Wavelet Sets, Fractal Surfaces and Coxeter Groups

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Workshop on Harmonic Analysis and Fractal Geometry

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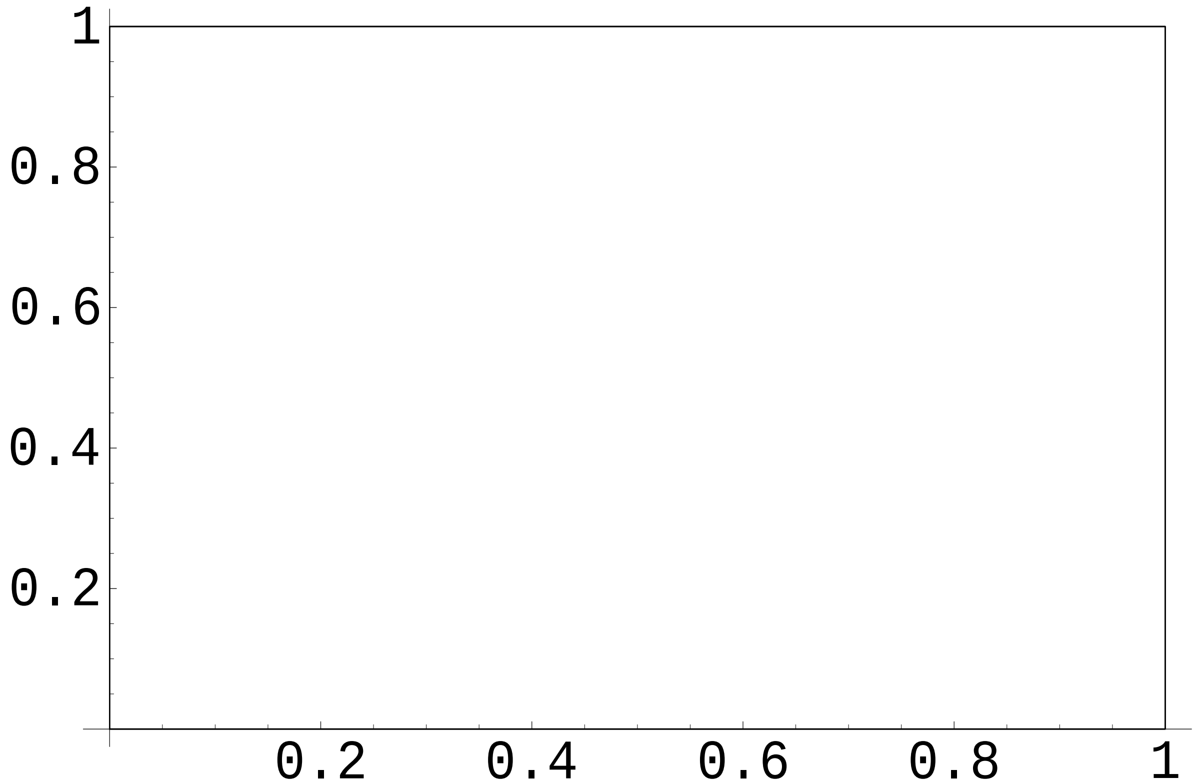
Outline

- Wavelet Sets based on Dilation/Translation Structure
- Fractal Functions and Fractal Surfaces
- Foldable Figures and Coxeter Groups
- Multiresolution Structures for $L^2(\mathbb{R}^n)$ on Foldable Figures
- Wavelet Sets based on Dilation/Reflection Structure

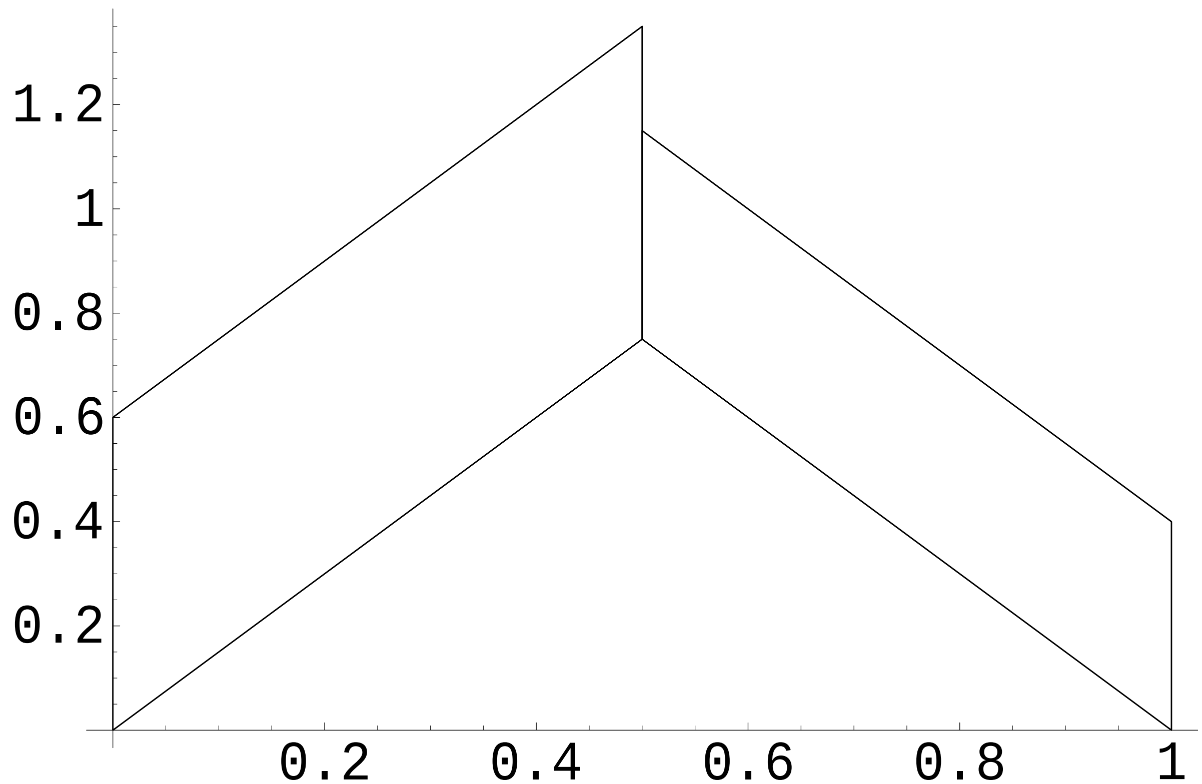
Fractal Interpolation Functions

- Fractal Interpolation Functions f introduced by Michael Barnsley in 1986
- Construction based on Affine Iterated Function Systems
- graph f is the limit (in the Hausdorff metric) of sequence of compact sets
- More general construction based on function spaces via Read-Bajraktarević operators (Dubuc, Bedford, M.)
- Fractal function is limit (in the metric of the underlying function space) of a sequence of functions (M. 1995)
- Regularity properties of fractal functions studied by M. in the setting of Besov and Triebel-Lizorkin spaces (1997, 2005)

