Generalizing Cographs to 2-cographs

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Abstract

A graph in which every connected induced subgraph has a disconnected complement is called a cograph. Such graphs are precisely the graphs that do not have the 4-vertex path as an induced subgraph. We define a 2-cograph to be a graph in which the complement of every 2-connected induced subgraph is not 2-connected. We show that, like cographs, 2-cographs can be recursively defined and are closed under induced minors. We characterize the class of non-2-cographs for which every proper induced minor is a 2-cograph. We further find the finitely many members of this class whose complements are also induced-minor-minimal non-2-cographs.

Mathematics Subject Classifications: 05C40, 05C83

1   Introduction

In this paper, we only consider simple graphs. Except where indicated otherwise, our notation and terminology will follow [6]. An **induced minor** of a graph G is any graph H that can be obtained from G by a sequence of operations each consisting of a vertex deletion or an edge contraction. If $H \neq G$, then H is a **proper induced minor** of G. Let e be an edge of G. Since we consider only simple graphs, we let G/e denote the simple graph obtained from the multigraph that results from contracting the edge e by deleting all but one edge from each class of parallel edges.

A **cograph** is a graph in which every connected induced subgraph has a disconnected complement. By convention, the graph K_1 is taken to be a cograph. Replacing connectedness by 2-connectedness, we define a graph G to be a **2-cograph** if G has no induced subgraph H such that both H and its complement, $\overline{H}$, are 2-connected. Note that K_1 is a 2-cograph. Cographs have been extensively studied over the last fifty years (see, for example, [7, 14, 5]). They are also called P_4-free graphs due to the following characterization [4].

Theorem 1. *A graph G is a cograph if and only if G does not contain the path P_4 on four vertices as an induced subgraph.*

In Section 2, we show that 2-cographs can be recursively defined, that every induced minor of a 2-cograph is also a 2-cograph, and that the complement of every 2-cograph is also a 2-cograph. In addition, we correct a result of Akiyama and Harary [1] that claims to characterize when the complement of a 2-connected graph is 2-connected.

Because the class of 2-cographs is closed under induced minors, our initial goal was to find all non-2-cographs with the property that every proper induced minor is a 2-cograph. But, as we show in Section 3, in contrast to Theorem 1, there are infinitely many such non-2-cographs. However, we were able to determine all infinite families of such graphs. For all $m \geqslant 1$, let M_m and N_m be the graphs shown in Figures 3 and 4, respectively. Let M_m' and N_m' be obtained from M_m and N_m by adding the edge st. Further, let N_m'' be the graph obtained from N_m' by adding the edge uz; let L_m be the graph shown in Figure 2; and, for all $j \geqslant 0$, let F_j be the graph shown in Figure 1. The next two theorems are the main results of the paper.

Theorem 2. *Let G be a graph that is not a 2-cograph such that every proper induced minor of G is a 2-cograph. Then*

(i) $|V(G)| \leqslant 16$; or

(ii) G is the complement of a cycle of length at least five; or

(iii) for some positive integer m, the complement of G is isomorphic to $F_{m-1}, L_m, M_m, M_m', N_m, N_m'$, or N_m''.

As we were unable to improve this bound of 16 vertices and the task of finding induced-minor-minimal non-2-cographs with at most 16 vertices seemed computationally infeasible, we were prompted to try to determine those graphs G for which both G and $\overline{G}$ are induced-minor-minimal non-2-cographs. The following theorem proves that, up to isomorphism, there are finitely many such graphs G. Its proof occupies most of Section 4.

Theorem 3. *Let G be a graph. Suppose that G is not a 2-cograph but that every proper induced minor of each of G and $\overline{G}$ is a 2-cograph. Then $5 \leqslant |V(G)| \leqslant 10$.*

The unique 5-vertex graph satisfying the hypotheses of the last theorem is C_5, the 5-vertex cycle. In the appendix, we list all of the other graphs that satisfy these hypotheses.

