LARGE W_k- OR $K_{3,t}$-MINORS IN 3-CONNECTED GRAPHS

GUOLI DING, STAN DZIOBIAK, AND HAIDONG WU

Abstract. There are numerous results bounding the circumference of certain 3-connected graphs. There is no good bound on the size of the largest bond (cocircuit) of a 3-connected graph, however. Oporowski, Oxley and Thomas [11] proved the following result in 1993. For every positive integer k, there is an integer $n = f(k)$ such that every 3-connected graph with at least n vertices contains a W_k- or $K_{3,k}$-minor. This result implies that the size of the largest bond in a 3-connected graph grows with the size of the graph. Oporowski *et al.* did not give a specific function $f(k)$. In this paper, we first improve the above authors' result and obtain a specific function $f(k)$. Then we use the result to obtain a lower bound for the largest bond of a 3-connected graph. In particular, we show the following: Let G be a 3-connected planar or cubic graph on n vertices. Then for any $\epsilon > 0$, (i) G has a W_k-minor with $k = O((\log n)^{1-\epsilon})$; and (ii) G has a bond of size at least $O((\log n)^{1-\epsilon})$.

1. Introduction

For a connected graph G, the *circumference of* G, denoted by $c(G)$, is the length of the longest cycle of G (with $c(G) = 0$ if G is acyclic). Similarly, the *co-circumference of* G, denoted by $c^*(G)$, is the size of the largest bond of G (with $c^*(G) = 0$ if G has no bonds, where a *bond*, or *co-circuit* of G is a minimal edge-cut of G). For the case of planar graphs, these two graph parameters are very closely related: if G is planar, then $c^*(G) = c(G^*)$, where G^* is the planar dual of G, because the bonds of G are precisely the cycles of G^*. There are numerous results in the literature bounding the circumference of certain 3-connected graphs (some of these results will be stated later in the paper). However, there are no good analogous results for co-circumference of 3-connected graphs. Oporowski, Oxley and Thomas [11] proved the following structure theorem for 3-connected graphs:

Theorem 1.1. [11] *For every positive integer k, there is an integer n such that every 3-connected graph with at least $n = f(k)$ vertices contains a W_k- or $K_{3,k}$-minor.*

A specific function $n = f(k)$ is not given in the paper, although a huge bound can be extracted from the proof. In this paper, we first improve Oporowski, Oxley and Thomas' result and obtain a specific function $f(k)$. Then we use the result to obtain a lower bound for the size of the largest bond of a 3-connected graph.

A specific good function $n = f(k)$ in the above theorem will be useful due to the following observation: if H is a minor of G, then:

i) $c(H) \leqslant c(G)$
ii) $c^*(H) \leqslant c^*(G)$

Therefore, if G has a minor with large cycle (bond), then G must also have a large cycle (bond). Clearly, both W_k and $K_{3,k}$ have a bond of size $\geqslant k$. Therefore it follows from Theorem 1.1 that for every 3-connected graph G on n vertices $c^*(G) \geqslant f^{-1}(n)$, where $n = f(k)$. Hence, one of goals of this paper is to find a good function $k = f^{-1}(n)$, and therefore a good lower bound for $c^*(G)$.

The problem of bounding the circumference of 3-connected graphs has been widely studied. In general, the best lower bound is not very impressive, for example, $c(K_{3,k}) = 6$. On the other hand, as W_k has a cycle of size $k + 1$, by Theorem 1.1, any 3-connected large planar graph has a large wheel minor, and thus has a large cycle. The following theorem of Chen and Yu give a specific bound for $c(G)$ for a 3-connected planar graph G.

Theorem 1.2. [3] *If G is a 3-connected planar graph on n vertices, then $c(G) \geqslant n^{\log_3 2}$.*

The next two results give further bounds on the circumference of 3-connected graphs.

