcal{F}(W_5 + e) = \mathcal{W} \cup \mathcal{K} \cup \{$*3-connected minors of graphs in Figure 4.6*$\}$.

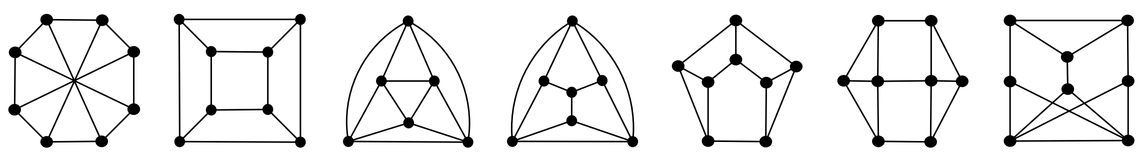

Figure 4.6: Maximal 3-connected $(W_5 + e)$-free graphs

Proof. The first four graphs in Figure 4.6 are V_8, Cube, Octahedron, and Pyramid. We denote the next three graphs by A_1, A_2, and A_3, respectively. To simplify our notation, we denote $W_5 + e$ by J. First, we prove that all graphs listed in the theorem are J-free. By Theorem 3.8, Cube, Octahedron, Pyramid, and graphs in $\mathcal{K}$ are W_5-free and thus they are also J-free. Since all 3-connected minors of a wheel is a wheel, every W_n is J-free. Next, since $\rho(V_8) = \rho(A_1) = 4 < 5 = \rho(J)$, it follows from Lemma 2.5 that V_8 and A_1 are J-free. For A_2 and A_3, since $\rho(A_2) = \rho(A_3) = \rho(J)$, we deduce from Lemma 2.5 that if J is a minor of A_2 or A_3 then the minor is obtained by contracting two and deleting zero edges. In particular, no edge in a triangle is contracted and no two edges from a 4-cycle are both contracted. Therefore, by inspecting A_2 and A_3 we see that the two contracted edges are not incident and all their ends have to be cubic. It follows that the maximum degree of the contracted graph must be four, which is different from that of J, and thus A_2 and A_3 are J-free as well.

Next we prove that every $G \in \mathcal{F}(J)$ is a minor of a graph listed in the theorem. By Theorem 2.1, G can be constructed from some wheel W_n by adding edges and splitting vertices. Let n be the largest such number. We first establish that either $n \le 5$ or $G = W_n$ or A_2. Suppose $n \ge 6$ and $G \ne W_n$. Then G has a 3-connected minor G' that is obtained from W_n by either adding an edge or splitting a vertex. Since $W_n + e$ has a J-minor, G' must be obtained from W_n by splitting v, its degree-n vertex. Let C be the cycle $W_n - v$ and let x, y be the two new vertices such that $d_{G'}(x) \le d_{G'}(y)$. If $d_{G'}(y) \ge 5$, we may choose two neighbors $x_1, x_2 \in V(C)$ of x and four neighbors $y_1, y_2, y_3, y_4 \in V(C)$ of y. Clearly, there are three possible distributions (up to isomorphism) of $x_1, x_2, y_1, y_2, y_3, y_4$ on cycle C. In each of these cases it is easy to see that a $C + \{xy, xx_1, xx_1, yy_1, yy_2, yy_3, yy_4\}$ (and hence of G) contains a J-minor. Thus $d_{G'}(y) < 5$, which implies that $n = 6$ and $d_{G'}(x) = d_{G'}(y) = 4$. Again, there are three cases, one results in A_2 while the other two (Cube $+e$ and $A_1 + e$) contain a J-minor. Finally, it is routine to verify that adding any edge or splitting any vertex in A_2 will result in a J-minor, which implies $G = A_2$, as required.

If G is W_5-free, by Theorem 3.8, G is in $\mathcal{K}$ or G is a minor of Cube, Octahedron, Pyramid, or $K_5^{\perp}$ (a minor of A_3), and thus we are done. In the following we assume that $G \succeq W_5$. From Theorem 2.2 and our discussion in the last paragraph we may further assume that G is W_6-free and so G can be constructed from W_5 by repeatedly adding edges and splitting vertices. We prove that the process terminates at $\{V_8, A_1, A_3\}$.

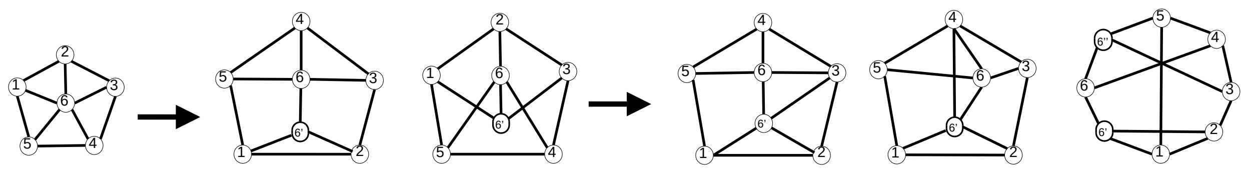

Figure 4.7: Γ_{11}^a, Γ_{11}^b and Γ_{12}^a, Γ_{12}^b, Γ_{12}^c

Adding any edge to W_5 results in a J-minor. Only vertex 6 of W_5 can be split and there are two ways to do it, which give raise to Γ_{11}^a and Γ_{11}^b. In Γ_{11}^a, adding any edge not incident with $6'$ results in a J-minor. There are two ways of adding an edge incident with $6'$, which give raise to Γ_{12}^a and Γ_{12}^b. In Γ_{11}^a, only vertex 6 can be split, which can be done in two ways and the results are Γ_{12}^c and A_1. Similarly, in Γ_{11}^b, adding any edge not incident with $6'$ results in a J-minor. There are two ways of adding an edge incident with $6'$, one gives raise to Γ_{12}^b and the other contains a J-minor. The only vertex that can be split in Γ_{11}^b is 6 and there are two ways to do it, which give raise to Γ_{12}^c or V_8.

