# NOTES ON HILBERT SPACES

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## 1. Hilbert spaces

A *Hilbert space* is a Banach space where the norm comes from an inner product $(\cdot,\cdot)$.

**Definition 1.** *Let* $\mathbf{V}$ *be a real or complex vector space. An inner product on* $\mathbf{V}$ *is a map*

$$\mathbf{V}\times\mathbf{V}\mapsto\mathbb{K}$$

*with the following properties:*

1. *For fixed* $\mathbf{y}\in\mathbf{V}$ *the map* $\mathbf{V}\ni\mathbf{v}\mapsto(\mathbf{v},\mathbf{y})\in\mathbb{K}$ *is linear;*
2. $(\mathbf{x},\mathbf{y})=\overline{(\mathbf{y},\mathbf{x})}$ *for all* $\mathbf{x},\mathbf{y}\in\mathbf{V}$*;*
3. *For all* $\mathbf{x}\in\mathbf{V}$ *we have* $(\mathbf{x},\mathbf{x})\geq 0$ *and* $(\mathbf{x},\mathbf{x})=0$ *if and only if* $\mathbf{x}=\mathbf{0}$.

*A space* $\mathbf{V}$ *with inner product* $(\cdot,\cdot)$ *is called an* ***inner product space****.*

Let us make few remarks on this:

**Remark 1.** *If* $\mathbb{K}=\mathbb{R}$*, then (2) reads* $(\mathbf{x},\mathbf{y})=(\mathbf{y},\mathbf{x})$ *and the map* $\mathbf{x}\mapsto(\mathbf{y},\mathbf{x})$ *is linear. If* $\mathbb{K}=\mathbb{C}$*, then it follows from (2) and (1) that the map* $\mathbf{x}\mapsto(\mathbf{y},\mathbf{x})$ *is conjugate linear, i.e.,*

$$(\mathbf{y},\lambda\mathbf{x}_1+\mathbf{x}_2)=\overline{\lambda}\,(\mathbf{y},\mathbf{x}_1)+(\mathbf{y},\mathbf{x}_2)$$

**Remark 2.** *Condition (3) says that we can define a norm on* $\mathbf{V}$ *by* $||\mathbf{x}||:=\sqrt{(\mathbf{x},\mathbf{x})}$ *.*

**Lemma 1** (Cauchy-Schwarz)**.** *Let* $\mathbf{V}$ *be an inner product space. Then for all* $x,\mathbf{y}\in\mathbf{V}$ *:*

$$|(\mathbf{x},\mathbf{y})|\leq||\mathbf{x}||\;||\mathbf{y}||$$

*and* $|(\mathbf{x},\mathbf{y})|=||\mathbf{x}||\;||\mathbf{y}||$ *if and only if* $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly dependent.*

*Proof.* The claim is obvious if $\mathbf{y}=\mathbf{0}$. So assume that $\mathbf{y}\neq\mathbf{0}$. Let $t\in\mathbb{C}$. Then

$$\begin{aligned}||\mathbf{x}-t\mathbf{y}||^2 &= (\mathbf{x}-t\mathbf{y},\mathbf{x}-t\mathbf{y})\\ &= ||\mathbf{x}||^2-t\,(\mathbf{y},\mathbf{x})-\overline{t\,(\mathbf{y},\mathbf{x})}+|t|^2\,||\mathbf{y}||^2\;.\end{aligned}$$

Take $t=(\mathbf{x},\mathbf{y})\,/\,||\mathbf{y}||^2$. Then

$$0\leq||\mathbf{x}-t\mathbf{y}||^2=||\mathbf{x}||^2-\frac{|(\mathbf{x},\mathbf{y})|^2}{||\mathbf{y}||^2}\,. \tag{1.1}$$

Hence $||\mathbf{x}||^2\,||\mathbf{y}||^2\geq|(\mathbf{x},\mathbf{y})|^2$. Assume that we have equality. Then (1.1) implies that $||\mathbf{x}-t\mathbf{y}||^2=0$ or $t\mathbf{y}=\mathbf{x}$. ∎

