MEASURE AND INTEGRATION

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1. MEASURE AND INTEGRATION

1.1. **Measure Spaces.** See the book

1.2. **Measurable Functions.** In this section $(X, \mathcal{B})$ is a fixed measurable space.

Lemma 1. *Let $f : X \to \overline{\mathbb{R}}$ and let $\alpha \in \mathbb{R}$. Then the following holds:*

1. $\{x \in X \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) < \alpha + 1/n\}$;
2. $\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > \alpha + 1/n\}$.

Proof. 1) It is clear that the left hand side is contained in the right hand side. Assume that $f(x) < \alpha + 1/n$ for all n. Then $f(x) \leq \lim_{n\to\infty}(\alpha + 1/n) = \alpha$. Hence the sets are equivalent. The proof of (2) is the same. ∎

Lemma 2. *Let $f : X \to \overline{\mathbb{R}}$. Then the following are equivalent:*

1. *For all $\alpha \in \mathbb{R}$ we have $\{x \in X \mid f(x) < \alpha\} \in \mathcal{B}$;*
2. *For all $\alpha \in \mathbb{R}$ we have $\{x \in X \mid f(x) \leq \alpha\} \in \mathcal{B}$;*
3. *For all $\alpha \in \mathbb{R}$ we have $\{x \in X \mid f(x) > \alpha\} \in \mathcal{B}$;*
4. *For all $\alpha \in \mathbb{R}$ we have $\{x \in X \mid f(x) \geq \alpha\} \in \mathcal{B}$.*

Proof. We proof that "(1)⇒(2)⇒(3)⇒(4)⇒(1)". Then the theorem follows.

"(1)⇒(2)": We have $\{x \in X \mid f(x) \leq \alpha\} = \cap \{x \in X \mid f(x) < \alpha + 1/n\} \in \mathcal{B}$ by (1), the above lemma and the fact that a countable intersection of sets in $\mathcal{B}$ is again in $\mathcal{B}$.

"(2)⇒(3)": $\{x \in X \mid f(x) > \alpha\} = X \setminus \{x \in X \mid f(x) \leq \alpha\} \in \mathcal{B}$.

"(3)⇒(4)": $\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > \alpha + 1/n\} \in \mathcal{B}$.

"(4)⇒(1)": $\{x \in X \mid f(x) < \alpha\} = X \setminus \{x \in X \mid f(x) \geq \alpha\}$. ∎

Definition 1. *The function $f : X \to \overline{\mathbb{R}}$ is called measurable if any one of the statements in the above lemma holds.*

We denote by $\mathcal{M}(X)$, $\mathcal{M}^+(X)$, $\mathcal{M}(X, \mathbb{R})$, and $\mathcal{M}^+(X, \mathbb{R})$ the set of measurable functions, nonnegative measurable functions, real valued measurable functions, and nonnegative real valued measurable functions respectively.

Lemma 3. *Let $f, g : X \to \mathbb{R}$ be measurable. Then the following holds:*

1. *Let $c \in \mathbb{R}$. Then the functions cf, $f + c$ are measurable.*
2. *The functions $f \cdot g$, $f \pm g$, $|f|$, $f \wedge g$ and $f \vee g$ are measurable.*

Proof. 1) Assume first that $c = 0$. Then $cf = 0$. Hence $\{x \in X \mid cf(x) < \alpha\} = \emptyset \in \mathcal{B}$ if $\alpha \leq 0$ and $\{x \in X \mid cf(x) < \alpha\} = X \in \mathcal{B}$ if $\alpha > 0$. If $c > 0$. Then

$$\{x \in X \mid cf(x) < \alpha\} = \{x \in X \mid f(x) < \alpha/c\} \in \mathcal{B}$$

and if $c < 0$ then

$$\{x \in X \mid cf(x) < \alpha\} = \{x \in X \mid f(x) > \alpha/c\} \in \mathcal{B}$$

We have $\{x \in X \mid f(x) + c < \alpha\} = \{x \in X \mid f(x) < \alpha - c\} \in \mathcal{B}$. Hence $f + c$ is measurable.

2) Let us first show that $f + g$ is measurable. Let $\alpha \in \mathbb{R}$. Then we claim that

$$\{x \in X \mid f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \{x \mid f(x) < r\} \cap \{x \mid g(x) < \alpha - r\} \in \mathcal{B}$$

To see this assume first that $f(y) + g(y) < \alpha$. Then there is a $r \in \mathbb{Q}$ such that

$$f(y) < r < \alpha - g(y)$$

Hence $y \in \{x \mid f(x) < r\} \cap \{x \mid g(x) < \alpha - r\}$. If $f(x) < r$ and $g(x) < \alpha - r$ then obviously $f(x) + g(x) < r + (\alpha - r) = \alpha$. Hence the claim. It follows that $f + g$ is measurable.

If $\alpha \leq 0$ then $\{x \mid f(x)^2 < \alpha\} = \emptyset \in \mathcal{B}$. For $\alpha > 0$ we have

$$\{x \mid f(x)^2 > \alpha\} = \{x \mid f(x) < -\sqrt{\alpha}\} \cup \{x \mid f(x) > \sqrt{\alpha}\} \in \mathcal{B}$$

Hence f^2 is measurable. But then

$$fg = \frac{1}{2}(f+g)^2 - f^2 - g^2$$

is also measurable.

Define the function $h : X \to \mathbb{R}$ by $h(x) = 0$ if $f(x) = 0$ otherwise $h(x) = \text{sign}(f(x))$. Then

$$\{x \mid h(x) < \alpha\} = \{x \mid f(x) < 0\} \in \mathcal{B} \qquad \text{if } a \leq 0$$

$$\{x \mid h(x) < \alpha\} = \{x \mid f(x) \leq 0\} \in \mathcal{B} \qquad \text{if } 0 < \alpha \leq 1$$

and

$$\{x \mid h(x) < \alpha\} = \{x \mid f(x) < 0\} \in \mathcal{B} \qquad \text{if } 1 < \alpha$$

Hence h is measurable. It follows that $|f| = hf$ is measurable. Hence the functions

$$f \wedge g = \frac{1}{2}((f+g) - |f - g|)$$

and

$$f \wedge g = \frac{1}{2}((f+g) + |f - g|)$$

are measurable. ∎

Lemma 4. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of real-valued measurable functions. Then the functions*

$$\sup f_n, \ \inf f_n, \ \overline{\lim} f_n \quad \textit{and} \quad \underline{\lim} f_n$$

are also measurable.

Proof. Define $g(x) = \sup f_n(x)$. Let $\alpha \in \mathbb{R}$. If $g(x) > \alpha$. Then there exists at least one $n \in \mathbb{N}$ such that $f_n(x) > \alpha$. On the other hand if $f_n(x) > \alpha$ for one n then $g(x) > \alpha$. Hence

$$\{x \in X \mid g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in X \mid f_n(x) > \alpha\} \in \mathcal{B}\,.$$

As $\inf f_n = -\sup(-f_n)$ it follows that $\inf f_n$ is measurable. We have

$$\overline{\lim} f_n = \inf_k \left[\sup_{n\geq k} f_n\right]$$

and

$$\underline{\lim} f_n = \sup_k \left[\inf_{n\geq k} f_n\right]$$

which are measurable by the first part. ∎

Suppose that $(X, \mathcal{B}, \mu)$ is a measure space. Let f and g be measurable functions. Then

$$\{x \mid f(x) \neq g(x)\} = \{x \mid (f-g)(x) \leq 0\} \cap \{x \mid (f-g)(x) \geq 0\} \in \mathcal{B}$$

We say that $f = g$ almost every where (a.e.) if

$$\mu \{x \mid f(x) \neq g(x)\} = 0$$

Lemma 5. *If μ is a complete measure and f is a measurable functions, then $f = g$ a.e. implies that g is measurable.*

2. Simple Functions

In this section $(X, \mathcal{B}, \mu)$ is a measure space. A *simple function* on X is a function $\varphi : X \to \mathbb{R}$ of the form

$$\varphi = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where the sets $E_i \subset X$ are measurable, χ_{E_i} denotes the characteristic function of E_i and $c_i \in \mathbb{R}$. We call the above representation of φ standard if the sets E_i are disjoint.

Proposition 1. *Let f be a nonnegative measurable function on X. Then there is a sequence $\{\varphi_n\}$ of simple functions such that*

1. *$\varphi_{n+1} \geq \varphi_n$;*
2. *$f(x) = \lim_{n\to\infty} \varphi_n(x)$ for each $x \in X$.*

If X is σ-finite, then we can choose the functions φ_n such that each of them vanishes outside a set of finite measure.

Proof. For $n \in \mathbb{N}$ and $k \leq n2^n$ let $F_n := \{x \mid f(x) \geq n\} \in \mathcal{B}$ and let

$$\begin{aligned} E_{n,k} \quad &: \quad = \left\{x \mid (k-1)2^{-n} \leq f(x) < k2^{-n}\right\} \\ &= \quad \left\{x \mid (k-1)2^{-n} \leq f(x)\right\} \cap \left\{x \mid f(x) < k2^{-n}\right\} \in \mathcal{B} \end{aligned}$$

Define

$$\varphi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n}$$

Then $\varphi_n \leq f$ for all n.

Next we show that $\varphi_{n+1} \geq \varphi_n$: Assume that $x \in F_n$. Then $\varphi_n(x) = n$. If $f(x) \geq n+1$ then $\varphi_{n+1}(x) = n+1 > \varphi_n(x)$: If $n \leq f(x) < n+1$, then there exists a $k \in [n2^{n+1}+1, (n+1)2^{n+1}] \cap \mathbb{N}$ such that

$$(k-1)2^{-(n+1)} \leq f(x) < k2^{-(n+1)}$$

Hence $x \in E_{n+1,k}$ and

$$\varphi_{n+1}(x) = \frac{k-1}{2^{n+1}} \geq \frac{n2^{n+1}}{2^{n+1}} = n = \varphi_n(x)\,.$$

Assume now that $x \in E_{n,k}$. Then

$$(k-1)2^{-n} \leq f(x) < k2^{-n}\,.$$

But

$$\frac{k-1}{2^n} = \frac{(2k-1)-1}{2^{n+1}}$$

and

$$\frac{k}{2^n} = \frac{2k}{2^{k+1}}$$

Hence for $k' = 2k-1$ or $k' = 2k$ we have

$$\frac{k'-1}{2^{n+1}} \leq f(x) < \frac{k'}{2^{n+1}}$$

It follows that

$$\varphi_{n+1}(x) = (k'-1)2^{-(n+1)} \geq (2k-2)2^{-(n+1)} = k2^{-n} = \varphi_n(x)\ .$$

$\varphi_n(x) \nearrow f(x)$: Let us first assume that $f(x) = \infty$. Then $x \in F_n$ for all n. Hence $\varphi_n(x) = n \nearrow \infty$. Assume now that $f(x) < \infty$. Then $\varphi_n(x)$ is a monotone increasing sequence and hence convergent. We have to show that $\lim_n \varphi_n(x) = f(x)$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$. Let $n \geq N$. Let k be such that $x \in E_{n,k}$. Then

$$0 \leq f(x) - \varphi_n(x) < k2^{-n} - (k-1)2^{-n} = 2^{-n} \leq 2^{-N} < \epsilon$$

∎

Theorem 1. *Let $f : X \to \overline{\mathbb{R}}$ be measurable. Then there exists simple functions $\{\varphi_n\}_n$ such that*

$$\lim_{n\to\infty} \varphi_n(x) = f(x)$$

for all $x \in X$.