It is routine to verify that adding any edge to V_8 or A_1 results a J-minor. Since these two are cubic graphs, it follows that if G contains either one of them then G is one of them. Thus we may assume that G contains Γ_{12}^a, Γ_{12}^b, or Γ_{12}^c. In Γ_{12}^a, adding any edge or splitting vertex 6 creates a J-minor; splitting vertex 3 either creates a J-minor or results in A_2. In Γ_{12}^b, adding any edge or splitting vertex 6 creates a J-minor; splitting vertex $6'$ either creates a J-minor or results in A_3. In Γ_{12}^c, adding edge $46''$ gives raise to A_3 while adding any other edge creates a J-minor. In conclusion, G has a A_3-minor. Finally, it is routine to verify that adding any edge or splitting any vertex in A_3 creates a J-minor, which implies $G = A_3$, and that proves the theorem. ■

4.3 Excluding *Octahedron*$\backslash e$

Recall that a 3-sum of two 3-connected graphs G_1, G_2 is obtained by identifying a triangle of G_1 with a triangle of G_2, and then deleting some of the common edges, as long as no degree-two vertices are created. The last graph in Figure 4.6 is a 3-sum of K_5 and Prism, where the common edges are all deleted. We will denote this graph by K_5^{Δ}. Let $\mathcal{S}$ be the set of graphs obtained by 3-summing wheels and Prisms over a common triangle. In other words, every graph in $\mathcal{S}$ is constructed from a set of wheels and Prisms, each with a specified triangle, by identifying all these specified triangles. Edges of these triangles could be deleted after the identification. It is worth pointing out that every 3-connected minor of a graph in $\mathcal{S}$ remains in $\mathcal{S}$, because 3-connected minors of a wheel are till wheels and 3-connected minors of a Prism are also wheels.

Theorem 4.5 *$\mathcal{F}(Octahedron \backslash e)$ consists of graphs in $\mathcal{S}$ and 3-connected minors of V_8, Cube, and K_5^{Δ}.*

Proof. In this proof we denote $Octahedron \backslash e$ by J. We first show that every graph listed in the theorem is J-free. Since $\rho(V_8) = \rho(Cube) = 4 < 5 = \rho(J)$ we deduce from Lemma 2.5 that V_8 and Cube are J-free. If K_5^{Δ} has a J-minor, since $\rho(K_5^{\Delta}) = \rho(J)$, this minor is obtained by contracting two edges and deleting none. It follows that edges in a triangle cannot be contracted. Up to isomorphism there is only one choice of such two edges yet the result of contracting these two edges leads to $K_5^{\perp}$, not J, so K_5^{Δ} is J-free. If $G \in \mathcal{S}$ has a J-minor, since J is 3-connected and all 3-connected minors of G are in $\mathcal{S}$, we deduce that J must be in $\mathcal{S}$. However, each graph in $\mathcal{S}$ has at most three vertices of degree > 3, yet J has four such vertices, so J is not in $\mathcal{S}$ and thus every graph in $\mathcal{S}$ is J-free.

Next we prove that every graph $G \in \mathcal{F}(J)$ is a minor of a graph listed in the theorem. In this proof we denote $W_5 + e$ by A_1, and the fifth and sixth graphs in Figure 4.6 by A_2, A_3, respectively. Notice that $A_1, A_2 \in \mathcal{S}$ as A_1 is a 3-sum of W_3 and W_4, and A_2 is a 3-sum of W_4 and the Prism. We first consider the case that G is A_1-free. In this case G is a minor of a graph H listed in Theorem 4.4. If H is V_8, Cube, A_2, K_5^{Δ}, W_n, or $K_{3,n}$ (which belongs to $\mathcal{S}$ as it is a 3-sum of n copies of W_3 over a common triangle), then it is trivial that G is a minor of a graph listed in Theorem 4.5. Thus H has to be Octahedron, Pyramid, or A_3. In Section 3 and the beginning of Section 4 we have listed all 3-connected graphs with at most eleven edges. It is easy to see that, other than J, they are either in $\mathcal{S}$ or minors of K_5^{Δ}. Thus we may assume that G has at least twelve edges. Since Octahedron and Pyramid are not J-free and they have twelve edges, H cannot be either one of them and so $H = A_3$. Notice that $A_3 \succeq J$ has thirteen edges and its only 3-connected J-free minor on twelve edges is W_6, so $G = W_6 \in \mathcal{S}$, as required.

From now on we assume $G \succeq A_1$ and we prove that G belongs to $\mathcal{S}$. By Theorem 2.2, G is constructed from A_1 by repeatedly adding edges and splitting vertices. Clearly, since A_1 is in $\mathcal{S}$, we only need to show that: if G is obtained from $H \in \mathcal{S}$ by adding an edge or splitting a vertex, then G either belongs to $\mathcal{S}$ or has a J-minor. Let H be the 3-sum of $H_1, H_2, ..., H_k$ over a common triangle with vertex set $X = \{x_1, x_2, x_3\}$, where each H_i is either a wheel or a Prism, and edges of the form $x_i x_j$ may or may not exist.

Suppose $G = H + e$, where $e = uv$. If both ends of e are in X then it is clear that $G \in \mathcal{S}$. Now we distinguish among the following three cases:

Case 1: $u \in V(H_1) - X$ and $v \in V(H_2) - X$;
Case 2: $u, v \in V(H_1) - X$; and
Case 3: $u \in V(H_1) - X$ and $v \in X$.

Case 1. We first consider a subcase that both H_1 and H_2 are W_3. Since $H \succeq A_1$, some H_i must have five or more vertices and moreover, H contains a minor H', which is obtained from H_1, H_2, W_4 by taking 3-sum over X such that at least one edge x_1x_2, x_2x_3, x_1x_3 remains in H'. Then $G \succeq H' + e \succeq J$ (see Firgure 4.8), which settles this subcase.