**Example 1.** *Let $I = [a, b]$ a closed interval in $\mathbb{R}$ or $I = \mathbb{R}$. Let $C_c(I)$ be the set of continuous functions on $I$ such that Supp$(f)$ is compact. Let $\varphi : I \to \mathbb{R}$ such that $\varphi(x) > 0$ for all $x \in I$. Then*

$$(f, g)_\varphi := \int_I f(x)\overline{g(x)}\,\varphi(x)\,dx$$

*defines an inner product on $C_c(I)$. Here $\int_I f(x)\,dx$ stands for the usual Riemann integral.*

**Example 2.** *Let $\mathbf{V} = \mathbb{C}^n$. Define an inner product on $\mathbf{V}$ by*

$$((z_1, \ldots, z_n), (y_1, \ldots, y_n)) = \sum_{j=1}^{n} z_j \overline{y_j}\,.$$

*Then*

$$||\mathbf{z}|| = \sqrt{\sum_{j=1}^{n} |z_j|^2} \quad \text{for } \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n\,.$$

**Definition 2.** *Let $\mathbf{V}$ be a vector space with inner product $(\cdot, \cdot)$. Then $\mathbf{V}$ is a Hilbert space if $(\mathbf{V}, ||\cdot||)$ is complete, i.e., a Banach space.*

**Example 3.** *The space $\mathbb{C}^n$ is an Hilbert space and any finite dimensional Hilbert space is isomorphic to $\mathbb{C}^n$.*

**Example 4.** *Let $X \subset \mathbb{R}$ be a measurable subset. Let $\mathbf{H} = L^2(X)$ with the inner product*

$$(f, g) = \int_X f(t)\overline{g(t)}\,dt$$

*More generally let $\varphi : X \to \mathbb{R}^+$ be integrable. For measurable functions $f, g : X \to \mathbb{R}$ let*

$$(f, g)_\varphi := \int_X f(t)\overline{g(t)}\varphi(t)\,dt$$

*Let $\mathbf{H}_\varphi$ be the space of all measurable functions with $(f, f)_\varphi < \infty$. Then $\mathbf{H}_\varphi$ is a Hilbert space.*

**Example 5.** *Define an inner product on $\ell^2$ by*

$$((a_n)_n, (b_n)_n) = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

*Then $\ell^2$ becomes a Hilbert space (prove this).*

We will later show that each inner product space can be isometrically embedded into a Hilbert space such that the image is dense. But first we will recall some simple facts.

**Definition 3.** *Let $(\mathbf{H}, (\cdot, \cdot))$ be an inner product space:*

1. *The vectors $\mathbf{u}, \mathbf{v} \in \mathbf{H}$ are orthogonal if $(\mathbf{u}, \mathbf{v}) = 0$.*
2. *Let $S, T \subset \mathbf{H}$. Then $S$ and $T$ are orthogonal if $(\mathbf{s}, \mathbf{t}) = 0$ for all $\mathbf{s} \in S$ and all $\mathbf{t} \in T$.*
3. *A set $S \subset \mathbf{H}$ is orthogonal if for all $\mathbf{s}, \mathbf{t} \in S$, $\mathbf{s} \neq \mathbf{t}$ we have $(\mathbf{s}, \mathbf{t}0 = 0$.*
4. *A set $S \subset \mathbf{H}$ is orthogonormal if $S$ is orthogonal and $||\mathbf{s}|| = 1$ for all $\mathbf{s} \in S$.*

5. *Let* $S \subset \mathbf{H}$ *then*

$$S^{\perp} = \{\mathbf{x} \in \mathbf{H} \mid \forall s \in S : (\mathbf{x}, s) = 0\}$$

6. *Let* $S, T \subset \mathbf{H}$. *Then*

$$S + T = \{\mathbf{s} + \mathbf{t} \mid \mathbf{s} \in S,\ \mathbf{t} \in T\}$$

7. *Assume that* $\mathbf{H}$ *is an Hilbert space. Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be closed subspaces of* $\mathbf{H}$. *Then* $\mathbf{S}$ *and* $\mathbf{T}$ *are Hilbert spaces. If* $\mathbf{S} \cap \mathbf{T} = \{\mathbf{0}\}$ *then the sum* $\mathbf{S} + \mathbf{T}$ *is denoted by* $\mathbf{S} \oplus \mathbf{T}$ *and called the direct sum of* $\mathbf{S}$ *and* $\mathbf{T}$. *In this case each element* $\mathbf{s} + \mathbf{t} \in \mathbf{S} \oplus \mathbf{T}$ *has a unique expression of this form.*