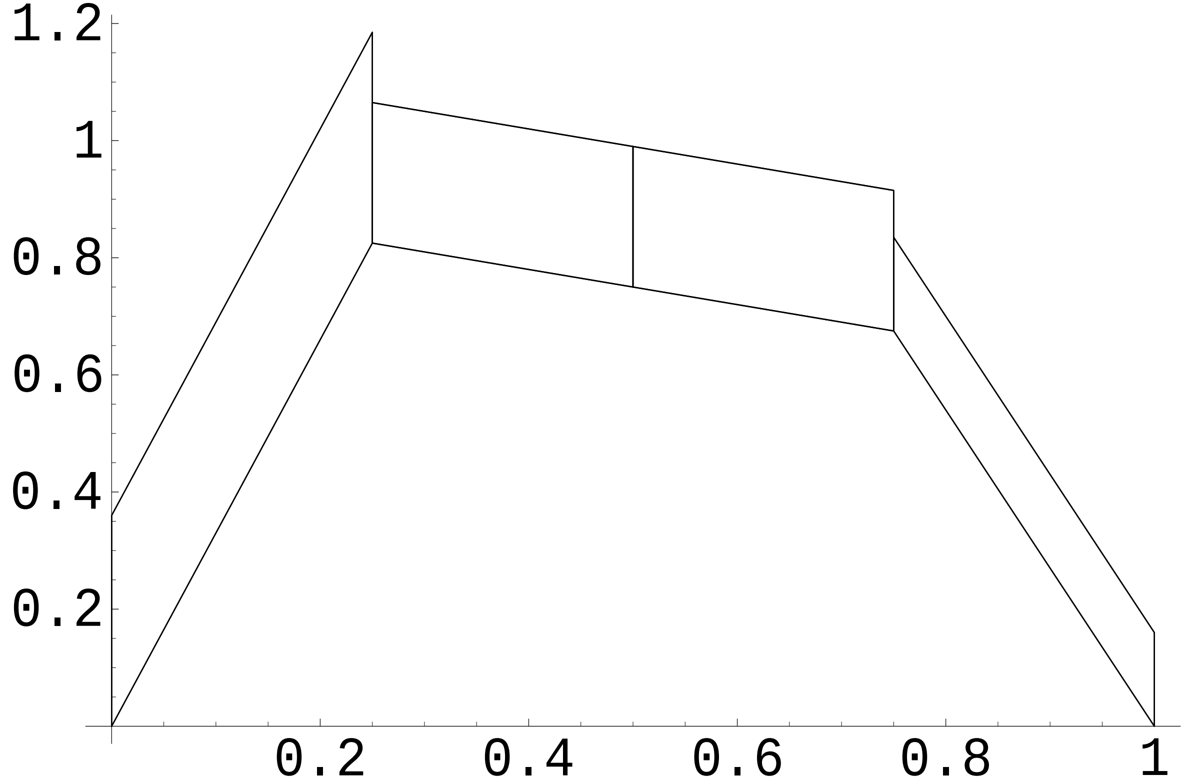
Geometric Construction



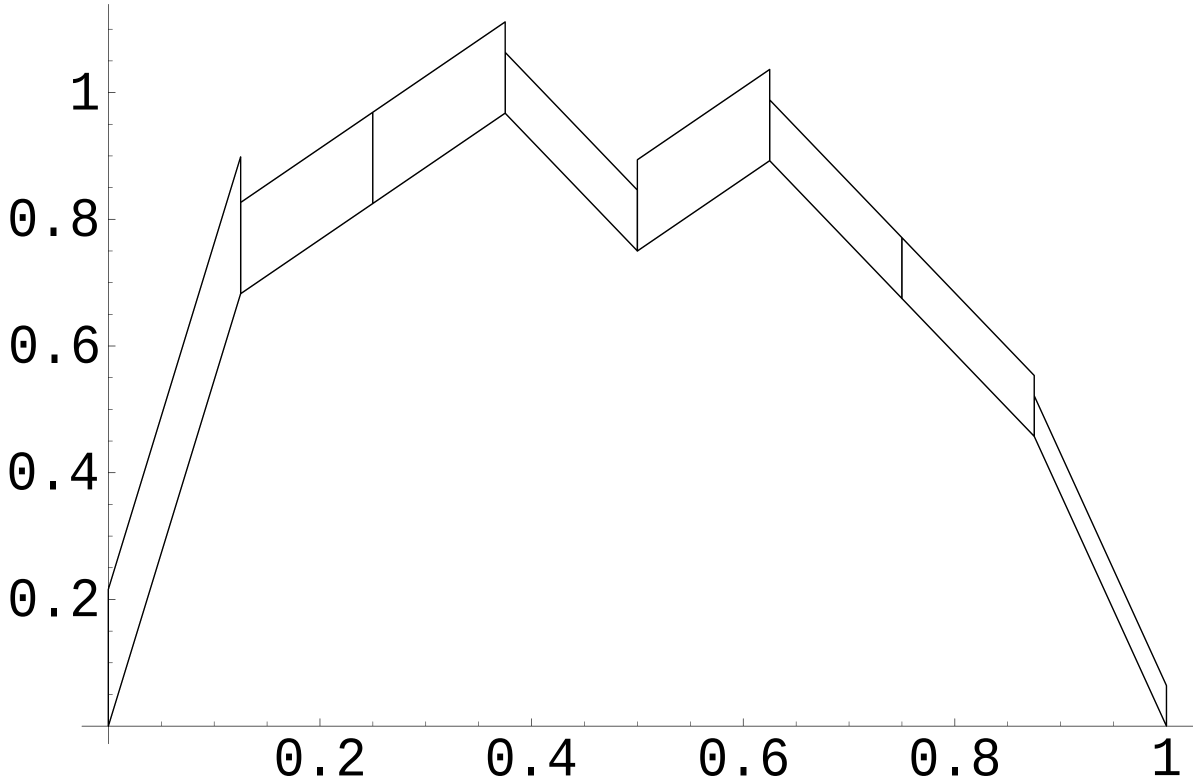
Geometric Construction



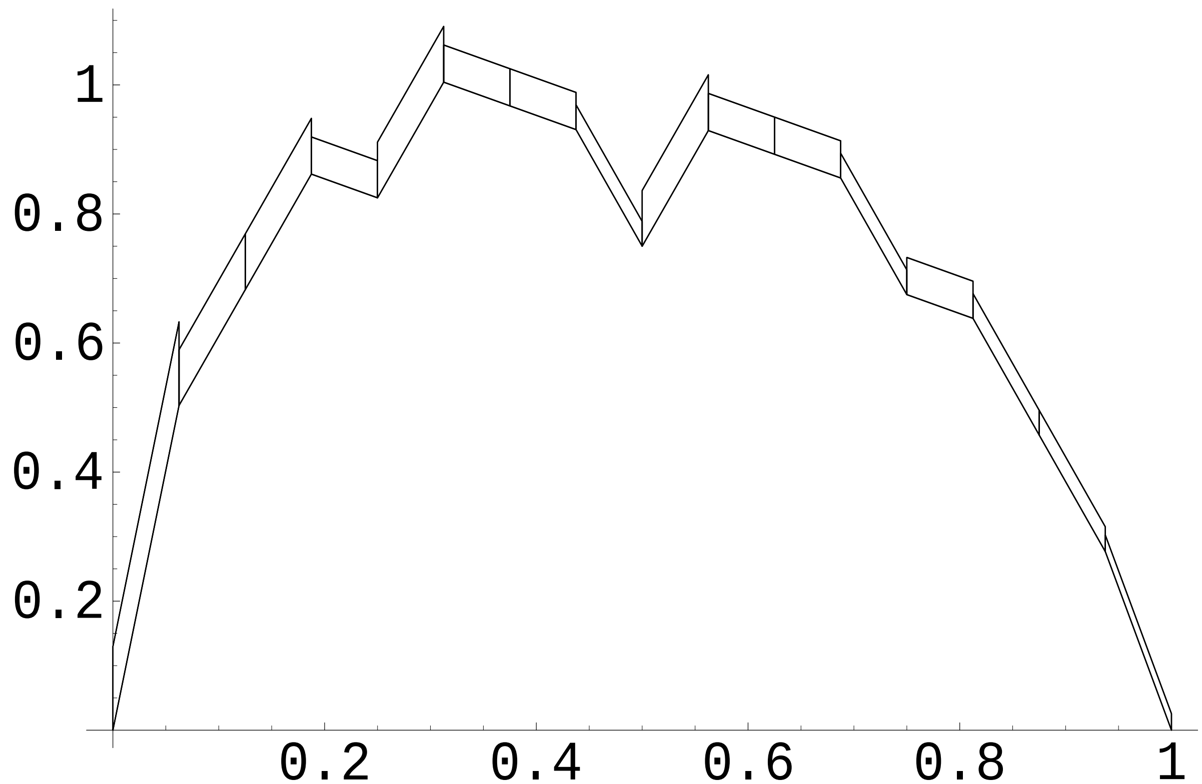
Geometric Construction



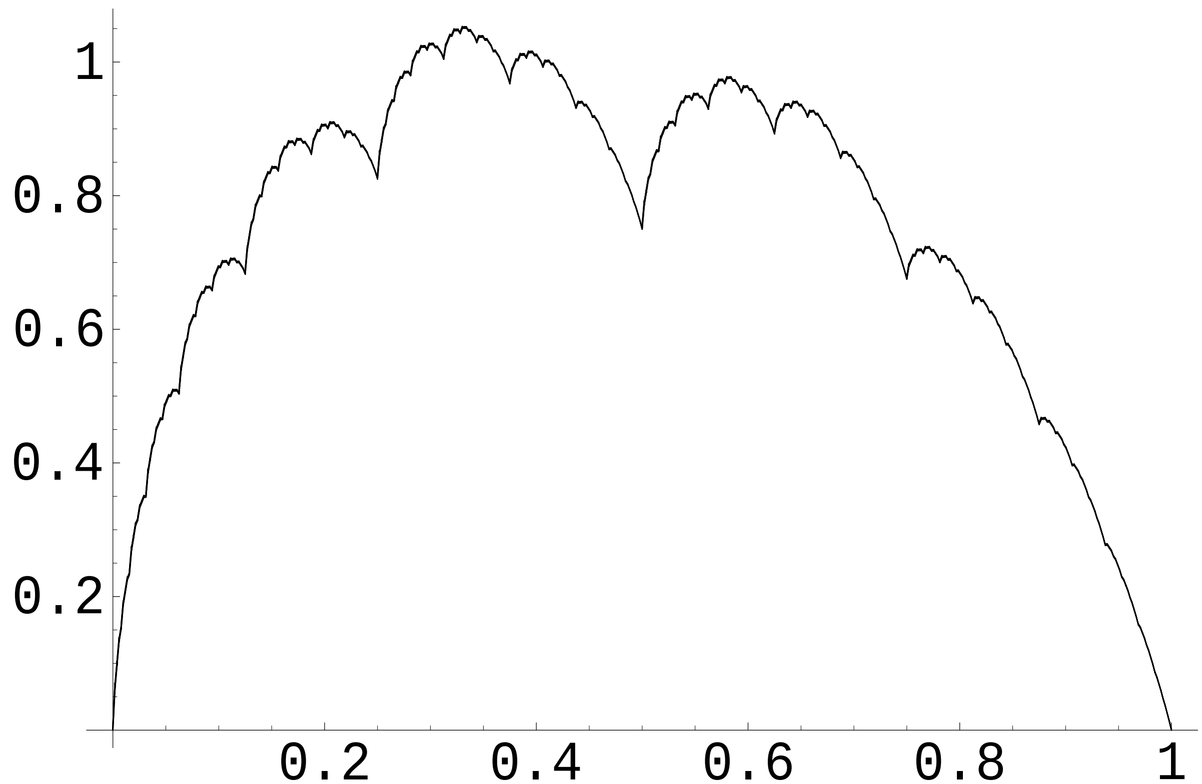
Geometric Construction



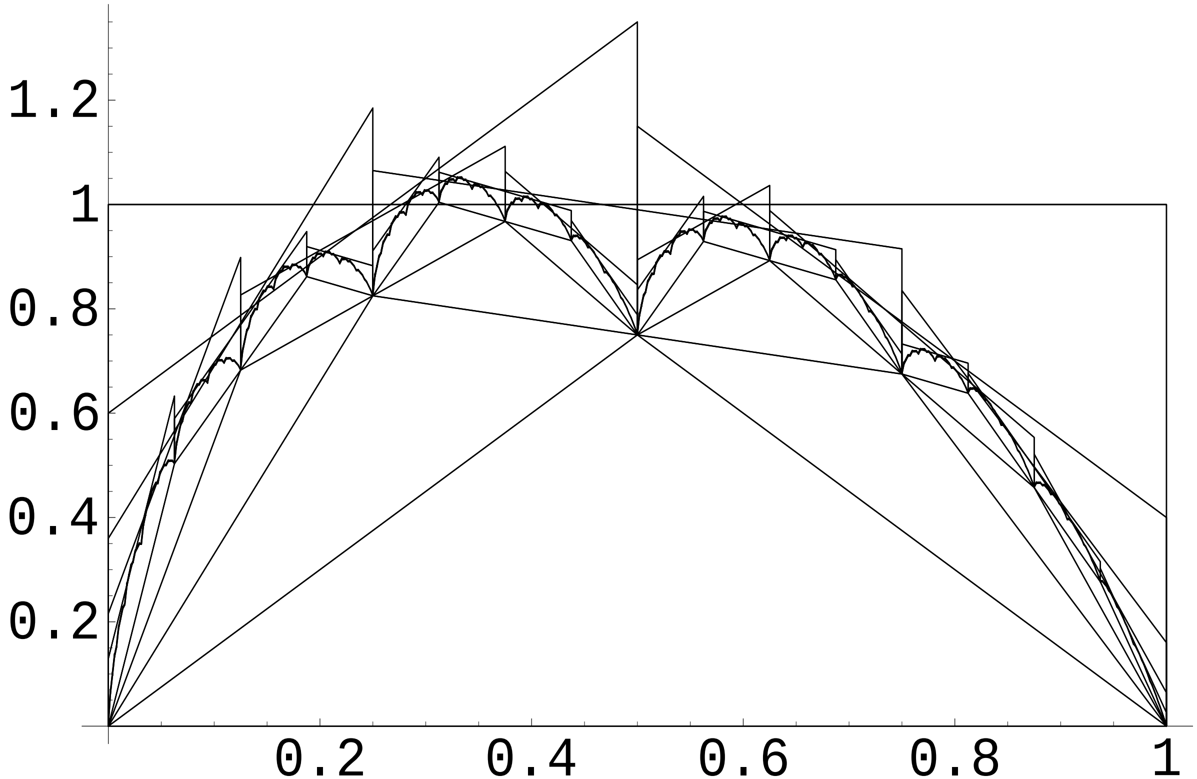
Geometric Construction



Geometric Construction



Geometric Construction



Fractal Functions

Theorem. Let $\Omega \subset \mathbb{R}$ be compact and $1 < N \in \mathbb{N}$. Assume $u_i : \Omega \to \Omega$ are contractive homeomorphisms, $\lambda_i : \mathbb{R} \to \mathbb{R}$ bounded functions and s_i real numbers, $i = 1, \ldots, N$. Let

$$\mathscr{T}(f) := \sum_{i=1}^{N} \left[\lambda_i \circ u_i^{-1} + s_i\, f \circ u_i^{-1}\right] \chi_{u_i(\Omega)}$$

If $\max\{|s_i|\} < 1$, then the operator $\mathscr{T}$ is contractive on $L^\infty(\Omega)$ and its unique fixed point $\mathfrak{F} : \Omega \to \mathbb{R}$ satisfies

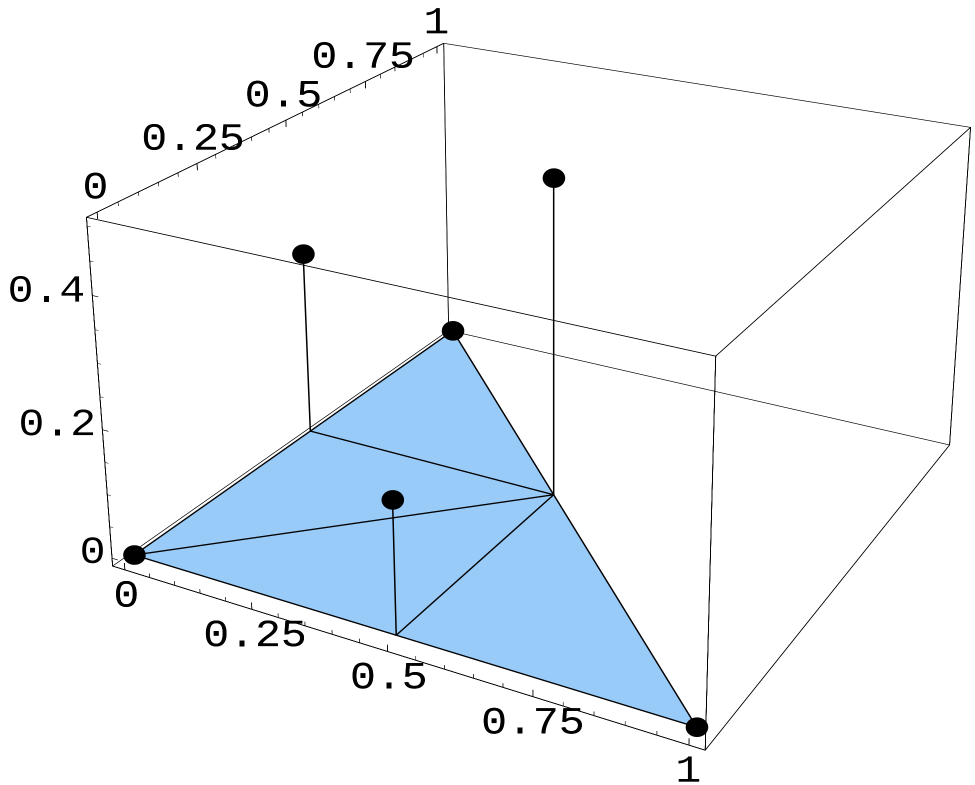
$$\mathfrak{F} = \sum_{i=1}^{N} \left[\lambda_i \circ u_i^{-1} + s_i\, \mathfrak{F} \circ u_i^{-1}\right] \chi_{u_i(\Omega)}$$

