NOTES ON BANACH SPACES

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1. Banach Spaces

1.1. Banach Spaces. Banach spaces are special cases of *topological vector spaces* which we will encounter later on. In short those are spaces that are endowed with two structure:

1. Algebraic structure: i.e., vector spaces.
2. Topological structure.

And those structures are compatible in the sense that the algebraic operations are continuous.

Definition 1. *Let* $\mathbf{V}$ *be a vector space over the field* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *A nonnegative real valued function* $\|\cdot\|$ *on* $\mathbf{V}$ *is called a norm if the following holds for all* $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ *and all* $\alpha \in \mathbb{K}$:

1. $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$;
2. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$;
3. $\|\alpha \mathbf{x}\| = |\alpha| \, ||\mathbf{x}||$.

A normed vector space becomes a metric space be defining

$$d(\mathbf{x}, \mathbf{y}) := ||\mathbf{x} - \mathbf{y}||$$

Definition 2. *Let* $\mathbf{V}$ *be a normed vector space of* $\mathbb{R}$ *or* $\mathbb{C}$. *Then* $\mathbf{V}$ *is called a Banach space if* $\mathbf{V}$ *is complete.*

From now on $\mathbb{K}$ will always stand for $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{V}$ be a vector space over $\mathbb{K}$. A subset $\mathbf{W} \subset \mathbf{V}$ is a *subspace* if for all $\mathbf{x}, \mathbf{y} \in \mathbf{W}$ and all $\alpha \in \mathbb{K}$ we have

$$\alpha \mathbf{x} + \mathbf{y} \in \mathbf{S}\,.$$

If the subspace $\mathbf{W}$ is closed in $\mathbf{V}$ then $\mathbf{W}$ is a *closed subspace*. If $\mathbf{V}$ is a Banach space, then every closed subspace is also a Banach space.

Let $A \subset \mathbf{V}$. Let $\mathcal{F}$ be the collection of all subspaces containing A. Then $\mathcal{F} \neq \emptyset$ because $\mathbf{V} \in \mathcal{F}$. We let

$$\operatorname{Span}(A) = \bigcap_{\mathbf{W} \in \mathcal{F}} \mathbf{W}$$

We denote $\operatorname{Span}(A)$ sometimes by $< A >$. Notice that $< A >$ is the smallest subspace of $\mathbf{V}$ containing A. We can also describe $< A >$ as the space of finite linear combinations $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$. The smallest closed subspace containing A is the space $\overline{< A >}$.

If $A, B \subset \mathbf{V}$ and $\lambda \in \mathbb{K}$. Then we use the following notation:

1. $A + B := \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \, \mathbf{b} \in B\}$;
2. $A - B := \{\mathbf{a} - \mathbf{b} \mid \mathbf{a} \in A, \, \mathbf{b} \in B\}$;
3. $\lambda A := \{\lambda \mathbf{a} \mid \mathbf{a} \in A\}$.

Here are few examples of Banach spaces:

Example 1. *Let $1 \leq p < \infty$. The spaces $L^p(\mathbb{R})$ are Banach spaces with norm*

$$||f||_p := \left(\int |f(t)|^p \, dt\right)^{1/p}$$

For $p = \infty$ the norm is the supremum norm.

Example 2. *Let $1 \leq p < \infty$. Define on the space of real sequences the norm*

$$||\{a_n\}_{n=0}^{\infty}||_p := \left(\sum_{j=0}^{\infty} |a_j|^p\right)^{1/p}$$

Then

$$\ell^p := \left\{\{a_n\} \mid ||\{a_n\}||_p < \infty\right\}$$

The spaces are Banach spaces.

1.2. **The Example $C(X)$ and the Stone-Weierstrass Theorem.** Let X be a compact Hausdorff space. In the following we will only deal with $\mathbb{K} = \mathbb{R}$ and leave the simple generalization to $\mathbb{C}$ to the reader. Define a *norm* $||\cdot||_\infty$ on $C(X)$ by

$$||f||_\infty := \sup_{x \in X} |f(x)|$$

This is finite because X is compact. The norm $||\cdot||_\infty$ is called the *supremum norm.*

Lemma 1. *The space $(C(X), ||\cdot||_\infty)$ is a Banach space.*

Proof. The only nontrivial statement is, that $C(X)$ is complete in the supremum norm. Let $\{f_n\}$ be a Cauchy sequence in $C(X)$. Then for $x \in X$ the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists. We have to show that $f \in C(X)$.