**Lemma 2.** *Let* $\mathbf{H}$ *be a Hilbert space and let* $\mathbf{S} \subset \mathbf{H}$ *be a closed subspace. Let* $\mathbf{u} \in \mathbf{H}$ *then there exits an* $\mathbf{v} \in \mathbf{S}$ *such that*

$$||\mathbf{u} - \mathbf{v}|| = \min_{\mathbf{s} \in \mathbf{S}} ||\mathbf{u} - \mathbf{s}|| \,.$$

**Lemma 3.** *Let* $\mathbf{S} \subset \mathbf{H}$. *Then* $\mathbf{S}^{\perp}$ *is a closed subspace of* $\mathbf{H}$ *and* $\left(\mathbf{S}^{\perp}\right)^{\perp}$ *is the closed subspace of* $\mathbf{H}$ *generated by* $\mathbf{S}$. *If* $\mathbf{S}$ *is a closed subspace of* $\mathbf{H}$, *then*

$$\mathbf{H} = \mathbf{S} \oplus \mathbf{S}^{\perp} \,.$$

*Proof.* Let $\{\mathbf{u}_n\}$ be a sequence in $\mathbf{S}^{\perp}$ converging to $\mathbf{u} \in \mathbf{H}$. Let $\mathbf{s} \in \mathbf{S}$, then

$$(\mathbf{u}, \mathbf{s}) = \lim\, (\mathbf{u}_n, \mathbf{s}) = 0 \,.$$

Hence $\mathbf{u} \in \mathbf{S}^{\perp}$ and $\mathbf{S}^{\perp}$ is closed. Let $\mathbf{u}, \mathbf{v} \in \mathbf{S}^{\perp}$, $\lambda \in \mathbb{K}$ and $\mathbf{s} \in \mathbf{S}$. Then

$$(\lambda \mathbf{u} + \mathbf{v}, \mathbf{s}) = \lambda (\mathbf{u}, \mathbf{s}) + (\mathbf{v}, \mathbf{s}) = \mathbf{0} \,.$$

Hence $\mathbf{S}^{\perp}$ is a subspace. We leave it as an exercise to show that $\mathrm{cl}\,(\mathbf{S}) = \mathbf{S}^{\perp\perp}$. Assume now that $\mathbf{S}$ is closed. Then $\mathbf{S} \oplus \mathbf{S}^{\perp}$ is a subspace of $\mathbf{H}$. Let $\mathbf{u} \in \mathbf{H}$. Let $\mathbf{v} \in \mathbf{S}$ be as in Lemma 2. We claim that $\mathbf{u} - \mathbf{v} \in \mathbf{S}^{\perp}$. Assume not, then there is an $\mathbf{s} \in \mathbf{S}$, $||\mathbf{s}|| = 1$, and $t = (\mathbf{u} - \mathbf{v}, \mathbf{s}) \neq 0$. Then

$$\begin{aligned} ||\mathbf{u} - \mathbf{v} - t\mathbf{s}||^2 &= (\mathbf{u} - \mathbf{v} - t\mathbf{s}, \mathbf{u} - \mathbf{v} - t\mathbf{s}) \\ &= ||\mathbf{u} - \mathbf{v}||^2 - |(\mathbf{u} - \mathbf{v}, \mathbf{s})|^2 \\ &< ||\mathbf{u} - \mathbf{v}||^2 \end{aligned}$$

contradicting our choice of $\mathbf{v}$.

