(Chapter head:)*
Introduction

0.1 Harmonic Analysis

The subject proper of *harmonic analysis* is to decompose functions into "simpler" functions. In the theory of differential equations that means to write an arbitrary functions as a sum or integral of eigenfunctions. If we have a symmetry group acting on the system, then we would like to write an arbitrary function as a sum of functions that transforms in a simple and controllable way under the symmetry group. The simplest example is the use of polar coordinates and radial functions for rotation symmetric equations.

The course begins with a short overview of classical Fourier analysis on the torus and R^n: Fourier series, Fourier transform, inversion formula, Plancherel theorem and extensions to distributions. The role of the Fourier transform in analysis of partial differential equations with constant coefficients and in the theory of *pseudodifferential operators* will also be discussed.

We can view R^n as a set or manifold. But we can also view it as an *abelian group.* In that sense R^n is a part of *abelian harmonic analysis.* The simplest example of *nonabelian harmonic analysis* is the Heisenberg group $H_n = R^{2n+1}$ (with a new group multiplication). The Heisenberg group is also a simple example of a *Lie group.* We will introduce the necessary structure of the Heisenberg group. The connection between classical Fourier analysis and harmonic analysis on H_n will be developed. Along with group and analytic structure, the *representation theory* of H_n will be presented. This includes the Stone-von Neumann theorem, connections to quantum mechanics, the Plancherel theorem, the Schrödinger model, the equivalent Fock–Bargmann representation, transforms of distributions, twisted convolutions, the strong Stone-von Neumann theorem and transforms of Schwartz functions as operators with smooth kernels.

0.2 Notation

We denote elements in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n \in \mathbb{R}$. The norm of x is the nonnegative number $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. For $x, y \in \mathbb{R}^n$ denote the inner product of x and y by $x \cdot y = (x, y) = x_1 y_1 + \cdots + x_n y_n$. Then $|x|^2 = x \cdot x$. We denote the space of continuous functions on $\mathbb{R}^n$ by $C(\mathbb{R}^n)$. If $r \in \mathbb{N}_0$ then $C^r(\mathbb{R}^n)$ is the space of r-times continuously differentiable functions on $\mathbb{R}^n$ and $C^\infty(\mathbb{R}^n)$ is the space of smooth functions. If $f : \mathbb{R}^n \to \mathbb{C}$ is continuous then

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}.$$

We say that f has compact support if $\operatorname{supp}(f)$ is compact. The space of compactly supported functions is denoted by $C_c^r(\mathbb{R}^n)$. We normalize the measure μ on $\mathbb{R}^n$ by

$$\mu = (2\pi)^{-n/2}\, dx_1 dx_2 \cdots dx_n$$

and let $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \mu)$. Then $L^p(\mathbb{R}^n)$ is a Banach space and $L^2(\mathbb{R}^n)$ is a Hilbert space with inner product

$$(f, g) := \int f(x)\overline{g(x)}\, d\mu(x)\,.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. For $x \in \mathbb{R}^n$ we let

$$x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n} \in \mathbb{R}$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then each polynomial function $p(x)$ on $\mathbb{R}^n$ of degree $\leq k$ can be written as

$$p(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha, \qquad a_\alpha \in \mathbb{C}\,.$$

Let $D_i = \partial/\partial x_i : C^r(\mathbb{R}^n) \to C^r(\mathbb{R}^n)$. For $\alpha \in \mathbb{N}_0^n$ let

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}\,.$$

Then we can define $p(D)$ for each polynomial $p(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha$ by

$$p(D) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha\,.$$

The operator $p(D)$ is called a *differential operator of degree k with constant coefficients.*

(Chapter head:)Fourier Series

1 Harmonic Functions on the Disk

In this section we discuss one of the problems that motivated the beginning of the theory of Fourier series and is close to Fourier's original work. Let $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ be the Laplace operator on $\mathbb{R}^n$. It is one of the most interesting differential operators on $\mathbb{R}^n$, partly because of the role it plays in partial differential equations arising in physics:

- **The heat equation**: $\Delta u = a^2 u_t$. Here $u(x, t)$ is a function of $n + 1$ variables, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, and the subscript $_t$ denotes the partial derivative with respect to t.
- **The wave equation:** $a^2 \Delta u = u_{tt}$.
- **The Schrödinger's equation:** $\frac{1}{i}\Delta u = u_t$.
- **The Helmholtz's equation:** $-\Delta u = \lambda u$.

Fourier analysis is *one* of the many tools to deal with the solutions to those equations and we will discuss that later in this book. As a motivation we start with the equation $\Delta u = 0$ on the unit disc

$$D := \{z \in \mathbb{C} \mid |z| < 1\} = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}$$

such that u takes prescribed values on the boundary. Thus we would like to solve the following *Diriclet's problem:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \in D \tag{1}$$

$$u(x, y, 0) = f(x, y) \qquad x^2 + y^2 = 1\,. \tag{2}$$

Here f is a given $\mathbf{L}^2$-function on the boundary and we will assume that $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$. That is u is twice continuously differentiable in D and continuous on the closed domain $\bar{D} = \left\{(x, y) \mid x^2 + y^2 \leq 1\right\}$.

Definition 1 *Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{C}$ is called **harmonic** on Ω if $\Delta u = 0$.*

Notice that a harmonic function can be viewed as a time independent solution to the heat equation. Let us rewrite (1) using polar-coordinates

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)\,.$$

The Laplacian then becomes

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)$$

and $u(r, \theta)$ is periodic in θ with period 2π, i.e., $u(r, \theta + 2\pi) = u(r, \theta)$. The Direclet's problem (1) and (2) is now

$$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0\,, \quad u(1, \theta) = f(\theta)\,. \tag{3}$$

One way to solve this is to use *separation of variables*, that is find first all solutions of the form:

$$u(r, \theta) = F(r)G(\theta).$$

Then the Laplace equation (1) is the:

$$\frac{1}{G(\theta)}\frac{d^2 G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right)\,.$$

The left hand side is independent of r and the right hand side is independent of θ. Hence there is a constant k such that

$$\frac{1}{G(\theta)}\frac{d^2 G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = k$$

which gives us the two ordinary differential equations:

$$\frac{d^2G}{d\theta^2}(\theta) = kG(\theta)$$
$$r\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = r^2F''(r) + rF'(r) = -kF(r)\,.$$

The general solution to these equations are:

$$G_k(\theta) = \begin{cases} a_0 + b_0\theta & \text{if} \quad k = 0; \\ a_k e^{\sqrt{k}\theta} + b_k e^{-\sqrt{k}\theta} & \text{if} \quad k \neq 0 \end{cases} \tag{4}$$

$$F_k(r) = \begin{cases} A_0 + B_0 \log(r) & \text{if} \quad k = 0; \\ A_k r^{\sqrt{-k}} + B_k r^{-\sqrt{-k}} & \text{if} \quad k \neq 0 \end{cases} \tag{5}$$

where we have indicated the dependence of F and G on the constant k by the index k. The function $G_k(\theta)$ has period 2π if and only if $k = -n^2 < 0$ or $k = 0$ and $b = 0$. The function F_k is defined on all of D if and only if $B_k = 0$ for all k. The same conclusion holds if we only assume that u is $\mathbf{L}^2$ on the disk. We must therefore have $F_k(r) = A_k r^n$. The solution is therefore

$$u_n(r,\theta) = r^n\left(a_n e^{in\theta} + b_n e^{-in\theta}\right), \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

Writing $b_n = a_{-n}$ we can then formally write our solution as::

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \tag{6}$$

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = f(\theta)\,. \tag{7}$$

To make the step from formal solution to actually a solution we still need to answer questions like:

1. Is it possible to choose the constants a_n such that the given function f can be written as $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$?

2. If the answer to (1) is yes, how can we actually find the constants a_n?

3. In what sense (pointwise, in $L^p, \dots$) does the series in 7 represent the function f?

4. Does the equation (6) then give a smooth function on the disk such that $\lim_{r\to 1} u(r,\theta) = f(\theta)$?

5. Is the solution to our problem unique?

6. Is every harmonic function in the disk given by a series as in 6?

To look for an answer we make few more formal calculations. Later we will show that those calculations can be made rigid. First multiply f by $e^{-im\theta}$ and then integrate. We interchange the summation and integration without further comments:

$$\begin{aligned}\int_0^{2\pi} f(\theta)e^{-im\theta}\,d\theta &= \int_0^{2\pi}\sum_{n=-\infty}^{\infty} a_n e^{in\theta}e^{-im\theta}\,d\theta\\ &= \sum_{n=-\infty}^{\infty} a_n \int e^{i(n-m)\theta}\,d\theta\\ &= 2\pi a_m\,.\end{aligned}$$

Because

$$\int_0^{2\pi} e^{ik\theta}\,d\theta = \begin{cases} 2\pi & \text{if} \quad k=0\,,\\ 0 & \text{if} \quad n\neq 0\,.\end{cases}$$

One of our first results on Fourier series says that if we set $a_m = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta}\,d\theta$ then $f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ holds in $\mathbf{L}^2([0,2\pi])$. The constant function $\theta\mapsto 1$ is in $\mathbf{L}^2([0,2\pi], \frac{d\theta}{2\pi})$ with norm one. Hence by the Hölder's inequality for $\mathbf{L}^2$-function we get

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\,d\theta \leq |f|_1 < \infty\,.$$

For $r\leq R<1$ we have:

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}} a_n r^{|n|}e^{in\theta}\right| &\leq \sum_{n\in\mathbb{Z}} |a_n|\, R^{|n|}\\ &\leq |f|_1\left(\sum_{n=1}^{\infty} R^n + \sum_{n=0}^{\infty} R^n\right)\\ &\leq \frac{2\,|f|_1}{1-R}\,.\end{aligned}$$

Therefore the series defining $u(r,\theta)$ converges uniformly on compact subsets of D and defines therefore a smooth function on the disk. To evaluate the limit $\lim_{t\to 1} u(x,y;t)$ we rewrite equation (7) as an integral over $[0,2\pi]$. Again interchanging summation and integration without further arguments. We also need the following simple fact

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \sum_{n=0}^{\infty}\left(re^{i\theta}\right)^n + \sum_{n=0}^{\infty}\left(re^{-i\theta}\right)^n - 1 = \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,. \qquad (8)$$

The function

$$P(r,\theta) := \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,,\quad 0\leq r<1,\,\theta\in\mathbb{R} \qquad (9)$$

is called the *Poisson kernel* for the unite disk. We will point out some properties of the Poisson kernel in the following exercises. But here is the graph of the Poisson kernel for $r = 0.5$ (red), $r = 0.7$ (black) and $r = 0.9$ (green):

Picture pipk

Inserting the definition of a_n we get:

$$\begin{aligned} u(r,\theta) &= \sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) r^{|n|} e^{in(\theta-\phi)}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) P(r, \theta - \phi)\, d\phi\,. \end{aligned} \tag{10}$$

Hence u is given by convoluting f with the Poisson kernel. This can be used to show that if f is continuous then $u(r,\theta) \to f(\theta)$ uniformly.