Let $x \in X$ and $\epsilon > 0$ be given. Let $0 < \epsilon' < \epsilon/3$. Chose N such that

$$||f_n - f_m||_\infty < \epsilon' \qquad \forall n, m \geq N$$

Then in particular $|f_n(y) - f_m(y)| < \epsilon'$ for all $y \in X$. It follows that

$$|f_n(y) - f(y)| \leq \epsilon' < \epsilon/3\,, \qquad \forall y \in X, \forall n \geq N$$

Thus

$$||f_n - f||_\infty < \epsilon/3$$

Chose $U \ni x$ open such that

$$|f_N(x) - f_N(y)| < \epsilon/3\,, \qquad \forall y \in U\,.$$

Then we have for $y \in U$:

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon$$

Thus f is continuous which shows that $C(X)$ is a Banach space. ∎

Suppose that $\mathbf{B}$ is a Banach space endowed with the structure of an algebra, i.e., a"multiplication" $\mathbf{B} \times \mathbf{B} \to \mathbf{B}$ satisfying the usual condition on the compatibility between the three operation addition, multiplication by scalars and the new multiplication. We call $\mathbf{B}$ for a *Banach algebra* if

$$||\mathbf{xy}|| \leq ||\mathbf{x}|| \, ||\mathbf{y}|| \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbf{B}$$

It follows by the definition that the multiplication is continuous.

Theorem 1. *Let X be a compact topological space. Then $C(X)$ is a Banach algebra.*

Proof. We know by the above lemma that $C(X)$ is a Banach space. Let $x \in X$. Then

$$|f(x)g(x)| = |f(x)| \, |g(x)| \leq ||f||_\infty \, ||g||_\infty$$

Taking the supremum it follows that $||fg||_\infty \leq ||f||_\infty \, ||g||_\infty$ ∎

Definition 3. *A family $\mathcal{F}$ of function on a topological space Y is said to separate points in Y if for every $x, y \in Y$, $x \neq y$ there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*

Lemma 2. *Let X be a compact Hausdorff space. Then the space $C(X)$ separates points.*

Proof. This follows form the fact that X is normal and the Urysohn's Lemma. ∎

The space $C(X)$ has also the structure of a *lattice*. Let $f, g \in C(X)$. Define

$$f \wedge g(x) := \min\{f(x), g(x)\} \quad \text{and} \quad f \vee g(x) := \max\{f(x), g(x)\}\,.$$

Then $f \wedge g, f \vee g \in C(X)$. A subspace $L \subset C(X)$ is called a *lattice* if for all $f, g \in L$ we have $f \wedge g$, $f \vee g \in L$. We leave the following to the reader:

Theorem 2 (Stone-Weierstrass)**.** *Let X be a compact space and let $A \subset C(X)$ be an algebra that separates points and contains the constant functions. Then A is dense in $C(X)$, i.e., given $f \in C(X)$ and $\epsilon > 0$, there exists a $g \in A$ such that $||g - f||_\infty < \epsilon$.*

Corollary 1. *Let $B \subset \mathbb{R}$ be closed and bounded and let $f : B \to \mathbb{R}$ be continuous. Let $\epsilon > 0$. Then there exists a polynomial function $p(x)$ such that $\sup_{x \in B} |p(x) - f(x)| < \epsilon$.*

We will proof the Stone-Weierstrass Theorem in few Lemmas.

Lemma 3. *There exists a sequence of polynomial functions $\{p_n(t)\}_n$ such that on the interval $[0, 1]$ the following holds:*

1. $0 \leq p_n(t) \leq p_{n+1}(t) \leq \sqrt{t}$.
2. $\lim_{n \to \infty} p_n(t) = \sqrt{t}$ *unifomly on* $[0, 1]$.

Proof. Define $p_n(t)$ inductively by $p_1(t) = 0$ and

$$p_{n+1}(t) := p_n(t) + \frac{1}{2}\left(t - p_n(t)^2\right)\,.$$

Then

$$\begin{aligned}\sqrt{t}-p_n(t) &= \sqrt{t}-p_n(t)-\frac{1}{2}\left(\sqrt{t}-p_n(t)\right)\left(\sqrt{t}+p_n(t)\right)\\ &= \left(\sqrt{t}-p_n(t)\right)\left(1-\frac{1}{2}(\sqrt{t}+p_n(t))\right)\end{aligned}$$

By induction hypothesis $\sqrt{t} \geq p_n(t)$. Hence $\sqrt{t}-p_n(t) \geq 0$ and $(\sqrt{t}+p_n(t)) \leq 2\sqrt{t} \leq 2$. It follows that

$$\sqrt{t}-p_n(t) \geq 0\,.$$

Furthermore

$$p_{n+1}(t)-p_n(t)=\frac{1}{2}\left(t-p_n(t)^2\right) \geq 0$$

because $t-p_n(t)^2 \geq 0$.