It follows now that $\mathbf{u} - \mathbf{v} \in \mathbf{S}^{\perp}$ and $\mathbf{u} = \mathbf{v} + (\mathbf{u} - \mathbf{v}) \in \mathbf{S} \oplus \mathbf{S}^{\perp}$. Hence $\mathbf{S} \oplus \mathbf{S}^{\perp} = \mathbf{H}$. ∎

From now on $\mathbf{H}$ will always stand for a Hilbert space with inner product $(\cdot, \cdot)$.

**Lemma 4.** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be orthogonal. Then* $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$. *In particular if* $\mathbf{x}$ *and* $\mathbf{y}$ *have norm one, then* $||\mathbf{x} + \mathbf{y}|| = \sqrt{2}$.

*Proof.* We have $||\mathbf{x} + \mathbf{y}||^2 = (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) = (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y})$ because $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. ∎

**Lemma 5.** *Assume that* $\mathbf{H}$ *is separable. Let* $S$ *be a orthonormal subset of* $\mathbf{H}$. *Then* $S$ *is finite or countable infinite.*

*Proof.* Assume that $S$ is not finite. Let $0 < \epsilon < 1$. Let $\mathbf{s}, \mathbf{t} \in S$, $\mathbf{s} \neq \mathbf{t}$. Suppose that $\mathbf{y} \in B_\epsilon(\mathbf{s}) \cap B_\epsilon(\mathbf{t})$. Then

$$\begin{aligned} 2 &= ||\mathbf{s} - \mathbf{t}||^2 \\ &= ||\mathbf{s} - \mathbf{y} + \mathbf{y} - \mathbf{t}||^2 \\ &\leq ||\mathbf{s} - \mathbf{y}||^2 + ||\mathbf{y} - \mathbf{t}||^2 \\ &< 2\epsilon < 2 \end{aligned}$$

A contradiction. Hence the balls $B_\epsilon(\mathbf{s})$, $\mathbf{s} \in S$, are disjoint. As $\mathbf{H}$ is assumed separable, we can at most have countable many disjoint open balls in $\mathbf{H}$ which implies that $S$ is at most countable. ∎

**Lemma 6** (Bessel's inequality). *Let $\{\mathbf{e}_j\}_{j\in I}$, $I$ a finite or countable index set. Let $\mathbf{x} \in \mathbf{H}$. Then*

$$\sum_{j\in I} (\mathbf{x}, \mathbf{e}_j)^2 \leq ||\mathbf{x}||^2$$

*Proof.* Let $J \subset I$ be finite. Then

$$\begin{aligned} 0 &\leq \left|\left|\mathbf{x} - \sum_{j\in J} (\mathbf{x}, \mathbf{e}_j)\mathbf{e}_j\right|\right|^2 \\ &= ||\mathbf{x}||^2 - 2\sum_{j\in J} (\mathbf{x}, \mathbf{e}_j)^2 + \sum_{i,j\in J} (\mathbf{x}.\mathbf{e}_i)(\mathbf{x}, \mathbf{e}_j)\,(\mathbf{e}_i, \mathbf{e}_j) \\ &= ||\mathbf{x}||^2 - \sum_{j\in J} (\mathbf{x}, \mathbf{e}_j)^2 \end{aligned}$$

because $(\mathbf{x}.\mathbf{e}_i)(\mathbf{x}, \mathbf{e}_j)\,(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{x}.\mathbf{e}_i)^2\delta_{ij}$. As this holds for any finite subset of $I$ the lemma follows. ∎

Our aim is to find a orthonormal subset $S = \{\mathbf{e}_j\}$ such that each $\mathbf{x} \in \mathbf{H}$ can be written in a unique way as

$$\mathbf{x} = \sum a_j \mathbf{e}_j$$

To avoid problems we will for simplicity assume that $\mathbf{H}$ is separable. Before we construct $S$ let us notice the following:

**Lemma 7.** *Let $S = \{\mathbf{e}_j\}_{j\in J}$ be a orthonormal set in $\mathbf{H}$. Then $\left|\left|\mathbf{x} - \sum_{j\in J} a_j \mathbf{e}_j\right|\right|$ is minimal for $a_j = (\mathbf{x}, \mathbf{e}_j)$.*