Proof. Let $f_+ := f \vee 0$ and $f_- := (-f)_+$. Then $f = f_+ - f_-$. Choose simple functions $\left\{\varphi_{+,n}\right\}_n$ and $\left\{\varphi_{-,n}\right\}$ such that

$$\lim_n \varphi_{+,n}(x) = f_+(x)$$

and

$$\lim_n \varphi_{-,n}(x) = f_-(x)$$

Notice that if $f(x) = 0$ then $\varphi_{+,n}(x) = \varphi_{-,n}(x) = 0$. Let $\varphi_n(x) = \varphi_{+,n}(x) - \varphi_{-,n}(x)$. Then φ_n is a simple function and

$$\lim_n \varphi_n(x) = f_+(x) - f_-(x) = f(x)\,.$$

∎

3. Integration

Let $\varphi = \sum c_i \chi_{E_i}$ be a simple measurable function. Assume that the representation is standard. If $\varphi \geq 0$ define

$$\int \varphi(x)\, d\mu(x) := \sum c_i \mu(E_i)$$

We sometimes simplify the notation by writing $\int \varphi(x)\, d\mu(x) = \int \varphi\, d\mu = \int \varphi$. We need to assume that $\varphi \geq 0$ to be sure that the sum on the right hand side exists, because otherwise we might end up with meaningless symbols as $\infty - \infty$. We also need to make sure that the integral is well defined, i.e., that the sum on the right hand side does not depend on the representation of φ.

Lemma 6. *Let $\varphi = \sum_{i=1}^n c_i \chi_{E_i} = \sum_{j=1}^m d_j \chi_{F_j}$. Then*

$$\sum_{i=1}^n c_i \mu(E_i) = \sum_{j=1}^m d_j \mu(F_j)$$

Proof. We notice first that $c_i = d_j$ if $E_i \cap F_j \neq \emptyset$. We can assume that $X = \cup E_i = \cup F_j$. Then $E_i = \dot{\cup}_j E_i \cap F_j$ and $F_j = \dot{\cup}_i E_i \cap F_j$. Hence

$$\begin{aligned} \sum_{i=1}^n c_i \mu(E_i) &= \sum_{i=1}^n c_i\, \mu(\cup_j (E_i \cap F_j)) \\ &= \sum_{i=1}^n \sum_{j=1}^m c_i\, \mu(E_i \cap F_j) \\ &= \sum_{j=1}^m \sum_{i=1}^n d_j\, \mu(E_i \cap F_j) \\ &= \sum_{j=1}^m d_j\, \mu(F_j) \end{aligned}$$

∎

Lemma 7. *Let φ and ψ be nonnegative simple functions. Let $c \in \mathbb{R}^+$. Then*

$$\int (c\varphi + \psi)\, d\mu = c \int \varphi\, d\mu + \int \psi\, d\mu$$

Proof. Let $\varphi = \sum_i c_i \chi_{E_i}$ and $\psi = \sum_j d_j \chi_{F_j}$ be a standard representation of φ respectively ψ. We can assume that $X = \cup E_i = \cup F_j$. (Otherwise we add $X \setminus \cup E_i$ with the constant 0 to the standard representation of φ, and similarly for ψ.) Then $c\varphi = \sum_{i,j} (cc_i)\, \chi_{E_i \cap F_j}$ and $\sum_{i,j} d_j\, \chi_{F_j \cap E_i}$ are also standard representations. Furthermore

$$c\varphi + \psi = \sum_{i,j} (cc_i + d_j) \chi_{E_i \cap F_j}$$

Hence

$$\begin{aligned}\int c\varphi + \psi\, d\mu &= \sum_{i,j}(cc_i + d_j)\,\mu(E_i\cap F_j)\\ &= c\sum_i c_i \sum_j \mu(E_i\cap F_j) + \sum_j d_j \sum_i \mu(E_i \cap F_j)\\ &= \sum_i c_i\mu(E_i) \,+\, \sum_j d_j\mu(F_j)\\ &= c\int \varphi\, du \,+\, \int \psi\, d\mu\end{aligned}$$

∎

Definition 2. *Let $f \in \mathcal{M}^+(X)$. We define the integral $\int f(x)\,d\mu(x)$ of f with respect to μ by*

$$\int f(x)\,d\mu(x) := \sup_{\varphi} \int \varphi(x)\,d\mu(x)$$

where the supremum is taken over all simple functions $\varphi \leq f$. If $E \in \mathcal{B}$ then

$$\int_E f(x)\,d\mu(x) := \int f(x)\chi_E(x)\,d\mu(x)$$

We will also use the following notations for the integral

$$\int_E f d\mu = \int_E f = \int_E f(x)\,d\mu(x)$$

If $E = X$ then we also write $\int f d\mu$ instead of $\int_X f\,du$.

Lemma 8. *Let $f, g \in \mathcal{M}^+(X)$ and $E, F \in \mathcal{B}$.*

1. *If $f \leq g$ then $\int f\,d\mu \leq \int g\,d\mu$;*
2. *If $E \subset F$ then $\int_E f\,d\mu \leq \int g\,d\mu$.*

Proof. 1) If φ is a simple function with $\varphi \leq f$. Then $\varphi \leq g$.
2) We have $f\chi_E \leq f\chi_F$. ∎

Corollary 1. *Let $f \in \mathcal{M}^+(X)$. Then $\int f d\mu = 0$ if and only if $f(x) = 0$ a.e.*

Proof. Assume first that $f = 0$ a.e. Let $\varphi = \sum c_j\chi_{E_j}$ be a simple functions $\varphi \leq f$. Then if $c_j \neq 0$ it follows that $\mu(E_j) = 0$. Hence $\int \varphi = 0$. It follows that $\int f d\mu = 0$.

Assume now that $\int f d\mu = 0$. Let

$$X_n := \{x \mid f(x) > 1/n\}\ .$$

Then $X_n \subset X_{n+1}$, $\mu(X_n) \leq n\int_{X_n} f d\mu \leq n\int_X d\mu = 0$. Furthermore

$$\bigcup_{n=1}^{\infty} X_n =: X_\infty = \{x \mid f(x) > 0\}$$

Hence $\mu(X_\infty) = \mu(\cup X_n) \leq \sum \mu(X_n) = 0$. ∎

Theorem 2 (Monotone Convergence)**.** *Let $\{f_n\}$ be a monotone increasing sequence of functions in $\mathcal{M}^+(X)$ which converges to f, then*

$$\lim_{n\to\infty} \int f_n(x)\,d\mu(x) = \int f(x)\,d\mu(x)$$

Proof. As $f = \lim f_n$ it follows that $f \in \mathcal{M}^+(X)$. We have $f_n \leq f_{n+1} \leq f$. Hence

$$0 \leq \int f_n d\mu \leq \int f_{n+1} d\mu \leq \int f d\mu.$$

Thus

$$\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$$

For the other direction it is enough to show that if φ is a simple function such that $0 \leq \varphi \leq f$, then $\int \varphi \leq \lim \int f_n$. This is because the integral of f is just the supremum over the integral of all such φ. Let $0 < \alpha < 1$. Let φ be a simple function with $0 \leq \varphi \leq f$. Let

$$X_n := \{x \in X \mid f_n(x) \geq \alpha\varphi(x)\}$$

Then X_n is measurable, $X_n \subset X_{n+1}$ and $\cup X_n = X$. Hence

$$\int_{X_n} \alpha\varphi \, d\mu \leq \int_{X_n} f \, d\mu \leq \int_X f \, d\mu$$

and - because φ is a simple function -

$$\lim_{n\to\infty} \int_{X_n} \alpha\varphi = \alpha \int_X \varphi \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

As α was arbitrary it follows that

$$\sup \int_X \varphi \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

which is what we had to show. ∎

Corollary 2. *Let $f, g \in \mathcal{M}^+(X)$ and let $c \in \mathbb{R}^+$. Then*

$$\int cf + g \, d\mu = c \int f \, d\mu + \int g \, d\mu$$

Proof. Let φ_n be a monotone sequence of simple functions converging to f and let ψ_n be a monotone increasing sequence of simple functions converging to g. Then $c\varphi_n + \psi_n$ is a monotone increasing sequence of simple functions converging to $cf + g$. Hence:

$$\begin{aligned}
\int cf + g \, d\mu &= \lim \left[\int c\varphi_n + \psi_n \, d\mu\right] \\
&= \lim \left[c \int \varphi_n d\mu + \int \psi_n \, d\mu\right] \\
&= c \lim \int \varphi_n d\mu + \lim \int \psi_n d\mu \\
&= c \int f d\mu + \int g d\mu
\end{aligned}$$

∎

Theorem 3 (Fatou's Lemma)**.** *Let $\{f_n\}$ be a sequence in $\mathcal{M}^+(X)$. Then*

$$\int_X \underline{\lim}(f_n) \, d\mu \leq \underline{\lim} \int_X f_n \, d\mu$$

Proof. Let

$$g_m := \inf_{k\geq m} \{f_k\} \,.$$

Then $\{g_m\}_m$ is an increasing sequence converging to $\underline{\lim}(f_n)$. Hence by the Monotone Convergence Theorem

$$\lim \int g_m \, d\mu = \int_X \underline{\lim}(f_n) \, d\mu$$

But if $n \geq m$ then $g_m \leq f_n$. Hence

$$\int_X g_m \, d\mu \leq \int_X f_n \, d\mu, \qquad n \geq m \,.$$

It follows that

$$\begin{aligned} \underline{\lim} \int_X f_n \, d\mu &= \lim_{m\to\infty} \left(\inf_{n\geq m} \int_X f_n \, d\mu \right) \\ &\geq \lim_{m\to\infty} \int_X g_m \, d\mu \\ &= \int_X \underline{\lim}(f_n) \, d\mu \end{aligned}$$

∎

Definition 3. *Let μ and ν be two measures defined on $\mathcal{B}$. Then ν is absolutely continuous with respect to μ if $\mu(E) = 0$ implies that $\nu(E) = 0$.*

Corollary 3. *Let $f \in \mathcal{M}^+(X)$. Then*

$$\mu_f : E \mapsto \mu_f(E) := \int_E f d\mu$$

is a measure, which is absolutely continuous with respect to μ.