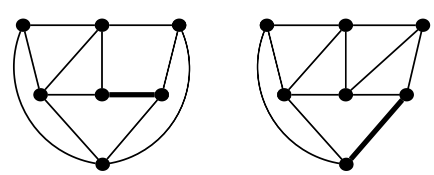

Figure 4.8: A J-minor can be obtained by contracting the heavy edge.

In the following proof, we will need to produce J-minors in almost every step. It would occupy too much space if we explain the constructions explicitly every time. Therefore, we will often simply present a graph on eight or nine vertices that contains a J-minor. With the help of different examples, the reader should be able to construct the minors without too much difficulty.

Now we assume that H_1 has five or more vertices. If $k > 2$, then a similar argument shows that H contains a minor H', which is a 3-sum of W_4 and W_3 over X such that $u \in V(W_4) - X$, $v \in V(W_3) - X$, and all three edges x_1x_2, x_2x_3, x_1x_3 remain in H' (so $H' = A_1$). It follows that $G \succeq H' + e \succeq J$. So we assume that $k = 2$. Figure 4.9 shows nine such graphs, where the middle three vertices are in X. The first graph is a 3-sum of two Prisms, the next three are 3-sums of a Prism and a wheel, and the last five are 3-sums of two wheels. It is straightforward to verify that H contains H', one of the first eight, as a minor, unless H equals the last graphs (with u, v as labeled). Then one can easily check that, if H equals the last graph then $H + e \in \mathcal{S}$, while in all other cases $H' + e$ and thus G contains J as a minor.

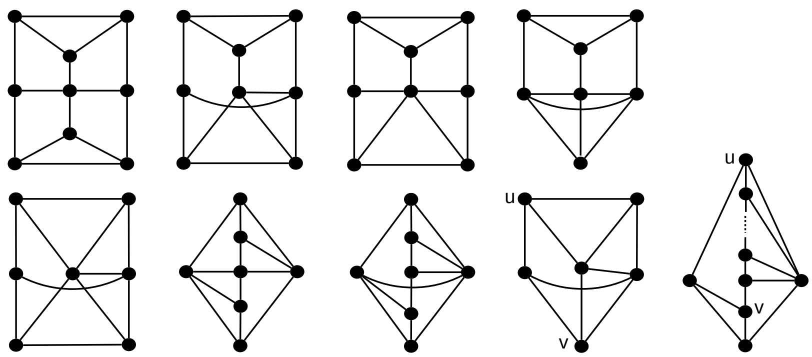

Figure 4.9: Relevant graphs in Case 1 when $k = 2$.

Case 2. It is clear that H_1 can only be a wheel with six or more vertices. Let Γ_1 and Γ_2 be the two graphs illustrated in Figure 4.10. If H has only one vertex outside H_1, then it is not difficult to see that either $H + e$ contains Γ_1, which contains J (by contracting the heavy edge), or $H + e = \Gamma_2$, which belongs to $\mathcal{S}$. So we assume that H has two or more vertices outside H_1. We claim that $H + e$ must contain Γ_1 and thus also J. If $k \geq 3$, then such a minor can be found easily by contracting $H_2 - X$ and $H_3 - X$. Thus we assume $k = 2$. It is straightforward to verify the claim if H_2 is a Prism, so we assume that H_2 is a wheel with at least five vertices. If the wheels H_1, H_2 have the same center vertex $x_1 \in X$, then x_2x_3 must be an edge of H (otherwise H would be a wheel, which does not contain A_1). So a Γ_1-minor can be found easily since H_2 has at least five vertices. If H_1, H_2 have different center vertices, say x_1, x_2, then x_3 must be adjacent to either x_1 or x_2. Now it is again routine to check that $H + e$ contains Γ_1 as a minor, which proves the claim and thus settles Case 2.

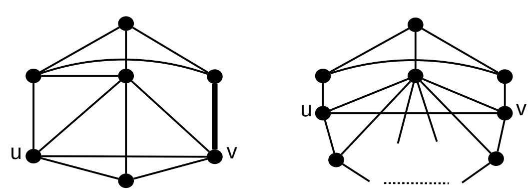

Figure 4.10: Relevant graphs in Case 2.

Case 3. The argument is very similar to that in the last two case so we only outline the proof and omit the details. If $k \geq 3$, then H contains a minor H', which is a 3-sum of W_4, W_3, W_3 over X such that

$u \in V(W_4) - X$, $v \in X$, and u, v are not adjacent in W_4. It follows that $G \succeq H' + e \succeq J$, so we assume that $k = 2$. If H_1 or H_2 is a Prism, then $H + e \succeq J$ (see the first four graphs in Figure 4.9), except for the third graph when H_1 is the Prism. In this exception case, H can also be expressed as a 3-sum of W_3, W_n, or a 3-sum of W_4, W_{n-1}. It is easy to check that, among the three additions, one contains J and the other two belong to $\mathcal{S}$. So we further assume that H_1 and H_2 are both wheels. If they have the same center (see the fifth and eighth graphs), then $H + e \succeq J$, except for the eighth graph with u as being labeled, which implies that G is a 3-sum of W_4 and W_n. So we assume that H_i $(i = 1, 2)$ has five or more vertices and has center x_i. We also assume that $d_H(x_i) \geq 4$ $(i = 1, 2)$ because otherwise H is also a 3-sum of a Prism and a wheel. If H can be expressed as the 3-sum of two other wheels (see the seventh graph), we assume that H_1 is as small as possible. If $v = x_3$, we may contract H_2 to W_4 and such that x_2 is adjacent to either x_1 or x_3, which implies that $H + e \succeq J$. If $v = x_2$, the minimality of H_1 implies $x_1x_3 \in E(H)$ and so $H + e \succeq J$, which completes Case 3.