*Proof.* We have

$$\begin{aligned} \left|\left|\mathbf{x} - \sum_j a_j \mathbf{e}_j\right|\right|^2 - \left|\left|\mathbf{x} - \sum_j (\mathbf{x}, \mathbf{e}_j)\mathbf{e}_j\right|\right|^2 &= \sum_j (\mathbf{x}, \mathbf{e}_j)^2 - 2a_j\,(\mathbf{x}, \mathbf{e}_j) + a_j^2 \\ &= \sum_j \left((\mathbf{x}, \mathbf{e}_j) - a_j\right) \\ &\geq 0 \end{aligned}$$

Hence $\left|\left|\mathbf{x} - \sum_j a_j \mathbf{e}_j\right|\right| \geq \left|\left|\mathbf{x} - \sum_j (\mathbf{x}, \mathbf{e}_j)\mathbf{e}_j\right|\right|$ and $\left|\left|\mathbf{x} - \sum_j a_j \mathbf{e}_j\right|\right| = \left|\left|\mathbf{x} - \sum_j (\mathbf{x}, \mathbf{e}_j)\mathbf{e}_j\right|\right|$ if and only if $a_j = (\mathbf{x}.\mathbf{e}_j)$ for all $j$. ∎

**Lemma 8.** *Let $S = \{\mathbf{e}_j\}_{j\in J}$ be a orthonormal set in $\mathbf{H}$. Then $\mathbf{x} - \sum_{j\in J} (\mathbf{x}, \mathbf{e}_j)\, \mathbf{e}_j$ is perpendicular to $S$.*

*Proof.* Let $\mathbf{e}_k \in S$. Then

$$\begin{aligned}\left(\mathbf{x} - \sum_{j\in J} (\mathbf{x}, \mathbf{e}_j)\, \mathbf{e}_j, \mathbf{e}_k\right) &= (\mathbf{x}, \mathbf{e}_k) - \sum_j (\mathbf{x}.\mathbf{e}_j)(\mathbf{e}_j, \mathbf{e}_k)\\ &= (\mathbf{x}, \mathbf{e}_k) - (\mathbf{x}, \mathbf{e}_k)\\ &= 0\end{aligned}$$

∎

It follows by the above lemma that $\mathbf{x} = \sum_{\mathbf{e}\in S} (\mathbf{x}, \mathbf{e})\mathbf{e}$ for each $\mathbf{x} \in \mathbf{H}$ if and only if $S$ is maximal orthonormal subset of $\mathbf{H}$. In that case we call $S$ for an *orthonormal basis* for $\mathbf{H}$. The existence of an orthonormal basis follows from the next theorem.

**Theorem 1.** *Let $T$ be a orthonormal subset of $\mathbf{H}$. Then there exists a orthonormal basis $S$ for $\mathbf{H}$ such that $T \subset S$.*

*Proof.* Let $X$ be the set of orthonormal subsets of $\mathbf{H}$ that contains $T$. Define an ordering on $X$ by inclusion. By the Hausdorff Maximum Principle there exists a maximal linearly ordered subset $\mathcal{T} \subset X$. Let

$$S = \bigcup_{L\in\mathcal{T}} L$$

Then $S$ is a maximal orthonormal subset containing $T$. ∎

**Proposition 1.** *Let $\mathbf{H}$ be a separable Hilbert space. Let $T$ be an orthonormal subset of $\mathbf{H}$. Then there exists an orthonormal basis $S = \{\mathbf{e}_j\}$ containing $T$. We have for each $\mathbf{x} \in \mathbf{H}$:*

1. $\mathbf{x} = \sum_j (\mathbf{x}, \mathbf{e}_j)\mathbf{e}_j$ *and;*
2. $||\mathbf{x}||^2 = \sum_j (\mathbf{x}, \mathbf{e}_j)^2$

The next theorem answer the question when a sum $\sum_j b_j \mathbf{e}_j$ converges in $\mathbf{H}$.