$\mathfrak{F}$ is called an *($\mathbb{R}$-valued) fractal function.*

(Affine) Fractal Surfaces

- Systematically defined first by M. (1990); Geronimo & Hardin 1993; Hardin & M. (1993)
- Defined on (triangular) regions Δ of $\mathbb{E}^n$
- Mappings $u_i : \Delta \to \Delta$, $x \mapsto A_i x + b_i$, $i = 1, \ldots, N$
- $\Delta = \bigcup_i u_i \Delta$ and $\overset{\circ}{u_i \Delta} \cap \overset{\circ}{u_j \Delta} = \varnothing$, $i \neq j$
- $\lambda_i : \mathbb{R}^n \to \mathbb{R}$, continuous (affine) functions, $i = 1, \ldots, N$
- $-1 < s_i < 1$ real numbers, $i = 1, \ldots, N$
- linear isomorphism $\Lambda := (\lambda_1, \ldots, \lambda_N) \to \mathfrak{F}_\Lambda$

Example I



Example II

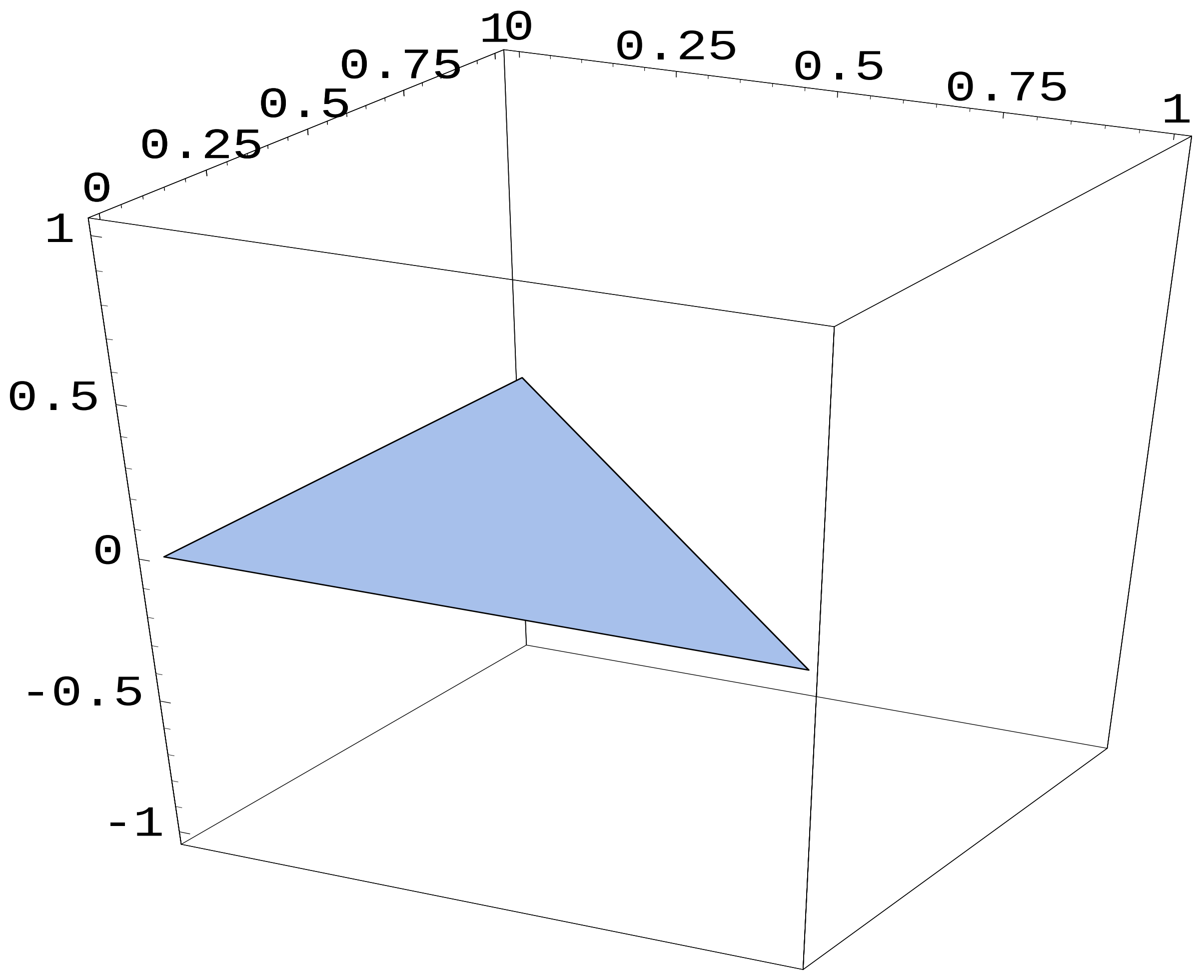
$$w_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ -z_1 & z_2 - z_1 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ z_1 \end{pmatrix}$$

$$w_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ -z_1 & z_2 - z_1 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ z_1 \end{pmatrix}$$

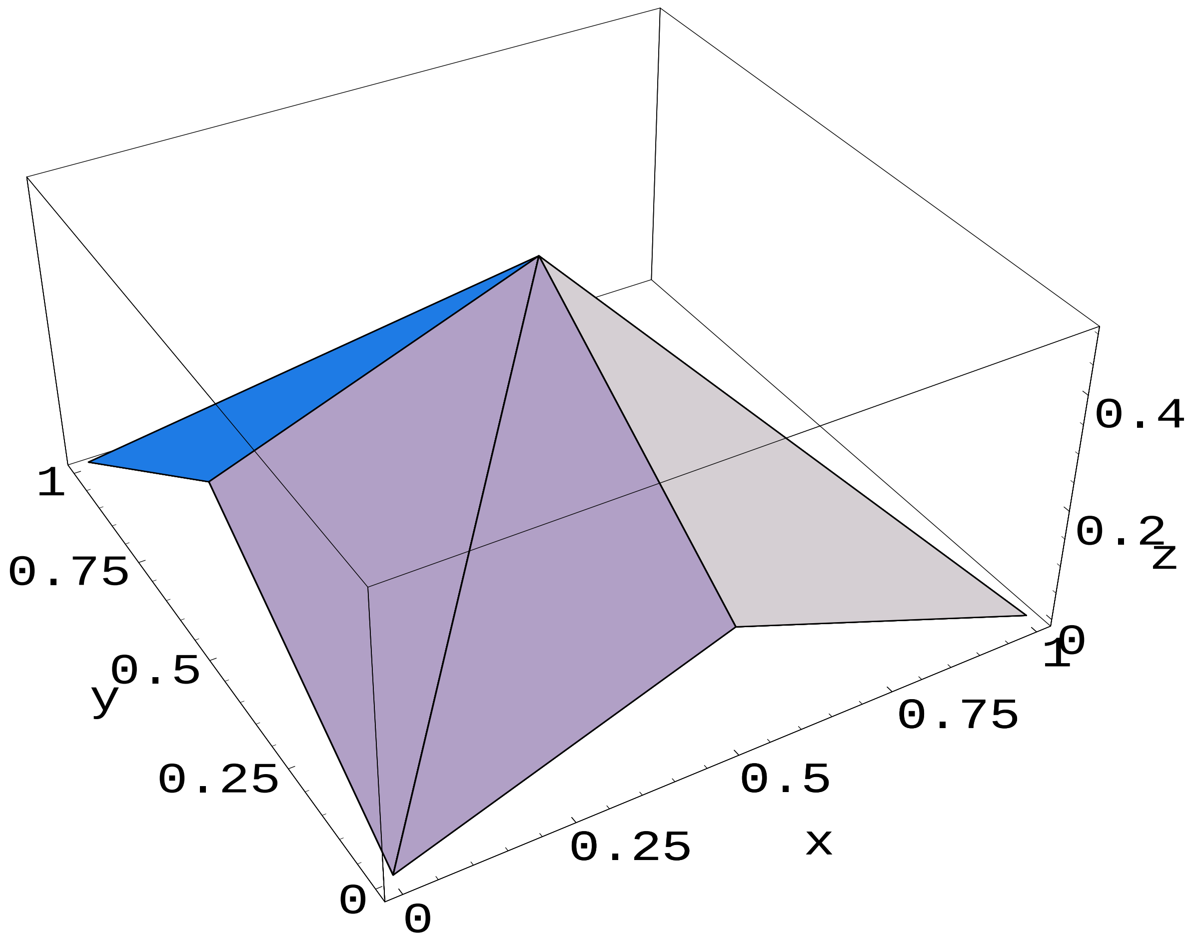
$$w_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ z_2 - z_3 & -z_3 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ z_3 \end{pmatrix}$$

$$w_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ z_2 - z_3 & -z_3 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ z_3 \end{pmatrix}$$