It follows that for each $t \in [0,1]$ the sequence $\{p_n(t)\}_n$ is monotone increasing and bounded. Hence convergent. Hence:

$$0=\lim_{n\to\infty} p_{n+1}(t)-\lim_{n\to\infty} p_n(t)=\frac{1}{2}\left(t-\lim_{n\to\infty} p_n(t)^2\right)$$

It follows that $\lim p_n(t) = \sqrt{t}$. But a monotone bounded function sequence is uniformly bounded on compact intervals. Hence the lemma ∎

Lemma 4. *There exists a sequence of polynomials $\{q_n(t)\}_n$ that converges uniformly on the interval $[-1,1]$ to the function $t \mapsto |t|$.*

Proof. Let $q_n(t) := p_n(t^2)$. ∎

Lemma 5. *Let A be as in the Stone-Weierstrass Theorem. Let $f \in \overline{A}$ then $|f| \in \overline{A}$ and $\overline{A}$ is a lattice.*

Proof. Let $f \in \overline{A}$. Assume first that $||f||_\infty \leq 1$. Let $q_n(t) = a_{n,m}t^m + a_{n.m-1}t^{m-1} + \cdots + a_{n,o}$ be as in the above lemma. Then

$$q_n(f) = a_{n,m}f^m + a_{n.m-1}f^{m-1} + \cdots + a_{n,o} \in \overline{A}$$

Hence $\lim_{n\to\infty} q_n(f) = |f| \in \overline{A}$ because $q_n(t)$ converges uniformly to $|t|$ on $[0,1]$. For general f we have $|f| = ||f||_\infty \left(\frac{|f|}{||f||_\infty}\right)$. Hence $|f| \in \overline{A}$.

Let now $f, g \in \overline{A}$. Then

$$\begin{aligned} f \wedge g &= \frac{1}{2}(f+g) - \frac{1}{2}\,|f-g| \in \overline{A}\\ f \vee g &= \frac{1}{2}(f+g) + \frac{1}{2}\,|f-g| \in \overline{A}\end{aligned}$$

by the first part and the fact that with $f, g \in \overline{A}$ we have $f \pm g \in \overline{A}$. ∎

Lemma 6. *Let A be as in the Stone-Weierstrass Theorem. Let $x, y \in X$, $x \neq y$, and let $a, b \in \mathbb{R}$ with $a \leq b$. Then there exists a $f \in \overline{A}$ such that: $f(x) = a$ and $f(y) = b$. Furthermore we can choose f such $f(z) \leq b$ for all $z \in X$.*

Proof. Choose $g \in A$ such that $g(x) \neq g(y)$. This is possible because A separates points. Let

$$f_1(t) := a + (b-a)\frac{g(t)-g(x)}{g(y)-g(x)}\,.$$

Then $f \in A$ because A contains the constants. Furthermore $f_1(x) = a$ and $f_1(y) = b$. We let now

$$f := (f_1 \wedge b)$$

Then $f \in \overline{A}$ because of the last lemma and $f(x) = a$ because $a \leq b$, $f(y) = b$ and $f(z) \leq b$. ∎

Lemma 7. *Let A be as before. Let $\epsilon > 0$ and $x \in X$ be given. Let $g \in C(X)$. Then there exists a $f \in \overline{A}$ such that*

$$g(x) = f(x)$$
$$f(y) \leq g(y) + \epsilon\,, \quad \forall y \in X$$

Proof. For $z \in X$ let $h_z \in \overline{A}$ be such that

$$h_z(x) = g(z)$$
$$h_z(z) = g(z) + \epsilon/2$$

This is possible by the last lemma. Both h_z and g are continuous. Hence there exists an open subset $U_z \ni z$ such that

$$h_z(y) < g(y) + \epsilon\ , \qquad \forall z \in U_z$$

We use now that X is compact. Hence there exists finitely many $U_i = U_{z_i}$ such that $X = \cup_{i=1}^n U_i$. Let $h_i = h_{x_i}$, and let

$$f = h_1 \wedge \cdots \wedge h_n \in \overline{A}$$

Then $f(x) = g(x)$. Let $y \in X$. Then there exists a i such that $y \in U_i$. Hence

$$f(y) \leq h_i(y) < g(y) + \epsilon$$

∎

We can now proof the Stone-Weierstrass Theorem: Let $g \in C(X)$. Let $\epsilon > 0$ be given. For each $x \in X$ we choose $f_x \in \overline{A}$ such that

$$\begin{aligned} g(x) &= f_x(x) \\ f_x(y) &< g(y) + \epsilon \end{aligned}$$

By the first equation there is an open $V_x \ni x$ such that

$$f_x(y) > g(y) - \epsilon\,, \qquad \forall y \in V_x\,.$$

Again we use that X is compact to find finitely many $V_i = V_{x_i}$ such that $X = \cup_{i=1}^k V_i$. Let $f_i = f_{x_i}$ and

$$f = f_1 \vee \cdots \vee f_k.$$

Let $z \in X$. Let i be such that $z \in V_i$. Then

$$g(z) - \epsilon < f_i(z) < g(z) + \epsilon\,.$$

Hence

$$g(z) - \epsilon \leq f(z) \leq g(z) + \epsilon\,.$$

It follows that $||f - g||_\infty \leq \epsilon$, which shows that $\overline{A}$ is dense.