Theorem 1.3. [9] *If G is a 3-connected cubic graph on n vertices, then $c(G) > n^a$, where $a = \log(1 + \sqrt{5}) - 1$ (≈ 0.69).*

Theorem 1.4. [4] *For every integer $t \geqslant 3$, if G is a 3-connected graph on n vertices with no $K_{3,t}$-minor, then $c(G) \geqslant (1/2)^{t(t-1)} n^{\log_{1729} 2}$.*

The main motivation behind this paper is the dual problem of bounding the co-circumference of 3-connected graphs, which has been less widely studied. One of the authors of this paper made the following conjecture:

Conjecture 1.5. *If G is a 3-connected graph on n vertices, then $c^*(G) = O(n^{\log_3 2})$.*

Using Theorem 1.2 and Euler's Formula, it is easy to verfy that the above Conjecture is true for planar graphs.

In the remainder of the paper, we denote $\log_2 x$ by $\log x$ for simplicity. The following are our main results, the proofs of which are given in Section 4.

Theorem 1.6. *Let G be a k-connected graph ($k \geqslant 3$) on n vertices. Then for any $\frac{1}{2} \leqslant c < 1$ and any integer $p \geqslant 1$, G has a W_s-minor with $s = \left\lfloor \frac{\sqrt{2c}}{12}\sqrt{\log(\log n)} \right\rfloor$, or a $K_{k,t}$-minor with $t = O((\log n)^p)$.*

In the case $k = 3$ in the Theorem above, we can sacrifice the size of the $K_{3,t}$ to obtain a larger wheel in the following sense.

Theorem 1.7. *Let G be a 3-connected graph on n vertices. Then G has a $K_{3,t}$-minor with $t = O(\sqrt{\log n})$, or a W_k-minor with $k = O(\sqrt{\log n})$.*

Theorem 1.8. *Let G be a 3-connected planar or cubic graph on n vertices. Then G has a W_k-minor with $k = \left\lfloor \frac{\sqrt{2c}}{12}\sqrt{\log n} \right\rfloor$, where $0.63 < c < 0.7$ is a fixed constant.*

Theorem 1.9. *If G is a 3-connected graph on n vertices, then $c^*(G) \geqslant \frac{1}{12}\sqrt{\log n}$.*

Remark 1.10. *The coefficient in the above Theorem can be improved. In fact, in the proof we show that for any fixed constant $\frac{1}{2} \leqslant c < 1$, $c^*(G) \geqslant \frac{\sqrt{2c}}{12}\sqrt{\log n}$ for sufficiently large n.*

Finally, we can force the unavoidable minors to contain any specified edge of the original graph as stated in the following Corollary, which is proved in Section 5. We use $K'''_{3,t}$ to denote the graph obtained from $K_{3,t}$ by adding three edges connecting each pair of vertices of the bipartition side of size three.

Corollary 1.11. *Let G be a 3-connected graph on n vertices and e be any edge of G. Then for any $\frac{1}{2} \leqslant c < 1$ and any integer $p \geqslant 1$, e is contained in a W_s-minor of G with $s = \left\lfloor \frac{\sqrt{2c}}{24}\sqrt{\log(\log n)} \right\rfloor$, or in a $K'''_{3,t}$-minor of G with $t = O((\log n)^p)$.*

A similar result can be obtained for large minors containing two edges of a 3-connected graph. In the last section, we will give asymptotically better bounds for sufficiently large $n = |V(G)|$.

2. Forcing a large $K_{3,t}$-minor

By the *length* of a path, we mean the number of edges in the path. Let T be rooted tree with root r. By the *height* of T, we mean the length of the longest path from any leaf to r. By the *depth* of a vertex v in T, we mean the length of the unique path from v to r in T.

Proposition 2.1. *Let G be a k-connected graph ($k \geqslant 3$) on n vertices with no path of length h. Then G has a $K_{k,t}$-minor with $t > \dfrac{1}{h^{k-1}} n^{1/h}$.*

Proof. Let T be a depth-first-search spanning tree of G (also called a normal spanning tree in [7]). Then the height of T is at most $h-1$. By definition of T, all of the remaining edges of G that are not in T join vertices to some of their descendants in T. Let v be a vertex of T with the maximum number d of children in T. Since the number of vertices at depth i in T is at most d^i, and the largest depth of any vertex in T is at most $h-1$, we have that $n \leqslant 1+d+d^2+\ldots+d^{h-1} < d^h$. Hence $d > n^{1/h}$.