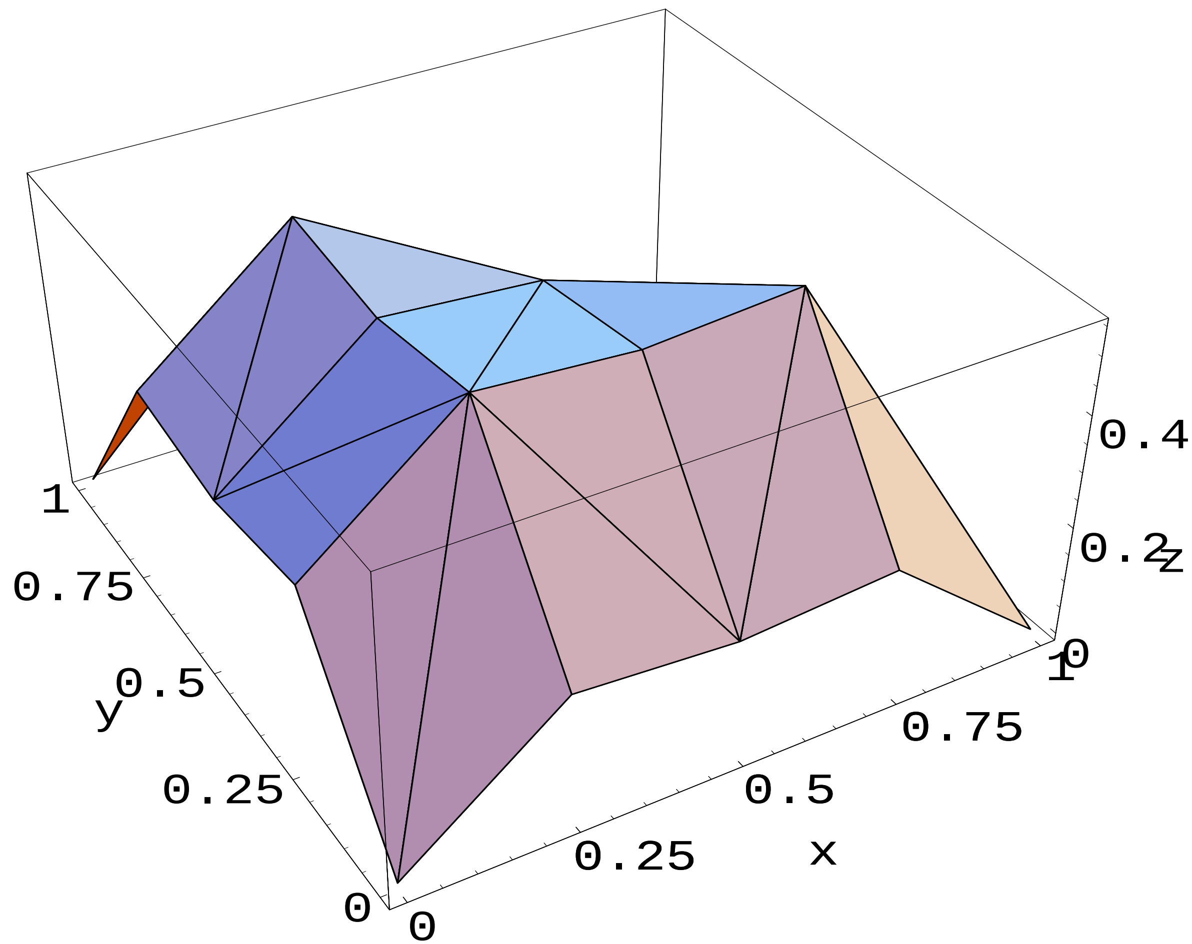
Example III



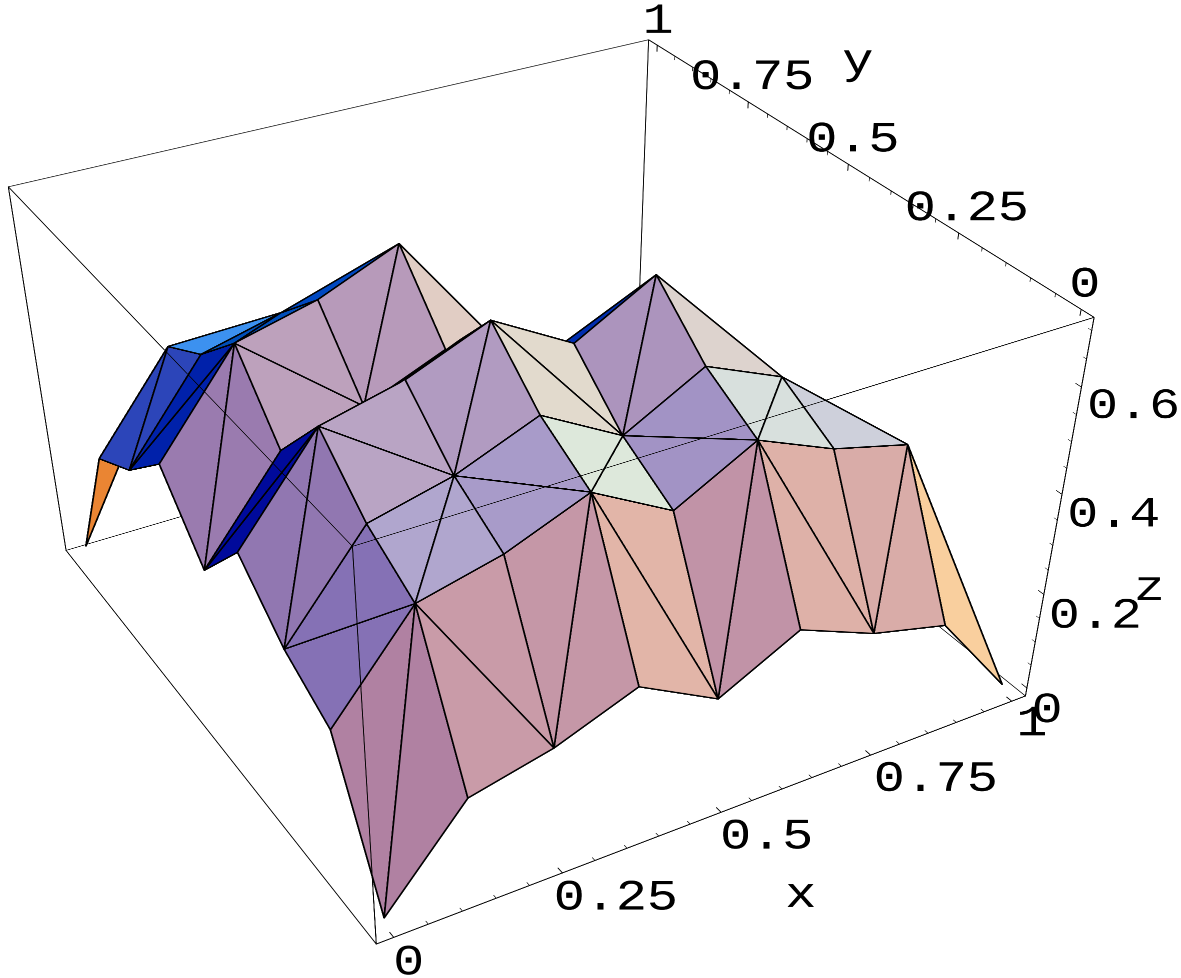
Example III



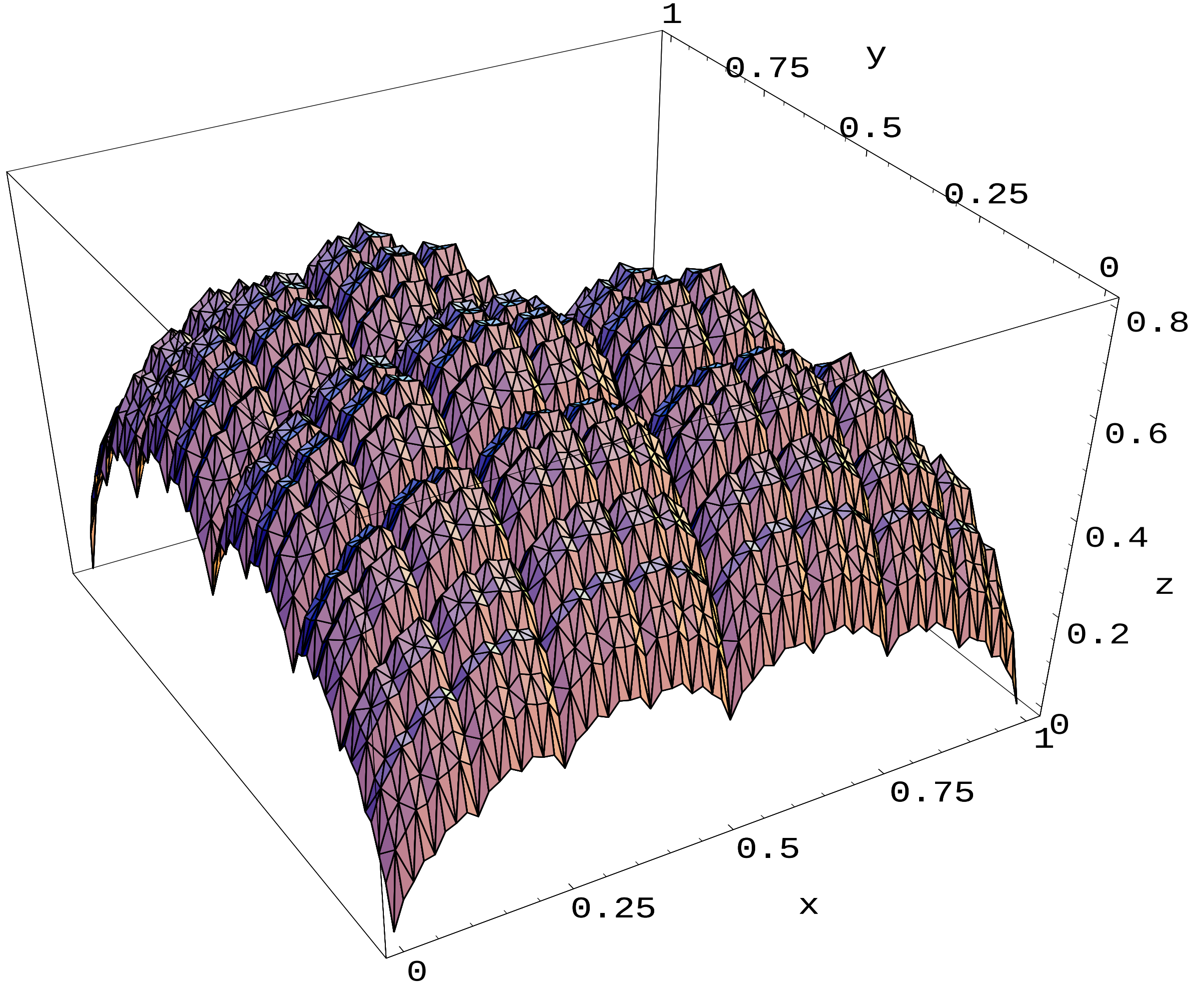
Example III



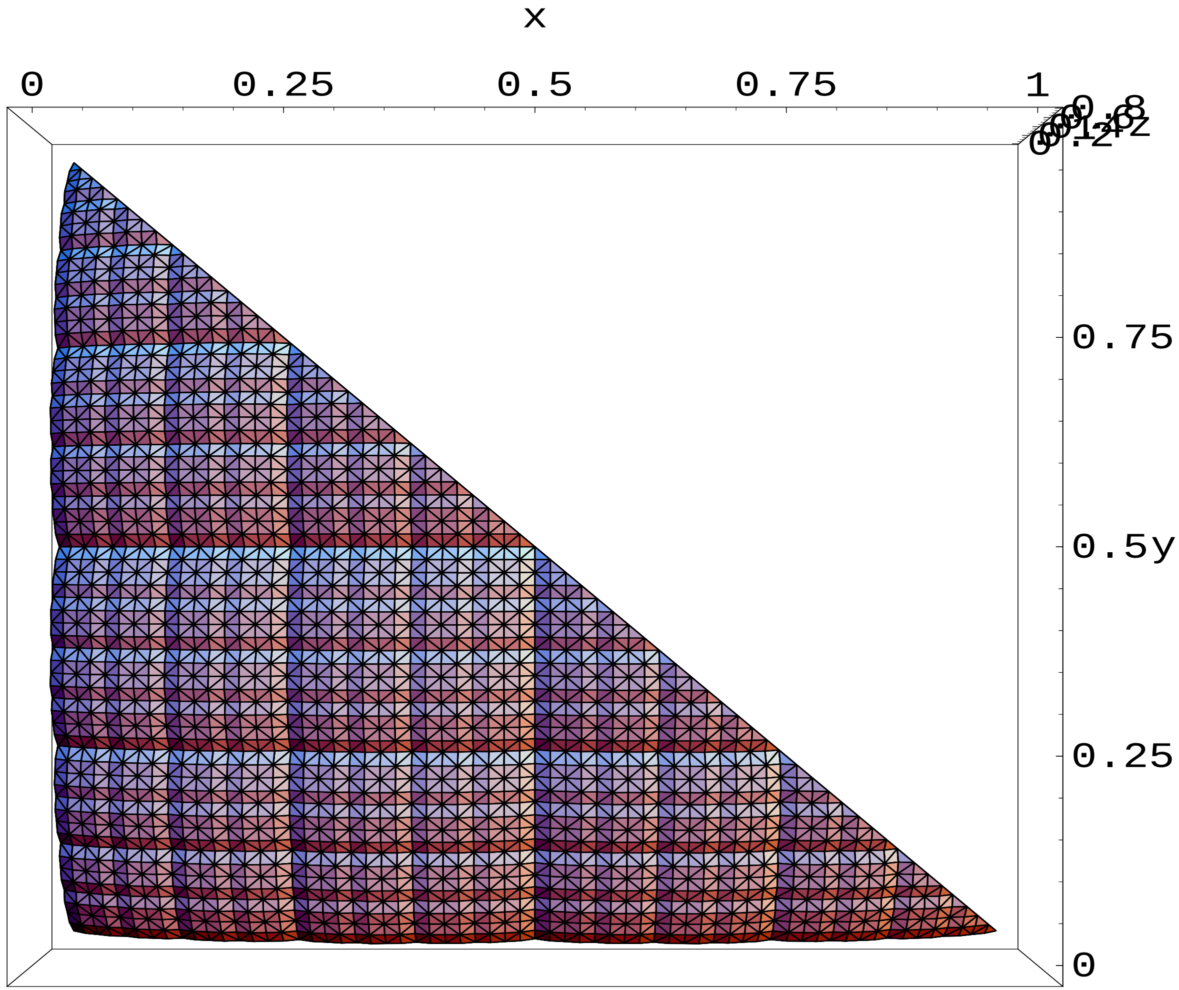