Exercise

1. Prove equation (9): $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$.

2. Prove the following:

   (a) $P(r,\theta) \geq 0$ and $\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)\, d\theta = 1$ for all $r \geq 0$.

   (b) The maximum of $\theta \mapsto P(r,\theta)$ occurs at $\theta = 0$ and $\max_\theta P(r,\theta) = \frac{1+r}{1-r}$. In particular $P(r,0) \to \infty$ as $r \to 1$.

   (c) The function $\theta \mapsto P(r,\theta)$ takes its minimum at $\theta = \pi$. Evaluate $P(r,\pi)$.

3. Suppose that f is 2π-periodic and piecewise continuous. Show that

   $$\lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) P(r, \theta - \phi)\, d\phi = f(\theta)$$

   if f is continuous at θ.

4. Write $u(x,y) = u(z)$ where $z = x + iy$. Suppose that f is continuous. Show that u is holomorphic on D if and only if $a_n = 0$ for all $n < 0$.

2 Periodic Functions

In this section we start with the theory of Fourier series of periodic functions. We show that the periodic functions corresponds to the functions on the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Using the torus has many advantages, in particular $\mathbb{T}$ is a compact abelian group. Our results on function spaces on the torus will serve as guiding line for more complex groups later on.

Let $f : \mathbb{R} \to \mathbb{C}$. Define $P_f := \{L \in \mathbb{R} \mid \forall x : f(x + L) = f(x)\}$ and let $P_f^+ := \{L \in P_f \mid L > 0\}$.

Definition 2 *A function $f : \mathbb{R} \to \mathbb{C}$ is called periodic if $P_f \neq \emptyset$.*

Example 3 *Every constant function $f : x \mapsto c \in \mathbb{R}$ is periodic and $P_f = \mathbb{R}$.*

Example 4 *Let $L \in \mathbb{R}$, $L \neq 0$. Then the functions $x \mapsto \cos(Lx)$, $\sin(Lx)$, e^{iLx} are all periodic and $P_f = \frac{2\pi}{L}\mathbb{Z}$.*

Example 5 *Let $L > 0$. Here is a way to construct* ***all*** *L-periodic functions. Suppose first that f has period $L > 0$. Let I be* ***any*** *interval $[a, a + L)$ of length L, then f determines a function $g = f|_I$ on I and the map $f \mapsto g$ is linear and injective. Let $g : I \to \mathbb{R}$ be given. Let $I_n := I + nL = [a + nL, a + (n+1)L)$, $n \in \mathbb{Z}$. Then $\mathbb{R} = \dot{\bigcup}_{n \in \mathbb{Z}} I_n$. For $x \in \mathbb{R}$, choose n such that $x = t + nL \in I_n$, $t \in I$. Then both t and n are uniquely determined. Define $f(x) := g(t)$. Then $f : \mathbb{R} \to \mathbb{C}$ is well defined. Let $y \in \mathbb{R}$ and write $y = t + nL$ for some $t \in [0, L)$. Then $y + L = t + (n + 1)L$. Hence $f(y + L) = g(t) = f(y)$ and f has period L. It follows that the space of L-periodic functions is linearly isomorphic to the space of function on I. Assume that g is continuous. Then f is continuous if and only if*

$$g(a + L-) := \lim_{t \nearrow L} g(a + t) = g(a)\,.$$

Examples of this construction are the function defined on $[-\pi, \pi)$ and given by $g_1(x) = x$, $g_2(x) = \pi - |x|$ and $g(x) = -1$ for $-\pi < x < 0$ and 1 for $0 < x < \pi$.

If f is periodic let $L_f := \min P_f \geq 0$.

Lemma 6 *Assume that f is periodic. Then P_f is a group. If $L_f > 0$ then $P_f = L_f\mathbb{Z}$.*

Proof. Let $L, M \in P_f$. Then

$$\begin{aligned} f(x + (M - L)) &= f(x - L) \\ &= f((x - L) + L) \\ &= f(x)\,. \end{aligned}$$

Hence $M - L \in P_f$. It follows that P_f is a group.

Assume that $L_f > 0$. It follows by the first part, that $L_f\mathbb{Z} \subset P_f$. Let $L \in P_f$. We can assume that $L > 0$. Then $L_f \leq L$. Choose $n \in \mathbb{N}$ such that

$$nL_f \leq L < (n + 1)L_f$$

Then $0 \leq L - nL_f < L_f$, and $L - nL_f \in P_f$ by the first part. Therefore $L = nL_f$ by the definition of L_f. ■

Lemma 7 *Suppose that f is continuous. Then $L_f = 0$ if and only if f is constant.*

Proof. If f is constant, then $L_f = 0$. So assume that $L_f = 0$. Let $x \in \mathbb{R}$ and $c := f(x)$. Let $y \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta > 0$ such that $|y - z| < \delta$ implies that $|f(y) - f(z)| < \epsilon$. Let $L \in P_f$ be such that $0 < L < \delta$. This is possible as $L_f = 0$. Let $n \in \mathbb{Z}$ be such that

$$nL < y - x \leq (n+1)L\,.$$

Then $0 < y - (x + nL) \leq L < \delta$, and $|f(x) - f(y)| = |f(x + nL) - f(y)| < \epsilon$. It follows that $f(y) = f(x) = c$. Hence f is constant. ■

Lemma 8 *Let f be periodic with period $L > 0$. Then the function g defined by $g(x) := f(Lx/2\pi)$ has period 2π.*

Proof. Let $x \in \mathbb{R}$. Then $g(x + 2\pi) = f(\frac{L}{2\pi}x + L) = f(\frac{L}{2\pi}x) = g(x)$. ■

Because of this lemma we can always assume that the function f has period 2π. By simple change of variable all our results will then holds for functions of arbitrary periods. Let $\kappa : \mathbb{R} \to \mathbb{T}$ be the map $\theta \mapsto e^{i\theta}$. The following lemma shows that we can replace our study of functions of period 2π by the study of functions on the torus. We leave the proof as an exercise. This correspondence can then be used to integrate and differentiate functions on the torus. One of the advantages of viewing functions on the torus is the fact that $\mathbb{T}$ is a compact abelian group.

Lemma 9 *A function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if and only if there exists a function $F : \mathbb{T} \to \mathbb{C}$ such that $F = f \circ \kappa$, i.e.*

$$f(\theta) = F(e^{i\theta}) \qquad \text{for all } x \in \mathbb{R}\,.$$

The function F is continuous if and only if f is continuous.

Exercise

1. Find a measurable nonconstant function $f : \mathbb{R} \to \mathbb{R}$ such that $L_f = 0$.
2. Let f and g be continuous functions with $L_f, L_g > 0$ and $L_f/L_g \in \mathbb{Q}$. Show that $f + g$ and fg are periodic?
3. Let u be a 2π-periodic twice continuously differentiable function on $\mathbb{R}$. Define

$$g(x,t) = u(t + x) - u(t - x)\,.$$

(a) Show that g is periodic in both x and t.

(b) Show that g is a solution to the wave equation $\partial_x^2 g = \partial_t^2 g$ such that on the endpoints 0 and 2π we have $g(0,t) = g(2\pi, 0) = 0$.

4. Let $L > 0$. View $\mathbb{T}$ as a topological subspace of $\mathbb{C}$. Define $\kappa : \mathbb{R} \to \mathbb{T}$ by $\kappa(x) = e^{ix/L}$.

   (a) Show that κ is continuous, surjective and open and κ is $2\pi L$-periodic

   (b) Let $C(\mathbb{R}, 2\pi L)$ be the space of continuous L-periodic functions on $\mathbb{R}$. View $C(\mathbb{R}, 2\pi L)$ and $C(\mathbb{T})$ as Banach spaces with the supremum norm, $|f|_\infty = \sup |f(x)|$. Then $\kappa_* : C(\mathbb{T}) \to C(\mathbb{R}, 2\pi L)$, $f \mapsto f \circ \kappa$ is an isomorphism of Banach spaces.

5. The torus is a group under complex multiplication. Show that the map $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$, $(z, w) \mapsto z^{-1}w$ is continuous. Thus $\mathbb{T}$ is a topological group.

6. Let $\chi : \mathbb{T} \to \mathbb{C}$ be a continuous homomorphism, i.e., $\chi(zw) = zw$. Then there exists a $n \in \mathbb{Z}$ such that $\chi(z) = z^n$ for all $n \in \mathbb{Z}$.