Let A be the set of ancestors of v in T. Since the depth of v is at most $h-2$, it follows that $|A| \leqslant h-2$. Also, since deleting $A \cup \{v\}$ disconnects G, and G is k-connected, it follows that $|A| \geqslant k-1$.

Let $v_1, v_2, \ldots, v_d$ be the children of v. For $i = 1, \ldots, d$, let D_i be the set consisting of v_i and all of its descendants in T; let $N(D_i)$ denote the neighborhood of D_i in G; and let $A_i = A \cap N(D_i)$. Since G is k-connected, it follows that $|A_i| \geqslant k-1$. Let $t = \left\lfloor \dfrac{d}{\binom{h-2}{k-1}} \right\rfloor$. Then $\frac{d}{t} \geqslant \binom{h-2}{k-1} \geqslant \binom{|A|}{k-1}$. By the pigeon-hole principle, there is a subset $A' \subseteq A$ with $|A'| = k-1$ and a subset $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, \ldots, d\}$ such that $A' \subseteq A_{i_j}$ for $j = 1, 2, \ldots, t$. Now let G' be the subgraph of G induced by $\{v\} \cup A' \cup \bigcup_{j=1}^{t} D_{i_j}$. We obtain a $K_{k,t}$-minor of G' (and hence of G) by deleting all the edges between vertices in A' in G', and for each $j = 1, 2, \ldots, t$ contracting the subgraph of G' induced by D_{i_j} to a single vertex.

Finally, we establish a lower bound for t. We obtain:

$$t = \left\lfloor \frac{d}{\binom{h-2}{k-1}} \right\rfloor > \left\lfloor \frac{d}{\binom{h}{k-1}} \right\rfloor > \frac{d}{h^{k-1}} > \frac{1}{h^{k-1}} n^{1/h}.$$

□

Corollary 2.2. *Let G be a 3-connected graph on n vertices with no path of length h. Then G has a $K_{3,t}$-minor with $t > \frac{1}{h^2} n^{1/h}$.*

3. Long path implies large wheel

The objective of this section is to prove Proposition 3.8, for which we will need Lemmas 3.5 and 3.6, and Theorem 3.7, which is due to [8]. For the proof of Lemma 3.5, we will need Theorems 3.1, 3.2, 3.3, and 3.4.

If C is a Hamiltonian cycle of graph G, we let $\Omega(G, C)$ denote the chord graph (or crossing graph) of G with respect to C. It is defined as follows: the vertices of the chord graphs are the chords of C, and two such vertices are adjacent if and only if the two chords cross.

The classical theorem of Ramsey states that for every pair of positive integers (m, n) there exists an integer N such that every graph on at least N vertices contains either a clique of order m or an independent set of order n. The smallest such integer N is called the Ramsey number of (m, n), and is denoted by $R(m, n)$.

Theorem 3.1. *(Erdos 1947, see* [7]*) For every integer $t \geqslant 3$, the Ramsey number of (t, t) satisfies $R(t, t) > 2^{t/2}$.*

Theorem 3.2. *For every integer $t \geqslant 3$, the Ramsey number of (t, t) satisfies* $\log R(t, t) < 2t - 2$.

Proof. Since $R(t, t) < \binom{2(t-1)}{t-1} < 4^{t-1} = 2^{2t-2}$, it follows that $\log R(t, t) < 2t - 2$. (Note: here we used the fact that for any integer $n \geqslant 1$, $\binom{2n}{n} < 4^n$, which easily follows by induction.) □

Theorem 3.3. [8] *Let t be a positive integer, and let G be a connected graph on $(R(t, t))^t$ vertices. Then G has an induced subgraph isomorphic to K_t, $K_{1,t}$, or P_t.*

Theorem 3.4. [8] *Let $t \geqslant 2$ be an integer, and let C be a Hamiltonian cycle of a graph G. If $\Omega(G, C)$ is a path on at least $6t$ vertices, then G has a minor isomorphic to W_t.*

