RESOLUTION OF SINGULARITIES AND BLOW-UPS

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The T-shirt picture shows the blow-up of the xy-plane $\mathbb{A}^2$ and of a curve C at the point P. The process of "blowing-up" is used in mathematics to "resolve singularities". Below is a more detailed description of the picture and its mathematical meaning.

Singular and non-singular points

In the branch of mathematics called Algebraic Geometry, one is interested in studying the solutions of polynomial equations. As an example, the curve C in the picture (Figure 1)

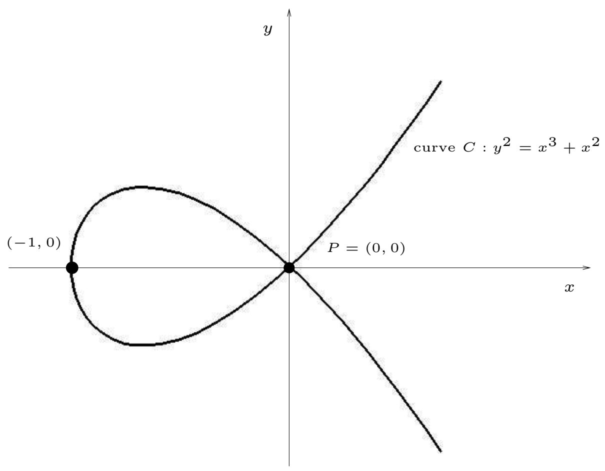

Figure 1

is the set of points with coordinates (x, y) such that $y^2 = x^3 + x^2$. The polynomial in question, for which C is the solution set, is $f(x, y) = y^2 - x^3 - x^2$. The points $(0, 0)$, $(-1, 0)$ and $(3, 6)$ are solutions of $f(x, y) = 0$ whereas $(1, 4)$ is not because $f(0, 0) = 0^2 - 0^3 - 0^2 = 0$, $f(-1, 0) = 0^2 - (-1)^3 - 1^2 = 0$, $f(3, 6) = 6^2 - 3^3 - 3^2 = 36 - 27 - 9 = 0$, and $f(1, 4) = 4^2 - 1^3 - 1^2 = 14 \neq 0$. Therefore, the points $(0, 0)$, $(-1, 0)$ and $(3, 6)$ lie on the curve C, but the point $(1, 4)$ does not.

This is an example of a singular curve; because it has "singularities". A point on a curve is singular if the curve at that point has a "corner", or "crosses itself", like P, for instance. A point on a curve is non-singular (or smooth) if the curve looks "smooth" around that point. Mathematically, "singularity" and "non-singularity" are defined in terms of tangents. The tangent at Q is the straight line through Q which best approximates the curve at that point. The precise definition is a little technical, and you can learn about it in a calculus course.

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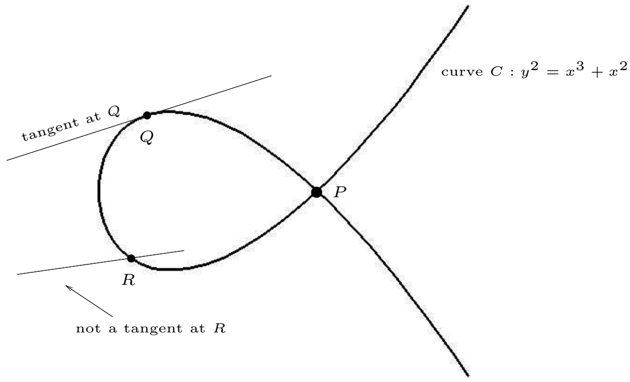

Figure 2

On our curve C, every point, except for P, has a well-defined tangent. At the point P, there seem to be two straight lines, called line 1 and line 2 in the picture (figure 3) that want to be a tangent, but neither qualifies as such, because each of them only approximates a piece of the curve at P but not the whole curve at P.

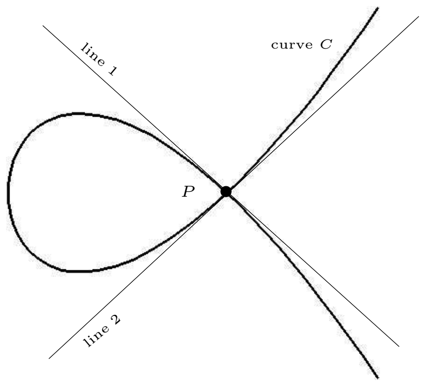

Figure 3

A non-singular (or smooth) point on a curve is a point at which the curve has a tangent. A singular point is a point at which the curve does not have a (well-defined) tangent. Therefore, every point on our curve C is non-singular, except for P, and P itself is a singular point.