3 Integration on the Torus

Let us recall some standard facts on integration on topological spaces. Let X be a locally compact σ-compact Hausdorf topological space. Denote the space of complex valued continuous function on X by $C(X)$. If $f \in C(X)$ then the support of f is the set

$$\mathrm{Supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}\,.$$

If Supp(f) is compact then we say that f has *compact support*. Let $C_c(X)$ denote the space of continuous functions with compact support. Let μ be a linear form on $C(X)$ such that $\mu(f) \geq 0$ if $f \geq 0$. Then there exists a unique regular Borel measure – also denoted by μ – on X such that

$$< \mu, f >= \int_X f(x)\, d\mu(x)$$

for all $f \in C_c(X)$. The L^p-norm is defined by

$$|f|_p := \left(\int_X |f(x)|^p \; d\mu(x) \right)^{1/p}$$

and $\mathcal{L}^p(X)$ is the space of measurable functions with finite L^p-norm. Let $\mathcal{N}(X)$ be the space of measurable functions that are zero almost every where. Then $\mathcal{N}(X) \subset \mathcal{L}^p(X)$ for all $p \geq 1$. Let

$$L^p(X) := \mathcal{L}^p(X)/\mathcal{N}(X)\,.$$

If $f \in \mathcal{L}^p(X)$ then we sometimes denote by $[f]$ the class in $L^p(X)$ containing f. By abuse of notation we sometimes simply write f instead of $[f]$. For $f \in \mathcal{L}^p(X)$ let $|[f]|_p = |f|_p$. Then $|[f]|_p$ is well defined. We recall the following standard facts:

Theorem 10 *Let μ and X be as above. Let $p \geq 1$ and let q be such that $1/p + 1/q = 1$. Then the following holds:*

1. ***(Minkowski's inequality)*** *Let $f, g \in \mathcal{L}^p(X)$ then $|f+g|_p \leq |f|_p + |g|_p$.*
2. *The space $L^p(X)$ is complete. In particular $L^p(X)$ is a Banach space.*
3. *The space $C_c(X)$ is dense in $L^p(X)$ for all $1 \leq p < \infty$.*
4. *Let $f \in L^p(X)$. If $\int fg\, d\mu = 0$ for all $g \in C_c(X)$ then $f = 0$.*
5. *Assume that $\{f_k\}_k$ is a sequence in $L^1(X)$ such that*
   (a) $\lim_k f_k(x)$ exists for almost all $x \in X$;
   (b) There exist $g \in L^1(X)$ such that $|f_k(x)| \leq g(x)$ for almost all $x \in X$.

   Then $\lim_k f_k \in L^1(X)$ and $\lim \int f_k\, d\mu = \int (\lim_k f_k)\, d\mu$.
6. ***(Hölder's inequality)*** *Let $f \in L^p(Y)$ and $g \in L^q(Y)$. Then $fg \in L^1(Y)$ and*
$$|fg|_1 \leq |f|_p\, |g|_q \ .$$
7. *If $1 < p < \infty$ then the dual space of $L^p(X)$ is $L^q(X)$, i.e. if $\nu : L^p(X) \to \mathbb{C}$ is continuous and linear, then there exists a unique $g \in L^q(X)$ such that for all $f \in L^p(X)$ we have $< \nu, f >= \int fg\, d\mu$.*
8. *The space $L^2(X)$ is a Hilbert space with inner product $(f,g) = \int_X f(x)\overline{g(x)}\, d\mu(x)$.*

Theorem 11 (Tonelli) *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and f nonnegative measurable function on $X \times Y$. Then*

1. *For almost all $x \in X$ the function $f_x : y \mapsto f(x,y)$ is a measurable function on Y.*
2. *For almost all $y \in Y$ the function $f^y : x \mapsto f(x,y)$ is a measurable function on X.*
3. *$\int f(x,y)\, d\nu(y)$ is a measurable function on X.*
4. *$\int f(x,y)\, d\mu(x)$ is a measurable function on Y.*

5. *We have*

$$\int_{X\times Y} f(x,y)\,d\,(\mu\times\nu) = \int_X \left[\int_Y f(x,y)\,d\nu(y)\right]\,d\mu(x)$$
$$= \int_Y \left[\int_X f(x,y)\,d\mu(y)\right]\,d\nu(x)$$

Theorem 12 (Fubini) *Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be two measure spaces and f integrable function on $X\times Y$ with respect to $\mu\times\nu$. Then*

1. *For almost all $x\in X$ the function $f_x : y\mapsto f(x,y)$ is an integrable function on Y.*
2. *For almost all $y\in Y$ the function $f^y : x\mapsto f(x,y)$ is an integrable function on X.*
3. *$\int f(x,y)\,d\nu(y)$ is an integrable function on X.*
4. *$\int f(x,y)\,d\mu(x)$ is an integrable function on Y.*
5. *We have*

$$\int_{X\times Y} f(x,y)\,d\,(\mu\times\nu) = \int_X \left[\int_Y f(x,y)\,d\nu(y)\right]\,d\mu(x)$$
$$= \int_Y \left[\int_X f(x,y)\,d\mu(y)\right]\,d\nu(x)$$

We define a regular Borel measure μ on $\mathbb{T}$ by

$$\int f(z)\,d\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{ix})\,dx = \frac{1}{2\pi}\int_0^{2\pi} f\circ\kappa(x)\,dx\,.$$

for $f\in C(\mathbb{T})$. The corresponding $\mathcal{L}^p$-spaces are the same as the space of $\mathcal{L}^p$-functions on $[0,2\pi]$ and the space of Lebesgue measurable functions on $\mathbb{R}$ that are 2π-periodic and integrable over $[0,2\pi]$. The measure μ is compatible with the groups structure on $\mathbb{T}$:

- μ is *left and right-invariant* in the sense that

$$\int f(y^{-1}z)\,d\mu(z) = \int f(zy)\,d\mu(z) = \int f(z)\,d\mu(z)\,,\quad f\in C(\mathbb{T})\,.$$

- μ is invariant under the map $z\mapsto z^{-1}$;

$$\int f(z^{-1})\,d\mu(z) = \int f(z)\,d\mu(z)\,,\quad f\in C(\mathbb{T})\,.$$

Lemma 13 *Let $1\le r\le s$. Then $L^s(\mathbb{T})\subset L^r(\mathbb{T})$ and the inclusion map $i : L^s(\mathbb{T})\to L^r(\mathbb{T})$ is a contraction, i.e., $|f|_r\le|f|_s$ for all $f\in L^s(\mathbb{T})$.*

Proof. Let $p = s/r \geq 1$. Choose q such that $1/p + 1/q = 1$. Then $|f|^r \in L^p(\mathbb{T})$ and the constant function $z \mapsto 1$ is in $L^q(\mathbb{T})$ because $\mu(\mathbb{T}) = 1 < \infty$. By the Hölder-inequality we get

$$|f|_r^r = \int |f|^r \, 1 \, d\mu \leq ||f|^r|_p \, |1|_q = |f|_s^r \, .$$

■

Let $f, g \in L^1(\mathbb{T})$. Then $(z, w) \mapsto f(z)g(z^{-1}w)$ is a measurable function on $\mathbb{T} \times \mathbb{T}$ and

$$\int\int \left|f(z)g(z^{-1}w)\right| \, d\mu(z)d\mu(w) = |f|_1 \, |g|_1 < \infty \, . \tag{11}$$

It follows that for almost all $w \in \mathbb{T}$ the function $z \mapsto f(z)g(z^{-1}w)$ is integrable and

$$f * g(w) := \int f(z)g(z^{-1}w) \, d\mu(z)$$

is in $L^1(\mathbb{T})$. The function $f * g$ is called the *convolution* of f and g. Notice that $f * g$ is defined for all $f \in L^p(\mathbb{T})$ and $g \in L^{p'}(\mathbb{T})$ as $L^p(\mathbb{T}), L^{p'}(\mathbb{T}) \subset L^1(\mathbb{T})$. For $w \in \mathbb{T}$ and $f \in L^p(\mathbb{T})$ define $\lambda(w)f \in L^p(\mathbb{T})$ by $[\lambda(w)f](z) = f(w^{-1}z)$ (see problems 1 and 2).

Lemma 14 *Let $f, g, h \in L^1(\mathbb{T})$. Then the following holds:*

1. $|f * g|_1 \leq |f|_1 \, |g|_1$.
2. $f * g \in C(\mathbb{T})$ *and* $|f * g|_\infty \leq |f|_1 \, |g|_1$.
3. $f * g = g * f$.
4. $f * (g * h) = (f * g) * h$.
5. *For $w \in \mathbb{T}$ we have* $[\lambda(w)f] * g = f * [\lambda(w)g] = \lambda(w)[f * g]$.

Proof. 1) This follows from (11)

2) Let $w, w_0 \in \mathbb{T}$. Assume first that $g \in C(\mathbb{T})$. Then g is bounded. Let $M > 0$ be such that $|g(z)| \leq M$ for all $z \in \mathbb{T}$. Then

$$|f(z)| \left|g(z^{-1}w) - g(z^{-1}w_0)\right| \leq M \, |f(z)| \, .$$

As $z \mapsto M \, |f(z)|$ is integrable it follows by the Lebesgue Dominated Convergence Theorem that

$$\lim_{w \to w_0} |f * g(w) - f * g(w_0)| \leq \int \lim_{w \to w_0} |f(z)| \left|g(z^{-1}w) - g(z^{-1}w_0)\right| \, d\mu(z) = 0$$

as g is assumed continuous. Let now g be arbitrary in $L^1(\mathbb{T})$. Let $\epsilon > 0$. Choose $k \in C(\mathbb{T})$ such that $|g-k|_1 < \frac{\epsilon}{3(|f|_1+1)}$. Let $\delta > 0$ be such that

$$|f * k(w) - f * k(w_0)| < \frac{\epsilon}{3} \qquad \text{for } |w - w_0| < \delta.$$

We also have for all $x \in \mathbb{T}$ and $f_1, f_2 \in L^1(\mathbb{T})$ that $\int \left|f_1(z^{-1}x) - f_2(z^{-1}x)\right| d\mu(z) = \int |f_1(z) - f_2(z)| \, d\mu = |f_1 - f_2|_1$. Hence for $|w - w_0| < \delta$:

$$\begin{aligned}
|f * g(w) - f * g(w_o)| &\leq |f * g(w) - f * k(w)| + |f * k(w) - f * k(w_o)| \\
&\qquad + |f * k(w_o) - f * g(w_o)| \\
&\leq 2\,|f|_1\,|g-k|_1 + \epsilon/3 \\
&< \epsilon\,.
\end{aligned}$$

By Hölder's inequality we get

$$\left|\int f(z)g(z^{-1}w)\,d\mu(z)\right| \leq \left(\int |f(z)|\,d\mu\right)\left(\int \left|g(z^{-1}w)\right|\,d\mu\right) = |f|_1\,|g|_1\,.$$

The claim follows as this holds for all $w \in \mathbb{T}$.