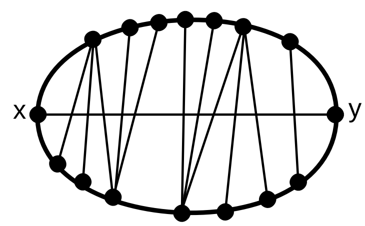
Lemma 3.5. *Let G be a 3-connected Hamiltonian graph on n vertices. Then G has a W_k-minor with $k = \left\lfloor \frac{1}{6\sqrt{2}} \sqrt{\log n} \right\rfloor$.*

Proof. Let C be a Hamiltonian cycle of G, and let $\Omega := \Omega(G, C)$ be the chord graph of G with respect to C. Let t be an integer such that $(R(t,t))^t \leqslant |V(\Omega)| < (R(t+1, t+1))^{t+1}$. By Theorem 3.3, Ω has an induced subgraph H isomorphic to K_t, $K_{1,t}$, or P_t. Let G' be the subgraph of G corresponding to H.

If $H = P_t$, then it follows by Theorem 3.4 that G', and hence G, has a minor isomorphic to W_k with $k = \left\lfloor \frac{t}{6} \right\rfloor$.

If $H = K_t$, then G' consists of a spanning cycle C' and t pairwise intersecting chords. Let xy be one such chord. Then by contracting the vertices in $C'(x, y)$ to a single vertex, we obtain a W_t-minor of G', and hence of G (here $C'(x, y)$ denotes the set of internal vertices of the path on C from x to y on any orientation).

Finally, if $H = K_{1,t}$, then G' consists of a spanning cycle C', a chord xy, and t chords that intersect xy, but not each other. Note that some of the chords may have common endpoints. Then again, by contracting the vertices in $C'(x, y)$ to a single vertex, or by contracting the vertices in $C'(y, x)$ to a single vertex, we obtain a W_k-minor of G', and hence of G, with $k = \left\lfloor \frac{t}{2} \right\rfloor$.



Since $\frac{n}{2} \leqslant |V(\Omega)| < (R(t+1, t+1))^{t+1}$, it follows by Theorem 3.2 that (for $t \geqslant 2$):

$$\log n - 1 < (t+1)(2t)$$
$$\log n < 2t^2 + 2t + 1 < 2(t+1)^2$$
$$t + 1 > \sqrt{\frac{\log n}{2}} \geqslant \left\lfloor \sqrt{\frac{\log n}{2}} \right\rfloor$$

Therefore $t \geqslant \left\lfloor \frac{1}{\sqrt{2}}\sqrt{\log n} \right\rfloor$. Finally, we let $k = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log n} \right\rfloor$, and the result follows. $\square$

Lemma 3.6. [1] *Let G be a 3-connected graph with a path of length L. Then G contains a cycle of length at least $\frac{2}{5}L + 2$.*

Theorem 3.7. [8] *Let G be a 3-connected graph and H be a topological minor of G without isolated vertices and without isthmuses. Then G has a 3-connected minor G_1 that has a subgraph H_1 that is isomorphic to a subdivision of H and $V(H_1) = V(G_1)$.*

Proposition 3.8. *Let G be a 3-connected graph with a path of length L. Then G has a W_k-minor with $k = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log(\frac{2}{5}L)} \right\rfloor$.*

Proof. By Lemma 3.6, G has a cycle C of length at least $\frac{2}{5}L$. By Theorem 3.7, G has a 3-connected minor G_1 with a spanning cycle C_1 that is a subdivision of C. Hence, C_1 also has length at least $\frac{2}{5}L$. Let $n = |V(G_1)|$, then by Lemma 3.5, G_1 (and hence G) has a W_l-minor with $l = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log n} \right\rfloor \geqslant \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log(\frac{2}{5}L)} \right\rfloor$. Therefore, we let $k = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log(\frac{2}{5}L)} \right\rfloor$, and the result follows. $\square$

4. Proofs of the Main Results

We first combine Corollary 2.2 and Proposition 3.8 to prove Theorem 1.6.