Now we turn to the second half of the proof, which is the case that G is obtained from H by splitting a vertex. From the construction of H we can see that every vertex in $V(H) - X$ has degree three. Thus G is obtained from H by splitting a vertex in X. By symmetry we assume that $x_1 \in X$ is split into x_1', x_2''. We group graphs H_i according to their adjacency with the two new vertices. Let $I' = \{i : G$ has an edge from x_1' to $V(H_i) - X\}$ and $I'' = \{i : G$ has an edge from x_1'' to $V(H_i) - X\}$. Let $n' = |I' - I''|$, $n'' = |I'' - I'|$, and $n_0 = |I' \cap I''|$. If there exist distinct indexes i_1, i_2, i_3, i_4 such that $i_1, i_2 \in I'$ and $i_3, i_4 \in I''$, then a J-minor can be found in G by contracting $E(H_{i_j} - X)$ $(j = 1, 2, 3, 4)$ and deleting $V(H_i) - X$ for all other i. So we assume that no such four indexes exist. Then it is not difficult to verify that at least one of the following inequalities holds: $n_0 + n' \leq 1$, $n_0 + n'' \leq 1$, $n_0 + n' + n'' \leq 3$. Now we organize the cases according to the values of n', n'', n_0.

Suppose $n_0 + n' \geq 2$ and $n_0 + n'' \geq 2$. Then $n_0 + n' + n'' \leq 3$ and thus $(n', n_0, n'') = (1, 1, 1)$, $(1, 2, 0)$, $(0, 2, 1)$, $(0, 2, 0)$, or $(0, 3, 0)$. If $(n', n_0, n'') = (1, 2, 0)$, $(0, 2, 1)$, or $(0, 3, 0)$, then a J-minor can be found in G by contracting $E(H_i - X)$ $(i = 1, 2, 3)$. If $(n', n_0, n'') = (0, 2, 0)$, then both H_1, H_2 are wheels with five or more vertices and x_1 is the center of both wheels. Since H contains A_1, $x_2x_3 \in E(H)$ and we may further assume that both wheels are W_4 and X contains at least two edges. Then it is routine to verify that G has a J-minor. Finally, when $(n', n_0, n'') = (1, 1, 1)$, a similar case checking proves that $G \succeq V_8 + e \succeq J$.

Therefore, we may assume by symmetry that $n_0 + n' \leq 1$. If $n_0 + n' = 0$ then G is the 3-sum of $H_0, H_1, ..., H_k$ over a common triangle on X, where H_0 is a 3-wheel. This implies $G \in \mathcal{S}$ and so we assume $n_0 + n' = 1$ and $I' = \{1\}$. In other words, x_1' is adjacent to at least one vertex in $V(H_1) - X$, but to no vertex in $V(H_i) - X$ $(i \geq 2)$. We first consider the case that $k \geq 3$. If H_1 is a prism then it is easy to see that G has a J-minor, so H_1 is a wheel. If H_1 is a 3-wheel, then either x_1' is adjacent to both x_2, x_3, which implies that G has a J-minor, or x_1' is adjacent to only one of x_2, x_3, which implies $G \in \mathcal{S}$. So we assume that H_1 has five or more vertices. If x_1 is the center of H_1 then it is straightforward to check that G has a J-minor unless H_1 is a 4-wheel and the split turns H_1 into a Prism (and thus $G \in \mathcal{S}$). If x_2 is the center of H_1, then either $x_1'x_3 \in E(G)$, which implies that G has a J-minor, or $x_1'x_3 \notin E(G)$, which implies that the split turns H_1 into a larger wheel (and thus $G \in \mathcal{S}$).

It remains to consider the case $k = 2$, under the assumptions that $n_0 + n' = 1$ and $I' = \{1\}$. The situation $\max\{|V(H_1)|, |V(H_2|\} \leq 6$ is handled by case checking. This part is tedious so we omit the details. We remark that we are ensured that we did not miss any cases since we wrote a computer program, which confirmed our checking [4]. Thus we assume in the following that $|V(H_i)| \geq 7$ for some $i \in \{1, 2\}$. Note that H_i is not a Prism, so we further assume that H_i is a wheel with center $x \in X$. If $x \neq x_1$, or $x = x_1$ but one of x_1', x_1'' is adjacent to all vertices in $H_i - X$, then the wheel structure remains intact. In such a situation we can replace H_i with W_5 since it does not change whether G has a J-minor or not, and neither it changes whether G belongs to $\mathcal{S}$ or not. This observation implies that $|V(H_2)| \leq 6$ and thus $|V(H_1)| \geq 7$, which in turn implies that x_1'' is also adjacent to at least one vertex of $H_1 - X$. Consequently, x_1 is the center of H_1. Furthermore, we can obtain a minor H' of H by contracting H_2 to W_3 and such that $x_2x_3, x_1x_i \in E(H')$, for $i = 1$ or 2, unless $H_2 = W_3$ and $x_1x_2, x_1x_3 \notin E(H)$. Let $\{x_1, x_2, x_3, y\}$ be the vertex set of this W_3. From the first two graphs in Figure 4.11 we conclude that x_1' is cubic. Then we deduce from the next two graphs that either $G \in \mathcal{S}$ or G equals the last graph. In the last case, notice that $G/zx_j \succeq A_2$ is a 3-sum of $W_3, H_1/zx_j, H_2$, so the result follows from an early case with $k > 2$. ∎

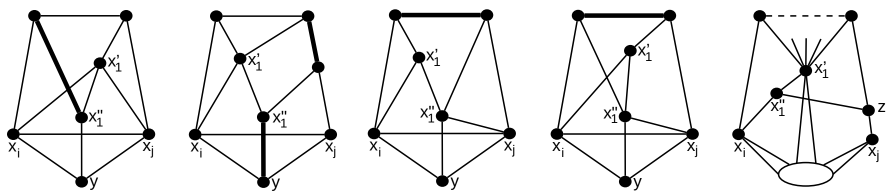

Figure 4.11: Contracting the heavy edges results in a J-minor.