3) Using the substitution $u = w^{-1}z$ we have $w = u^{-1}z$ and

$$\begin{aligned}
f * g(z) &= \int f(w)g(w^{-1}z)\,d\mu(w) \\
&= \int f(u^{-1}z)g(u)\,d\mu(u) \\
&= g * f(z)\,.
\end{aligned}$$

4) By the definition we get:

$$\begin{aligned}
f * (g * h)(z) &= \int f(w)g * h(w^{-1}z)\,d\mu(z) \\
&= \int f(w) \int g(u)h(u^{-1}w^{-1}z)\,d\mu(u)\,d\mu(w) \\
&= \int\int f(w)g(u)h((wu)^{-1}z)\,d\mu(u)\,d\mu(w)
\end{aligned}$$

Now we make the change of variable $s = wu$ or $u = w^{-1}s$. Then using Fubini's Theorem to interchange to order of the integrals we get:

$$\begin{aligned}
\int\int f(w)g(u)h((wu)^{-1}z)\,d\mu(u)\,d\mu(w) &= \int\int f(w)g(w^{-1}s)h(s^{-1}z)\,d\mu(u)d\mu(w) \\
&= \int \left[\int f(w)g(w^{-1}s)\,d\mu(w)\right] h(s^{-1}z)\,d\mu(s) \\
&= (f * g) * h(z)
\end{aligned}$$

5) This follows by

$$\int f(w^{-1}u)g(u^{-1}z)\,d\mu = \int f(u)g(u^{-1}w^{-1}z)\,d\mu = \int f(u)g(w^{-1}u^{-1}z)\,d\mu.$$

■

Exercise

1. For $w \in \mathbb{T}$ define $\lambda(w) : C(\mathbb{T}) \to C(\mathbb{T})$ by $[\lambda(w)f](z) = f(w^{-1}z)$.
   (a) Show that $\lambda(w)$ is continuous with norm one.
   (b) Let $1 \leq p < \infty$. Let $f \in L^p(\mathbb{T})$. Choose a sequence $g_j \in C(\mathbb{T})$ such that $g_j \to f$ in $L^p(\mathbb{T})$. Define $\lambda(w)f := \lim[\lambda(w)g_j]$. Show that $\lambda(w)$ is well defined and that $|\lambda(w)| = 1$.
   (c) Let $U(L^2(\mathbb{T}))$ be the group of unitary maps on $L^2(\mathbb{T})$. That is $U(L^2(\mathbb{T}))$ is the groups of all continuous linear maps L on $L^2(\mathbb{T})$ such that $L^*L = LL^* = \mathrm{id}$, where id is the identity map $u \mapsto u$. Show that $\lambda : \mathbb{T} \to U(L^2(\mathbb{T}))$ is a group homomorphism such that

$$\lim_{w\to 1} |\lambda(w)f - f|_2 = 0\,, \qquad \forall f \in L^2(\mathbb{T})\,.$$

2. For $w \in \mathbb{T}$ and $f \in \mathcal{L}^2(\mathbb{T})$ define $[\lambda_1(w)f](z) := f(w^{-1}z)$.
   (a) Let $f, g \in \mathcal{L}^2(\mathbb{T})$. Show that $\lambda_1(w)f = \lambda_1(w)g$ a.e. In particular if $f = 0$ a.e., then $\lambda_1(w)f = 0$ a.e.
   (b) Show that $\lambda_1(w)$ defines a isometry on $L^2(\mathbb{T})$ and that $\lambda_1(w) = \lambda(w)$.
3. For $1 \leq p < q$ find $f \in L^p(\mathbb{T})$ such that $f \notin L^q(\mathbb{T})$.
4. Let $h \in L^1(\mathbb{T})$.
   (a) Let $g \in L^2(\mathbb{T})$. Show that $\lambda(h)g := h * g$ is in $L^2(\mathbb{T})$. (**Hint:** Let $f \in L^2(\mathbb{T})$. Then $f\overline{\lambda(h)g}$ is integrable and $f \mapsto \int f(z)\overline{\lambda(h)g(z)}\,d\mu(z)$ is a continuous linear form on $L^2(\mathbb{T})$.
   (b) Show that $\lambda(h) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear map with $|\lambda(h)| \leq |h|_1$.

4 The Fourier Transform

The functions of the form

$$p(\theta) = \sum_{n=-N}^{M} a_n e^{in\theta}$$

are called *trigonometric polynomials.* The trigonometric polynomials are 2π-periodic, and as a functions in $\mathbb{T}$ they have the simple form

$$p(z) = \sum_{n=-N}^{M} a_n z^n$$

Our aim is to show that the trigonometric polynomials are dense in $L^2(\mathbb{T})$. Denote the function $z \mapsto z^n$ by e_n. We will show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthogonal basis for $L^2(\mathbb{T})$. Let $f \in L^1(\mathbb{T})$. Then fe_{-n} is integrable. Define the *Fourier transform* $\hat{f} : \mathbb{Z} \to \mathbb{C}$ of f by

$$\hat{f}(n) := \int f(z)z^{-n}\, d\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)\, e^{-in\theta} d\theta \tag{12}$$

where $F = f \circ \kappa$ as usually. Notice that $\hat{f}(n) = (f, e_n) = f * e_n(1)$ for $f \in L^2(\mathbb{T})$. The corresponding *Fourier series* is

$$\sum \hat{f}(n)\, e^{in\theta} = \sum \hat{f}(n) z^n \tag{13}$$

or

$$\sum_{n=-\infty}^{\infty} (f, e_n) e_n\,.$$

Notice that

$$(e_n, e_m) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \delta_{n,m}\,.$$

Hence $\{e_n \mid n \in \mathbb{Z}\}$ is a orthonormal subset of $L^2(\mathbb{T})$. For two set A and B such that $A \subset B$ we denote by χ_A the characteristic function of A, i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.We call $f : \mathbb{T} \to \mathbb{C}$ a for a *step function* if there exists disjoint intervals $I_j \subset [0, 2\pi]$ and complex numbers c_j such that

$$f(e^{i\theta}) = \sum c_j \chi_{I_j}(\theta)\,.$$

Definition 15 *Let $g \in C(\mathbb{T})$ and $r \in \mathbb{N}_0$. Then g is called r-times continuously differentiable if and only if the continuous periodic function $g \circ \kappa$ is r-times continuously differentiable.*

We denote the space of r-times continuously differentiable functions by $C^r(\mathbb{T})$. The space of smooth functions $C^\infty(\mathbb{T})$ is $\cap_r C^r(\mathbb{T})$, i.e. the space of functions that are r-times continuously differentiable for all r. Define the first order differential operator D by

$$[Df](e^{ix}) = \frac{df \circ \kappa}{dx}(x) = \frac{d}{dx} f(e^{ix})\,.$$

Notice that $Dz^n = inz^n$ for all n. Recall $\lambda(w)f(z) = f(w^{-1}z)$.

Lemma 16 *Let $f, g \in L^1(\mathbb{T})$. Let $p = \sum_{n=-N}^{M} a_n e_n$ be a trigonometric polynomial. Then the following holds:*

1. *$f * p = \sum_{n=-N}^{M} a_n \hat{f}(n) e_n$.*
2. *$\left|\hat{f}(n)\right| \leq |f|_1$ for all $n \in \mathbb{Z}$.*
3. *$\widehat{\lambda(w) f} = w^{-n} \widehat{f}$.*
4. *Denote by $f^\vee$ the function $f^\vee(z) = f(z^{-1})$. Then $\widehat{f^\vee(n)} = \hat{f}(-n)$.*
5. *$\lim_{n\to\infty} \hat{f}(n) = 0$ (Lebesgue Lemma).*
6. *$\widehat{f * g} = \widehat{f}\widehat{g}$.*
7. *Assume that $f \in C^r(\mathbb{T})$ then $\widehat{D^r} f(n) = (in)^r \hat{f}(n)$.*

Proof. 1) By definition we get:

$$\begin{aligned} f * p(z) &= \sum a_n \int f(w)(w^{-1}z)^n \, d\mu(w) \\ &= \sum a_n z^n \int f(w) w^{-n} \, d\mu(w) \\ &= \sum a_m \hat{f}(n) e_n(z)\,. \end{aligned}$$

2) This follows immediately from (1) and Lemma 14, 2, because $\left|\hat{f}(n)\right| = |f * e_n(1)| \leq |f * e_n|_\infty = |f|_1$ as $|e_n|_1 = 1$.

3) By definition we have

$$\begin{aligned} \widehat{\lambda(w) f}(n) &= \int [\lambda(w) f](z) z^{-n} \, d\mu(z) \\ &= \int f(w^{-1}z) z^{-n} \, d\mu(z) \\ &= \int f(z) [wz]^{-n} \, d\mu(z) \\ &= w^{-n} \int f(z) z^{-n} \, d\mu(z) \\ &= w^{-n} \hat{f}(n)\,. \end{aligned}$$

4) The follows in the same manner as (2):

$$\begin{aligned}\widehat{f^\vee(n)} &= \int f(z^{-1})z^{-n}\,d\mu(z)\\ &= \int f(z)z^{n}\,d\mu(z)\\ &= \hat f(-n)\,.\end{aligned}$$

5) Assume first that $F = f\circ\kappa = C\chi_{[a,b]}$ with $0 \le a < b \le 2\pi$. Then

$$\begin{aligned}\hat f(n) &= \frac{C}{2\pi}\int_a^b e^{-in\theta}\,d\theta\\ &= \frac{C}{2\pi in}\left[e^{-inb} - e^{-ina}\right] \to 0 \qquad n\to\infty\,.\end{aligned}$$

It follows that the claim holds for f a step function.