Proof of Theorem 1.6. Let c be a constant such that $\frac{1}{2} \leqslant c < 1$, and let $L = \frac{5}{2}(\log n)^c$. If G has a path of length L, then by Proposition 3.8, G has a W_s-minor with $s = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log(\frac{2}{5}L)} \right\rfloor = \left\lfloor \frac{1}{6\sqrt{2}}\sqrt{\log(\log n)^c)} \right\rfloor = \left\lfloor \frac{\sqrt{2c}}{12}\sqrt{\log(\log n)} \right\rfloor$.

On the other hand, if G has no path of length L, then by Proposition 2.1, G has a $K_{3,t}$-minor with:

$$t > \frac{1}{L^{k-1}}n^{1/L} = \frac{1}{L^{k-1}}e^{\frac{\ln n}{L}}$$

From the Maclaurin series of e^x, it follows that $e^x > \frac{x^j}{j!}$ for any positive real number x and any fixed positive integer j. Therefore:

$$t > \frac{1}{L^{k-1}} \cdot \frac{(\frac{\ln n}{L})^j}{j!} = \frac{(\ln n)^j}{j! L^{k-1+j}} = \frac{(\frac{\log n}{\log e})^j}{j!(\frac{5}{2})^{k-1+j}(\log n)^{c(k-1+j)}} = \frac{1}{j!(\frac{5}{2})^{k-1+j}(\log e)^j} \cdot (\log n)^{j(1-c)-c(k-1)}$$

Finally, for any integer $p \geqslant 1$, choose j such that $j(1-c) - c(k-1) \geqslant p$, and we have that $t = O((\log n)^p)$.

□

We now prove Theorem 1.7 using Theorem 1.4 of [4], Theorem 3.7, and Lemma 3.5.

Proof of Theorem 1.7. Let $\beta = \log_{1729} 2$, let $t = c \cdot \sqrt{\log n}$, where c is a positive constant such that $c^2 < \beta$, and let $\alpha = 2^{-t(t-1)}$. By Theorem 1.4, either G has a $K_{3,t}$-minor, or a cycle C_s with $s \geqslant \alpha n^\beta$. Note that:

$$\log s \geqslant \log \alpha + \beta \log n = -t^2 + t + \beta \log n > -t^2 + \beta \log n = (\beta - c^2) \log n = O(\log n).$$

Therefore $\log s = O(\log n)$. If G does not have a $K_{3,t}$-minor, and therefore has a cycle $C := C_s$, it follows by Theorem 3.7 that G has a 3-connected minor G_1 with a spanning cycle C_1 that is a subdivision of C. Hence, C_1 also has length at least s. Then by Lemma 3.5, G_1 (and hence G) has a W_k-minor with $k = O(\sqrt{\log s}) = O(\sqrt{\log n})$ and the conclusion of the Theorem follows.

□

Proof of Theorem 1.8. Let G be a 3-connected planar or cubic graph on n vertices. It follows from Theorems 1.2 and 1.3 that G has a cycle C of length at least n^c, where $c := \log_3 2$ (≈ 0.63) if G is planar, or $c := \log(1 + \sqrt{5}) - 1$ (≈ 0.69) if G is cubic. Then, by Theorem 3.7, G has a 3-connected minor G_1 with a spanning cycle C_1 that is a subdivision of C. Hence, C_1 also has length at least n^c. Therefore, by Lemma 3.5, G_1 (and hence G) has a W_k-minor with $k = \left\lfloor \frac{\sqrt{2}}{12} \sqrt{\log n^c} \right\rfloor = \left\lfloor \frac{\sqrt{2c}}{12} \sqrt{\log n} \right\rfloor$.