Proof. We only have to show that μ_f is σ-additive. Let $\{E_n\}_n$ be a sequence of disjoint sets in $\mathcal{B}$. Let $F_m = \cup_{n=1}^m E_m$. Then

$$\chi_{F_m} = \sum_{n=1}^m \chi_{E_n}$$

and

$$f\chi_{F_m} \leq f\chi_{F_\infty}$$

Furthermore we have $\lim f\chi_{F_m} = f\chi_{F_\infty}$. Hence by the Monotone Convergence Theorem and the finite additivity of the integral:

$$\begin{aligned} \mu_f(\cup E_n) &= \int_X f\chi_{F_\infty} \\ &= \lim_{m\to\infty} \int f\chi_{F_m} \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} \int_X f\chi_{E_n} \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} \mu_f(E_n) \\ &= \sum_{n=1}^{\infty} \mu_f(E_n) \ . \end{aligned}$$

Assume that $\mu(E) = 0$. Then $f\chi_E = 0$ a.e. Hence $\int f\chi_E = \mu_f(E) = 0$. ∎

Lemma 9. *Suppose that $\{f_n\}$ is a monotone sequence in $\mathcal{M}^+(X)$ that converges almost everywhere to a function $f \in \mathcal{M}^+(X)$. Then*

$$\lim \int f_n\, d\mu = \int f\, d\mu$$

Proof. Let $M := x \mid \lim f_n(x) = f(x)\}$. Then $\{f_n\chi_M\}$ is a monotone sequence that converges to $f\chi_M$. By the Monotone Convergence Theorem we have

$$\int f\chi_M d\mu = \lim \int f_n\chi_M\, d\mu$$

On the other hand we have that $f = f\chi_M$ a.e. and $f_n\chi_M = f_n$ a.e. Hence

$$\int f = \int f\chi_M \qquad \text{and} \qquad \int f_n = \int f_n\chi_M$$

This implies now the Lemma. ∎

Theorem 4. *Let $\{g_n\}$ be a sequence in $\mathcal{M}^+(X)$. Then*

$$\int \left(\sum_{n=0}^{\infty} g_n\right) d\mu = \sum_{n=0}^{\infty} \int_X g_n\, d\mu$$

Proof. Let $f_m := \sum_{n=0}^{m} g_m$. Then $f_m \in \mathcal{M}^+(X)$ is an increasing sequence with $\lim f_m = \sum_{n=0}^{\infty} g_n$. It follows by the Monotone Convergence Theorem that

$$\begin{aligned} \int \left(\sum_{n=0}^{\infty} g_n\right) d\mu &= \int \lim f_m\, d\mu \\ &= \lim \int f_m\, d\mu \\ &= \lim \sum_{n=1}^{m} \int g_n\, d\mu \\ &= \sum_{n=0}^{\infty} \int g_n\, d\mu \end{aligned}$$

∎

Definition 4. *A nonnegative measurable function f is integrable over $E \in \mathcal{B}$ if $\int_E f < \infty$. A function $f : X \to \overline{\mathbb{R}}$ is integrable over E if f is measurable and both f^+ and f^- are integrable over E. The integral of f is then defined as*

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

The set of integrable functions is denoted by $\mathcal{L}^1(X) = \mathcal{L}^{`}(X, \mu)$..

Notice that a function is integrable over E if and only if $f \in \mathcal{L}^1(E, \mu_E)$ where μ_E.

Lemma 10. *The function f is integrable over E if and only if $|f|$ is integrable over E and in that case $\left|\int f\right| \leq \int |f|$.*

Proof. We have $|f| = f^+ + f^-$. Hence $|f|$ is integrable if and only if f^+ and f^- are integrable and

$$\int_E |f| = \int_E f^+ + \int_E f^- \geq \left|\int_e f^+ - \int_E f^-\right| = \left|\int_E f\right|$$

∎

The next lemma show that the integral of f is independent of the representation of f as a difference of positive functions.

Lemma 11. *Let $f = f_1 - f_2$ with f_1, f_2 nonnegative. Then f is integrable if f_1 and f_2 are integrable and in that case $\int f = \int f_1 - \int f_2$.*

Proof. We have $f = f_1 - f_2 = f^+ - f^-$. It follows that $f(x) \leq f_1(x)$. So in particular it follows that $f^+(x) \leq f_1(x)$. Hence f^+ is integrable if f_1 is integrable. Similarly it follows that f^- is integrable if f_2 is integrable. Thus f is integrable if f_1 and f_2 are integrable. We have $f_1 + f^- = f_2 + f^+$. Thus

$$\int f_1 + f^- = \int f_2 + f^+$$

But we have already seen that the integral is additive for positive functions. Thus

$$\int f_1 + \int f^- = \int f_1 + \int f^+ .$$

Hence - as all the integrals are finite- we have

$$\int f_1 - \int f_2 = \int f^+ - \int f^- = \int f$$

∎

Proposition 2. *Let $E \in \mathcal{B}$, $f, g \in \mathcal{M}(X)$ integrable over E, and $c \in \mathbb{R}$, Then the following holds.*

1. *$cf + g$ is integrable over E and $\int_E cf + g = c \int f + \int g$ (i.e., the space of integrable functions over E is a vector space and $\int$ is a linear functional).*
2. *If $h \in \mathcal{M}(X)$ and $|h| \leq |f|$, then h is integrable.*
3. *If $g \leq f$ a.e. then $\int g \leq \int f$.*
4. *If $g = 0$ a.e. then $\int g = 0$.*

Proof. 1) The claim is obvious if $c = 0$. Assume that $c > 0$. Then

$$cf + g = c(f^+ + g^+) - (f^- + g^-).$$

The claim follows by the last lemma.

2) This follows by Lemma 8 and Lemma 10.

3) $f - g \geq 0$ (a.e.). Then $(f - g)^- = 0$ almost everywhere. By Corollary 1 it follows that $\int (f-g)^- = 0$. Hence

$$\int f - g = \int (f-g)^+ \geq 0$$

Hence by (1) $\int f - \int g \geq 0$ or $\int f \geq \int g$.

4) Follows by (3). ∎

Remark 1. *If f is integrable, then the set where f is infinite is of measure zero.*

Theorem 5 (Lebesgue Dominated Convergence Theorem). *Let $\{f\}_n$ be a sequence of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n. Then f is integrable and*

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proof. We remark first that $\int |f|$ is finite because of Fatou's Lemma. Hence $\left|\int f\right| < \infty$. Be redefining the functions f_n and f on a set of measure zero we can assume that the convergence is everywhere on X. The function f is measurable and f is integrable by proposition 2. As $g + f_n \geq 0$ it follows by Fatou's Lemma that

$$\begin{aligned} \int f + \int g &= \int (f+g) \\ &\leq \underline{\lim} \int (g + f_n) \\ &= \int g + \underline{\lim} \int f_n \end{aligned}$$

Hence $\int f \leq \underline{\lim} \int f_n$. But we also have $g - f_n \geq 0$. Hence

$$\begin{aligned} \int g - \int f &= \int (g-f) \\ &\leq \underline{\lim} \int (g - f_n) \\ &= \int g - \overline{\lim} \int f_n \end{aligned}$$

Hence $\overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$. It follows that $\lim \int f_n$ exists and is equal to $\int f$. ∎

4. $\mathcal{L}^p$-Spaces

For $1 \leq p < \infty$ let $\mathcal{L}^p(X) = \mathcal{L}^p(X, \mu)$ be the space of measurable functions on X such that $|f|^p$ is in $\mathcal{L}^1(X)$. Thus a measurable function is in $\mathcal{L}^p$ if and only if

$\int |f|^p < \infty$. Let

$$||f||_p = \sqrt[p]{\int |f|^p \, d\mu}$$

We have $||f||_p = 0$ if and only if $f = 0$ a.e. We will now show that $(\mathcal{L}^p, ||\cdot||_p)$ satisfies the axioms for a Banach space except that $||\cdot||_p$ is only a pseudonorm.

Theorem 6 (Hölder's Inequality). *If $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ with $1 < p < \infty$ and $1/p + 1/q = 1$ then $fg \in \mathcal{L}^1$ and*

$$||fg||_1 \leq ||f||_p \, ||g||_q$$

Proof. The statement is clear if $||f||_p = 0$ or $||g||_q = 0$ because then $fg = 0$ a.e. Let us therefore assume that the right hand side is > 0. Let $\alpha \in (0, 1)$. Define $\phi(t) := \alpha t - t^\alpha$. Then $\phi'(t) = \alpha - \alpha t^{\alpha-1} = \alpha(1 - t^{\alpha-1})$. Hence $\phi'(t) < 0$ for $0 < t < 1$, $\phi'(t) > 0$ for $1 < t$ and $\phi'(1) = 0$. It follows that $\phi(t) > \phi(1) = \alpha - 1$. This implies that

$$t^\alpha \leq \alpha t + (1 - \alpha), \qquad t \geq 0$$

Take $t = a/b$ with a and b positive. Then $[a/b]^\alpha \leq \alpha[a/b] + (1 - \alpha)$. Multiply this by b to get

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha) b$$

Let now p and q be as in the lemma and let $\alpha = 1/p$. Let $a = |f(x)|^p \, / \, ||f||_p^p$ and let $B = |g(x)|^q \, / \, ||g||_q^q$. Then

$$\frac{|f(x)g(x)|}{||f||_p \, ||g||_q} \leq \frac{|f(x)|^p}{p \, ||f||_p^p} + \frac{|g(x)|^q}{q \, ||g||_q^q}$$