The process in mathematics that allows one to "improve" singularities, and eventually to "get rid" of them is called "blowing-up".

Blowing-up a plane at its origin

The picture in Figure 4 below shows the "blow-up of a plane at a point P", where P is the origin of the plane. What is labelled $\mathbb{A}^2$ in the picture and looks like an ellipse, is supposed to be (a part of) the xy-plane embedded in 3-space. It is a little tilted for esthetic reasons. The blow-up $\widetilde{\mathbb{A}}^2$ is the surface that lies above (and a little to the left of) the plane $\mathbb{A}^2$ in the picture. It contains the z-axis which lies above our point P.

Loosely speaking, the blow-up $\widetilde{\mathbb{A}}^2$ of the plane $\mathbb{A}^2$ at the point P is obtained from the plane $\mathbb{A}^2$ by removing the point P, and replacing it with a line in such a way that the resulting surface $\widetilde{\mathbb{A}}^2$ is smooth. It can be described as follows. Recall that $\mathbb{A}^2$ was our xy-plane. We think of the plane $\mathbb{A}^2$ as the collection of lines through P in that plane. These are the dashed lines of the ellipse in Figure 4. Except for P, every point Q in the plane $\mathbb{A}^2$ lies on exactly one straight line through P, namely the line PQ which goes through P and Q. For every line PQ through P in the plane $\mathbb{A}^2$ there is exactly one line $\tilde{P}\tilde{Q}$ in the blow-up $\widetilde{\mathbb{A}}^2$ lying parallel above the line PQ. The height of the line $\tilde{P}\tilde{Q}$ depends on the angle in the plane $\mathbb{A}^2$ between the x-axis and the line PQ. If the angle is 0, then the height of the line is 0 and the line $\tilde{P}\tilde{Q}$ is just the x-axis. As the angle goes from 0 to 90°, then the heights of the lines $\tilde{P}\tilde{Q}$ go from 0 to ∞, and the lines $\tilde{P}\tilde{Q}$ sweep out the "upper part" of a spiral. As the angle goes from 0 to $-90°$, then the heights of the lines $\tilde{P}\tilde{Q}$ go from 0 to $-\infty$, and they sweep out the "lower part" of a spiral. For those who know about slopes, the height of the line $\tilde{P}\tilde{Q}$ is the slope of the line PQ, that is, if Q has coordinates (a, b) then the height of the line $\tilde{P}\tilde{Q}$ is $\frac{b}{a}$. So our blow-up surface $\widetilde{\mathbb{A}}^2$ looks like a spiral along the z-axis that makes one turn when going from $-\infty$ to ∞.

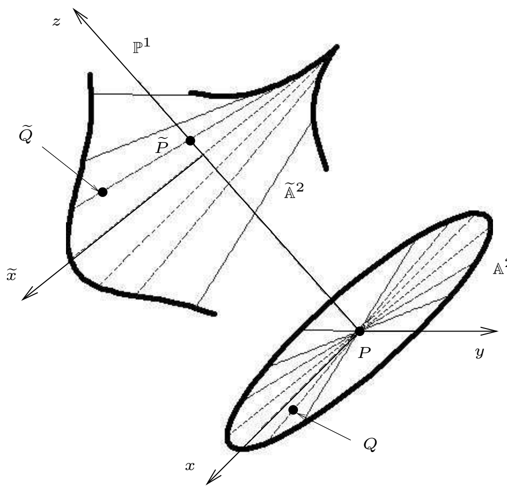

Figure 4

You may have noticed that the line $x = 0$ in $\mathbb{A}^2$ (which consists of the points $Q = (a, b)$ with $a = 0$) has not been assigned a line in $\widetilde{\mathbb{A}}^2$ yet, as its slope would be $\pm\infty$. For this line, we add an extra line at infinity, so that $\widetilde{\mathbb{A}}^2$ is actually the "spiral along the z-axis" plus an extra line at infinity. Over each point in $\mathbb{A}^2$, except for P, there lies exactly one point in $\widetilde{\mathbb{A}}^2$. Over the point P, however, we have the whole z-axis plus a point at infinity. In algebraic geometry, a line plus a point at infinity is called the projective line, and is denoted by $\mathbb{P}^1$. Therefore, the set of points in $\widetilde{\mathbb{A}}^2$ lying over P is the projective line $\mathbb{P}^1$, and we can say that in the blow-up of the plane $\mathbb{A}^2$, the point P has been replaced by the (projective) line $\mathbb{P}^1$.