□

Finally, to prove Theorem 1.9, we use the following result of Lemos and Oxley [10]:

Theorem 4.1. [10] *If M is a connected matroid, then $c(M)c^*(M) \geqslant 2|E(M)|$.*

Proof of Theorem 1.9. Let G be a 3-connected graph on n vertices. Then, it follows from Theorem 4.1 that $c^*(G) \geqslant \frac{2|E(G)|}{c(G)} \geqslant \frac{3n}{c(G)}$. Let $\frac{1}{2} \leqslant c < 1$ be a fixed constant. If $c(G) < n^c$, then $c^*(G) \geqslant \frac{3n}{c(G)} \geqslant 3n^{1-c}$. And if $c(G) \geqslant n^c$, then it follows by arguments similar to those in the proof of Theorem 1.8, that G has a W_k-minor with $k = \left\lfloor \frac{\sqrt{2}}{12}\sqrt{\log n^c} \right\rfloor = \left\lfloor \frac{\sqrt{2c}}{12}\sqrt{\log n} \right\rfloor$. Since W_k has a bond of size $k+1$, it follows that $c^*(G) \geqslant \frac{\sqrt{2c}}{12}\sqrt{\log n}$. We conclude that

$$c^*(G) \geqslant max\{3n^{1-c}, \frac{\sqrt{2c}}{12}\sqrt{\log n}\}.$$

When $c = \frac{1}{2}$, we obtain Theorem 1.9. In general, it follows that $c^*(G) \geqslant \frac{\sqrt{2c}}{12}\sqrt{\log n}$ for sufficiently large n.

$\square$

5. Large minors containing a specified element

We finally prove Corollary 1.11, which follows immediately from Lemma 5.2 and Theorem 1.6.

The following is the graph version of a matroid theorem first proved by Brylawski [2].

Theorem 5.1. *Let H be a 2-connected loopless minor of a 2-connected loopless graph G, and suppose that $f \in E(G)-E(H)$. Then at least one of G/f and $G\backslash f$ is loopless, 2-connected and contains H as a minor.*

Lemma 5.2. *Suppose that G is a 3-connected graph containing a W_t- or a $K_{3,t}$-minor and let e be any edge of G. Then e is contained in a $W_{t'}$, $K_{3,t'}$, or a $K'''_{3,t-1}$-minor of G with $t' \geqslant \lceil \frac{t}{2} \rceil$.*

This lemma can be obtained from a series of more general matroid results in [5]. We give a pure graph proof here.

Proof. Let H be a fixed W_t- or a $K_{3,t}$-minor of G. If $e \in E(H)$, then we are done. Therefore $e \in E(G) - E(H)$. By repeatedly applying Theorem 5.1 for each edge $f \in E(G) - (E(H) \cup \{e\})$, we obtain a 2-connected minor H' of G such that either $H'/e = H$ or $H'\backslash e = H$. In the latter case, if e is in parallel with an edge f in H', by swapping e and f, we deduce that e is in a W_t- or a $K_{3,t}$-minor of G. So we may assume that H' is a simple graph. We divide the proof into two cases: (1) $H = W_t$, and (2) $H = K_{3,t}$.

(1) Suppose that $H = W_t$. Let x be the hub vertex of H, and let C be the cycle of H consisting of vertices $v_1, v_2, \ldots, v_n$ in the cyclic order.

If $H' \backslash e = H$, then $V(H') = V(H)$ and the endpoints of e are v_i and v_j for some $i \neq j$. At least one of the paths $C[v_i, v_j]$ or $C[v_j, v_i]$ contains at least $\lceil \frac{t}{2} \rceil$ vertices, so without loss of generality, suppose that the first one does. Then by contracting the vertices in $C(v_i, v_j]$ to a single vertex and removing parallel edges, we obtain a $W_{t'}$-minor of G containing e with $t' \geqslant \lceil \frac{t}{2} \rceil$.

If $H'/e = H$, then let u and v be the endpoints of e in H' and let w be the vertex in H obtained by contracting e in H'. There are two cases: either $w = v_i$ for some $i = 1, 2, \ldots, t$, or $w = x$. First, if $w = v_i$, then $N(\{u, v\}) = \{v_{i-1}, v_{i+1}, x\}$, and since $H'/e = H$, $N(u) \cap N(v) = \varnothing$, where all neighborhoods are understood to be in H' and, for clarity, they do not include vertices u and v. Also, since H' is 2-connected, $|N(u)|, |N(v)| \geqslant 1$. Therefore, without loss of generality, we have the following two subcases:

(a) $N(u) = \{v_{i-1}, v_{i+1}\}$ and $N(v) = \{x\}$. Then by contracting edge vx, we obtain a W_t-minor of G containing e.