1.3. **Linear Operators.** Let $\mathbf{V}$ and $\mathbf{W}$ be two vector spaces. A map $T : \mathbf{V} \to \mathbf{W}$ is linear if

$$T(\lambda \mathbf{x} + \mathbf{y}) = \lambda T(\mathbf{x}) + T(\mathbf{x}), \quad \forall \boldsymbol{\lambda} \in \mathbb{R}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{V}.$$

If $\mathbf{V}$ and $\mathbf{W}$ are normed vector spaces then the linear operator T is *bounded* if there exists a constant $M > 0$ such that

$$||T(\mathbf{x})|| \leq M\, ||\mathbf{x}||, \qquad \forall \mathbf{x} \in \mathbf{V}.$$

If T is bounded we set

$$||T|| := \sup_{\mathbf{x} \in \mathbf{X}} \frac{||T(\mathbf{x})||}{||\mathbf{x}||} \leq M < \infty.$$

The number $||T||$ is called the *norm* of T. A simple calculation shows that

$$||T|| = \sup_{||\mathbf{x}||=1} ||T(\mathbf{x})|| = \sup_{||\mathbf{x}|| \leq 1} ||T(\mathbf{x})||$$

Lemma 8. *If $T : \mathbf{V} \to \mathbf{W}$ and $S : \mathbf{W} \to \mathbf{Z}$ are both bounded. Then the composition $ST : \mathbf{V} \to \mathbf{Z}$. is also bounded.*

Proof. Let $\mathbf{x} \in \mathbf{V}$. Then $||(ST)(\mathbf{x})|| \leq ||S(T\mathbf{x})|| \leq ||S||\; ||T(\mathbf{x})|| \leq ||S||\; ||T||\; ||\mathbf{x}||$. ∎

A bounded operator $T : \mathbf{V} \to \mathbf{W}$ is called *invertible* if there exists a bounded linear operator $S : \mathbf{W} \to \mathbf{V}$ such that

$$ST = I_{\mathbf{V}} \qquad \text{and} \quad TS = I_{\mathbf{W}}.$$

Here $I_{\mathbf{X}}$ stands for the identity map $\mathbf{x} \mapsto \mathbf{x}$. The kernel $Ker(T)$ of a linear operator is

$$Ker(T) = \{\mathbf{x} \in \mathbf{V} \mid T(x) = 0\}.$$

$Ker(T)$ is a subspace of $\mathbf{V}$.

Example 3. *In the finite dimensional case a linear map $T : \mathbf{V} \to \mathbf{V}$ then the following three properties are equivalent:*

1. *T is invertible;*
2. *T is injective;*
3. *T is surjective.*

This is different in the infinite dimensional case. Let $1 \leq p < \infty$. Let $\mathbf{V} = \ell_p$. Define the shift operators S and T by:

$$S((r_n)) := (0, r_0, r_1, \dots)$$

and

$$T((r_n)) := (r_1, r_2, r_3, \dots)$$

Then S is injective, T is surjective, and $TS = I$. But the kernel of ST is exactly

$$Ker(TS) = \{(r, 0, 0, \dots) \mid r \in \mathbb{R}\}.$$

Proposition 1. *A bounded linear operator is uniformly continuous. If a linear operator is continuous at one point it is bounded.*

Let $\mathbf{V}, \mathbf{W}$ be normed spaces. Let $\mathcal{B}(\mathbf{V}, \mathbf{W})$ be the space of bounded operators from $\mathbf{V}$ to $\mathbf{W}$. Let $S, T \in \mathcal{B}(\mathbf{V}, \mathbf{W})$ and $\mathbf{x} \in \mathbf{V}$. Then

$$\begin{aligned} ||(S+T)(\mathbf{x})|| &= ||S(\mathbf{x}) + T(\mathbf{x})|| \\ &\leq ||S(\mathbf{x})|| + ||T(\mathbf{x})|| \\ &\leq (||S|| + ||T||)\, ||\mathbf{x}|| \end{aligned}$$

Hence $||S+T|| \leq ||S|| + ||T||$. If $||T|| = 0$ then $||T(\mathbf{x})|| = 0$ for all $\mathbf{x} \in \mathbf{V}$. Hence $T = 0$. We also have $||\lambda T|| = |\lambda|\, ||T||$. Hence $||\cdot||$ is a norm on $\mathcal{B}(\mathbf{V}, \mathbf{W})$.