Integrating over X gives

$$||fg||_1 \, / \, ||f||_p \, ||g||_q \leq \frac{1}{p} + \frac{1}{q} = 1$$

or $||fg||_1 \leq ||f||_p \, ||g||_q$. ∎

Theorem 7 (Cauchy-Bunyakovskii-Schwarz Inequality). *Let $f, g \in \mathcal{L}^2$ Then fg is integrable and*

$$\left| \int fg \, d\mu \right| \leq \int |fg| \, d\mu \leq ||f||_2 \, ||g||_2$$

Theorem 8 (Minkowski's Inequality). *Let $1 \leq p < \infty$. Then $\mathcal{L}^p$ is a vector space, i.e., If $f, g \in \mathcal{L}^p$, and $\lambda \in \mathbb{R}$, then $\lambda f + g \in \mathcal{L}^p$. Furthermore*

$$||f + g||_p \leq ||f||_p + ||g||_p$$

Proof. This is clear for $p = 1$, so let us assume that $1 < p$. Let q be given by $1/p + 1/q = 1$. This is the case if and only if $p + q = pq$, or $p = pq - q$. That $\lambda f \in \mathcal{L}^p$ is clear. Let $x \in X$. Then

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq 2 \max\{|f(x)|, |g(x)|\}$$

Hence

$$|f(x) + g(x)|^p \leq 2^p \max\{|f(x)|^p, |g(x)|^p\} \leq 2^p \left(|f(x)|^p + |g(x)|^p\right)$$

It follows that $|f(x)+g(x)|^p$ is integrable. Hence $f+g \in \mathcal{L}^p$. Now $(p-1)q = p$. Hence $|f+g|^{p-1} \in \mathcal{L}^q$ and: We get now from Hölder's inequality:

$$|f+g|^p = |f+g|\,|f+g|^{p-1} \leq |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$$

∎

Lemma 12. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^p$, $1 \leq p < \infty$, such that for all $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $||g_n - g_m|| < \epsilon$. Then there is a $g \in \mathcal{L}^p$ such that $\lim_n ||g_n - g|| = 0$.*

Proof. To be added. ∎

Let

$$\mathcal{N} = \{f \in \mathcal{M}(X) \mid f = 0 \text{ a.e.}\}$$

Let $L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\mathcal{N}$. Then $||\cdot||_p$ is a well definite nonnegative function on L^p. Notice that the elements in L^p are **classes of functions** but all the elements in a class are equal except on a set of measure zero. Because of this will will most of the time just work with the elements of L^p as if they were functions. If $f \in \mathcal{L}^p$ then we will sometimes write $[f]$ for the corresponding class. Definition of $||\cdot||_\infty$, $\mathcal{L}^\infty(X,\mu)$ and $\mathbf{L}^\infty(X,\mu)$ will be added. It follows from what we have done so far that L^p is Banach space:

Theorem 9. *Let $1 \leq p < \infty$. Then $L^p(X,\mu)$ is a Banach space.*

4.1. **Signed Measures.** To motivate the following constructions and theorems let us assume that $f \in \mathbf{L}^1(X)$. Define a function

$$\mu_f : \mathcal{B} \to \mathbb{R}$$

by

$$\mu_f(E) := \int_E f(x)\, d\mu(x)$$

Then map μ_f has the following properties:

1. $\mu_f(\emptyset) = 0$.
2. μ_f is σ-additive: Let $\{E_n\}$ be a collection of disjoint measurable sets. Then
$$\mu_f\left(\bigcup E_n\right) = \sum \mu_f(E_n)$$
and the series on the right converges absolutely. This follows from the fact, that with $f \in \mathbf{L}^1$ we also have $|f| \in \mathbf{L}^1$ and
$$\mu_{|f|}(\cup E_n) = \sum \int_{E_n} |f|\; d\mu \geq \sum \left| \int_{E_n} f\, d\mu \right| = \sum \left|\mu_f(E_n)\right|.$$
We notice that this is necessary because the left hand side does not depend on the order we number the sets E_j but only on the union. Hence the left hand side has to be independent of the ordering that we sum. But that is the same as saying that the sum converges absolutely.
3. The map μ_f is not a measure, because it can take negative values. But we have
$$\mu_f = \mu_{f^+} - \mu_{f^-}$$
and bot μ_{f^+} and μ_{f^-} are measures. Thus we can decompose the "signed measure" μ_f into the difference of two measures.

4. Let $P := \{x \mid f(x) > 0\}$, $N := \{x \mid f(x) < 0\}$ and $Z := \{x \mid f(x) = 0\}$. Then
$$X = P\dot{\cup}N\dot{\cup}Z$$
and
(a) If $E \subset P$ then $\mu_f(E) \geq 0$. Thus P is a "positive set" for μ_f;
(b) If $E \subset N$ then $\mu_f(E) \leq 0$. Thus N is a "negative set" for μ_f;
(c) If $E \subset Z$ then $\mu_f(E) = 0$. Thus Z is a "null set" for μ_f.

We would like to generalize this to other set-functions. We start with the following definition:

Definition 5. *By a signed measure on the measurable space $(X, \mathcal{B})$ we mean an function $\mu : \mathcal{B} \to \overline{\mathbb{R}}$ such that*

1. *μ assume at most one of the values ∞ and $-\infty$.*
2. *$\mu(\emptyset) = 0$.*
3. *Let $\{E_j\}$ be a disjunct collect of sets in $\mathcal{B}$. Then*
$$\mu(\cup E_j) = \sum_j \mu(E_j)$$
where the series on the right hand side converges absolutely.

Definition 6. *Let μ be a signed measure. Let $A \in \mathcal{B}$. Then we call A for:*

1. *positive if for all $B \subseteq A$ we have $\mu(B) \geq 0$.*
2. *negative if it is positive for $-\mu$.*
3. *null set if it is both positive and negative.*

Notice that A is a null set if and only if $\mu(B) = 0$ for all $B \subseteq A$. Notice that it is necessary to request each subset to be positive, negative respectively zero. For example take $X = [-1, 1]$ and $f(x) =$sign(x). Let $\mu = \lambda_f$ where λ is the Lebesgue measure on $[-1, 1]$. Then $\mu(X) = 0$, but X is not a null set. If $A = [-0.5, 1]$ then $\mu(A) = 0.5 > 0$, but $\mu([-0.5, 0]) = -0.5 < 0$ whereas $\mu([0, 1]) = 1 > 0$. Thus A is not positive.

Lemma 13. *Let μ be a signed measure on $(X, \mathcal{B})$. Let $E \in \mathcal{B}$ be such that $0 < \mu(E) < \infty$. Then there is a positive set $A \subseteq E$ such that $\mu(A) > 0$.*

Proof. If E is positive, then we are done. So let us assume that E is not positive. Then there exists a measurable set $E_1 \subseteq E$ such that $\mu(E_1) < 0$. Let $n_1 \in \mathbb{N}$ be the smallest integer such that there exists a set $E_1 \subseteq E$ with
$$\mu(E_1) < -1/n_1.$$
If $E \setminus E_1$ is not positive then we can find a smallest integer n_2 such that there exists a set $E_2 \subseteq E \setminus E_1$ such that
$$\mu(E_2) < -1/n_2.$$
Inductively we construct disjoint measurable sets $\{E_k\}$ such that if $E \setminus \cup_{j=1}^k E_j$ is not positive, then n_{k+1} is the smallest integer such that there exists a measurable set $E_{k+1} \subseteq E \setminus \cup_{j=1}^k E_j$ and $\mu(E_{k+1}) \leq -1/n_{k+1}$. Notice that by construction we have
$$n_j \leq n_{j+1}\,.$$

Furthermore

$$\begin{aligned}\mu(E) &= \mu(E \setminus \cup_{j=1}^{k} E_j) + \sum_{j=1}^{k} \mu(E_j) \\ &\leq \mu(E \setminus \cup_{j=1}^{k} E_j) - \sum_{j=1}^{k} \frac{1}{n_j}\end{aligned}$$

In particular $0 < \mu(E) \leq \mu(E \setminus \cup_{j=1}^{k} E_j)$. Notice that if $E \setminus \cup_{j=1}^{k} E_j$ is positive then this implies our statement. Thus we can assume that the sequence $\{E_j\}$ is infinite. Let $F := \cup E_j$ and $A := E \setminus F$. Then both A and F have finite measures and $\mu(A) \geq \mu(E) > 0$. Both statements follows by $\mu(E) = \mu(A) + \mu(F) < \infty$ and the fact that μ can only take one of the values ∞ or $-\infty$. But

$$\mu(F) = -\sum \frac{1}{n_j}$$

Hence $\sum \frac{1}{n_j} < \infty$. In particular we get that n_j is a monotone sequence converging to ∞.

We will now show that A is positive. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{n_k - 1} < \epsilon \qquad \forall k \geq N$$

This is possible because $n_k \leq n_{k+1}$. As $A \subseteq E \setminus \cup_{j=1}^{k} E_j$ it follows that A can not contain a set B such that

$$\mu(B) \leq -1/n_{k+1} \leq \frac{-1}{n_k - 1} < -\epsilon\,.$$

Hence $\mu(B) \geq -\epsilon$. As ϵ was arbitrary it follows that $\mu(B) \geq 0$. ∎

Theorem 10 (Hahn Decomposition Theorem)**.** *Let μ be a signed measure on a measurable space $(X, \mathcal{B})$. Then there is a positive set P and a negative set N such that*

$$X = P \dot{\cup} N\,.$$

Proof. We can assume that $\mu(L) < \infty$ for all $L \in \mathcal{B}$. Otherwise we replace μ by $-\mu$. We notice that the set of positive sets is not empty as $\emptyset$ is positive. Let

$$\lambda := \sup_{P \text{ positive}} \mu(P) \leq \infty\,.$$

Let $\{P_n\}$ be a sequence of positive sets such that $\lim \mu(P_n) = \lambda$, and let $P := \cup P_n$ and let $F_1 := P_1$ and

$$F_{n+1} := P_{n+1} \setminus \cup_{j=1}^{n} P_j$$

We claim that P is positive. Let $E \subseteq P$. Then

$$E = \cup E \cap P_n = \dot{\cup} E \cap F_n$$

As P_n is positive it follows that $\mu(E \cap F_n) \geq 0$. Hence

$$\mu(E) = \sum \mu(E \cap F_n) \geq 0$$

In particular we get

$$\mu(P) \leq \lambda\,.$$

But on the other hand we have $\mu(P) \geq \mu(P_n)$ as $P_n \subseteq P$ and P is positive. Hence

$$\lambda \geq \mu(P) \geq \sup \mu\,(P_n) = \lim \mu(P_n) = \lambda$$

It follows that

$$\mu(P) = \lambda\,.$$

In particular

$$0 \leq \lambda < \infty\,.$$

Let $N = X \setminus P$. Assume that N is not negative. Then there is $E \subseteq N$ such that $\mu(E) > 0$. The set E can not be positive because then $P \cup E$ is positive and

$$\lambda \geq \mu(P \cup E) = \mu(P) + \mu(E) = \lambda + \mu(E) > \lambda\,.$$

But by the last lemma we can find a positive set $A \subseteq E$ such that $\mu(E) > 0$. That again would give a contradiction. Hence N is negative. The theorem follows now because P and N are disjoint by construction. ∎