**Theorem 2.** *Let $S = \{\mathbf{e}_j\}_{j\in J}$ be an orthonormal set. Then $\sum_{j\in J} b_j \mathbf{e}_j$ converges in $\mathbf{H}$ if and only if $\sum_{j\in J} b_j^2$ converges.*

*Proof.* We can assume that $J = \mathbb{N}$ because the statement is obviously true for $J$ finite. Assume that $\sum_{j=1}^{\infty} b_j \mathbf{e}_j$ converges. Then $\mathbf{s}_n := \sum_{j=1}^{n} b_j \mathbf{e}_j$ is a Cauchy sequence in $\mathbf{H}$. Let $\epsilon > 0$ be given. Then there exists a $N \in \mathbb{N}$ such that $||\mathbf{s}_n - \mathbf{s}_m||^2 < \epsilon$ for all $n, m \geq N$ and assume that $m > n$. Then

$$\begin{aligned}||\mathbf{s}_n - \mathbf{s}_m||^2 &= \sum_{j=n+1}^{m} b_j^2\\ &= \left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{j=1}^{m} b_j^2\right)\end{aligned}$$

It follows that $\left\{\left(\sum_{j=1}^{n} b_j^2\right)\right\}_n$ is a Cauchy sequence, and hence convergent.

Assume now that $\left\{\left(\sum_{j=1}^{n} b_j^2\right)\right\}_n$ is a Cauchy sequence. Then the above calculation shows that $\{\mathbf{s}_n\}_n$ is a Cauchy sequence. As $\mathbf{H}$ is complete it follows that $\lim_{n\to\infty}\mathbf{s}_n = \sum_{j=1}^{\infty} b_j\mathbf{e}_j$ exists. ∎

We define a metric on $\mathbf{V}$ by $d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||$. This metric defines a topology on $\mathbf{V}$ such that the maps

$$\begin{aligned}\mathbf{V}\times\mathbf{V} &\ni (\mathbf{x},\mathbf{y})\mapsto \mathbf{x}+\mathbf{y}\in\mathbf{V}\\ \mathbb{K}\times\mathbf{V} &\ni (\lambda,\mathbf{x})\mapsto \lambda\mathbf{x}\in\mathbf{V}\end{aligned}$$

are continuous (prove this!). Hence $\mathbf{V}$ is a topological vector space. Recall that the (continuous) dual of $\mathbf{V}$ – denoted by $\mathbf{V}^*$ – is the space of all linear maps $f:\mathbf{V}\to\mathbb{K}$. The space $\mathbf{V}^*$ becomes a Banach space if we define

$$||f|| := \sup_{||\mathbf{x}||=1} |f(\mathbf{x})| \,.$$

Lemma 1 implies that for a given $\mathbf{u}\in\mathbf{V}$ the map $\mathbf{x}\mapsto(\mathbf{x},\mathbf{u})$ is linear and satisfies $|(\mathbf{x},\mathbf{u})|\le ||\mathbf{x}||\,||\mathbf{u}||$. Hence it is in $\mathbf{V}^*$. Define $\iota^*:\mathbf{V}\to\mathbf{V}^*$ and $\iota^{**}:\mathbf{V}\to\mathbf{V}^{**}$ by

$$i^*(\mathbf{u})(\mathbf{v}) := (\mathbf{v},\mathbf{u})$$

and

$$\iota^{**}(\mathbf{u})(f) := f(\mathbf{u})\,.$$

We notice the following simple facts:

**Lemma 9.** *Let the notation be as above. Then the following holds:*

1. *$\iota^*$ is conjugate linear and $\iota^{**}$ is linear.*
2. *Let $\mathbf{u}\in\mathbf{V}$ then $||\iota^*(\mathbf{u})|| = ||\mathbf{u}||$. In particular it follows that $\iota^*$ is injective.*
3. *Let $\mathbf{u}\in\mathbf{V}$ then $||\iota^{**}(\mathbf{u})|| = ||\mathbf{u}||$. In particular it follows that $\iota^{**}$ is injective.*
4. *$||\iota^*|| = 1$. In particular it follows that$\iota^*$ is continuous.*
5. *$||i^{**}|| = 1$. In particular it follows that $\iota^*$ is continuous.*

*Proof.* (1) This is clear.