Proposition 2. *Let $\mathbf{V}, \mathbf{W}$ be normed spaces. If $\mathbf{W}$ is a Banach space, then $\mathcal{B}(\mathbf{V}, \mathbf{W})$ is also a Banach space.*

1.4. **Linear Functionals and the Hahn-Banach Theorem.** Before we proof the Hahn-Banach Theorem recall the *Hausdorff Maximum Principle*: A relation R on a set X is *antisymmetric* and *transitive* relation on a set X is called a *partial ordering* (on X). Partial ordering will usually be denoted by $\prec$. A partial ordering $\prec$ is called a *linear ordering* if for any two elements $x, y \in X$ we have $x \prec y$ or $y \prec x$. If $x \prec y$ then we say that x *precedes* y. Sometimes we say in this case that x is *smaller* than y. If $\prec$ is a linear a partial ordering $x \preceq y$ means that either $x \prec y$ or $x = y$. The following maximum principle is equivalent to the axiom of choice:

Axiom 1 (Hausdorff Maximal Principle)**.** *Let $\prec$ be a partial ordering on a set X. Then there is a maximal linearly ordered subset $S \subset X$*

Proposition 3 (Hahn-Banach)**.** *Let p be a real-valued function defined on the vector space $\mathbf{V}$ satisfying*

$$p(\mathbf{x} + \mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y}) \quad \textit{and} \quad p(\lambda \mathbf{x}) = \lambda p(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ and all $\lambda \geq 0$. Suppose that f is a linear functional on defined on an subspace $\mathbf{W} \subset \mathbf{V}$, and satisfying

$$f(\mathbf{x}) \leq p(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{W}.$$

Then there is a linear functional F defined on $\mathbf{V}$ such that

$$F(\mathbf{x}) \leq p(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{V}$$

and such that

$$F(\mathbf{x}) = f(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{W}.$$

Proof. Let X be the set of all pair $(\mathbf{S}, g)$ such that $\mathbf{S} \subset \mathbf{V}$ is a subspace containing $\mathbf{W}$ and $g : \mathbf{S} \to \mathbb{R}$ is linear such that $g(\mathbf{x}) \leq p(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{S}$ and $g|\mathbf{W} = f$. Then $X \neq \emptyset$ because $(\mathbf{W}, f) \in X$. Define a linear ordering $\prec$ on X by

$$(\mathbf{S}, g) \prec (\mathbf{R}, h) \Leftrightarrow \mathbf{S} \subset \mathbf{R} \quad \text{and} \quad h|\mathbf{S} = g$$

By the maximum principle there is a maximal linearly ordered family $\mathcal{S}$ in X. Let $\mathcal{S} = \{(\mathbf{S}_i, g_i)\}_{i \in I}$. Let

$$\mathbf{S} := \bigcup_{i \in I} \mathbf{S}_i\,.$$

We claim that $\mathbf{S}$ is a subspace of $\mathbf{V}$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{S}$, $\lambda \in \mathbb{R}$. There exists $\mathbf{S}_i$ and $\mathbf{S}_j$ such that $\mathbf{x} \in \mathbf{S}_i$ and $\mathbf{y} \in \mathbf{S}_j$. Because $\mathcal{S}$ is linearly ordered we can assume that $i = j$. Hence $\lambda \mathbf{x} + \mathbf{y} \in \mathbf{S}_i \subset \mathbf{S}$.

Define $g : \mathbf{S} \to \mathbb{R}$ in the following way: If $\mathbf{x} \in \mathbf{S}$ then there is a $\mathbf{S}_j$ such that $\mathbf{x} \in \mathbf{S}_j$. Let

$$g(\mathbf{x}) := g_j(\mathbf{x}) .$$

We have to show that g is well defined. Assume that we also have $\mathbf{x} \in \mathbf{S}_k$. Then we can assume that $\mathbf{S}_k \subset \mathbf{S}_j$. Hence $g_j|\mathbf{S}_j = g_k$. In particular $g_j(\mathbf{x}) = g_k(\mathbf{x})$. That g is proved in the same way as $\mathbf{S}$ is a vector space. By construction we have

$$\forall (\mathbf{R}, h) \in \mathcal{S} \quad : \quad (\mathbf{R}, h) \preceq (\mathbf{S}, g)$$

Because of the maximality of $\mathcal{S}$ it follows that $(\mathbf{S}, g) \in \mathcal{S}$. As in the book it is now shown that actually $\mathbf{S} = \mathbf{V}$. Hence the theorem. ∎

Proposition 4. *Let the notation be as in the Hahn-Banach Theorem. Let G be an Abelian semigroup of linear operators on $\mathbf{V}$ such for every $A \in G$ we have*

$$p(A\mathbf{x}) \le p(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbf{V}$$

and

$$f(A\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbf{W}$$

Then there exists a linear extension of f such that $F(A\mathbf{x}) = F(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{V}$.