Notice that the Hahn-decomposition is not unique. For example in the case μ_f then P and $P \cup Z$ are positive sets and the negative sets in the Hahn decomposition are $N \cup Z$ respectively N. But the restriction of μ to P and N is unique:

Lemma 14. *Let $(X, \mathcal{B}, \mu)$ be as in the last theorem. Let $X = P_1 \dot\cup N_1 = P_2 \dot\cup N_2$ be two decompositions of X into positive and negative set. Let $E \in \mathcal{B}$. Then $\mu(E \cap P_1) = \mu(E \cap P_2)$ and $\mu(E \cap N_1) = \mu(E \cap N_2)$.*

We keep our setup $(X, \mathcal{B}, \mu)$. This lemma allows us to define two measures μ^+ and μ^- in the following way: Let $X = P \cup N$ be a Hahn decomposition for X. Let $E \in \mathcal{B}$. Then

$$\begin{aligned} \mu^+(E) &: = \mu(E \cap P) \\ \mu^-(E) &: = -\mu(E \cap N)\,. \end{aligned}$$

Then μ^+ and μ^- are measures and $\mu = \mu^+ - \mu^-$. We call μ^+ the *positive part* of μ and μ^- the *negative part* of μ. The measure $|\mu| := \mu^+ + \mu^-$ is called the *total variation* of μ.

Definition 7. *Let μ and λ be measure on the same sigma algebra $\mathcal{B}$. Then μ and λ are mutually singular (denoted by $\mu \perp \lambda$) if there exists two disjoint sets $A, B \in \mathcal{B}$ such that $X = A \cup B$ and*

$$\mu(A) = \lambda(B) = 0.$$

We say that μ is absolutely continuous with respect to λ (denoted by $\mu << \lambda$) if $\lambda(E) = 0$ implies $\mu(E) = 0$.

Remark 2. *We can also use the above phrases for signed measure. If μ and λ are signed measure then $\mu \perp \lambda$ if $|\mu| \perp |\lambda|$ and $\mu << \lambda$ if $|\mu| << |\lambda|$.*

Notice that $\mu(E) = \mu(E \cap A)$ and $\lambda(E) = \lambda(E \cap B)$ for all $E \in \mathcal{B}$. Thus μ and λ are supported by disjoint sets. The consequence of the above discussion is the following theorem:

Theorem 11 (The Jordan Decomposition)**.** *Let μ be a signed measure defined on $\mathcal{B}$. Then there are two mutually singular measure μ^+ and μ^- on $\mathcal{B}$ such that*

$$\mu = \mu^+ - \mu^-\,.$$

Moreover there is only one such set of mutually singular measures.

4.2. **The Radon-Nikodym Theorem.** Our first application of the Hahn decomposition if the proof of the Radon-Nikodym theorem, i.e., that $\mu << \lambda$ if and only if $\mu = \lambda_f$ for some $f \in M^+(X)$.

Theorem 12 (Radon-Nikodym)**.** *Let μ and λ be measures defined on $(X, \mathcal{B})$. Assume that X is $\sigma-$finite with respect to λ and that $\mu << \lambda$. Then there exists a $f \in M^+(X)$ such that $\mu = \lambda_f$. The function f is unique up to set of measure zero.*

Proof. Assume first that $\lambda(X) < \infty$. For $c > 0$ let

- $P(c)$: The positive set for the signed measure $\mu - c\lambda$;
- $N(c)$: The negative set for the signed measure $\mu - c\lambda$.

Let

- $A_1 := N(c)$;
- $A_{k+1} := N((k+1)c) \setminus \bigcup_{j=1}^{k} A_j$.

Then

$$A_k = N(kc) \cap \bigcap_{j=1}^{k-1} P(jc)\,.$$

Let $E \subset A_k$ be measurable. Then

- $(\mu - kc\lambda)(E) \leq 0$ or $\mu(E) \leq kc\lambda(E)$;
- For $j < k$: $(\mu - jc\lambda)(E) \geq 0$ or $\mu(E) \geq jc\lambda(E)$.

In particular by taking $j = k - 1$ we get:

$$(k-1)c\lambda(E) \leq \mu(E) \leq k\mu(E)\,.$$

Define now

$$A := \bigcup_{j=1}^{\infty} A_j \qquad \text{and} \qquad B = X \setminus A$$

Then $\mu = \mu_A + \mu_B$ and the same holds for λ. Let $E \in B$ be contained in B. Then E is a positive set for $\mu - kc\lambda$ for all k. Thus $\mu(E) \geq kc\lambda(E)$ for all k. Thus if $\lambda(E) \neq 0$ then $\mu(E) = \infty$. On the other hand if $\lambda(E) = 0$ then $\mu(E) = 0$ because $\mu << \lambda$. Hence we can define $f(x) = \infty$ for all $x \in B$.

Let us now work on A. Notice that A and B do not depend on the choice of $c > 0$! We will write $A(c)$ and $A_k(c)$ if we need the dependence of those sets on c. Take $c = 2^{-n}$, $n \in \mathbb{N}$. Notice that in this case we can choose the sets A and A_k such that

$$A_k(2^{-n}) = A_{2k-1}(2^{-(n+1)}) \cup A_{2k}(2^{-(n+1)})$$

Define

$$f_n(x) := (k-1)/2^n\,, \quad x \in A_k\;.$$

Then the above shows that the sequence $\{f_n\}$ is monotonic increasing. Define the function f on A by $f = \sup f_n$. Then be the monotone convergence theorem it follows that

$$\int_E f\, d\lambda = \sup \int_E f_n\, d\lambda$$

for all $E \in \mathcal{B}$, $E \subset A$. We also have

$$\begin{aligned}\int_{E\cap A_k} f_n\,d\lambda &= (k-1)2^{-n}\lambda(E\cap A_k)\\ &\leq \mu(E\cap A_k)\\ &\leq k2^{-1}\lambda(E\cap A_k)\\ &= \int_{E\cap A_k} f_n\,d\lambda + 2^{-n}\lambda(E\cap A_k)\,.\end{aligned}$$

Thus

$$\int_{E\cap A_k} f_n\,d\lambda \leq \mu(E\cap A_k) \leq \int_{E\cap A_k} f_n\,d\lambda + 2^{-n}\lambda(E\cap A_k)\,.$$

Let $\epsilon > 0$ be given. Choose N such that $2^{-N}\lambda(E\cap A_k) < \epsilon$. (Notice that we are using the A_k defined by $c = 2^{-N}$ so this set is fixed.) Then

$$\int_{E\cap A_k} f_n\,d\lambda \leq \mu(E\cap A_k) \leq \int_{E\cap A_k} f_n\,d\lambda + \epsilon \qquad \forall n \geq N.$$

Thus

$$\int_{E\cap A_k} f d\lambda \leq \mu(E\cap A_k) \leq \int_{E\cap A_k} f\,d\lambda + \epsilon$$

As ϵ was arbitrary we finally get

$$\mu(E\cap A_k) = \int_{E\cap A_k} f\,d\lambda$$

This implies:

$$\begin{aligned}\mu(E\cap A) &= \sum \mu(E\cap A_k)\\ &= \sum \int_{E\cap A_k} f\,d\lambda\\ &= \int_{E\cap A} f\,d\lambda\end{aligned}$$

This shows that $\mu = \lambda_f$.

If $\lambda(X) = \infty$, then we choose a disjoint sequence $\{X_n\}$ such that $X = \cup X_n$ and $\mu(X_n) < \infty$ for all n. Let f_n be the function constructed above for λ and μ restricted to X_n. We view f_n as a function on X be letting $f|_{X\setminus X_n} = 0$. Let $f = \sum f_n$. Then $\mu = \lambda_f$. ∎

Theorem 13 (Lebesgue Decomposition). *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let ν be a σ-finite measure defined on $\mathcal{B}$. Then we can find measures ν_0 and ν_1 such that:*

1. *ν_0 is singular with respect to μ;*
2. *$\nu_1 << \mu$;*
3. *$\nu = \nu_0 + \nu_1$.*

The measures ν_0 and ν_1 are unique.

Proof. Let $\lambda := \nu + \mu$. Then λ is σ-finite and $\nu, \mu << \lambda$. Choose $f, g \in M^+$ such that

$$\mu(E) = \int_E f \, d\lambda \quad \text{and} \quad \nu(E) = \int_E g \, d\lambda \, .$$

Let $A := \{x \mid f(x) > 0\}$ and $B := \{x \mid f(x) = 0\}$. Then $A, B \in \mathcal{B}$ and $X = A \dot{\cup} B$. Define ν_0 and ν_1 by

$$\nu_0(E) := \nu(E \cap B) \quad \text{and} \quad \nu_1(E) := \nu(E \cap B) \, .$$

Then $\nu = \nu_0 + \nu_1$. By construction we obviously have $\nu_0 \perp \mu$. On the other hand assume that $\mu(E) = 0$. Then $f(x) = 0$ for λ-almost all $x \in E$. We also have $E = (E \cap A) \cup (E \cap B)$. Hence $\lambda(E \cap B) = 0$. It follows that

$$\begin{aligned} \nu_0(E) &= \nu(E \cap B) \\ &= \int_{E \cap B} g \, d\lambda \\ &= 0 \end{aligned}$$

Hence $\nu_0 << \mu$. The uniqueness is obvious and is left to the reader. ∎

4.3. **Riesz Representation Theorem.** The next application is the proof of the Riesz Representation Theorem:

Theorem 14 (The Riesz Representation Theorem)**.** *Let (X, B, μ) be a σ-finite measure space. Let $1 \leq p < \infty$ and let $F \in \mathbf{L}^p(X)^*$. Then there exists a $g \in \mathbf{L}^q(X)$, where $1/p + 1/q = 1$, such that*

$$< f, F >= \int fg \, d\mu \qquad \forall f \in \mathbf{L}^p(X) \, .$$

We need two lemmas before we start the proof and the space $L^\infty(X)$.

