2007 LSU Math Contest
Open Session

Questions 1 - 12 are worth 1 point each and questions 13 - 24 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 12 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Erase clearly any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1
When you roll a single ordinary dice, which of the following is the most likely to be true about your score?

A it is odd B it is a factor of 18 C it is prime

D it is a factor of 12 E it is even

2
Suppose that b and c are constants and

$$(x+2)(x+b) = x^2 + cx + 6.$$

What is c ?

A -5 B -3 C -1 D 3 E 5

3
If the equations $ax + 3y = 5$ and $2x + by = 3$ represent the same line in a coordinate plane, then ab is equal to:

A -6 B -1 C 2 D 3 E 6

4
What is the number of pairs (x, y) of real numbers satisfying

$$|\tan \pi y| + (\sin \pi x)^2 = 0 \quad \text{and} \quad x^2 + y^2 \le 2\ ?$$

A 1 B 4 C 5 D 8 E 9

5
Suppose that $P(x) = ax^4 + bx^2 + x + 5$ and that $P(-3) = 2$. What is $P(3)$?

A -5 B -2 C 1 D 3 E 8

6
Suppose that $3 = k \cdot 2^r$ and that $15 = k \cdot 4^r$. What is r ?

A $-\log_2 5$ B $\log_5 2$ C $\log_{10} 5$ D $\log_2 5$ E $\frac{5}{2}$

7
For all real numbers x, except $x = 0$ and $x = 1$, the function f is defined by

$$f\left(\frac{x}{x-1}\right) = \frac{1}{x}\ .$$

Suppose $0 < \theta < \pi/2$. What is $f\left((\sec\theta)^2\right)$?

A $(\sin\theta)^2$ B $(\cos\theta)^2$ C $(\tan\theta)^2$ D $(\cot\theta)^2$ E $(\csc\theta)^2$

8
If $f(x) = 2^x$, then 16^8 is equal to

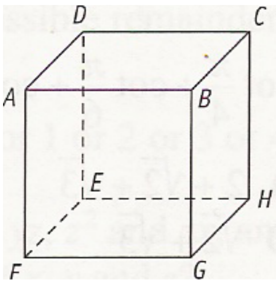
A $f(7)$ B $f(12)$ C $f(f(5))$ D $f(f(3))$ E $f(f(f(f(3))))$

9
Suppose that $\log_2(\log_3(\log_5(\log_7 N))) = 11$.
How many different prime numbers are factors of N ?

A 1 B 2 C 3 D 4 E 7

10

$ABCDEFGH$ is a cube with edges of length 2. L is the mid-point of FE and M is the mid-point of GH.
What is the area of the triangle ALM ?



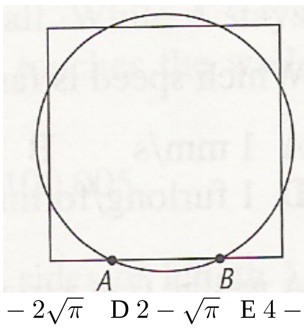
A $\dfrac{3\sqrt{2}}{2}$ B $\dfrac{3\sqrt{10}}{4}$ C $\sqrt{5}$ D 3 E $\dfrac{3\sqrt{5}}{2}$

11
The first two terms of a sequence are a, b. From then on, each term is equal to the negative of the previous term plus the term before that. What is the sixth term?

A $2b - 3a$ B $b - a$ C $2a - 3b$ D $5b - 3a$ E $-3a + b$

12

The circle and the square have the same center and the same area. If the circle has radius 1, what is the length of AB?



A $\sqrt{4-\pi}$ B $2\sqrt{1-\pi}$ C $4 - 2\sqrt{\pi}$ D $2 - \sqrt{\pi}$ E $4 - \pi$

Questions 13 - 24 Exact Answers

These next twelve questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ or 6^{40}.

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13 Calculate

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}\,.$$

Express your answer as a ratio of relatively prime positive integers.

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14 If $x=\left(1+\frac{1}{n}\right)^n$ and $y=\left(1+\frac{1}{n}\right)^{n+1}$ express y^x as a power of x.

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15 A rectangle has height two units less than its base.
A square has side one unit less than the base of the rectangle. The rectangle has area A.
What is the area of the square, in terms of A ?

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16 A function $f(x)$ satisfies $f(1-x)=f(1+x)$ for every number x. The equation $f(x)=0$ has 8 roots.
What is the sum of all these 8 roots?

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17 Find the exact value of $\sin x$ if $\tan x=\cos x$.

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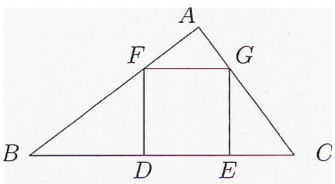
18 In triangle ABC, $AB=4$, $BC=5$ and $AC=3$. $DEGF$ is a square with DE on BC and F on AB and G on AC.



What is the area of the square $DEGF$?

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19 How many elements are there in the set

$$\{x:\ x \text{ a positive integer such that } -1\le \log_{\frac{1}{x}} 10 < -\frac{1}{2}\}\ ?$$

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20 Find the value of

$$2006^2-2005^2+2004^2-2003^2+\cdots+4^2-3^2+2^2-1^2$$

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21 Three judges for a talent quest have to vote publicly on three performers A, B, and C, listing their order of preference. In how many ways can the judges vote so that two of them agree in their order of preference, while the third differs?

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22 Let ABC be a triangle with altitudes CD and AE, with $BD=3$, $DA=5$, $BE=2$. Find EC.

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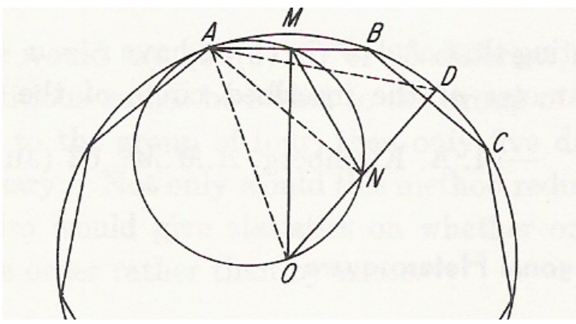
23 Let AB, BC be two adjacent sides of a regular *nonagon* (a polygon with nine sides) inscribed in a circle with center O. Let M be the midpoint of AB and N the midpoint of the radius perpendicular to BC. Find the angle OMN (in degrees).



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24 Consider the following two statements:

(P) Given that the planes α and β in 3-space intersect along the line ℓ, the lines a on α and b on β are *skew* lines (i.e., a and b are not both in the same plane), then ℓ can intersect at most one of a and b.

(Q) There do not exist infinitely many lines which are pairwise skew.

Which of them is or are true?

Tie Breaker requiring Full Solution

Please give a detailed explanation on the answer sheet to your solution to Question **24** above.

This tie breaker question is graded as an essay question,

i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit.

It is graded only to separate first, second, and third place ties.