Let now $f \in L^1(\mathbb{T})$ be arbitrary. Let $\epsilon > 0$. Choose a step function h such that $|f-h|_1 < \epsilon/2$. Then choose $N \in \mathbb{N}$ such that $\left|\widehat{h}(n)\right| < \epsilon/2$ for all $n \ge N$. Then for all $n \ge N$ we have:

$$\begin{aligned}\left|\widehat{f}(n)\right| &\le \left|\widehat{f}(n) - \widehat{h}(n)\right| + \left|\widehat{h}(n)\right|\\ &\le |f-h|_1 + \left|\widehat{h}(n)\right|\\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

6) Using the invariance of the measure μ and Fubini we get:

$$\begin{aligned}\widehat{f*g}(n) &= \int f*g(z)\,z^{-n}\,d\mu(z)\\ &= \int\left[\int f(w)g(w^{-1}z)\,d\mu(w)\right]\,z^{-n}\,d\mu(z)\\ &= \int\int f(w)g(w^{-1}z)z^{-n}\,d\mu(z)d\mu(w)\\ &= \int f(w)\int g(z)\,[wz]^{-n}\,d\mu(z)d\mu(w)\\ &= \int f(w)w^{-n}\int g(z)z^{-n}\,d\mu(z)d\mu(w)\\ &= \widehat{f}(n)\widehat{g}(n)\,.\end{aligned}$$

7) For $r = 1$ this follows immediately by integration by parts:

$$\widehat{Df}(n) = \frac{1}{2\pi}\int_0^{2\pi}(f\circ\kappa)'(\theta)\,e^{-in\theta}\,d\theta = \frac{-1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{d}{d\theta}e^{-in\theta}\,d\theta = in\hat f(n)$$

■

Lemma 17 *Let $g \in C(\mathbb{T})$ and $\epsilon > 0$. Then there exists a trigonometric polynomial p such that*

$$|g - p|_\infty < \epsilon\,.$$

Proof. Identify $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z = x + iy$. As $\mathbb{T}$ is a compact subset of $\mathbb{R}^2$ it follows by the Stone Weierstrass Theorem that there is a polynomial $q(x,y) = \sum a_{n,m} x^n y^m$ such that $|g - q|_\infty < \epsilon$. If $x + iy = e^{i\theta} \in \mathbb{T}$ we get

$$x = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad y = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,.$$

Define $p(e^{i\theta}) := q(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i})$. Then p is a trigonometric polynomial and $|g - p|_\infty < \epsilon$ ■

Theorem 18 (Plancherel Theorem) *The set of functions $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$. In particular we have for arbitrary $f \in L^2(\mathbb{T})$.*

1. $f = \sum_{n=-\infty}^{\infty} (f, e_n) e_n$ *in* $L^2(\mathbb{T})$.
2. $|f|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$

Proof. We have already noticed that the set $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal. Let $\epsilon > 0$ and let $f \in L^2(\mathbb{T})$. Choose $g \in C(\mathbb{T})$ such that $|f - g|_2 < \epsilon/2$. By Lemma 17 there is a trigonometric polynomial p such that

$$|g - p|_\infty < \epsilon/2\,.$$

Thus

$$\begin{aligned} |g - p|_2^2 &= \int |g(z) - p(z)|^2 \, d\mu(z) \\ &\le \int |g - p|_\infty^2 \, d\mu(z) \\ &< (\epsilon/2)^2\,. \end{aligned}$$

Or $|g - p|_2 < \epsilon/2$. It follows that

$$|f - p|_2 \le |f - g|_2 + |g - p|_2 < \epsilon\,.$$

From this the theorem follows. ■

Let ℓ^2 be the set of sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that

$$|\{a_n\}|_{\ell^2} := \sqrt{\sum_{n=-\infty}^{\infty} |a_n|^2} < \infty.$$

Lemma 19 *Let $\{a_n\}, \{b_n\} \in \ell^2$. Then $\sum a_n \overline{b_n}$ converges absolutely.*

Proof. Let $M, N \in \mathbb{Z}$, $N < M$. Then

$$\begin{aligned}\left|\sum_{n=N}^{M} a_n \overline{b_n}\right| &\leq \sum_{n=N}^{M} |a_n|\,|b_n| \\ &\leq \sqrt{\sum_{n=N}^{M} |a_n|^2}\sqrt{\sum_{n=N}^{M} |b_n|^2} \\ &\leq |\{a_n\}|\,|\{b_n\}| \\ &< \infty\end{aligned}$$

This implies the claim. ■

Define an inner product $(\cdot,\cdot)$ on ℓ^2 by

$$(\{a_n\}, \{b_n\}_n) := \sum a_n \overline{b_n}\,.$$

For $\{a_n\}, \{b_n\} \in \ell^2$ and $\lambda \in \mathbb{R}$ define

$$\{a_n\} + \{b_n\} := \{a_n + b_n\}$$

and

$$\lambda\{a_n\} := \{\lambda a_n\}$$

Lemma 20 *The space ℓ^2 with the inner product $(\cdot,\cdot)$ is a Hilbert space.*

Proof. 1) ℓ^2 is a normed vector space: Let $\{a_n\}, \{b_n\} \in \ell^2$. Then $|a_n + b_n|^2 \leq |a_n|^2 + 2|a_n|\,|b_n| + |b_n|^2$. Hence

$$\begin{aligned}\sum |a_n + b_n|^2 &\leq |\{a_n\}|^2 + 2\sum |a_n|\,|b_n| + |\{b_n\}|^2 \\ &\leq |\{a_n\}|^2 + 2|\{a_n\}|\,|\{b_n\}| + |\{b_n\}|^2 \\ &= (|\{a_n\}| + |\{b_n\}|)^2 \\ &< \infty\end{aligned}$$

It follows that $\{a_n\} + \{b_n\} \in \ell^2$ and $|\{a_n\} + \{b_n\}| \leq |\{a_n\}| + |\{b_n\}|$.

It is clear that $\lambda\{a_n\} \in \ell^2$ and $|\lambda\{a_n\}| = |\lambda|\,|\{a_n\}|$. Assume that $|\{a_n\}| = 0$. Then $a_n = 0$ for all n so $\{a_n\} = 0$.

2) ℓ^2 is complete: Let $\{A_n\}$ be a Cauchy sequence in ℓ^2. Write $A_n = \{a_{j,n}\}_{j\in\mathbb{Z}}$. Then for each $\epsilon > 0$ there is a $N = N(\epsilon) \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$|A_n - A_m|^2 = \sum_{j=-\infty}^{\infty} |a_{j,n} - a_{j,m}|^2 < \epsilon\,.$$

This implies that each of the sequences $\{a_{j,n}\}_n$ is a Cauchy sequence. Let

$$b_j := \lim_{n\to\infty} a_{j,n}$$

and $B = \{b_j\}$. We have to show that $B \in \ell^2$ and $\lim A_n = B$. We notice that $|a_{j,n} - a_{j,m}| \geq 0$. Hence we can interchange the limit and summation to get

$$\sum_{j=-\infty}^{\infty} \left| b_j - a_{j,m} \right| \leq \epsilon$$

if $m \geq N(\epsilon)$. It follows that $\{b_j - a_{j,m}\} \in \ell^2$ and hence $B = A_m + \{b_j - a_{j,m}\} \in \ell^2$. The last inequality also says that $\lim A_n = B$. ∎

We can now reformulate the Plancherel Theorem:

Theorem 21 *The Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2$ is an isomorphism of Hilbert spaces.*

Proof. We have by the Plancherel Theorem that

$$\sum \left| \hat{f}(n) \right|^2 = |f|^2 < \infty \,.$$

Hence $\mathcal{F}f \in \ell^2$ and $\mathcal{F}$ is an isometry into ℓ^2. Let $A = \{a_n\} \in \ell^2$. Define a sequence

$$f_n(z) = \sum_{j=-n}^{n} a_n z^n \in L^2(\mathbb{T}).$$

Thus $f_n = \sum_{j=-n}^{n} a_n e_n$. For $m \geq n$ we now get

$$|f_n - f_m|^2 = \sum_{n<|j|\leq m} |a_j|^2 \,.$$

But $\sum |a_j|^2 < \infty$. Hence if $\epsilon > 0$ then we can find an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sum_{n\leq|j|} |a_j|^2 < \epsilon.$$

But this obviously implies that $\{f_n\}$ is a Cauchy sequence in $L^2(\mathbb{T})$. Then $f = \sum a_j e_j$ in L^2. Furthermore we get

$$\begin{aligned} \hat{f}(n) &= (f, e_n) \\ &= \sum a_j (e_j, e_n) \\ &= a_n \,. \end{aligned}$$

Hence $\mathcal{F}f = A$ and $\mathcal{F}$ is surjective. ∎

Let $C^\infty(\mathbb{T})$ be the space of smooth function on $\mathbb{T}$, i.e., $C^\infty(\mathbb{T}) = \cap_{r\in\mathbb{N}} C^r(\mathbb{T})$. Define a topology on $C^\infty(\mathbb{T})$ by the seminorms

$$\sigma_n(f) := |D^n f|_\infty \,.$$

We leave it as an exercise to show that $C^\infty(\mathbb{T})$ with this topology is a locally convex complete topological vector space. We denote $C^\infty(\mathbb{T})$ with this topology by $\mathcal{D}(\mathbb{T})$. To find the image of $\mathcal{D}(\mathbb{T})$ under the Fourier transform let PW($\mathbb{T}$) be the space of sequences $a = \{a_n\}_{n\in\mathbb{Z}}$ such that for each k

$$\rho_k(a) := \sup_n (1+|n|)^k |a_n| < \infty .$$

Then PW($\mathbb{T}$) becomes a locally convex complete topological vector space. Notice that

$$\frac{|n|^k}{(1+|n|)^k} \to 1 , \quad n \to \infty .$$

Hence there exists positive constants C_k, D_k such that for all $n \neq 0$.

$$C_k(1+|n|)^k |a_n|_k \le |n|^k |a_n| \le D_k (1+|n|)^k |a_k| .$$

The topology on PW($\mathbb{T}$) can therefore also be defined by the seminorms ρ'_k where $\rho'_0 = \rho_0$ and

$$\rho_k(a) := \sup |n|^k |a_n| , \quad k \neq 0 .$$

This topology can also be defined by using the seminorms

$$\rho''_k(a) = \sum |n|^k |a_n| .$$

Theorem 22 (Paley-Wiener Theorem) *The Fourier transformation is a topological isomorphism $\mathcal{F} : \mathcal{D}(\mathbb{T}) \to \ell_\infty$. Furthermore we have*

$$D^k f(z) = \sum (in)^k \hat{f}(n) z^n$$

for all k and all $f \in \mathcal{D}(\mathbb{T})$.