Proposition 5. *Let $\mathbf{0} \neq \mathbf{x} \in \mathbf{V}$, then there exists a bounded linear functional $f : \mathbf{V} \to \mathbb{R}$ such that*

$$-||\mathbf{y}|| \le f(\mathbf{y}) \le ||\mathbf{y}|| \qquad \forall \mathbf{y} \in \mathbf{V}$$

and $f(\mathbf{x}) = ||\mathbf{x}||$. In particular $||f|| = 1$.

Proof. As in the book. ∎

Lemma 9. *Let $\mathbf{V}$ be a normed vector space. Then the set of bounded linear functionals on $\mathbf{V}$ separates points in $\mathbf{V}$.*

Proof. Let $\mathbf{s}, \mathbf{t} \in \mathbf{V}$, such that $\mathbf{s} \neq \mathbf{t}$. Take $\mathbf{x} = \mathbf{s} - \mathbf{t}$. Then the above lemma says that there exists a linear functional f such that $f(\mathbf{s} - \mathbf{t}) \neq 0$. But that is the same as $f(\mathbf{s}) \neq f(\mathbf{t})$. ∎

Let X be a metric space and $A \subset X$. For $y \in X$ we define

$$d(y, A) := \inf_{a \in A} d(y, a).$$

Then $0 \le d(y, A) < \infty$. We call the number $d(y, A)$ the *distance* of y to A. Assume that $d(y, A) = 0$. Then we can find a sequence $\{a_n\}_n$ in A such that

$$d(y, a_n) \to 0 \qquad n \to \infty$$

But then $a_n \to y$ and $y \in \overline{A}$. Hence

$$\overline{A} = \{y \in X \mid d(y, A) = 0\} .$$

In particular we have defined the notation of a the distance of a point from a subspace in $\mathbf{V}$.

Proposition 6. *Let $\mathbf{W} \subset \mathbf{V}$ be a subspace. Let $\mathbf{y} \in \mathbf{V}$ with $d(\mathbf{y}, \mathbf{W}) \ge \boldsymbol{\delta} > 0$. Then there is a $f : \mathbf{V} \to \mathbb{R}$ such that $f(\mathbf{y}) = \delta$, $f|\mathbf{W} = 0$ and $||f|| \le 1$.*

Let $\mathbf{V}$ be a normed vector space. The *dual* $\mathbf{V}^*$ of $\mathbf{V}$ is the Banach space of all bounded linear maps $\mathbf{V} \to \mathbb{R}$. If $\mathbf{v} \in \mathbf{V}$ and $f \in \mathbf{V}^*$ then we will sometimes write

$$< f, \mathbf{x} >= f(\mathbf{x})$$

The dual $\mathbf{V}^{**}$ of $\mathbf{V}$ is defined as $\mathbf{V}$ is a Banach space. Define a map $\varphi : \mathbf{V} \to \mathbf{V}^{**}$ by

$$< \varphi(\mathbf{x}), f >=< f, \mathbf{x} > .$$

Then φ is linear. Assume that $\varphi(\mathbf{x})$ =0. Then $< f, \mathbf{x} >$=0 for all $f \in \mathbf{V}^*$. By the above lemma it follows that $\mathbf{x} = 0$. Thus φ is injective. We have

$$\begin{aligned} |< \varphi(\mathbf{x}), f >| &= |< f, \mathbf{x} >| \\ &\leq ||f|| \, ||\mathbf{x}|| \end{aligned}$$

Hence $||\varphi(\mathbf{x})|| \leq ||\mathbf{x}||$. So in particular $||\varphi|| \leq 1$. But if $\mathbf{x} \neq 0$ then there exists a $f \in \mathbf{V}^*$ of norm $||f|| = 1$ such that $f(\mathbf{x}) = ||\mathbf{x}||$. Hence

$$< \varphi(\mathbf{x}), f >= ||\mathbf{x}||$$

which implies that $||\varphi(\mathbf{x}|| = \|\mathbf{x}\|$ and $\|\varphi\| = 1$.

Definition 4. *Let $\mathbf{V}, \mathbf{W}$ be normed spaces. Let $T : \mathbf{V} \to \mathbf{W}$ be linear. Then T is an isometry if $||T(\mathbf{x})|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbf{V}$. We say that $\mathbf{V}$ and $\mathbf{W}$ are isometric if there exists an isometry of $\mathbf{V}$ onto $\mathbf{W}$.*

Let us remark the following:

1. If T is an isometry, then T is injective.
2. If T is an isometry, then $||T|| = 1$.
3. If T is an isometry onto $\mathbf{W}$ then T is invertible and $T^{-1} : \mathbf{W} \to \mathbf{V}$ is also an isometry. (Define $S : \mathbf{W} \to \mathbf{V}$ by

$$S(\mathbf{y}) = \mathbf{x} \quad \text{if} \quad T(\mathbf{x}) = \mathbf{y}$$

S is well defined because T is bijective. It is also clear that S is linear. Furthermore we have

$$||S(\mathbf{y})|| = ||\mathbf{x}|| = ||T(\mathbf{x})|| = ||\mathbf{y}||$$

Hence S is an isometry. We have $S(T(\mathbf{x})) = \mathbf{x}$ be definition. We also have $T(S(\mathbf{y})) = T(\mathbf{x}) = \mathbf{y}$. Hence T is invertible with inverse S.