(b) $N(u) = \{v_{i-1}, x\}$ and $N(v) = \{v_{i+1}\}$. Then by contracting edge vv_{i+1}, we obtain a W_t-minor of G containing e.

Second, if $w = x$, then similarly to the previous case, $N(\{u, v\}) = \{v_1, v_2, \ldots, v_n\}$, $N(u) \cap N(v) = \varnothing$, and $|N(u)|, |N(v)| \geqslant 1$. Then, at least one of $N(u)$ or $N(v)$ contains at least $\lceil \frac{t}{2} \rceil$ vertices, so without loss of generality, suppose that $N(u)$ does. Let v_j be any neighbor of v in H', then by contracting edge vv_j, we obtain a $W_{t'}$-minor of G containing e with $t' \geqslant \lceil \frac{t}{2} \rceil$. This proves (1).

(2) Suppose now that $H = K_{3,t}$. Let $(\{a_1, a_2, a_3\}, \{b_1, b_2, \ldots, b_t\})$ be the bipartition of H.

If $H' \backslash e = H$, then $V(H') = V(H)$ and the endpoints of e are, without loss of generality, either a_1 and a_2, or b_1 and b_2. In the first case, by contracting edge $b_1 a_3$, we obtain a $K'''_{3,t-1}$-minor of G containing e. In the second case, by contracting edge b_1a_1, and removing unnecessary edges, we obtain a $K_{3,t-1}$-minor of G containing e.

If $H'/e = H$, then let u and v be the endpoints of e in H' and let w be the vertex in H obtained by contracting e in H'. There are two cases: either $w = b_i$ for some $i = 1, 2, \dots, t$, or $w = a_j$ for some $j = 1, 2, 3$. First, if $w = b_i$, then as in Case (1) above, $N(\{u, v\}) = \{a_1, a_2, a_3\}$, $N(u) \cap N(v) = \varnothing$, and $|N(u)|, |N(v)| \geqslant 1$. Therefore, without loss of generality, we have that $N(u) = \{a_1\}$ and $N(v) = \{a_2, a_3\}$. Then by contracting edge a_1u, we obtain a $K_{3,t}$-minor of G containing e. Second, if $w = a_j$, then similarly, $N(\{u, v\}) = \{b_1, b_2, \dots, b_t\}$, $N(u) \cap N(v) = \varnothing$, and $|N(u)|, |N(v)| \geqslant 1$. Then, at least one of $N(u)$ or $N(v)$ contains at least $\lceil \frac{t}{2} \rceil$ vertices, so without loss of generality, suppose that $N(u)$ does. Let b_k be any neighbor of v in H', then by contracting edge vb_k, we obtain a $K_{3,t'}$-minor of G containing e with $t' \geqslant \lceil \frac{t}{2} \rceil$. This proves (2) and the Lemma. □

Corollary 1.11 extends a result of Chun, Oxley, and Whittle (Corollary 1.4 in [5]). In [6], Chun and Oxley proved a binary matroid result with the following consequence for graphs:

Theorem 5.3. *For every integer n exceeding two, there is an integer $j(n)$ so that if G is a simple 3-connected graph having at least $j(n)$ edges, and $\{e, f\} \subseteq E(G)$, then e and f are edges f a minor of G that is isomorphic to W_n or $K'''_{3,n}$.*

Using some of the lemmas in [6] and our main result, we can again give a specific bound $j(n)$ in the above result.

6. Improving the bounds asymptotically

In the proof of Lemma 3.5, we use the fact that a graph with $R(t,t)^t$ vertices has a $K_{1,t}, K_t$, or P_t minor (Theorem 3.3). This can be improved by the following result for chord graphs.