Lemma 15. *Let $1 \leq p < \infty$ and let q be given by $1/p + 1/q = 1$. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $g \in L^1(X)$ such that there exists a constant $M > 0$ with*

$$\left| \int g\varphi \, d\mu \right| \leq M \, ||\varphi||_p$$

for all simple functions φ. Then $g \in \mathbf{L}^q(X)$.

Proof. See Book p. 283. ∎

Lemma 16. *Let $\{E_n\}$ be a sequence of disjoint measurable sets, and for each n let f_n be a function in $\mathbf{L}^p(X)$, $1 \leq p < \infty$, that vanishes outside E_n. Let $f := \sum f_n$. Then $f \in \mathbf{L}^q(X)$ if and only if $\sum ||f_n||_p^p < \infty$. In this case $f = \sum f_n$ in $\mathbf{L}^p(X)$ and*

$$||f||_p = \sqrt[p]{\sum ||f_n||_p^p} \, .$$

Proof. We can assume that $X = \cup E_n$. Let $x \in X$ then there is an unique n such that $x \in E_n$. Hence $f(x) = f_n(x)$. In particular $|f(x)|^p = |f_n(x)|^p$. It follows (using the monotone convergence theorem) that

$$\begin{aligned}\int |f(x)|^p \; d\mu(x) &= \sum \int_{E_n} |f(x)|^p \; d\mu(x) \\ &= \sum \int_{E_n} |f_n(x)|^p \; d\mu(x) \\ &= \sum ||f_n||^p \\ &< \infty\end{aligned}$$

In the same way

$$\begin{aligned}\left|\left| f - \sum_{j=1}^{N} f_n \right|\right|_p^p &= \left|\left| \sum_{j=N+1}^{\infty} f_n \right|\right|_p^p \\ &\leq \sum_{j=N+1}^{\infty} ||f_n||_p^p\end{aligned}$$

and $\sum_{j=N+1}^{\infty} ||f_n||_p^p$ can be made arbitrary small. From this the Lemma follows. ∎

We can now proof the Riesz Representation Theorem:

Proof. (Of the Riesz Representation Theorem) We can obviously assume that $\mu(X) > 0$. Assume first that $\mu(X) < \infty$. Define

$$\nu(E) := < F, \chi_E > , \qquad E \in \mathcal{B} .$$

Then $|\nu(E)| \leq ||F|| \, ||\chi_E|| \leq ||F|| \; \mu(X)^{1/p}$ for all $E \in \mathcal{B}$. Moreover:

1. $\nu(\emptyset) = 0$ because $\chi_\emptyset = 0$.
2. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union. Then

$$\chi_E = \sum_{n=1}^{\infty} \chi_{E_n} \qquad (\text{in } \mathbf{L}^p)$$

Let $f = \sum (\text{sign}(\nu(E_n)))\chi_{En}$. Then $f \in \mathbf{L}^p$ and

$$< F, f > = \sum_{n=1}^{\infty} |\nu(E_n)| \; .$$

Hence $\sum_{n=1}^{\infty} |\nu(E_n)|$ converges absolutely and it follows that

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E_n) \, .$$

Hence ν is a signed measure. Let $\nu = \nu^+ - \nu^-$. Suppose that $\mu(E) = 0$. Then $\chi_E = 0$ almost everywhere. Let P (respectively N) be the positive (respectively negative) set for ν. Then $\mu(E \cap P) = \mu(E \cap N) = 0$. In particular $< F, \chi_{E\cap P} > = < F, \chi_{E\cap N} > = 0$. Thus

$$\nu^+(E) = \nu(E \cap P) = < F, \chi_{E\cap P} > = 0$$

and similarly $\nu^-(E) = 0$. It follows that ν^+ and ν^- are absolutely continuous with respect to μ. Thus there are $g^+, g^- \in M^+$ such that

$$\nu^+ = \mu_{g^+} \quad \text{and} \quad \nu^- = \mu_{g^-} \, .$$

Hence $\nu = \mu_g$ where $g = g^+ - g^-$. As $\nu^+(X), \nu^-(X) < \infty$ it follows that $g \in \mathbf{L}^1$. We need to show that $g \in \mathbf{L}^q(X)$. Let

$$f(x) = \frac{\operatorname{sign}(g(x))}{\mu(X)^{1/p}} |g(x)|^{p-1} .$$

∎

5. Measure and Outer Measure

5.1. Outer Measures and Measurability.

5.2. The Extension Theorem.

Definition 8. *A collection $\mathcal{A}$ of subsets of X is called an algebra of sets if*

1. *If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$;*
2. *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.*

Lemma 17. *Let $\mathcal{A}$ be an algebra of sets. Let $E_j \in \mathcal{A}$, $j = 1, \ldots, n$ be a finite collection of sets in $\mathcal{A}$. Then $\cup_{j=1}^n E_j$, $\cap_{j=1}^n E_j \in \mathcal{A}$.*

Proof. It suffice to show that with $A, B \in \mathcal{A}$ it follows that $A \cap B \in \mathcal{A}$. The Lemma follows then by induction. But $A \cap B = (X \setminus [A^c \cup B^c]) \in \mathcal{A}$. ∎

Definition 9. *A measure μ on an algebra $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that*

1. *$\mu(\emptyset) = 0$;*
2. *If $\{E_i\}$ is a disjoint sequence of sets in $\mathcal{A}$ such that $\cup E_i \in \mathcal{A}$. Then*

$$\mu(\cup E_i) = \sum \mu(E_i) .$$

Thus μ is a measure if and only if $\mathcal{A}$ is a σ-algebra. We will show that if μ is a measure on a algebra $\mathcal{A}$ then there exists a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$ and a measure $\bar{\mu}$ on $\mathcal{B}$ that extends μ, i.e., $\bar{\mu}(E) = \mu(E)$ for all $E \in \mathcal{A}$. We do this by first constructing an outer measure μ^* that extends μ. Then the extension theorem from the last section guaranties the existence of $\bar{\mu}$. We define μ^* by

$$\mu^*(B) = \inf \sum_{j=1}^{\infty} \mu(A_i)$$

where the inf is taking over all sequences $\{A_i\}$ in $\mathcal{A}$ such that $B \subset \cup A_i$.

Lemma 18. *The function $\mu^* : \mathcal{P}(X) \to \bar{\mathbb{R}}$ satisfies the following:*

1. *$\mu^*(\emptyset) = 0$;*
2. *$\mu^*(B) \geq 0$;*
3. *If $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$;*
4. *If $A \in \mathcal{A}$ then $\mu^*(A) = \mu(A)$;*
5. *If $\{A_n\}$ is a sequence of sets then*

$$\mu^*(\cup A_i) \leq \sum \mu^*(A_i) .$$

In particular it follows that μ^* is an outer measure.

Proof. (1)-(3) follows directly from the definition.

4) As $A \subset A \cup \emptyset \cup \emptyset \cdots$ if follows that $\mu(A) \geq \mu^*(A)$. On the other hand if $A \subset \cup E_n$ then $A = \cup(A \cap E_n) = \cup(A \cap F_n)$ where $F_1 := E_1$, and

$$F_{n+1} = E_{n+1} \setminus \cup_{j=1}^n F_j = E_{n+1} \setminus \cup_{j=1}^n E_j$$

Hence

$$\mu(A) \leq \sum \mu(A \cap F_n) \leq \sum \mu(A \cap E_n) \leq \sum \mu(E_n).$$

Hence $\mu(A) \leq \mu^*(A)$.

5) Let $\epsilon > 0$. For each n choose $\{E_{n,k}\}$ such that

$$A_n \subset \cup E_{n,k}$$

and

$$\sum_k \mu(E_{n,k}) \leq \mu^*(A_n) + \epsilon 2^{-n}.$$

Then

$$\cup A_n \subset \cup_n \cup_k E_{n,k}$$

and hence

$$\begin{aligned} \mu^*(\cup A_n) &\leq \sum_{n,k} \mu(E_{n,k}) \\ &\leq \sum_n \mu^*(A_n) + \epsilon \end{aligned}$$

Since ϵ was arbitrary it follows that $\mu(\cup A_n) \leq \sum \mu^*(A_n)$. ∎

Lemma 19. *Let $\mathcal{B}$ be the σ-algebra given by μ^*. Then $\mathcal{A} \subset \mathcal{B}$.*

Proof. Let $E \in \mathcal{A}$ and let $A \subset X$. Let $\epsilon > 0$ be given. Let $\{A_n\}$ be a sequence in $\mathcal{A}$ such that

$$A \subset \bigcup_{n=1}^{\infty} A_n$$

and

$$\sum \mu(A_n) \leq \mu^*(A) + \epsilon.$$

Then, since $A \cap E \subset \cup(A_n \cap E)$ and $A \cap E^c \subset \cup(A_n \cap E^c)$ and $A_n \cap E_n$, $A_n \cap E_n^c \in \mathcal{A}$ it follows that

$$\mu^*(A \cap E) \leq \sum \mu(A_n \cap E)$$

and

$$\mu^*(A \cap E^c) \leq \sum \mu(A_n \cap E^c).$$

Thus

$$\begin{aligned} \mu^*(A) &\leq \mu^*(A \cap E) + \mu^*(A \cap E^c) \\ &\leq \sum \mu(E \cap A_n) + \sum \mu(E^c \cap A_n) \\ &= \sum \mu(A_n) \\ &\leq \mu^*(A) + \epsilon \end{aligned}$$

As $\epsilon > 0$ was arbitrary it follows that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Hence E is measurable. ∎

By the extension theorem we get now:

Theorem 15 (Carathéodory Extension Theorem)**.** *Let μ be a measure on the algebra $\mathcal{A}$. Then there exists a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$ and a complete measure $\bar{\mu}$ on $\mathcal{B}$ such that $\bar{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$. If μ is finite then so is $\bar{\mu}$.*

Theorem 16 (Hahn Extension Theorem)**.** *Let μ be a σ-finite measure on the algebra $\mathcal{A}$. Let $\mathcal{B}$ be the σ-algebra of μ^*-measurable sets. Then the extension $\bar{\mu}$ of μ is unique and $\bar{\mu}$ is σ-finite.*

Proof. Suppose first that μ is finite. Let ν be a measure on $\mathcal{B}$ that agrees with μ on $\mathcal{A}$. We notice that both $\bar{\mu}$ and ν are finite. Let $E \in \mathcal{B}$. Let $\{E_n\}$ be a sequence in $\mathcal{A}$ such that $E \subset \cup E_n$. Then

$$\begin{aligned} \nu(E) &\leq \nu(\cup E_n) \\ &\leq \sum \nu(E_n) \\ &= \sum \nu(E_n)\,. \end{aligned}$$

Hence

$$\nu(E) \leq \bar{\mu}(E)\,.$$

But we also have

$$\begin{aligned} \infty &> \mu(X) = \bar{\mu}(E) + \bar{\mu}(E^c) \\ &= \nu(E) + \nu(E^c) \end{aligned}$$

because $X = E \dot{\cup} E^c$ and both $\bar{\mu}$ and ν are additive. But by the first part we have

$$\nu(E) \leq \bar{\mu}(E) \quad \text{and} \quad \nu(E^c) \leq \bar{\mu}(E^c)$$

Hence $\nu(E) = \bar{\mu}(E)$ and $\nu(E^c) = \bar{\mu}(E^c)$. This complete the proof for $\mu(X) < \infty$.