What we have just seen is the following:

Lemma 10. *φ is an isometry of $\mathbf{V}$ into $\mathbf{V}^{**}$.*

We know that $\mathbf{V}^{**}$ is a Banach space. Hence $\overline{\varphi(\mathbf{V})}$ is also a Banach space. Thus:

Lemma 11. *Let $\mathbf{V}$ be a normed vector space. Then there exists a Banach space $\mathbf{B}$ and an isometry $T : \mathbf{V} \to \mathbf{B}$ such that $T(\mathbf{V})$ is dense in $\mathbf{B}$, i.e., each normed vector space can be isometrically embedded into a Banach space.*

Definition 5. *The normed vector space $\mathbf{V}$ is reflexive if $\varphi : \mathbf{V} \to \mathbf{V}^{**}$ is an isometric isomorphism.*

Notice that if $\mathbf{V}$ is reflexive, then $\mathbf{V}$ is a Banach space.

Example 4. *The spaces* $\mathbf{L}^p([0,1])$ *are reflexive for* $1 < p < \infty$*: Recall that* $(\mathbf{L}^p)^* \simeq \mathbf{L}^q$ *if* $1 \leq p < \infty$ *and* $1/p + 1/q = 1$*. Thus for* $1 < p < \infty$ *we get*

$$\mathbf{L}^{p**} \simeq \mathbf{L}^{q*} \simeq \mathbf{L}^p$$

Hence $\mathbf{L}^p$ *is reflexive for* $1 < p < \infty$*.*

For $p = 1$ *we have* $\mathbf{L}^{1**} = \mathbf{L}^{\infty *} \supset \mathbf{L}^1$*. We claim that* $\mathbf{L}^{\infty *} \neq \mathbf{L}^1$*. Let* $\mathbf{V} = \mathbf{L}^\infty([0,1])$ *and let* $\mathbf{W} = C([0,1])$*. Then* $\mathbf{W}$ *is a closed subspace of* $\mathbf{V}$*. Define* $\delta \in \mathbf{W}^*$ *by*

$$< \delta, g >= g(0)$$

Then $\delta(g) \leq ||g||_\infty =: p(g)$*. By the Hahn-Banach theorem Theorem there exists a* $F \in \mathbf{L}^\infty([0,1])^*$ *such that* $||F|| \leq 1$ *and* $F(g) = g(0)$ *for each* $g \in C([0,1])$*.*

Suppose we have $y \in \mathbf{L}^1([0,1])$ *such that*

$$< F, g >= \int_0^1 g(t)y(t)\,dt \qquad \forall g \in \mathbf{L}^\infty([0,1])$$

Define $x_n(t) := (1-t)^n$*. Then pointwise (and in* $\mathbf{L}^1$*)* $x_n \to 0$*. We have* $|x_n(t)y(t)| \leq |y(t)|$ *and* y *is integrable. We have* $< F, x_n >= x_n(0) = 1$ *for all* n*. Hence*

$$\begin{aligned} 1 &= \lim < F, x_n > \\ &= \lim \int_0^1 x_n(t)y(t)\,dt \\ &= \int_0^1 \lim x_n(t)y(t)\,dt \\ &= 0 \end{aligned}$$

which is impossible. Hence $F \notin \varphi(\mathbf{L}^1([0,1])) \subset \mathbf{L}^1([0,1])^{**} = \mathbf{L}^\infty([0,1])^*$*. Hence* $\mathbf{L}^1([0,1])$ *is not reflexive.*

2. Baire Category Theorem

Definition 6. *Let* X *be a topological space and let* $E \subset X$*.*

1. E *is nowhere dense if* $\overline{E}^o = \emptyset$*. (This is the same as* $\overline{X \setminus \overline{E}} = X \setminus \overline{E}^o = X$*, or* $X \setminus \overline{E}$ *is dense.)*
2. E *is of the first category if and only if* E *is a finite or countable union of nowhere dense sets.*
3. E *is of the second category if* E *is not of the first category.*