4.4 Excluding $(W_5 + e)^*$

Theorem 4.6 *$\mathcal{F}((W_5{+}e)^*) = \mathcal{W} \cup \{$3-connected minors of K_6, $K_{4,4}$, Petersen, and graphs in Figure 4.12$\}$.*

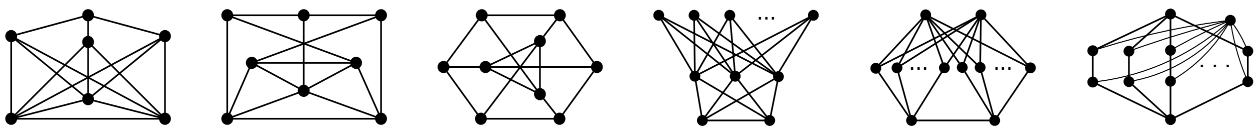

Figure 4.12: Some maximal 3-connected $(W_5 + e)^*$-free graphs

Proof. Let A_1, A_2, A_3 denote the first three graphs in Figure 4.12, respectively. The next two graphs in Figure 4.12 are denoted by $K^{+}_{3,n}$ and $K^{\perp}_{3,n}$, respectively, since they are obtained from $K_{3,n}$ $(n \geq 4)$ by adding an edge and splitting a vertex, respectively. Let $\widehat{\Theta}_n$ denote the last graph in Figure 4.12, where n is the number of triangles in the graph. This graph is so named because a subdivision of $K_{2,n}$ is usually called a Θ graph and $\widehat{G}$ stands for a graph obtained from G by adding a new vertex that is joined to vertices of G arbitrarily. In this proof we will denote $(W_5 + e)^*$ by J.

We first prove in three paragraphs that all graphs listed in the theorem are J-free. K_6 is J-free because it has fewer vertices than J. Since $K_{4,4}$ is bipartite while J is not, if $K_{4,4}$ has a J-minor then at least one edge is contracted. Since $K_{4,4}$ has only one more vertex than J, only one edge can be contracted. However, the new vertex of $K_{4,4}/e$ meets all its triangles but J does not have a vertex with this property, which implies that J is not a subgraph of $K_{4,4}/e$ and thus J is not a minor of $K_{4,4}$. Suppose the Petersen graph, denoted by P_{10}, has a J-minor. Since P_{10} has three more vertices than J, we may assume that three edges are contracted and thus one edge is deleted. Notice that $P_{10}\backslash e$ is a subdivision of V_8 and J has min-degree > 2, so V_8 has a J-minor. Clearly, one edge f of V_8 has to be contracted. However, the new vertex meets all triangles of V_8/f, which implies V_8/f is not J, so V_8, and thus also P_{10}, is J-free.

Observe that A_1 has a vertex that does not belong to any triangle while J does not have such a vertex. Hence J is not a spanning subgraph of A_1, which implies J is not a minor of A_1 since they have the same number of vertices. If A_2 has a J-minor then exactly one edge is contracted. If the middle vertical edge is contracted then the new vertex meets all triangles of the contracted graph, which is impossible since J does not have such a vertex. If any other edge is contracted, at least one of the top three vertices does not belong to any triangle, which is again impossible, so A_2 is J-free. If A_3 has a J-minor then we may assume that an edge e is deleted. By symmetry there are three choices for e. In each case, it is routine to check that $A_3\backslash e$ is a subdivision of a graph that is a minor of P_{10}. Since P_{10} is J-free, it follows that A_3 is also J-free.

Now we consider the four infinite families. W_n is J-free since all its 3-connected minors are wheels. Notice that $K^{+}_{3,n}$ has a set of ≤ 3 vertices whose deletion results in at most one edge. This is a property preserved under taking minors. Moreover, it is straightforward to verify that deleting any ≤ 3 vertices from J results in two or more edges and thus every minor of $K^{+}_{3,n}$ is J-free. Let us call a forest a *double-star* if it has a set of ≤ 2 vertices that meets all edges of the forest. Notice that $K^{\perp}_{3,n}$ has a set of ≤ 2 vertices whose deletion results in a double-star. This is a property preserved under taking minors. In addition, it is routine to check that J does not have this property, which implies that all minors of $K^{\perp}_{3,n}$ are J-free. In this proof let us call a graph a Θ-*graph* if it is the union of internally vertex-disjoint paths between two specified vertices such that each path has at most three edges. Notice that $\widehat{\Theta}_n$ has a vertex whose deletion results in a Θ-graph. Moreover, all its 3-connected minors also have this property. Since J does not have this property, which is easy to verify, it follows that all the 3-connected minors of $\widehat{\Theta}_n$ are J-free.

Next we prove the second half of the theorem that every 3-connected J-free graph G is a minor of one of the graphs listed in the theorem. By Theorem 2.1, G can be constructed from some wheel W_n by adding edges and splitting vertices. Let n be the largest such number. We first establish that either $G = W_n$ or $n \leq 6$. Suppose otherwise that $G \neq W_n$ and $n \geq 7$. Then G has a 3-connected minor G' that is obtained from W_n by either adding an edge or splitting a vertex. It is easy to see that $W_7 + e$ has a J-minor, which implies that $G' = W_n + e$ has a J-minor, a contradiction. Hence G' is obtained from W_n by splitting v, its degree-n vertex. Let C be the cycle $W_n - v$ and let x, y be the two new vertices such that $d_{G'}(x) \leq d_{G'}(y)$. Choose two neighbors $x_1, x_2 \in V(C)$ of x such that they are as close (on C) as possible. Then $C - \{x_1, x_2\}$ consists of two paths (one would be empty if x_1, x_2 are adjacent), and the longer one must contain (at least) three neighbors y_1, y_2, y_3 of y. Now it is clear that $C + \{xx_1, xx_2, yy_1, yy_2, yy_3, xy\}$ contains a J-minor, again a contradiction.

If $G = W_n$ then we are done, so we assume that $G \neq W_n$. From what we proved in the last paragraph we deduce that $n \leq 6$. If $n \leq 4$ then G is W_5-free. In this case the result follows from Theorem 3.8 immediately. Therefore, G is obtained from W_5 or W_6 by adding edges and splitting vertices, which we call a *growing process*. From the proofs of the previous theorems we have seen how this process works. Since everything is routine and since the process for the current problem is even longer, we are not going to go through all the details. Instead, we only provide a summary of each iteration, where the actual computation was done using computer. A more detailed supplement can be found in [4] and that can help the reader to verify the whole process.