(2) We have

$$||\iota^*(\mathbf{u})|| = \sup_{||\mathbf{x}||=1} |(\mathbf{x},\mathbf{u})| \le \sup_{||\mathbf{x}||=1} ||\mathbf{x}||\,||\mathbf{u}|| = ||\mathbf{u}||\,.$$

So $||\iota^*(\mathbf{u})||\le||\mathbf{u}||$. If $\mathbf{u}=\mathbf{0}$ then there is nothing to prove as $\iota^*(\mathbf{0})=\mathbf{0}$. So assume that $\mathbf{u}\neq\mathbf{0}$. Let $\mathbf{x} = ||\mathbf{u}||^{-1}\mathbf{u}$. Then $||\mathbf{x}||=1$ and

$$\iota^*(\mathbf{u})(\mathbf{x}) = ||\mathbf{u}||$$

which implies that $||\iota^*(\mathbf{u})||\ge||\mathbf{u}||$. Thus $||\iota^*(\mathbf{u})|| = ||\mathbf{u}||$.

(3) We have

$$||\iota^{**}(\mathbf{u})|| = \sup_{||f||=1} |f(\mathbf{u})| \le ||f||\,||\mathbf{u}||\,.$$

We may assume that $\mathbf{u}\neq\mathbf{0}$. As we have already proven that $\iota^*(\mathbf{u})\in\mathbf{V}^*$ and $||\iota^*(\mathbf{u})|| = ||\mathbf{u}||$ we can take $f = ||\mathbf{u}||^{-1}\iota^*(\mathbf{u})$. Then $||f||=1$ and

$$|\iota^{**}(\mathbf{u})(f)| = ||\mathbf{u}||\,.$$

Hence $||\iota^{**}(\mathbf{u})|| = ||\mathbf{u}||$.

(4) and (5) follows from (2) respectively (3). ∎

**Proposition 2** (Riesz representation Theorem). *Let* $\mathbf{H}$ *be a separable Hilbert space. Then* $\iota^* : \mathbf{H} \to \mathbf{H}^*$ *is bijective, i.e. let* $f \in \mathbf{H}^*$*, then there exists an unique* $\mathbf{u} \in \mathbf{H}$ *such that for all* $\mathbf{x} \in \mathbf{H}$ *we have:*

$$f(\mathbf{x}) = (\mathbf{x}, \mathbf{u}) .$$

*Proof.* Let $S = \{\mathbf{e}_j\}_{j \in J}$ be an orthonormal basis for $\mathbf{H}$. Let $f \in \mathbf{H}^*$. To find $\mathbf{u}$ let us notice that if such an $\mathbf{u}$ exists then $\mathbf{u} = \sum (\mathbf{u}, \mathbf{e}_j)\, \mathbf{e}_j$ and $f(\mathbf{e}_j) = \overline{(\mathbf{e}_j, \mathbf{u})}$. Hence $\mathbf{u} = \sum \overline{f(\mathbf{e}_j)}\, \mathbf{e}_j$. So we have to show that

$$\mathbf{u} := \sum \overline{f(\mathbf{e}_j)}\, \mathbf{e}_j$$

exists in $\mathbf{H}$. If $\mathbf{H}$ is finite dimensional there is nothing to proof, so let us assume that $J = \mathbb{N}$. Let $n \in \mathbb{N}$. Let $\mathbf{u}_n := \sum_{j=1}^{n} \overline{f(\mathbf{e}_j)} \mathbf{e}_j$. Then

$$\begin{aligned} ||\mathbf{u}_n||^2 &= \sum_{j=1}^{n} |f(\mathbf{e}_j)|^2 \\ &= f\left( \sum_{j=1}^{n} \overline{f(\mathbf{e}_j)}\, \mathbf{e}_j \right) \\ &= f(\mathbf{u}_n) \\ &\leq ||f||\, ||\mathbf{u}_n|| \end{aligned}$$