Example 5. *Let* $a \in \mathbb{Q}$*. Then* $E_p := \{p\}$ *is nowhere dense. It follows that* $\mathbb{Q} = \cup_{q \in \mathbb{Q}} E_q$ *is of the first category, even if* $\mathbb{Q}$ *is dense in* $\mathbb{R}$*.*

Lemma 12. *If* E *is of the second category, then* $\overline{E}$ *contains an open set.*

Proof. If $\overline{E}$ does not contain an open set, then $\overline{E}^o = \emptyset$. Thus E would be nowhere dense. ∎

Theorem 3 (Bair)**.** *Let* X *be a complete metric space and* $\{O_k\}$ *a countable collection of dense open sets. Then* $\bigcap O_k$ *is dense.*

Theorem 4 (Bair Category Theorem)**.** *Let* X *be a complete metric space. Then no nonempty open subset of* X *is of first category.*

An application of the theorem is the following *uniformly boundness principle*:

Theorem 5. *Let X be a metric space and let $\mathcal{F} \subset C(X)$. Suppose that for each $x \in X$ there is a number M_x such that $|f(x)| \le M_x$ for all $f \in \mathcal{F}$. Then there is a nonempty set $O \subset X$ and a constant $M > 0$ such that*

$$|f(x)| \le M \qquad \forall f \in \mathcal{F}\,, \forall x \in O\,.$$

Proof. For $N \in \mathbb{N}$ let

$$E_N = \{x \in X \mid \forall f \in \mathcal{F} \,:\, |f(x)| \le N\}$$

Then

$$E_N = \bigcap_{f \in \mathcal{F}} \{x \in X \mid |f(x)| \le N\}$$

a intersection of closed sets. Hence E_N is closed. Let $x \in X$ and choose $N \ge M_x$. Then $x \in E_N$. It follows that $X = \cup E_N$. As X is not of the first category it follows that one of the E_N contains an open set O. Hence $|f(x)| \le N$ for all $x \in O$. ∎

3. The Closed Graph Theorem

Definition 7. *Let X and Y be topological spaces. A map $f : X \to Y$ is open if $f(O)$ is open in Y for all $O \subset X$ open.*

The main result in this section is the following *open mapping theorem*:

Proposition 7. *Let $\mathbf{V}$ and $\mathbf{W}$ be Banach spaces. A continuous surjective linear transformation $T \in B(\mathbf{V}, \mathbf{W})$ is open. In particular if T is bijective, then T is an isomorphism.*

Let us name few applications before we proof this:

Proposition 8. *Let $\mathbf{V}$ be a vector space which is complete in the norms $||\cdot||_1$ and $||\cdot||_2$. Suppose there exists a constant C_2 such that*

$$||\mathbf{x}||_1 \le C_2\, ||\mathbf{x}||_2 \qquad \forall \mathbf{x} \in \mathbf{V}\,.$$

Then the norm are equivalent, i.e., there is a constant C_1 such that

$$||\mathbf{x}||_2 \le C_1\, ||\mathbf{x}||_1 \qquad \forall \mathbf{x} \in \mathbf{V}$$

Proof. The identity map $\mathbf{V} \ni \mathbf{x} \mapsto \mathbf{x} \in \mathbf{V}$ is a continuous map $(\mathbf{V}, ||\cdot||_2) \to (\mathbf{V}, ||\cdot||_1)$. Hence an isomorphs, thus the inverse $(\mathbf{V}, ||\cdot||_1) \to (\mathbf{V}, ||\cdot||_2), \mathbf{x} \mapsto \mathbf{x}$ is bounded, which is exactly the statement. ∎

Let $T : \mathbf{V} \to \mathbf{W}$ be linear map. Then the *graph of T* is the subspace $G(T) \subset \mathbf{V} \times \mathbf{W}$ defined by:

$$G(T) := \{(\mathbf{v}, T(\mathbf{v}) \mid \mathbf{v} \in \mathbf{V}\}$$

Proposition 9 (Closed Graph Theorem)**.** *Let $\mathbf{V}$ and $\mathbf{W}$ be Banach spaces, and let $T : \mathbf{V} \to \mathbf{W}$ be linear. Then T is continuous if and only if $G(T)$ is closed.*

Proposition 10 (Uniform boundness Theorem)**.** *Let $\mathbf{V}$ be a Banach space, and let $\mathbf{W}$ be a normed space. Let $\mathcal{F} \subset B(\mathbf{V}, \mathbf{W})$ be such that for each $\mathbf{v} \in \mathbf{V}$ there exists a constant $M_{\mathbf{v}}$ such that*

$$T(\mathbf{v}) \le M_{\mathbf{v}}\, ||\mathbf{v}|| \qquad \forall T \in \mathcal{F}$$

Then the operators in $\mathcal{F}$ are uniformly bounded; i.e., there is a constant M such that

$$||T|| \le M \qquad \forall T \in \mathcal{F}\,.$$

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