To be honest, the picture in Figure 4 does not show exactly what I have just described. In the picture, I have actually moved the blow-up surface $\widetilde{\mathbb{A}}^2$ a little up in the z-direction, so that the relevant features won't overlap with the plane $\mathbb{A}^2$. In the picture, the line $\tilde{P}\tilde{Q}$ above the line PQ has height $10 + \frac{b}{a}$, for Q the point with coordinates (a, b). In particular, the line $\tilde{x}$ lying above the x-axis has height 10 in the picture...

Resolution of a curve singularity

The resolution of singularities $\widetilde{C}$ is a curve obtained from C by replacing the singular point P with a bunch of other points such that the result is a smooth curve. In our case, it is obtained by "blowing up the curve C at its singular point P".

The new curve $\widetilde{C}$ will lie in the blow-up surface $\widetilde{\mathbb{A}}^2$ described in the previous section. To construct $\widetilde{C}$, imagine putting a light under the $\mathbb{A}^2$-plane and directing it upwards in the z-direction. The curve C in the xy-plane will produce a shadow on the blow-up surface $\widetilde{\mathbb{A}}^2$ which consists of a curve $\widetilde{C}$ plus the whole z-axis (and the point at infinity). This shadow is not a smooth curve since it crosses itself at P_0 and P_1. But if we remove the z-axis (and the point at ∞) from the shadow, and fill in the two holes at P_0 and P_1, then we obtain a smooth curve $\widetilde{C}$ which is a resolution of singularities of C. The heights of the points P_0 and P_1 are the slopes of the two lines line 1 and line 2 of figure 3 which pretended to be tangents of C at P.

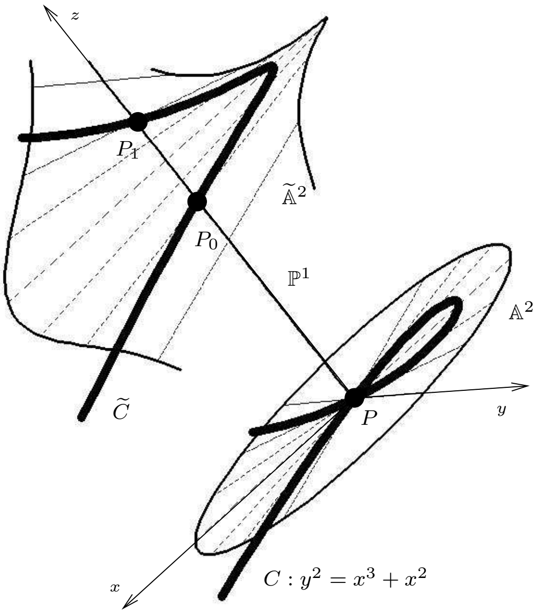

Figure 5

Another way to think of $\widetilde{C}$ is as follows. Imagine walking along the curve C, but instead of staying in the xy-plane $\mathbb{A}^2$, you slowly walk upwards. You will walk twice through P, but each time at a different height, once at P_0 and another time at P_1. The result is the smooth curve $\tilde{C}$. Unfortunatedly, this works only for curves, and not in higher dimensions. The blow-up process, however, works in all dimensions!

Hironaka's theorem

Besides curves, there are surfaces and higher-dimensional objects which may have singularities. One of the greatest achievements in 20th century mathematics is Hironaka's result that resolutions of singularities also exist in higher dimensions. They are constructed by successive blow-ups. For his result, Hironaka was awarded the Fields medal in 1970, which is the equivalent of the Nobel prize in mathematics.

An important problem which is still open is the question whether or not resolutions of singularities always exist in "positive characteristic". Hironaka's theorem only works for number systems for which $p \cdot a = 0$ can only happen for a prime number p when $a = 0$. In characteristic $p > 0$ (where p is some prime number), one works with number systems where $p \cdot a = 0$ for any a, regardless of whether $a = 0$ or $a \neq 0$. In this world, the existence of resolutions of singularities is still unknown!