Hence

$$||\mathbf{u}_n|| \leq ||f||$$

It follows that $\sum |f(\mathbf{e}_j)|^2$ exists and hence also $\lim \mathbf{u}_n = \sum_{j=1}^{\infty} \overline{f(\mathbf{e}_j)} \mathbf{e}_j$. Let $\mathbf{x} = \sum (\mathbf{x}, \mathbf{e}_j) \mathbf{e}_j \in \mathbf{H}$. Then

$$\begin{aligned} f(\mathbf{x}) &= f(\textstyle\sum (\mathbf{x}, \mathbf{e}_j) \mathbf{e}_j) \\ &= \sum (\mathbf{x}, \mathbf{e}_j)\, f(\mathbf{e}_j) \qquad f \text{ continuous and linear} \\ &= \left( \mathbf{x}, \sum \overline{f(\mathbf{e}_j)} \mathbf{e}_j \right) \\ &= (\mathbf{x}, \mathbf{u}) \end{aligned}$$

∎

Let now $\mathbf{V}$ be an inner product space and $\iota^{**} : \mathbf{V} \to \mathbf{V}^{**}$. Recall that $\mathbf{V}^{**}$ is a Banach space. Let $\mathbf{H}$ be the closure of $\iota^{**}(\mathbf{V})$ in $\mathbf{V}^{**}$. Then $\mathbf{H}$ is a Banach space. Let $\mathbf{u}, \mathbf{v} \in \mathbf{H}$. Let $\{\mathbf{u}_n\}_n$, $\{\mathbf{v}_n\}_n$ be Cauchy sequence in $\mathbf{V}$ such that $\lim \iota^{**}(\mathbf{u}_n) = \mathbf{u}$ and $\lim \iota^{**}(\mathbf{v}_n) = \mathbf{v}$. Then there is an $M > 0$ such that $||\mathbf{u}_m||, ||\mathbf{u}_n|| \leq M$ for all $m, n$ and hence

$$\begin{aligned} |(\mathbf{u}_n, \mathbf{v}_n) - (\mathbf{u}_m, \mathbf{v}_m)| &= |(\mathbf{u}_n - \mathbf{u}_m, \mathbf{v}_n)| + |(\mathbf{u}_m, \mathbf{v}_n - \mathbf{v}_m)| \\ &\leq ||\mathbf{u}_n - \mathbf{u}_m||\, ||\mathbf{v}_n|| + ||\mathbf{v}_n - \mathbf{v}_m||\, ||\mathbf{u}_m|| \\ &\leq M \left( ||\mathbf{u}_n - \mathbf{u}_m|| + ||\mathbf{v}_n - \mathbf{v}_m|| \right) . \end{aligned}$$

Hence $\{(\mathbf{u}_n, \mathbf{v}_n)\}_n$ is a Cauchy sequence in $\mathbb{R}$. Define

$$(\mathbf{u}, \mathbf{v}) := \lim_{n \to \infty} (\mathbf{u}_n, \mathbf{v}_n) .$$

We leave it to the reader to show that $(\mathbf{u}, \mathbf{v})$ is independent of the Cauchy sequences used in the definition and that $(\cdot, \cdot)$ is an inner product on $\mathbf{H}$ such that $||\mathbf{u}||^2 = (\mathbf{u}, \mathbf{u})$

for all $\mathbf{u} \in \mathbf{H}$, where $||\cdot||$ is the norm on $\mathbf{H}$ coming from $\mathbf{V}^{**}$. As $\iota^{**}$ is an isometry and $\operatorname{Im}(\iota^{**})$ is dense in $\mathbf{H}$ we have actually proved the following.

**Theorem 3.** *Let $\mathbf{V}$ be an inner product space with inner product $(\cdot,\cdot)_{\mathbf{V}}$. Then there exists a Hilbert space $(\mathbf{H},(\cdot,\cdot)_{\mathbf{H}})$ and a linear map $\varphi : \mathbf{V} \to \mathbf{H}$ such that the following holds:*

1. *For all $\mathbf{u},\mathbf{v} \in \mathbf{V}$ we have $(\mathbf{u},\mathbf{v})_{\mathbf{V}} = (\varphi(\mathbf{u}),\varphi(\mathbf{v}))_{\mathbf{H}}$. In particular $\varphi$ is an isometry.*
2. *$\varphi(\mathbf{V})$ is dense in $\mathbf{H}$.*

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