From W_5 we can get two 11-edge J-free graphs: one on six vertices and one on seven vertices. From these two we obtain eight 12-edge J-free graphs: two on six vertices, five on seven vertices, and one on eight vertices. From these eight and W_6 we obtain fifteen 13-edge J-free graphs: two on six vertices, nine on seven vertices, and four on eight vertices. From these fifteen we obtain seventeen 14-edge J-free graphs, nine of which are shown in Figure 4.13. Among the other eight, one is on six vertices, three are on seven vertices, three are on eight vertices, and one is on nine vertices. From these eight we obtain seven 15-edge J-free graphs, including K_6, Petersen, A_1, A_2, and A_3, while the other two have seven and eight vertices, respectively. From the first five we do not get any new J-free graphs, which means that they are maximal. From the last two we get only one 16-edge J-free graph, $K_{4,4}$. Finally, from $K_{4,4}$ do not get any new J-free graphs, so $K_{4,4}$ is also maximal, which terminates the growing process.

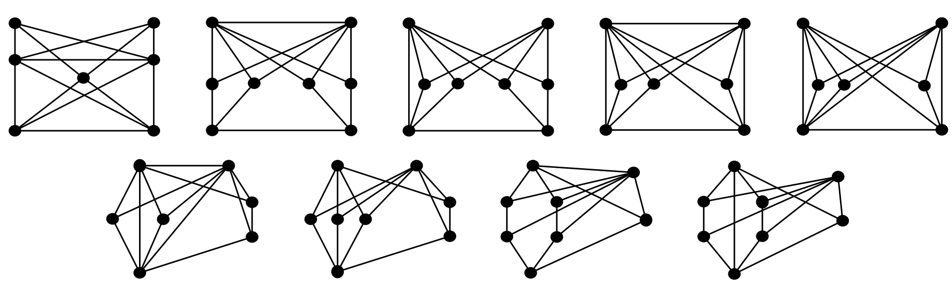

Figure 4.13: Seeds for the last three infinite families

It remains to consider the nine graphs in Figure 4.13. We prove that the growing process starting form these nine graphs will only lead to a minor of $K_{3,n}^{+}$, $K_{3,n}^{\perp}$, or $\widehat{\Theta}_n$, which will complete the whole proof. Let us denote these nine graphs by $\Gamma_1, \Gamma_2, \cdots, \Gamma_9$, respectively. Notice that Γ_1 is a minor of $K_{3,n}^{+}$; $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ are minors of $K_{3,n}^{\perp}$; and $\Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9$ are minors of $\widehat{\Theta}_n$. We consider these three cases separately.

We first consider Γ_1. Observe that there are three ways of adding an edge to Γ_1, two of which lead to a J-minor and the other one, adding an edge between the center vertex and a degree-5 vertex, leads to a minor of $K_{3,5}^{+}$. Moreover, there are nine ways of splitting a vertex in Γ_1, all lead to a J-minor [4]. We claim that if G is obtained by growing from Γ_1 and G is a minor of $K_{3,n}^{+}$, then adding an edge or splitting a vertex in G only results in a minor of $K_{3,n+1}^{+}$, as long as the resulting graph is J-free. The claim holds if $G = \Gamma_1$ since it is a restatement of our observation. In general, since G is a 3-connected minor of $K_{3,n}^{+}$, its vertices can be partitioned into X, Y, Z such that X consists of cubic vertices on the top, Z consists of two adjacent degree-4 vertices at the bottom, and Y consists of three vertices in the middle (see the drawing of $K_{3,n}^{+}$ in Figure 4.12). Our observation on Γ_1 implies that $G + e$ has a J-minor, unless e is between two vertices in Y. So the claim holds for edge additions. The same argument also proves the claim if we split a vertex in Z. Since all vertices in X are cubic, we only need to consider how to split a vertex in Y. Suppose the three vertices in Y are y_1, y_2, y_3, and suppose G' is obtained by splitting y_1 into y_1', y_1'' such that y_1'' has as many neighbors in $X \cup Z$ as y_1'. We may assume that y_1' has at least one neighbor in $X \cup Z$, for otherwise G' is a minor of $K_{3,n}^{+}$. Then it is routine to verify that G' contains a split of Γ_1 as a minor. Thus our observation again implies that G' has a J-minor, which proves the claim. As a consequence, we assume in the following that all graphs appeared in the growing process are Γ_1-free.

Now we consider $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$. We claim that if G is obtained from growing these four graphs and G is a minor of $K_{3,n}^{\perp}$ then adding an edge or splitting a vertex in G only results in a minor of $K_{3,n+1}^{\perp}$, as long as the resulting graph is $\{J, \Gamma_1\}$-free. Observe that the assumptions on G imply that G can be expressed as $K_{3,n'}^{\perp}$ ($n' \leq n$) together with a few extra edges. To be more precise, let x_1, x_2 be the top two vertices of $K_{3,n'}^{\perp}$ (see the drawing in Figure 4.12), z_1, z_2 be the bottom two vertices of $K_{3,n'}^{\perp}$, and $Y_1 \cup Y_2$ (where $Y_1 \cap Y_2 = \emptyset$) be the set of middle vertices such that z_i ($i = 1, 2$) is adjacent to all vertices in Y_i. Other than edges of $K_{3,n'}^{\perp}$ the only edges in G are between vertices in $\{x_1, x_2, z_1, z_2\}$. Moreover, if $|Y_i| = 1$ then z_i is adjacent to both x_1, x_2. We make the following observations [4] when G equals one of $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$.

(i) If $G \in \{\Gamma_2, \Gamma_3\}$ and $G + e$ is J-free, then both ends of e belong to $\{x_1, x_2, z_2, z_2\}$ and $G + e$ is a minor of $K_{3,6}^{\perp}$. If $G \in \{\Gamma_4, \Gamma_5\}$ and $G + e$ is $\{J, \Gamma_1\}$-free, then both ends of e belong to $\{x_1, x_2, z_2, z_2\}$ and $G + e$ is a minor of $K_{3,6}^{\perp}$.