Proof. We have $\mathcal{F}(D^k f)(n) = (in)^k \mathcal{F}(f)(n)$. Hence $\mathcal{F}(f) \in \ell_\infty$ and $\rho'_k(\mathcal{F}f) \le \sigma_k(f)$. Hence $\mathcal{F}$ is continuous. Assume that $\mathcal{F}f = 0$. Then $\hat{f}(n) = 0$ for all n. By the Plancherel theorem it follows that $f = 0$ a.e. But f is continuous, so $f = 0$.

Let now $a = \{a_n\} \in \ell_\infty$. As $|a_n z^n| \le \rho_2(a) n^{-2}$ for all $n \neq 0$ it follows that the series $\sum a_n z^n$ converges uniformly and defines a continuous function. Define

$$f(z) := \sum_{n\in\mathbb{Z}} a_n z^n .$$

Then f is continuous. Let $F(\theta) = f(e^{i\theta}) = \sum a_n e^{in\theta}$. As $\left|n a_n e^{in\theta}\right| \le \rho_3(a) n^{-2}$ it follows that F is continuously differentiable and

$$F'(\theta) = \sum n a_n e^{in\theta} .$$

Repeating the argument shows that F is smooth and

$$F^{(k)}(\theta) = i^k \sum n^k a_n e^{in\theta} .$$

Thus f is smooth and

$$D^k f(z) = i^k \sum n^k a_n z^n . \tag{14}$$

If $n \neq 0$ then $|a_n z^{n-m}| \leq C_2 n^{-2}$. It follows that we can interchange the integration and summation to get

$$\begin{aligned} \hat{f}(m) &= \int \sum_n a_n z^{n-m} \, d\mu \\ &= \sum_n a_n \int z^{n-m} \, d\mu \\ &= a_m . \end{aligned}$$

It follows that the Fourier transform is surjective. By the Open Mapping Theorem it follows that $\mathcal{F}$ is a topological isomorphism. The last claim follows form (14). ∎

Exercise

1. Let f be the function on $\mathbb{T}$ such that $f \circ \kappa|_{[-\pi,\pi)}(t) = t$.
   (a) Evaluate $\hat{f}(n)$.
   (b) Evaluate $\lim_{t\to\pi} \sum \hat{f}(n)e^{int}$.
2. Show that $f \in L^2([0, 2\pi], dx)$ can be written in the form
   $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) .$$
   Find an expression for a_n and b_n.
3. Let g be the function on the torus corresponding to $f(\theta) = |\theta|$, $\theta \in [-\pi, \pi)$.
   (a) Find $\hat{g}(n)$.
   (b) Show that the Fourier series converges uniformly.
4. Use the Fourier transform to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
5. Let f be the periodic function corresponding to $\chi_{[-\pi,0)} - \chi_{[0,\pi)}$. Evaluate $\hat{f}(n)$.

6. A function f on the torus is even if $f(z) = f(z^{-1})$ and odd if $f(z) = -f(z^{-1})$. Suppose that $f \in C^2(\mathbb{T})$. Show the following:

   (a) If f is even then $f(e^{i\theta}) = \sum \hat{f}(n)\cos(n\theta)$;

   (b) If f is odd then $f(e^{i\theta}) = \sum \hat{f}(n)\sin(n\theta)$.

7. Let $g \in C^1(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Then $f * g \in C^1(\mathbb{T})$ and $D(f * g) = f * Dg$.

8. Let $L > 0$. Let f be a L-periodic function such that $\int_0^L |f(t)|^2\, dt < \infty$. Show that there are constants $a_n \in \mathbb{C}$ such that in $L^2([0, L))$ we have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t/L}.$$

   Find an expression for a_n.

9. Show that the seminorms ρ_k, ρ_k', and ρ_k'' all defines the same seminorm on $\mathrm{PW}(\mathbb{T})$.

10. Let $\mathbb{T}^k := \{\mathbf{z} = (z_1, \ldots, z_k) \mid z_j \in \mathbb{T}\}$ with the product topology. For $\mathbf{z}, \mathbf{w} \in \mathbb{T}^k$ let $\mathbf{zw} = (z_1 w_1, \ldots, z_k w_k)$. For $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ let $e_{\mathbf{n}}(\mathbf{z}) = z_1^{n_1} \cdots z_k^{n_k}$. Show the following:

    (a) $\mathbb{T}^k$ is a topological group, i.e., the map $\mathbb{T}^k \times \mathbb{T}^k \ni (\mathbf{z}, \mathbf{w}) \mapsto \mathbf{z}^{-1}\mathbf{w} \in \mathbb{T}^k$ is continuous.

    (b) $e_{\mathbf{n}} : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism.

    (c) IF $\chi : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism, then there exists a $\mathbf{n} \in \mathbb{Z}^k$ such that $\chi = e_{\ltimes}$.

11. **(Fourier series of $\mathbb{T}^k$)** Let $\mu_k = \mu \times \cdots \times \mu$ be the product measure on $\mathbb{T}^k$. For $f \in L^1(\mathbb{T}^k)$ define $\hat{f} : \mathbb{Z}^k \to \mathbb{C}$ by

$$\hat{f}(\mathbf{n}) := \int_{\mathbb{T}^k} f(\mathbf{z})\, e_{-\mathbf{n}}(\mathbf{z})\, d\mu_k.$$

    Show the following:

    (a) If $f \in L^2(\mathbb{T}^k)$ then $|f|_2 = \sqrt{\sum_{\mathbf{n} \in \mathbb{Z}^k} \left|\hat{f}(\mathbf{n})\right|^2}$.

    (b) If $f \in L^2(\mathbb{T}^k)$ then $f = \sum_{\mathbf{n} \in \mathbb{Z}^k} \hat{f}(\mathbf{n})\, e_{\mathbf{n}}$ in $L^2(\mathbb{T}^k)$.

12. Show that there is no differentiable function f on $\mathbb{T}$ such that $(Df)(e^{i\theta}) = 1$.

13. **(Differential equations on the torus)** Let $p(z) = \sum_{n=0}^{k} a_n z^n$ be a polynomial. Define $p(D) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ by

$$p(D)f = \sum_{n=0}^{k} a_n D^n f\,.$$

Show that if $g \in C^\infty(\mathbb{T})$ is such that $\hat{g}(n) = 0$ if $p(in) = 0$ then the differential equation $p(D)f = g$ has a solution.

14. **(The Hilbert transform on the torus)** Let $f \in L^3(\mathbb{T})$ show that there exists a unique $g \in L^2(\mathbb{T})$ such that

$$\hat{g}(n) = -i\,\mathrm{sgn}(n)\,\hat{f}(n)\,, \qquad \forall n \in \mathbb{Z}\,.$$

Define $Hf = g$. Then $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is linear. Prove the following:

(a) $|Hf|_2 = |f|_2$ if and only if $\int f\,d\mu = 0$.

(b) If $f \in C^\infty(\mathbb{T})$ then $Hf \in C^\infty(\mathbb{T})$.

(c) If $f \in C^\infty(\mathbb{T})$ then $Hf(0) = -2\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(e^{i\theta})\,\sin(n\theta)\,d\theta\,.$

5 Convergence of the Fourier Series

We saw in the last section that the Fourier transform converges in L^2 if f is an L^2-function. Also if f is smooth then the Fourier series converges uniformly to f. The proof actually shows that $f(z) = \sum \hat{f}(n)z^n$ uniformly if f is in $C^2(\mathbb{T})$. But in general it does not hold that the Fourier series $\sum a_n z^n$ converges to $f(z)$. In this section we will deal with the question how to recover the function from its Fourier series. We start by stating two negative results:

Lemma 23 *There exists a dense subset $D \subset L^1(\mathbb{T})$ such that the Fourier series does not converges in $L^1(\mathbb{T})$ for $f \in D$.*

Define the partial Fourier sum $s_N(f)$ to be

$$s_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e_n\,.$$

Lemma 24 *Let $\{z_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{T}$. Then there exists a function $f \in C(\mathbb{T})$ such that $\lim_{N\to\infty} |s_N(f)(z_k)| = \infty$ for all $k \in \mathbb{N}_0$.*

Let $D_N(z) = \sum_{n=-N}^{N} z^n$. The function D_N is called *Dirichlet's kernel.*

Lemma 25 *Let $N \in \mathbb{N}$. Then the following holds:*

1. $D_N(z) = \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}}$ if $z \neq 1$ and $D_N(1) = 2N + 1$..

2. $D_N(e^{i\theta}) = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)}$ if $\theta \neq 0$ and $D_N(1) = 2N+1$

3. $D_N(z) = D_N(z^{-1})$.

4. $\int_{\mathbb{T}} D_N(z)\, d\mu(z) = 1$.

5. $s_N(f)(z) = f * D_N(z) = D_N * f(z)$.

Proof. 1) We have

$$\begin{aligned} D_N(z) &= z^{-N} \sum_{n=0}^{2N} z^n \\ &= z^{-N} \frac{1 - z^{2N+1}}{1-z} \\ &= \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}} . \end{aligned}$$

Also $D_N(1) = \sum_{n=-N}^{N} 1 = 2N+1$.

2) This follows immediately by using that $\sin(\psi) = (e^{i\psi} - e^{-i\psi})/2i$.

3) This follows from (1).

4) Using $\int_{\mathbb{T}} z^n\, d\mu = 0$ for $n \neq 0$ we get $\int_{\mathbb{T}} D_N(z)\, d\mu(z) = \sum_{n=-N}^{N} \int z^n\, d\mu(z) = \int 1\, d\mu(z) = 1$.