Lemma 6.1. *If G is a connected chord graph of a Hamiltonian graph H with t^{2t} vertices, then G has an induced subgraph isomorphic to $K_t, K_{1,t}$, or P_t.*

Proof. First suppose that G has a vertex x of degree at least t^2. Let X be the set of chords of H that cross x. We can define a partial order on X as follows: first draw the Hamiltonian cycle C of H and then draw x from top to bottom. Then chords in X are from the left to right. For any two chords c_1, c_2 in X, define $c_1 < c_2$ if they do not cross and c_1 is below c_2. One can easily check that this is a poset. Note that a chain (with respect to $<$) is a set of pairwise non-crossing chords in X, and an anti-chain is a pairwise crossing matching. By Dilworth's theorem, X has either a chain or an anti-chain of size t. Thus G has either $K_{1,t}$ or K_{1+t} as an induced subgraph.

Now suppose that the max-degree of G is less than t^2. If G has P_t then we are done. Else, the breadth-first search tree of G has height less than t. It follows that $|V(G)| < 1 + t^2 + (t^2)^2 + (t^2)^3 + \ldots + (t^2)^{t-1} < (t^2)^t = t^{2t}$, a contradiction. □

We now improve Lemma 3.5 for sufficiently large values of n.

Lemma 6.2. *Let G be a 3-connected Hamiltonian graph on n vertices. For any $\epsilon > 0$, G has a W_k-minor with $k = O((\log n)^{1-\epsilon})$.*

Proof. The proof proceeds exactly like the one of Lemma 3.5, except for the bound computation. We let t be the largest integer such that $t^{2t} \leqslant |V(\Omega)| < (t+1)^{2(t+1)}$. By Lemma 6.1, Ω has an induced subgraph H isomorphic to K_t, $K_{1,t}$, or P_t.

Since $\frac{n}{2} \leqslant |V(\Omega)| < (t+1)^{2(t+1)}$, it follows that $\log n - 1 \leqslant 2(t+1)\log(t+1)$. Let $x = t+1$, so that $x \log x \geqslant \frac{1}{2}(\log n - 1)$. Let $\epsilon > 0$, we will show that $x = O((\log n)^{1-\epsilon})$, and therefore $t = O((\log n)^{1-\epsilon})$. Let p be a positive integer such that $0 < \frac{1}{p+1} \leqslant \epsilon$. Since $x^{\frac{1}{p}} \geqslant \log x$, for sufficiently large x (and hence for sufficiently large t, and hence for sufficiently large n), it follows that:

$$x^{1+\frac{1}{p}} \geqslant x \log x \geqslant \frac{1}{2}(\log n - 1)$$

$$x^{\frac{p+1}{p}} \geqslant \frac{1}{2}(\log n - 1)$$

$$x \geqslant [\frac{1}{2}(\log n - 1)]^{\frac{p}{p+1}} = [\frac{1}{2}(\log n - 1)]^{1-\frac{1}{p+1}} \geqslant [\frac{1}{2}(\log n - 1)]^{1-\epsilon} = O((\log n)^{1-\epsilon})$$

Therefore $t = O((\log n)^{1-\epsilon})$.

Finally, since $k \geqslant \lfloor \frac{t}{6} \rfloor$, just as in the proof of Lemma 3.5, it follows that $k = O((\log n)^{1-\epsilon})$. □

Using the above Lemma in place of Lemma 3.5, we can improve the bounds in Theorem 1.6, 1.8 and 1.9 for sufficiently large n. The proofs are similar and will be omitted here.

Theorem 6.3. *Let G be a k-connected graph on n vertices ($k \geqslant 3$). Then for any $\epsilon > 0$ and any integer $p \geqslant 1$, G has a W_s-minor with $s = O((\log(\log n))^{1-\epsilon})$, or a $K_{k,t}$-minor with $t = O((\log n)^p)$.*

Theorem 6.4. *Let G be a 3-connected planar or cubic graph on n vertices. Then for any $\epsilon > 0$, G has a W_k-minor with $k = O((\log n)^{1-\epsilon})$.*

Theorem 6.5. *If G is a 3-connected graph on n vertices, then for any $\epsilon > 0$, G has a bond of size at least $O((\log n)^{1-\epsilon})$.*

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Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana

E-mail address: ding@math.lsu.edu

Department of Mathematics, University of Mississippi, University, Mississippi

E-mail address: smdziobi@olemiss.edu

Department of Mathematics, University of Mississippi, University, Mississippi

E-mail address: hwu@olemiss.edu