(ii) If $G \in \{\Gamma_2, \Gamma_3\}$ then no splitting of G is J-free. If $G \in \{\Gamma_4, \Gamma_5\}$ and if G', obtained from G by splitting a vertex, is $\{J, \Gamma_1, \Gamma_3\}$-free, assuming that x_1, x_2 and z_1, z_2 are enumerated from left to right in Figure 4.13, then either the splitting is at x_1 in Γ_4 with the neighborhood partition $\{x_2, z_1\}$-$\{\text{rest}\}$, or the splitting is at z_1 in Γ_5 with the neighborhood partition $\{x_1, x_2\}$-$\{\text{rest}\}$. In both cases, we end up with the same graph G', which is a minor of $K_{3,5}^{\perp}$. This graph will be referred to as the *special slitting* of Γ_4 and Γ_5.

For a general graph G, from (i) it follows that either $G + e$ is a minor of $K_{3,n+1}^{\perp}$ or $G + e$ has a J- or Γ_1-minor because $G + e$ contains some $\Gamma_t + e$ ($2 \leq t \leq 5$) as a minor. Now suppose that G' is obtained from G by splitting a vertex v. Since all vertices in $Y_1 \cup Y_2$ are cubic, v must belong to $\{x_1, x_2\}$ or $\{z_1, z_2\}$. We consider these two cases separately.

Suppose $v = x_i$. Let x_i', x_i'' be the two new vertices. Let Y_1', Y_2' be neighbors of x_i' in Y_1, Y_2, respectively, and let Y_1'', Y_2'' be defined similarly. Let us assume $|Y_1' \cup Y_2'| \geq |Y_1'' \cup Y_2''|$. If $Y_1'' \cup Y_2'' = \emptyset$, then all neighbors of x_i'' are among x_i', x_j, z_1, z_2, where $x_j \in \{x_1, x_2\} - \{x_i\}$. If x_i'' is not adjacent to both z_1, z_2, then G' is a minor of $K_{3,n+1}^{\perp}$; if x_i'' is adjacent to both z_1, z_2 then G' has a J-minor (by considering the subgraph of G' induced on $x_i', x_i'', x_j, z_1, z_2$, any two vertices from Y_1', and one vertex from Y_2'). Thus we assume that $|Y_1'' \cup Y_2''| \geq 1$. Now we claim that G' contains a non-special splitting of Γ_t ($2 \leq t \leq 5$) as a minor, which will settle the case $v = x_i$, since they all have a J-minor. For $k = 1, 2$, by contracting edges of the form $z_k y$ we may assume that: if $Y_k' \neq \emptyset \neq Y_k''$, then $|Y_k'| = |Y_k''| = 1$; if one of Y_k', Y_k'' is empty, then the other has size $\min\{|Y_k|, 2\}$. At this point, the claim can be verified directly.

Suppose $v = z_i$. Let z_i', z_i'' be the two new vertices such that z_i'' is adjacent to $z_j \in \{z_1, z_2\} - \{z_i\}$. Let Y_i be partitioned into Y_i' and Y_i'' according to the adjacency with z_i', z_i''. If $Y_i' = \emptyset$ then z_i' is adjacent to only x_1, x_2, z_i'', which implies that G' is a minor of $K_{3,n}^{\perp}$. Thus we assume $Y_1' \neq \emptyset$. As in the last case, we claim that G' contains a non-special splitting of Γ_t $(2 \leq t \leq 5)$ as a minor, which will settle the case $v = z_i$. The proof of the claim is also similar to that in the last case. We may assume that: if $|Y_j| \geq 2$, then $|Y_j| = 2$; if $Y_i' \neq \emptyset \neq Y_i''$, then $|Y_i'| = |Y_i''| = 1$; if one of Y_i', Y_i'' is empty, then the other has size $\min\{|Y_i|, 2\}$. Again, the claim can be verified directly.

Finally, we analyze $\Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9$, the last four graphs in Figure 4.13. Based on what we have proved so far we may exclude Γ_5 as well. That is, we only need to consider $\{J, \Gamma_5\}$-free graphs. Let Γ_0 be obtained from $\widehat{\Theta}_3$ by adding three edges zx_1, zx_2, x_1x_2, where z is its degree-six vertex. Then Γ_0 is a minor of $\widehat{\Theta}_4$. We observe [4] that all $\{J, \Gamma_5, \Gamma_7\}$-free graphs generated from $\{\Gamma_6, \Gamma_8, \Gamma_9\}$ are minors of Γ_0. This process takes four iterations: from $\{\Gamma_6, \Gamma_8, \Gamma_9\}$ we obtain three 15-edge graphs, two with eight vertices and one with nine vertices; then we obtain three 16-edge graphs, one with eight vertices and two with nine vertices; then we obtain two 17-edge graph, both with nine vertices; and finally we obtain the 18-edge graph Γ_0, which cannot be extended anymore.

Because of the last observation, we only need to start the growing process from Γ_7. As before, we claim that if G is obtained from growing Γ_7 and G is a minor of $\widehat{\Theta}_n$, then adding an edge or splitting a vertex in G only leads to a minor of $\widehat{\Theta}_{n+1}$, provided that the new graph is $\{J, \Gamma_5\}$-free. Note that G has a vertex z such that $G - z$ consists of internally vertex-disjoint paths between two vertices x_1, x_2 such that each path has at most three edges and all the internal vertices of these paths are adjacent to z. In the following, by a *path* of G we will mean an x_1x_2-path of $G - z$ with two or three edges. We also denote $Y = V(G) - \{x_1, x_2, z\}$. Again, it is routine [4] to verify that the claim holds when $G = \Gamma_7$. In particular,

(i) if $e \neq x_1x_2$ is a missing edge of Γ_7 and e is not incident with z, then $\Gamma_7 + e$ as a J- or Γ_5-minor;
(ii) splitting any vertex of Γ_7 leads to either a J- or Γ_5-minor;
(iii) if $\Gamma_7' = \Gamma_7 + x_1z$, then any splitting of z in Γ_7' leads to a J- or Γ_5-minor.