Example III



Example III



Example III



Fractal Surface Basis I

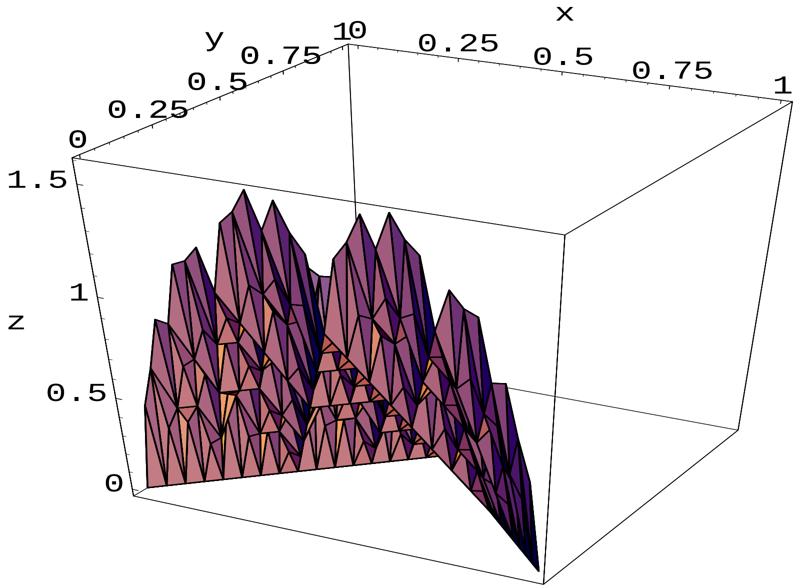
A fractal surface basis is obtained by using the isomorphism $\Lambda \to \mathfrak{F}_\Lambda$:

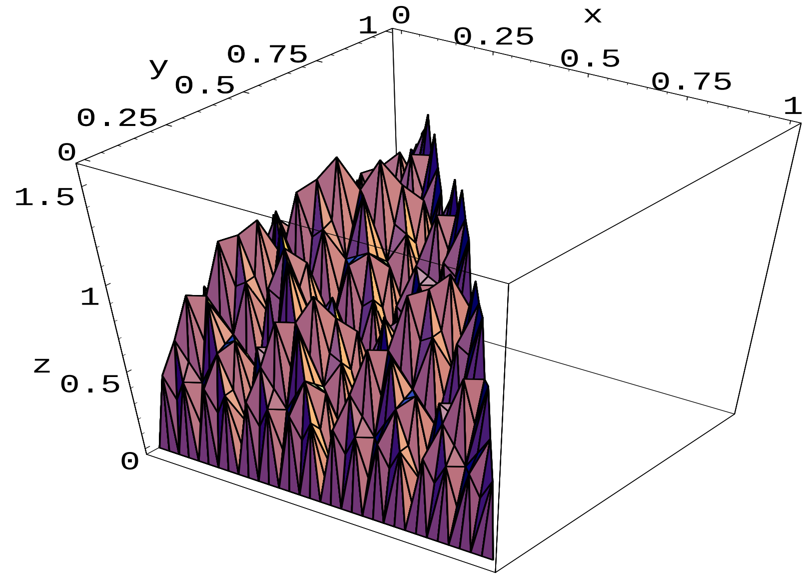
If $\mathfrak{F}$ interpolates the set $\{(x_i, y_j, z_{ij})\}$ on Δ, then

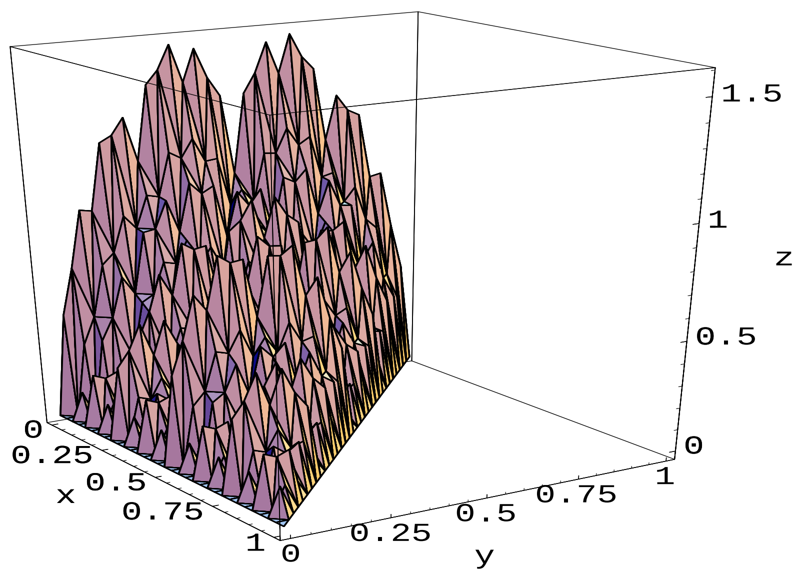
$$\mathfrak{F} = \sum_{i,j} z_{ij}\, \varphi_{ij},$$

where $\varphi_{ij}(x,y) = \begin{cases} 1, & (x,y) = (x_i, y_j) \\ 0, & \text{otherwise} \end{cases}$

Fractal Surface Basis II

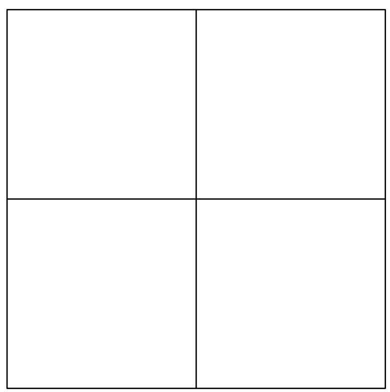


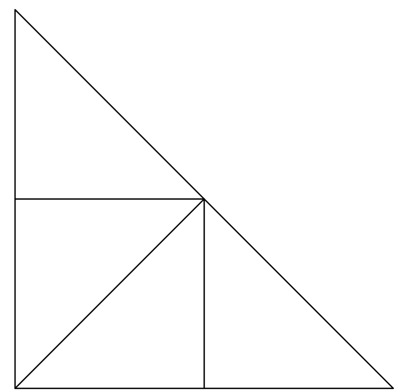




Foldable Figures

Definition. [Hoffman & Withers 1988] A compact connected subset F of $\mathbb{R}^n$ is called a *foldable figure* iff $\exists$ finite set $\mathcal{S}$ of hyperplanes that cuts F into finitely many congruent subfigures $F_1, \ldots, F_m$, each similar to F, so that reflection in any of the hyperplanes in $\mathcal{S}$ bounding F_k takes it into some F_ℓ.





Theorem. [Hoffman & Withers 1988] A foldable figure $F \subset \mathbb{E}^n$ is a convex polytope that tessellates $\mathbb{E}^n$ by reflections in hyperplanes.

Coxeter Groups

- Let $H \subset \mathbb{E}^n$ be a (linear) hyperplane. A linear transformation ρ is called a reflection about H if $\rho(H) = H$ and $\rho(x) = -x$, if $x \in H^\perp$.
- $\rho_r(x) = x - \dfrac{2\langle x, r\rangle}{\langle r, r\rangle} r$, for fixed $0 \neq r \in H^\perp$.
- **Coxeter Group**: A discrete group with a finite set of generators $\{r_i \,:\, i = 1, \ldots, k\}$ satisfying

$$\mathcal{C} := \left\langle r_1, \ldots, r_k \,\middle|\, (r_i r_j)^{m_{ij}} = 1,\ 1 \leq i, j \leq k \right\rangle$$

  where $m_{ii} = 1$, for all i, and $m_{ij} \geq 2$, for all $i \neq j$. ($m_{ij} = \infty$ is used to indicate that no relation exists.)
- Finite Coxeter groups $\cong$ finite Euclidean reflection groups

Root Systems and Weyl Groups I

Root System $\mathcal{R}$: finite set of nonzero vectors $r_1, \ldots, r_k \in \mathbb{E}^n$ satisfying

- $\mathbb{E}^n = \text{span}\,\{r_1, \ldots, r_k\}$
- $r, \alpha r \in \mathcal{R}$ iff $\alpha = \pm 1$
- $\forall r, s \in \mathcal{R}$: $s - \dfrac{2\langle s, r\rangle}{\langle r, r\rangle} r \in \mathcal{R} \Longleftrightarrow$
  $\forall r \in \mathcal{R}$, the root system $\mathcal{R}$ is closed with respect to the reflection through the hyperplane orthogonal to r.
- $\forall r, s \in \mathcal{R}$: $\dfrac{2\langle s, r\rangle}{\langle r, r\rangle} \in \mathbb{Z} \Longleftrightarrow \rho_r(s) - s \in \mathbb{Z}$

Weyl Group $\mathcal{W}$ of $\mathcal{R}$: group generated by the set of reflections $\{\rho_r \, : \, r \in \mathcal{R}\}$. $|\mathcal{W}| < \infty$.

Root Systems and Weyl Groups II

- $r \in \mathcal{R}$ is positive (negative) $\iff \langle r, x\rangle > 0$ ($\langle r, x\rangle < 0$) for some $x \in \mathbb{E}^n$.
- Every $\mathcal{R}$ has a basis $\mathcal{B} = \{b_i\}$ consisting of positive (negative) roots.
- **Weyl Chamber**: $C_i := \{x \in \mathbb{E}^n \,:\, \langle x, b_i\rangle > 0\}$
- $\mathcal{W}$ acts simply transitive on the Weyl chambers.
- $C := \overline{\bigcap_i C_i}$ is a noncompact fundamental domain for the Weyl group $\mathcal{W}$. C is convex and connected.
- Every Weyl group = finite Coxeter group has a fundamental domain that is a simplicial cone.