5) This follows by Lemma 16, part 1. ∎

Here are few example, $N = 1$, $N = 5$, $N = 10$ and $N = 100$:

Picture

What we see is that the function D_N get more and more localized around $z = 1$ and then oscillates. The problem with D_N is this oscillation and the following fact which we will not proof:

Lemma 26 *$\int_{\mathbb{T}} |D_N(z)|\, d\mu(z) = \frac{4}{\pi^2} \log(N) + O(1)$ as $N \to \infty$.*

Definition 27 *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Then $\{a_n\}$ is **Cesáro summable to L** if the average $1/(N+1)\sum_{k=0}^{N} s_k$ of the partial sums $\sum_{n=0}^{k} a_k$ converges to L. The sequence $\{a_n\}_n$ is **Abel summable to L** if $\sum_{n=0}^{\infty} a_k r^k$ exists for all $0 \leq r < 1$ and $\lim_{r\to 1} \sum a_n r^n = 1$.*

To recover $f \in L^1(\mathbb{T})$ from its Fourier transform one uses the average, i.e., Cesáro summability. Define

$$\sigma_N(f)(z) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(f)(z) .$$

Let

$$\Sigma_N(z) = \sum_{n=0}^{N} D_N(z)$$

so that

$$\sigma_N(f) = f * \Sigma_N\,.$$

Then the following holds:

Lemma 28 *1.* $\Sigma_N(e^{i\theta}) = \frac{1}{N+1}\left[\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right]^2$ *if* $e^{i\theta} \neq 1$ *and* $\Sigma_N(1) = N+1$*. In particular we have* $\Sigma_N \geq 0$*.*

2. $\Sigma(z) = \Sigma(z^{-1})$*.*

3. $\int_{\mathbb{T}} \Sigma_N \, d\mu = 1$*.*

Proof. 1) Using that $D_N(z) = (z^{N+1} - z^{-N})/(z-1)$ it follows that

$$\begin{aligned}(z-1)\Sigma_N(z) &= \sum_{n=0}^{N}\left(z^{n+1} - z^{-n}\right)\\ &= z\frac{1-z^{N+1}}{1-z} - \frac{1-1/z^{N+1}}{1-1/z}\\ &= \frac{2z}{z-1}\left[\frac{z^{N+1}+z^{-(N+1)}}{2} - 1\right]\end{aligned}$$

Let $z = e^{i\theta}$ and use that $\cos(2\psi) - 1 = -[\sin(\psi)]^2$, and

$$\begin{aligned}\frac{2e^{i\theta}}{(e^{i\theta}-1)^2} &= \frac{2}{e^{i\theta}+e^{-i\theta}-2}\\ &= \frac{1}{\cos(\theta)-1}\\ &= \frac{-1}{[\sin(\theta/2)]^2}\end{aligned}$$

then it follows that $\Sigma_N(e^{i\theta}) = \frac{1}{N+1}\left[\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right]^2$. The claim for $z = 1$ follows either by continuity or by

$$\frac{1}{N+1}\sum_{n=0}^{N} 2n+1 = N+1\,.$$

2) This follows by the fact that sin is odd.

3) By lemma 25 we have

$$\begin{aligned}\int \Sigma_N(z)\,d\mu(z) &= \frac{1}{N+1}\sum_{n=0}^{N}\int D_n(z)\,d\mu(z)\\ &= \frac{1}{N+1}\sum_{n=0}^{N} 1\\ &= 1\end{aligned}$$

■

We see that Σ_N is concentrated more and more around $z = 1$ as $N \to \infty$.

Lemma 29 *Let $0 < \delta < \pi$. Then there exists a constant $C = C(\delta)$ independent of N such that $\left|\Sigma_N(e^{i\theta})\right| \leq \frac{C}{N+1}$ for $\delta \leq |\theta| \leq \pi$.*

Proof. Choose $C > 0$ such that $|\sin(\theta/2)| \geq 1/\sqrt{C}$ for $\delta < |\theta| \leq \pi$. As $|\sin(N+1)\theta/2| \leq 1$ it follows that

$$\left|\Sigma_N(e^{i\theta})\right| \leq \frac{C}{N+1}\,.$$

■

Theorem 30 (Fejér) *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{T})$. Then $f * \Sigma_N$ is a trigonometric polynomial and*

$$\lim_{N\to\infty} |f * \Sigma_N - f|_p = 0\,.$$

If $f \in C(\mathbb{T})$ then

$$\lim_{N\to\infty} |f * \Sigma_N - f|_\infty = 0\,.$$

In particular the sequence $a_0 := \hat{f}(0)$, $a_n := \hat{f}(n)z^n + \hat{f}(-n)z^{-n}$ is uniformly Cesáro summable to $f(z)$.

Proof. Let $\epsilon > 0$ be arbitrary. Let q be conjugate to p, i.e. $1/p + 1/q = 1$. Let $h \in L^q(\mathbb{T})$ with $|h|_q = 1$. By the Hölder-inequality and the fact that $\int \Sigma_N\,d\mu = 1$ we get:

$$|\Sigma_N * f(z) - f(z)| = \left|\int_{\mathbb{T}} \Sigma_N(w)(f(w^{-1}z) - f(z))\,d\mu(w)\right|\,.$$

Hence

$$\begin{aligned}
|(\Sigma_N * f - f)h|_1 &= \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \Sigma_N(w)(f(w^{-1}z) - f(z))h(w)\, d\mu(w) \right| d\mu(z) \\
&\leq \int \int \Sigma_N(w) \left| f(w^{-1}z) - f(z) \right| |h(w)|\, d\mu(w) d\mu(z) \\
&= \int \Sigma_N(w) \int \left| f(w^{-1}z) - f(z) \right| |h(z)|\, d\mu(z) d\mu(w) \qquad \text{(Tonelli)} \\
&\leq \int \Sigma_N(w) \left| \lambda(w)f - f \right|_p d\mu(w) \qquad \text{(Hölder)} \\
&= \int_{|w-1|<\delta} \Sigma_N(w) \left| \lambda(w)f - f \right|_p d\mu(w) \\
&\quad + \int_{|w-1|\geq\delta} \Sigma_N(w) \left| \lambda(w)f - f \right|_p d\mu(w) \\
&= A + B\,.
\end{aligned}$$

To estimate the first integral we choose $\delta > 0$ so that $|\lambda(w)f - f|_p < \epsilon/2$ (see problem 1 in section 3). For the second one we choose N_0 so that

$$\int_{|w-1|\geq\delta} \Sigma_N(w)\, d\mu(w) < \frac{\epsilon}{2(|f|_p + 1)} \qquad \text{for } N \geq N_0\,.$$

Then $A + B < \epsilon$ for $N \geq N_0$. Hence $|(\Sigma_N * f - f)\, h|_1 < \epsilon$ for all $h \in L^q(\mathbb{T})$, $|h|_q = 1$ and all $N \geq N_)$. But $L^q(\mathbb{T}) = L^p(\mathbb{T})^*$ as $p < \infty$. Therefore

$$|\Sigma_N * f - f|_p = \sup_{h \in L^q(\mathbb{T}),\ |h|_q = 1} |(\Sigma_N * f - f)\, h|_1 < \epsilon\,.$$

If f is continuous then (using that $\int \Sigma_N\, d\mu = 1$)

$$\begin{aligned}
|\Sigma_N * f(z) - f(z)| &\leq \int_{\mathbb{T}} \Sigma_N(w) \left| f(w^{-1}z) - f(z) \right| d\mu(w) \\
&= \int_{|w-1|<\delta} \Sigma_N(w) \left| f(w^{-1}z) - f(z) \right| d\mu \\
&\quad + \int_{|w-1|\geq 1} \Sigma_N(w) \left| f(w^{-1}z) - f(z) \right| d\mu \\
&= A + B\,.
\end{aligned}$$

Now we use that f is uniformly continuous on $\mathbb{T}$ to make A less than $\epsilon/2$, and we use that f is bounded on $\mathbb{T}$ to make B less than $\epsilon/2$ for $N \geq N_0$. Then it follows that $|\Sigma_N * f - f|_\infty < \epsilon$ for $N \geq N_0$. ∎

Let $P(r,\theta) = (1-r^2)/(1-2r\cos(\theta)+r^2)$ be the Poisson kernel as in the introduction. Then for $z = re^{i\theta}$ we have

$$\begin{aligned} f *_\theta P(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\psi})\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,d\mu(\psi) \\ &= \hat{f}(0) + \sum_{n=1}^{\infty} r^n(\hat{f}(n)z^n + \hat{f}(-n)z^{-n})\,. \end{aligned}$$

Lemma 31 *Let $0 < \delta < \pi$. Then there exists a constant $C > 0$ and $0 < r_0 < 1$ such that for all $\delta \le |\theta| \le \pi$ and all $r_0 \le r \le 1$ we have*

$$P(r,\theta) < C(1-r^2)\,.$$

Proof. As $\cos(\theta)$ is decreasing for $\theta \in [0,\pi]$ it follows that $1-2r\cos(\theta)+r^2 \ge 1-2r\cos(\delta)+r^2$. Let $g(r) = 1-2r\cos(\delta)+r^2$. Then $g'(r) = 2(r-\cos(\delta))$. Hence g is decreasing for $0 \le r < \cos(\delta) < 1$ and increasing for $\cos(\delta) < r \le 1$. It follows that

$$1-2r\cos(\theta)+r^2 \ge 1-2\cos^2(\delta)+\cos^2(\delta) = 1-\cos^2(\delta)\,.$$

The claim follows now by letting $C := 1/(1-\cos^2(\delta))$. ■

Repeating the arguments in the proof of Fejér's Theorem we get:

Theorem 32 (Poisson Theorem) *Let $P_r = P(r,\cdot)$. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T})$. Then*

$$\lim_{r\to 1} |f * P_r - f|_p = 0\,.$$

If $f \in C(\mathbb{T})$ then

$$\lim_{r\to 1} |f * P_r - f|_\infty = 0\,.$$

In particular the sequence $a_0 := \hat{f}(0)$, $a_n := \hat{f}(n)z^n + \hat{f}(-n)z^{-n}$ is uniformly Abel summable to $f(z)$.