For a general G, we deduce from (i) that, if $e \neq x_1x_2$ is not incident with z, then $G + e$ contains either J or Γ_5 as a minor, which proves the claim for edge additions. Next, suppose G' is obtained from G by splitting a vertex v. Since all vertices in Y are cubic, v must be z or x_i $(i = 1, 2)$. We consider these two cases separately. Let v', v'' be the two new vertices and let v' have as many neighbors in Y as v''. We first assume $v = z$. If z'' has no neighbor in Y, then G' is a minor of $\widehat{\Theta}_{n+1}$. If z'' has two or more neighbors in Y, then G' has a minor that is obtained from Γ_7 by splitting z, which implies by (ii) that G' has a J- or Γ_5-minor. Hence z'' has exactly one neighbor in Y. Since z'' has degree at least three, z'' is adjacent to at least one of x_1, x_2. It follows that G' has a minor that is obtained from Γ_7' by splitting z, which implies by (iii) that G' has a J- or Γ_5-minor. Therefore, G' contains either J or Γ_5 as a minor if $v = z$.

In the case $v = x_1$ or x_2, we assume by symmetry that $v = x_1$. If x_1'' is not adjacent to any vertex in Y, then G' is a minor of $\widehat{\Theta}_{n+1}$. Similarly, if x_1'' is adjacent to only one y in Y and y is in a 2-edge path of G then G' is also a minor of $\widehat{\Theta}_{n+1}$. Hence we assume that either x_1'' has two or more neighbors in Y or x_1'' has exactly one neighbor y in Y such that y is in a 3-edge path of G. In the first case G' has a minor that is obtained from Γ_7 by splitting x_1, which implies by (ii) that G' has a J- or Γ_5-minor. In the second case G' has a J-minor, which can be seen by choosing three paths of G, including the one that contains y, and then deleting all internal vertices of all other paths from G'. This proves our claim that that completes the proof of the theorem. ∎

5 Appendix

The purpose of this section is to list, in a concise form, characterizations of H-free graphs, for all the sixteen 3-connected graphs on at most eleven edges. Hopefully, those who are only interested in applying these results would find this Appendix useful.

By Lemma 2.4, H-free graphs are precisely those that are constructed by repeatedly taking 0-, 1-, and 2-sums, starting from K_1, K_2, K_3, and 3-connected H-free graphs. Therefore, we only need to describe 3-connected H-free graphs.

Special graphs: Graphs K_n, $K_{m,n}$, W_n (wheel), Prism, Cube, Oct (Octahedron), and Petersen are defined as usual. Other necessary graphs are illustrated in figures indicated below:
Figure 3.1: $K_5^{\perp}$, Pyramid
Figure 3.2: V_8
Figure 4.1: $K_{3,3}^{\nabla}$, $K_{3,3}^{\ddagger}$, $(W_5+e)^*$
Figure 4.6: K_5^{Δ} (the last graph)

Graph families:
$\{W_n\} = \{W_n : n \geq 3\}$
$\{K_{3,n}\} = \{K_{3,n} : n \geq 3\}$
$\mathcal{S} = \{$3-sums of wheels and Prisms over a common triangle$\}$
$\mathcal{C}^{\downarrow} = \{$3-connected minors of graphs in $\mathcal{C}\}$
$\mathcal{G}_{m.n} = \{$3-connected minors of all graphs illustrated in Figure m.n$\}$
Note that $\{K_{3,n}\}^{\downarrow}$ consists of 3-connected graphs obtained from $K_{3,n}$ ($n \geq 1$) by adding edges to its color class of size three. A more detailed definition of each family mentioned below can be found right before the corresponding theorem is stated.

Theorems:

$\boldsymbol{H}$	$\lvert\boldsymbol{E(H)}\rvert$	**3-connected H-free graphs**	**Theorem**
K_4	6	$\emptyset$	3.1
W_4	8	$\{K_4\}$	3.2
$K_5\backslash e$	9	$\{K_{3,3}, Prism\} \cup \{W_n\}$	3.3
$Prism$	9	$\{K_5\} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$	3.4
$K_{3,3}$	9	$\{K_5\} \cup \{$3-connected planar graphs$\}$	3.5
$Prism{+}e$	10	$\{K_5, Prism\} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$	3.6
$K_{3,3}+e$	10	$\{K_{3,3}, K_5\} \cup \{$3-connected planar graphs$\}$	3.7
W_5	10	$\{K_5^{\perp}, Cube, Oct, Pyramid\}^{\downarrow} \cup \{K_{3,n}\}^{\downarrow}$	3.8
K_5	10	$\{V_8\} \cup \{$3-sums of 3-connected planar graphs$\}$	3.9
$Cube/e$	11	augmentations of graphs in $\mathcal{G}_{3.3}$	3.10
$K_5^{\perp}$	11	$\{K_5, V_8\} \cup \{$3-sums of 3-connected planar graphs$\}$	4.1
$K_{3,3}^{\nabla}$	11	$\{K_6\}^{\downarrow} \cup \{K_{3,n}\}^{\downarrow} \cup \{$3-connected planar graphs$\}$	4.2
$K_{3,3}^{\ddagger}$	11	$\mathcal{G}_{4.3} \cup \{$3-connected planar graphs$\}$	4.3
W_5+e	11	$\mathcal{G}_{4.6} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$	4.4
$Oct\backslash e$	11	$\{V_8, K_5^{\Delta}, Cube\}^{\downarrow} \cup \mathcal{S}$	4.5
$(W_5+e)^*$	11	$\{K_6, K_{4,4}, Petersen\}^{\downarrow} \cup \mathcal{G}_{4.12} \cup \{W_n\}$	4.6

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