Assume now that μ is σ-finite. Choose X_n such that $\mu(X_n) < \infty$ and $X = \cup X_n$. We may assume that the sets X_n are disjunct. By the first part it follows that $\bar{\mu}|X_n = \nu|X_n$. Let $E \in \mathcal{B}$. Then $E = \cup(X_n \cap E)$. Hence

$$\begin{aligned} \bar{\mu}(E) &= \sum \bar{\mu}(E \cap X_n) \\ &= \sum \nu(E \cap X_n) \\ &= \nu(E) \end{aligned}$$

Hence the claim. ∎

Let $\mathcal{A}$ be an algebra. Then $\mathcal{P}(X)$ is a σ-algebra containing $\mathcal{A}$. Let

$$\mathcal{A}^* = \bigcap_{\mathcal{A}\subset\mathcal{C}\ \sigma\text{-algebra}} \mathcal{C}$$

Then $\mathcal{A}^*$ is the smallest σ-algebra containing $\mathcal{A}$. We call $\mathcal{A}^*$ the *σ-algebra generated by $\mathcal{A}$*. We notice that $\mathcal{B}$ from the above theorem is a σ-algebra containing $\mathcal{A}$. Hence $\mathcal{A}^* \subset \mathcal{B}$, but $\mathcal{B}$ can be much bigger. But the Theorem guaranties that we can extend μ to a measure on $\mathcal{A}^*$, by restricting $\bar{\mu}$ to $\mathcal{A}^*$.

5.3. **Product Measures.** Our first application is the construction of a product measure and Fubini's Theorem. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}.\nu)$ be measure spaces. Let

$$Z := X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

We call the sets

$$A \times B\,, \qquad A \in \mathcal{A}\,,\ B \in \mathcal{B}$$

for *measurable rectangle.* Denote the collection of all finite union of measurable rectangles by $\mathcal{C}_0$.

Lemma 20. *Let $E \in \mathcal{C}_0$. Then E is a finite collection of disjoint measurable rectangles.*

Proof. By induction we can reduce the proof to the situation where

$$E = A \times B \cup C \times D\,.$$

But then

$$E = (A \cap C) \times (B \cap D) \dot{\cup} (A \setminus C) \times B \dot{\cup} (A \cap C) \times (B \setminus D) \dot{\cup} (A \cap C) \times (D \setminus B) \dot{\cup} (C \setminus A) \times D\,.$$

∎

Lemma 21. *Let the notation be as above. Then $\mathcal{C}_0$ is an algebra of sets.*

Proof. We have by definition that any finite union of sets in $\mathcal{C}_0$ is again in $\mathcal{C}_0$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$\begin{aligned} (A \times B)^c &= \{(x, y) \mid x \notin A \,\text{or}\, y \notin B\} \\ &= A^c \times Y \cup X \times B^c \end{aligned}$$

∎

We denote by $\mathcal{A} \times \mathcal{B}$ the σ-algebra generated by $\mathcal{C}_0$. Define a function on $\mathcal{C}_0$ by

$$\lambda(\dot{\cup}_{i=1}^n A_i \times B_i) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

We have to show that λ is well defined. Then λ is a measure on the algebra $\mathcal{C}_0$. Denote by $\mu \times \nu$ the measure on $\mathcal{A} \times \mathcal{B}$ generated by λ.

Theorem 17 (Product Measure)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Then there is a complete measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$ such that*

$$\mu \times \nu(A \times B) = \mu(A)\nu(B) \qquad \forall A \in \mathcal{A},\ \forall B \in \mathcal{B}\,. \tag{5.1}$$

If μ and ν are both σ-finite, then $\mu \times \nu$ is σ-finite and the measure $\mu \times \nu$ is uniquely determined by 5.1. The measure $\mu \times \nu$ is called the product of μ and ν.

Proof. Define λ as above. Let

$$E = \bigcup_{i=1}^n A_i \times B_i = \bigcup_{j=1}^k C_j \times D_j$$

be a disjoint union of rectangles. Then

$$\begin{aligned} \chi_E(x, y) &= \sum \chi_{A_i}(x)\chi_{B_i}(y) \\ &= \sum \chi_{C_j}(x)\chi_{D_j}(y) \end{aligned}$$

Keep x fixed and integrate over Y given (recall that we are dealing with finite sums)

$$\sum \chi_{A_i}(x)\nu(B_i) = \sum \chi_{C_j}(x)\nu(D_j)\,.$$

Integrating over X gives then

$$\sum \mu(A_i)\nu(B_i) = \sum \mu(C_j)\nu(D_j)\,.$$

Hence $\lambda(E)$ is well defined.

The properties $\lambda(\emptyset) = 0$ and $\lambda(A) \leq \lambda(B)$ if $A \subset B$ are obvious. Thus we only have to show that λ is σ-additive. Let $\{E_i\} \in \mathcal{C}_0$ be disjoint and such that $\cup_i E_i \in \mathcal{C}_0$. We have to show that

$$\lambda\left(\bigcup E_i\right) = \sum \lambda(E_i)\,.$$

As the union as well as each E_i can be written as finite union of disjoint measurable rectangle, we can assume that $E = A \times B = \bigcup_{i=1}^{\infty} A_i \times B_i$, disjoint union. Then again

$$\chi_E(x,y) = \chi_A(x)\chi_B(y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)\,.$$

Integrating and using Theorem 4 and the fact that all the functions and number occurring are nonnegative, we get

$$\chi_A(x)\nu(B) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\nu(B_i)$$

Integrating over X and using Theorem 4 again gives

$$\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)\,.$$

To generalize this to finite union is now clear.

Assume that μ and ν are σ-finite. Let $X = \cup X_n$ and $Y = \cup Y_m$ with $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. Then

$$X \times Y = \bigcup_{n,m} X_n \times Y_m$$

and $\lambda(X_n \times Y_m) = \mu(X_n)\nu(Y_m) < \infty$.

The theorem follows now from the Carathéodory Extension Theorem and Hahn Extension Theorem. ∎

We can now integrate functions on $X \times Y$ with respect to the measure $\mu \times \nu$. We would like to show that the integral can be evaluated by integrating first the one variable and then the other, i.e.,

$$\begin{aligned}\int_{X\times Y} h(x,y)\, d\left(\mu \times \nu\right)(x,y) &= \int_X \left(\int_Y h(x,y)\, d\nu(y)\right) d\mu(x)\\ &= \int_X \left(\int_Y h(x,y)\, d\mu(x)\right) d\nu(y)\,.\end{aligned}$$

As usually the strategy is to prove this first for the characteristic function χ_E. That implies the claim for simple functions. We can then proof it for positive functions h by choosing an increasing sequence of simple functions, $\varphi_n \nearrow h$. The general case follows then by applying this to h^+ and h^-.

Let $E \subset X \times Y$. Let $x \in X$ and $y \in Y$. We define the *x-section* and the *y-section* of E to be the sets:

$$\begin{aligned} E_x \quad &: \quad = \{y \in Y \mid (x,y) \in E\} \ ; \\ E^y \quad &: \quad = \{x \in X \mid (x,y) \in E\} \ . \end{aligned}$$

Similarly if f is a function defined on $X \times Y$ we define the *x-section* and the *y-section* of f to be the functions:

$$f_x(y) := f(x,y) \qquad \text{and} \qquad f^y(x) := f(x,y)\,.$$

Notice that $(\chi_E)_x = \chi_{E_x}$ and similarly for y. We remark the following simple fact:

Lemma 22. *Let the notation be as above. Then the following holds:*

1. $(E_x)^c = (E^c)_x$ *and* $(E^y)^c = (E^c)^y$.
2. $(\cup_i E_i)_x = \cup\,(E_i)_x$ *and* $(\cup E_i)^y = \cup\,(E_i)^y$.

Proof. We will only show the statement for x as the proof for y is exactly the same.

1) We have

$$y \in (E_x)^c \Leftrightarrow y \notin E_x \Leftrightarrow (x,y) \notin E \Leftrightarrow (x,y) \in E^c \Leftrightarrow y \in (E^c)_x\,.$$

2) Similarly we have

$$\begin{aligned} y \quad &\in \quad (\cup E_i)_x \Leftrightarrow (x,y) \in \cup E_i \\ &\Leftrightarrow \quad \exists i\,:\,(x,y) \in E_i \\ &\Leftrightarrow \quad \exists i\,:\,y \in (E_i)_x \\ &\Leftrightarrow \quad y \in \cup E_{ix} \end{aligned}$$

∎

Definition 10. *A class $\mathcal{M}$ of sets is called monotone if it is non-empty and $\mathcal{M}$ contains all union of increasing sequences of elements in $\mathcal{M}$ as well as the intersection of all decreasing sequences of elements in $\mathcal{M}$.*

Lemma 23. *Let $\mathcal{C}$ be an algebra of sets. Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$ and let $\mathcal{B}$ be the smallest algebra monotone class containing $\mathcal{C}$. Then $\mathcal{A} = \mathcal{B}$.*

Proof. Obviously $\mathcal{B} \subset \mathcal{A}$. To show the opposite we need to show that $\mathcal{B}$ is an algebra, because then each countable union can be written as union of increasing sets from $\mathcal{B}$.