Affine Reflection Groups

- Reflection about affine hyperplanes: $r \in \mathcal{R}, k \in \mathbb{Z}$, $H_{r,k} := \{x \in \mathbb{E}^n \,:\, \langle x, r \rangle = k\}$

$$\rho_{r,k}(x) = x - \frac{2\langle x, r \rangle - k}{\langle r, r \rangle} r = \rho_r(x) + k\, r^\vee$$

- **Affine Weyl Group**: $\widetilde{\mathcal{W}} := \big\langle \rho_{r,k} \,|\, r \in \mathcal{R}, k \in \mathbb{Z} \big\rangle$
- $|\widetilde{\mathcal{W}}| = \infty$
- **Theorem.** The affine Weyl group $\widetilde{\mathcal{W}}$ of a root system $\mathcal{R}$ is the semi-direct product $\mathcal{W} \ltimes \Gamma$, where Γ is the abelian group generated by the coroots $r^\vee$. Moreover, Γ is the subgroup of translations of $\widetilde{\mathcal{W}}$ and $\mathcal{W}$ the isotropy group (stabilizer) of the origin.

Essential Reflection Groups

Let $\mathcal{G}$ be a reflection group and $\mathcal{O}_n$ the group of linear isometries of $\mathbb{E}^n$. There exists a homomorphism $\phi : \mathcal{G} \to \mathcal{O}_n$

$$\phi(g)(x) = g(x) - g(0), \quad g \in \mathcal{G},\ x \in \mathbb{E}^n.$$

- $\mathcal{G}$ is called essential if $\phi(\mathcal{G})$ only fixes $0 \in \mathbb{E}^n$.
- The elements of $\ker \phi$ are called translations.

Connection to Foldable Figures

Theorem. [Bourbaki,1968] The reflection group corresponding to a foldable figure F is the affine Weyl group of some root system.

Theorem. [Bourbaki,1968] If F is a foldable figure then F is the fundamental domain for the group generated by reflections through its bounding hyperplanes.

Theorem. [Bourbaki,1968] Let $\mathcal{G}$ be a reflection group with fundamental domain C. Then C is compact if $\mathcal{G}$ is essential and without fixed points.

Theorem. There exists a bijection between foldable figures and reflection groups that are essential and without fixed points.

Fractal Surfaces on Foldable Figures I

- Let $F \subset \mathbb{E}^n$ be a foldable figure with 0 as one of its vertices.
- Let $\mathcal{H}$ be the set of hyperplanes associated with F.
- Let Σ be the tessellation of F induced by $\mathcal{H}$.
- Let $\widetilde{\mathcal{W}}$ be the affine Weyl group generated by $\mathcal{H}$.

Then the following properties hold.

- $\mathcal{H}$ consists of the translates of a finite set of linear hyperplanes.
- $\widetilde{\mathcal{W}}$ is simply-transitive on Σ, i.e., $\forall(\sigma, \tau \in \Sigma)\ \exists! r \in \widetilde{\mathcal{W}} : \tau = r\sigma$.
- $\forall \varkappa \in \mathbb{N} : \varkappa\mathcal{H} \subset \mathcal{H}$.

Fractal Surfaces on Foldable Figures II

- Fix $1 < \varkappa \in \mathbb{N}$ and let $\Delta := \varkappa F$.
- Δ is also a foldable figures, whose N subfigures $\Delta_i \in \Sigma$. Let $\Delta_1 := F$.
- Tessellation and set of hyperplanes for Δ are $\varkappa\Sigma$ and $\varkappa\mathcal{H}$, resp.
- By simple transitivity of $\widetilde{\mathcal{W}}$, define similitudes $u_i : \Delta \to \Delta_i$ by:

$$u_1 := (1/\varkappa)(\cdot) \quad \text{and} \quad \forall j = 2, \ldots, N : \qquad u_j := r_{j,1} \circ u_1.$$

- Choose functions $\lambda_1, \ldots, \lambda_N \in C(\mathbb{R}^n, \mathbb{R})$ satisfying $\lambda_i(x) = \lambda_j(x)$, whenever $x = u_i^{-1}(e_{ij}) = u_j^{-1}(e_{ij})$, where $e_{ij} = u_i(\Delta) \cap u_j(\Delta)$.

Fractal Surfaces on Foldable Figures III

- Construct a fractal function $\mathfrak{F}_\Lambda$ on Δ.
- Let $C^{\widetilde{\mathcal{W}}} := \prod \left\{ \underbrace{C(\mathbb{R}^n, \mathbb{R}) \times \ldots \times C(\mathbb{R}^n, \mathbb{R})}_{N \text{ factors}} \, : \, r \in \widetilde{\mathcal{W}} \right\}$
- For $\mathbf{\Lambda} \in C^{\widetilde{\mathcal{W}}}$, define $\mathfrak{F}_{\mathbf{\Lambda}}$ by

$$\mathfrak{F}_{\mathbf{\Lambda}}\big|_{r\overset{\circ}{\Delta}} := \mathfrak{F}_{\mathbf{\Lambda}(r)} \circ r^{-1}, \quad r \in \widetilde{\mathcal{W}},$$

where $\mathbf{\Lambda}(r) = (\mathbf{\Lambda}(r)_1, \ldots, \mathbf{\Lambda}(r)_N)$ is the r-th coordinate of $\mathbf{\Lambda}$.

MRA on Foldable Figures I

Let V be a linear space of $\mathbb{R}$-valued functions on $\mathbb{R}^n$, $1 < \varkappa \in \mathbb{N}$, and $\widetilde{W}$ an affine Weyl group.

- V is dilation - invariant with scale $\varkappa \iff$ ($f \in V \implies D_\varkappa f := f(\cdot/\varkappa) \in V$)
- V is $\widetilde{W}$ - invariant $\iff (f \in V \implies f \circ r \in V$, $\forall r \in \widetilde{W})$

$D_\varkappa$ - invariance of a global fractal function $\mathfrak{F}_{\boldsymbol{\Lambda}}$ can be expressed in terms of an associated $\delta_\varkappa$ - invariance of $\boldsymbol{\Lambda} \in C^{\widetilde{W}}$.

MRA on Foldable Figures II

- Choose a finite-dimensional subspace U of $C(\mathbb{R}^n, \mathbb{R})$ such that $U^{\widetilde{\mathcal{W}}}$ is $\delta_\varkappa$ - invariant.
- Define $V_0 := \left\{ \mathfrak{F}_{\mathbf{\Lambda}} \,:\, \mathbf{\Lambda} \in U^{\widetilde{\mathcal{W}}} \right\}$ and $V_m := D_\varkappa^{-m} V_0$
- If $\dim U = d$, then $\dim V_0|_\Delta = Nd$.
- Gram-Schmidt Orthogonalization $\Longrightarrow \exists$ orthogonal basis $\{\varphi_j \,:\, j = 1, \ldots, Nd\}$ of $V_0|_\Delta$.
- Let $\Phi := (\varphi_1, \ldots, \varphi_{Nd})^T$. Then $V_1 \subset V_0 \Longrightarrow \exists$ sequence of $(Nd \times Nd)$-matrices $\{P(r) \,:\, r \in \widetilde{\mathcal{W}}\}$, only a finite number of which are nonzero, such that

$$\Phi(x/\varkappa) = \sum_{r \in \widetilde{\mathcal{W}}} P(r)\,(\Phi \circ r)(x)$$

MRA on Foldable Figures III

- For $m \in \mathbb{N}$, define the wavelet spaces $W_m := V_{m-1} - V_m$.
- $\dim W_0|_\Delta = \dim V_{-1}|_\Delta - \dim V_0|_\Delta = (\varkappa^n - 1)(Nd) =: K$
- Gram-Schmidt Orthogonalization $\Longrightarrow \exists$ orthogonal basis $\{\psi_\ell : \ell = 1, \ldots, K\}$ of $W_0|_\Delta$.
- Let $\Psi := (\psi_1, \ldots, \varphi_K)^T$. $W_1 \subset V_0 \Longrightarrow \exists$ sequence of $K \times Nd$-matrices $\{Q(r) : r \in \widetilde{\mathcal{W}}\}$, only a finite number of which are nonzero, such that

$$\Psi(x/\varkappa) = \sum_{r \in \widetilde{\mathcal{W}}} Q(r)\,(\Phi \circ r)(x)$$