Exercise

1. Show that the sequence $a_n = (-1)^n$ is Abel and Cesáro summable to $1/2$.
2. Show that if $\sum a_n$ converges to L then $\{a_n\}$ is Abel and Cesáro summable to L.
3. Let $\delta_0 \ge \delta_1 \ge \cdots \ge 0$ be a decreasing sequence with $\lim \delta_n = 0$. Define $a_0 := \delta_0$ and $a_n := \delta_n - \delta_{n-1}$ for $n \ge 1$. Let $g(z) := \sum a_n z^n$ .Show that g is continuous, $g * D_N$ converges uniformly to g, and $|g * D_N - g|_\infty \ge \delta_N$.

6 Applications

In this section we discuss some applications of the Fourier transform. The first two are examples how to use the Fourier transform to solve differential equations, and the last one is an example of its application in geometry.

6.1 The Wave Equation

The genear form of the wave equation is

$$\partial_x^2 u(x,t) = a^2 \partial_t^2 u(x,t)\,, \quad u(x,0) = f(x)\,,\, u_t(x.0) = g(x)\,. \tag{15}$$

Here $x \in [0, L]$, f, g are functions on $[0, L]$ and $a > 0$ is a constant. For simplicity we will assume that $L = \pi$. In the case where $u(0, t) = u(\pi, t) = 0$ for all t, and f and g are real valued, this equation describs the vibration of homogeneous string, fastened at both ends and starting at position $u(x, 0) = f(x)$ with initial velocity $u_t(x, t) = g(x)$. Then constant $a = T/\rho$ is given by the tension T and the linear density ρ.

As we are motivated by the vibration of a string let us assume that $f, g \in C^2((0, \pi))$, both real valued, and $f(0) = f(\pi) = g(0) = g(\pi) = 0$. Let us look for a smooth solution $x \mapsto u(x, t)$. First we extend $u(\cdot, t)$, f and g to an odd function on $[-\pi, \pi]$ by $f(-x) = -f(x)$, $g(-x) = -g(x)$, and $u(-x, t) = -u(x, t)$. Then f and g are continuous on $[-\pi, \pi]$ and smooth on $(-\pi, \pi)$. Take the Fourier transform in the x-variable and denoting it by $\hat{u}(n, t)$ we get, using that all functions are real valued and odd:

$$\begin{aligned}
\hat{u}(n,t) &= \overline{\hat{u}(-n,t)} = \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(\theta,t)\sin(n\theta)\,d\theta\,; \\
\hat{f}(n) &= \overline{\hat{f}(-n)} = \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta\,; \\
\hat{g}(n) &= \overline{\hat{g}(-n)} = \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta\,.
\end{aligned}$$

Next we notice that differentiation in the t-variable commutes with taking the Fourier transform in the x-variable. Thus for all $n \in \mathbb{Z}$:

$$-n^2\,\hat{u}(n,t) = a^2\,\hat{u}_{tt}(n,t)\,, \quad \hat{u}(n,0) = \hat{f}(n),\, \hat{u}_t(n,0) = \hat{g}(n)\,.$$

This is an ordinary second order initial value problem for the function $t \mapsto \hat{u}(n, t)$ with unique solution

$$\hat{u}(n,t) = \hat{f}(n)\cos\left(\frac{n}{a}t\right) + a\hat{g}(n)\frac{\sin\left(\frac{n}{a}t\right)}{n}\,.$$

Summing up we get

$$u(x,t) = 2i\sum_{n=1}^{\infty}\left(\hat{f}(n)\cos\left(\frac{n}{a}t\right) + a\hat{g}(n)\frac{\sin\left(\frac{n}{a}t\right)}{n}\right)\sin(nx)\,\,. \tag{16}$$

This solution also fullfills $u(0,t) = u(\pi,t) = 0$ for all $t \in \mathbb{R}$. We notice that this solution is periodic in t with period $2\pi a$.

If $L \neq \pi$ let us replace $x \mapsto u(x,t)$ by $v(x,t) := u\left(\frac{L}{\pi}x\right)$ and similarily for f and g. Then the new Cauchy problem is:

$$\partial_x^2 v(x,t) = a_1^2 u_{tt}(x,t)\,, \quad v(x,0) = f_1(x),\ v_t(x,0) = g_1(x)$$

with $f_1(x) = f(Lx/\pi)$, $g_1(x) = g(Lx/\pi)$ and $a_1 = \pi a/L$. Using the above result for v we get:

$$u(x,t) = 2i\sum_{n=1}^{\infty}\left(\hat{f}(n)\cos\left(\frac{Ln}{a\pi}t\right) + \frac{\pi a}{L}\hat{g}(n)\frac{\sin\left(\frac{Ln}{a\pi}t\right)}{n}\right)\sin\left(\frac{\pi}{L}nx\right)\,.$$

6.2 The Heat Equation

In this section we discuss the heat equation

$$a^2\partial_x^2 u(x,t) = u_t(x,t)\,, \quad u(x,0) = f(x)\,, \, 0 \leq x \leq L\,.$$

In the case where f is real valued this corresponds to the differential equation describing the heat flow in a homogeneous cylindrical rod of length L, whose lateral surface is insulated from the surrounding medium, initial temperature at a point $x \in [0,L]$ given by $f(x)$. The constant a is in this case given by $a^2 = K/c\rho$ where K is the *termal conductivity* of the material from which the rod is made, c is the *heat capacity* and ρ is the *density*. We will only consider solutions that are fixed by the same constant at the endpoints $x = 0$ and $x = L$. we can then assume that $u(0,t) = u(L,t) = 0$ for all t. As for the wave equation we will assume that $L = \pi$ and then extend all functions depending on the variable x as odd functions on $[-\pi,\pi]$. Taking the Fourier transform in the x-variable we get:

$$-a^2n^2\hat{u}(n,t) = \hat{u}_t(n,t)\,, \quad \hat{u}(n,0) = \hat{f}(n)\,.$$

Thus $\hat{u}(n,t) = \hat{f}(n)e^{-a^2n^2t}$. Now using that f and $u(\cdot,t)$ are odd and real valued we get:

$$u(x,t) = \sum_{n\in\mathbb{Z}}\hat{f}(n)e^{-inx-a^2n^2t} = 2i\sum_{n=1}^{\infty}\hat{f}(n)e^{-n^2a^2t}\sin(nt)\,.$$

Similarily for general $L > 0$ we get

$$u(x,t) = 2i\sum_{n=1}^{\infty}\hat{f}(n)e^{-n^2\pi^2a^2t/L^2}\sin\left(\frac{nL}{\pi}t\right)\,.$$

6.3 The Isoperimetric Problem

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be a continuous smooth simple and closed curve. Then $C := \gamma([0, 2\pi])$ defines a bounded domain $\Omega(\gamma) \subset \mathbb{C}$. Assume for simplicity that the length of C is one. Question: For which curve is the area $a(\Omega(\gamma))$ is maxima?

Theorem 33 (Hurwitz) *We have $a(\Omega(\gamma)) \leq 1/4\pi$ and $a(\Omega(\gamma)) = 1/2\pi$ if and only if $\Omega(\gamma)$ is a circle.*

Proof. Let $\Omega = \Omega(\gamma)$. Let $G(e^{i\theta}) = \gamma(\theta) = x(\theta) + iy(\theta)$. Then G is a smooth function on the torus. Hence

$$G(z) = \sum_{n=-\infty}^{\infty} \widehat{G}(n) z^n$$

and

$$\gamma'(\theta) = DG(z) = i \sum_{n=-\infty}^{\infty} n\widehat{G}(n) z^n \,.$$

We assume that the curve is parametrized such that

$$|\gamma'(\theta)|^2 = x'(\theta)^2 + y'(\theta)^2 = \frac{1}{2\pi} \,.$$

Then, by the Plancherel formula

$$\begin{aligned} \frac{1}{4\pi^2} &= |DG|_2^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} x'(t)^2 + y'(t)^2 \, dt \\ &= \sum_{n=-\infty}^{\infty} n^2 |\widehat{G}(n)|^2 \,. \end{aligned}$$

Or

$$\pi \sum_{n=-\infty}^{\infty} n^2 \left|\widehat{G}(n)\right|^2 = \frac{1}{4\pi} \,.$$

Apply now the Stoke's Theorem to the boundary of Ω to get

$$\begin{aligned}
a(D(\gamma)) &= \frac{1}{2}\int\int_{\Omega} d\omega \\
&= \frac{1}{2}\int_{\partial\Omega} xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} x(\theta)y'(\theta) - y(\theta)x'(\theta)d\theta \\
&= \frac{-1}{2i}\int_0^{2\pi} \gamma(\theta)\overline{\gamma'(\theta)}\,d\theta \\
&= \pi\sum_{n=-\infty}^{\infty} n\widehat{G}(n)\overline{\widehat{G}(n)} \\
&= \pi\sum_{n=-\infty}^{\infty} n|\widehat{G}(n)|^2 .
\end{aligned}$$

It follows now that

$$\frac{1}{4\pi} - a(D(\gamma)) = \pi\sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 .$$

Consider the polynomial $p(x) = x(x-1)$. Then $p(0) = p(1) = 0$ and $p'(x) = 2x - 1$. Thus p is decreasing for $x < \frac{1}{2}$ and increasing for $x > \frac{1}{2}$. It follows that $n(n-1) > 0$ for all $n \neq 0$ or $n \neq 1$. Hence

$$\frac{1}{4\pi} - a(\Omega) = \pi\sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 \geq 0$$

and

$$\frac{1}{4\pi} - a(\Omega) = 0$$

if and only if $\widehat{G}(n) = 0$ for $n \neq 0$ or $n \neq 1$. But in that case

$$G(e^{i\theta}) = \widehat{G}(0) + \widehat{G}(1)e^{-i\theta}$$

which is a circle. ∎