Let $E \in \mathcal{B}$. Define

$$\mathcal{B}(E) := \{F \in \mathcal{B} \mid E \setminus F, F \setminus E, E \cap F \in \mathcal{B}\}\ .$$

Then $E, \emptyset \in \mathcal{B}(E)$ and $\mathcal{B}(E)$ is a monotone class. Moreover $F \in \mathcal{B}(E)$ if and only if $E \in \mathcal{B}(F)$.

Suppose that $E \in \mathcal{C}$. Then - because $\mathcal{C}$ is an algebra- we have $\mathcal{C} \subset \mathcal{B}(E)$. Hence - as $\mathcal{B}$ was the smallest monotone class generated by $\mathcal{C}$ - we get $\mathcal{B}(E) = \mathcal{B}$.

Let now $F \in \mathcal{B}$ be arbitrary. Then $F \in \mathcal{B}(E)$. Hence $E \in \mathcal{B}(F)$. It follows that $\mathcal{C} \subset \mathcal{B}(F)$. So again we get $\mathcal{B}(F) = \mathcal{B}$. Thus $\mathcal{B}$ is closed under taking complement and finite intersection. Hence it is also closed under taking finite union. Hence it is an algebra and then also a σ-algebra. ∎

Lemma 24. *Let the notation be as above. Let $x \in X$ and $y \in Y$. Then the following holds:*

1. *If* $E \in \mathcal{A} \times \mathcal{B}$ *then* $E_x \in \mathcal{B}$ *and* $E^y \in \mathcal{A}$.

2. *If f is measurable then f_x and f^y are measurable.*

Proof. 1) Assume first that $E = A \times B$. Then $E_x = \emptyset \in \mathcal{B}$ if $x \notin A$ and $E_x = B \in \mathcal{B}$ if $x \in A$. Also if

$$E = \bigcup_{i=1}^{n} A_i \times B_i \in \mathcal{C}_0$$

is a disjoint union of measurable rectangles, then

$$E_x = \bigcup_{i,\ x\in A_i} B_i \in \mathcal{C}_0 \, .$$

Define $\mathcal{E} = \{E \in \mathcal{A} \times \mathcal{B} \mid E_x \in \mathcal{B}\}$. Then $\mathcal{C}_0 \subset \mathcal{E}$. Let $\{E_n\}$ be an increasing sequence in $\mathcal{E}$. Then

$$\left(\cup E_n\right)_x = \cup E_{nx} \in \mathcal{B} \, .$$

Similarly we show that the intersection of a decreasing sequence is back in $\mathcal{E}$. It follows that $\mathcal{E}$ is a monotone class containing $\mathcal{C}_0$. Hence $\mathcal{E} = \mathcal{A} \times \mathcal{B}$.

The claim for E^y is shown in the same way.

2) We have for $\alpha \in \mathbb{R}$:

$$\begin{aligned} \{y \in Y \mid f_x(y) > \alpha\} &= \{y \in Y \mid f(x,y) > \alpha\} \\ &= \{(z,y) \mid f(z,y) > \alpha\}_x \in \mathcal{B} \, . \end{aligned}$$

The claim for f^y follows in the same way. ∎

Lemma 25. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $E \in \mathcal{A} \times \mathcal{B}$, $x \in X$ and $y \in Y$. Define*

$$f(x) = \nu(E_x)$$

and

$$g(y) = \mu(E^y) \, .$$

Then f and g are measurable and

$$\int f(x)\, d\mu(x) = \int g(y)\, d\nu(y) = \mu \times \nu(E) \, .$$

Proof. Assume first that X and Y are finite. Let

$$\mathcal{E} := \left\{ E \in \mathcal{A} \times \mathcal{B} \mid f \text{ is measureable and } \int f = \mu \times \nu(E) \right\}$$

We will show that $\mathcal{E} = \mathcal{A} \times \mathcal{B}$. We do that in few steps.

1) If $E = A \times B$ then $f(x) = \chi_A(x)\nu(B)$. Hence $E \in \mathcal{E}$. In particular it follows that $\mathcal{E} \neq \emptyset$.

2 Let $\{E_n\}$ be an increasing sequence in $\mathcal{E}$ then $\cup E_n \in \mathcal{E}$: So let $\{E_n\}_n$ be a sequence with $E_n \subset E_{n+1}$. Then

$$\left(\bigcup_{j=1}^{N} E_i \right)_x = (E_N)_x$$

so $\left(\bigcup_{j=1}^{N} E_i\right) \in \mathcal{E}$. Let $f_N(x) = \nu(E_{Nx})$ and $f(x) = \nu((\cup E_n)_x)$. Then $\{f_N\}$ is an increasing sequence with

$$f(x) = \lim_{N\to\infty} f_N(x)$$

so f is measurable. By the monotone convergence theorem we get

$$\int f\,d\mu = \lim \int f_N\,d\mu = \lim \mu\times\nu(E_N) = \mu\times\nu(\cup E_N)$$

by Problem p. 257. Hence $\cup E_n \in \mathcal{E}$.

3) Similarly we show that if $\{E_n\}$ is a decreasing sequence in $\mathcal{E}$ then $\cap E_n \in \mathcal{E}$.

Thus $\mathcal{E}$ is a monotone class containing the algebra $\mathcal{C}_0$. Hence $\mathcal{E} = \mathcal{A}\times\mathcal{B}$.

The theorem now follows in the usual way writing $X\times Y = \cup X_n\times Y_n$ with X_n and Y_n finite and by using the monotone convergence theorem. ∎

Theorem 18 (Tornelli's Theorem)**.** *Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be σ-finite measure spaces and let $h : X\times Y \to [0,\infty]$ be measurable. Then the functions*

$$f(x) := \int h_x(y)\,d\nu(y) \quad = \int h(x,y)\,d\nu(y)$$

and

$$g(y) := \int h^y(x)\,d\mu(x) = \int h(x,y)\,d\mu(x)$$

are measurable and

$$\int f(x)\,d\mu(x) = \int g(y)\,d\nu(y) = \int h(x,y)\,d\mu\times\nu(x,y)$$

Proof. Suppose first that $h = \chi_E$. Then $h(x,y) = \chi_{E_x}(y)$ and

$$\begin{aligned}\int_Y h(x,y)\,d\nu(y) &= \int_Y \chi_{E_x}(y)\,d\nu(y)\\ &= \nu(E_x)\,.\end{aligned}$$

By the last lemma it follows that $x\mapsto \nu(E_x)$ is measurable and

$$\int_X\left(\int_Y h(x,y)\,d\nu(y)\right)\,d\mu(x) = \int_X \nu(E_x)\,d\mu(x) = \mu\times\nu(E) = \int_{X\times Y} h\,d(\mu\times\nu)\,.$$

Similarly it follows that

$$\int_Y\int_X h(x,y)\,d\mu(x)d\nu(y) = \int_{X\times Y} h(x,y)\,d(\mu\times\nu)(x,y)\,.$$

From this it follows that the claim holds for simple functions. Let now $h\in M^+$. Choose an increasing sequence $\{\varphi_n\}$ of simple function approximating h. Then (by the monotone convergence theorem)

$$\begin{aligned}\int_{X\times Y} h &= \lim\int_{X\times Y}\varphi_n\\ &= \lim\int_X\left(\int_Y \varphi_n(x,y)\,d\nu(y)\right)\,d\mu(x)\\ &= \lim\int_Y\left(\int_X \varphi_n(x,y)\,d\mu(x)\right)\,d\nu(x)\,.\end{aligned}$$

As $\varphi_n \nearrow h$ it follows that $\int \varphi_n(x,y)\,d\nu(y)$ is an increasing sequence. Hence

$$\lim\int_X\left(\int_Y \varphi_n(x,y)\,d\nu(y)\right)\,d\mu(x) = \int_X \lim\left(\int_Y \varphi_n(x,y)\,d\nu(y)\right)\,d\mu(x)$$

But $\varphi_{nx} \nearrow h_x$. Hence by the monotone convergence theorem we have

$$\begin{aligned}\lim \int_Y \varphi_n(x,y)\, d\nu(y) &= \int_Y \lim \varphi_n(x,y)\, d\nu(y) \\ &= \int_Y h(x,y)\, d\nu(y)\,.\end{aligned}$$

As each of the functions $x \mapsto \int_Y \varphi_n(x,y)\, d\nu(y)$ are measurable it follows that $x \mapsto \int_Y h(x,y)\, d\nu(y)$ is measurable. Furthermore

$$\begin{aligned}\int_{X\times Y} h &= \lim \int_X \left(\int_Y \varphi_n(x,y)\, d\nu(y)\right) d\mu(x) \\ &= \int_X \left(\lim \int_Y \varphi_n(x,y)\, d\nu(y)\right) d\mu(x) \\ &= \int_X \left(\int_Y \lim \varphi_n(x,y)\, d\nu(y)\right) d\mu(x) \\ &= \int_X \left(\int_Y h(x,y)\, d\nu(y)\right) d\mu(x)\,.\end{aligned}$$

We show in the same way that $\int_{X\times Y} h = \int_Y \int_X h(x,y) d\mu(x) d\nu(y)$. From that the theorem follows. ∎

Theorem 19 (Fubini's Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $h \in \mathbf{L}^1(X \times Y, \mu \times \nu)$. Then the extended real-valued functions defined almost everywhere by*

$$f(x) := \int h_x(y)\, d\nu(y) \;= \int h(x,y)\, d\nu(y)$$

and

$$g(y) := \int h^y(x)\, d\mu(x) = \int h(x,y)\, d\mu(x)$$

are integrable and

$$\int f(x)\, d\mu(x) = \int g(y)\, d\nu(y) = \int h(x,y)\, d\mu \times \nu(x,y)$$

Proof. We use Tonelli's Theorem for the functions h^+ and h^-. ∎

We remark that it is necessary to assume that the function h is measurable to make sure that the integrals involving the functions h_x^+, h^{+y}, h_x^- and h^{-y} are finite for almost all y and x. But Tonelli's theorem can be used to actually proof that the function h is integrable because

$$\begin{aligned}\int_{X\times Y} |h(x,y)| &= \int_X \left(\int_Y |h(x,y)|\; d\nu(y)\right) d\mu(x) \\ &= \int_Y \left(\int_X |h(x,y)|\; d\mu(x)\right) d\nu(y)\,.\end{aligned}$$

Hence h is integrable if and only if the integral

$$\int_Y |h(x,y)|\; d\nu(y)$$

is integrable as a function of x (so in particular it has to be finite for almost all $x \in X$).

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