2 Preliminaries

Let G be a graph. A vertex u of G is a **neighbour** of a vertex v of G if uv is an edge of G. The **neighbourhood $N_G(v)$** of v in G is the set of all neighbours of v in G. Viewing G as a subgraph of K_n where $n = |V(G)|$, we colour the edges of G green while assigning the colour red to the non-edges of G. In this paper, we use the terms **green graph** and
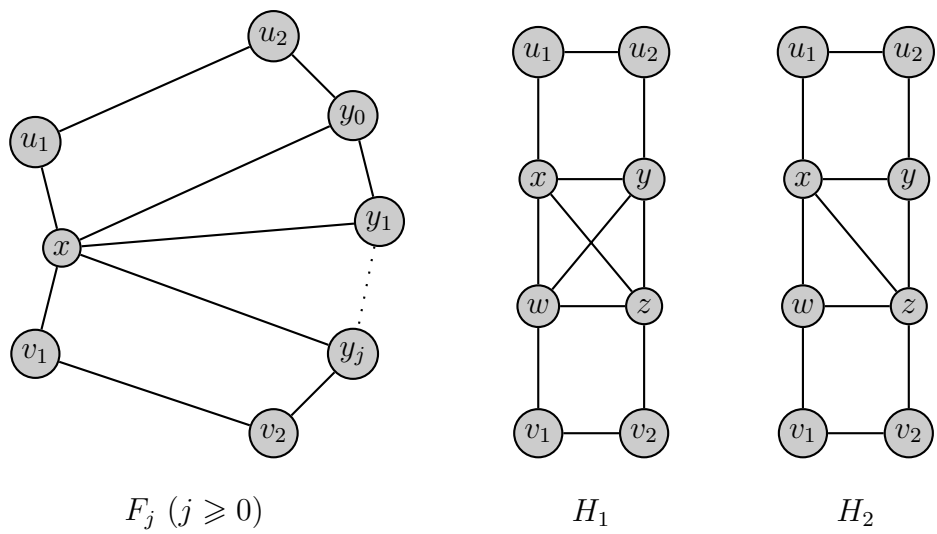
Figure 1: The complements of the induced-minor-minimal non-2-cographs that are critically 2-connected.
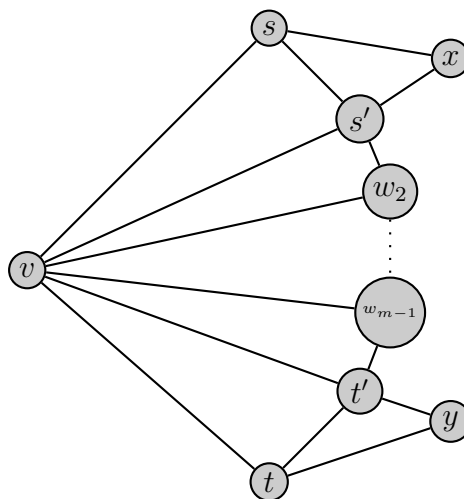

Figure 2: For each $m \geqslant 1$, the complement of the above graph L_m is an induced-minor-minimal non-2-cograph.
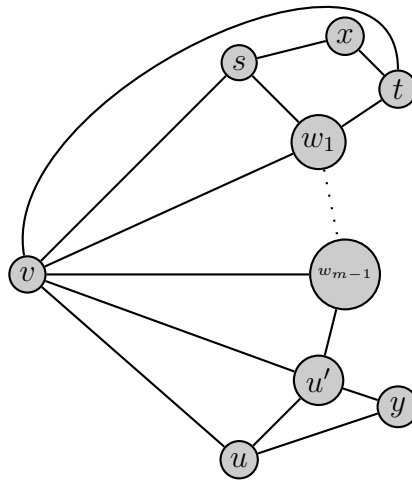
Figure 3: M_m, a graph whose complement is an induced-minor-minimal non-2-cograph.
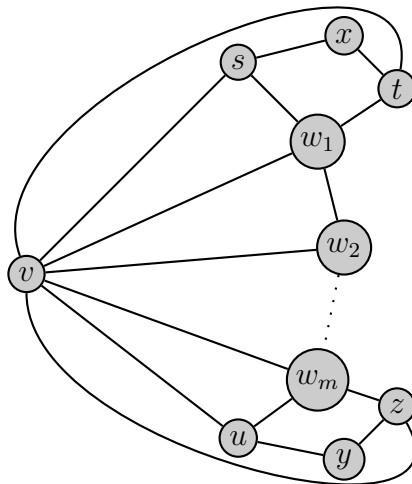

Figure 4: N_m, a graph whose complement is an induced-minor-minimal non-2-cograph.

red graph for G and its complementary graph $\overline{G}$, respectively. An edge of G is called a **green edge**, while a **red edge** refers to an edge of $\overline{G}$. The **green degree** of a vertex v of G is the number of **green neighbours** of v, while the **red degree** of v is its number of **red neighbours**.

Let G_1 and G_2 be graphs. If their vertex sets are disjoint, their **0-sum**, $G_1 \oplus_0 G_2$, is their disjoint union. Now suppose that $V(G_1) \cap V(G_2) = T$, that $G_1[T] = G_2[T]$, and that $|T| = t$. Then the union of G_1 and G_2, which has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, is a **t-sum**, $G_1 \oplus_t G_2$, of G_1 and G_2.

For $k \geqslant 1$, a graph G is a **k-cograph** if, for every induced subgraph H of G, at least one of H and $\overline{H}$ is not k-connected. Thus a 1-cograph is just a cograph. Clearly, every k-cograph is also a $(k + 1)$-cograph.

We omit the straightforward proofs of the next three results.

Lemma 4. *Let G be a k-cograph. Then*

(i) every induced subgraph of G is a k-cograph, and

(ii) $\overline{G}$ is a k-cograph.

Lemma 5. *For $0 \leqslant t < k$, a t-sum of two k-cographs is a k-cograph.*

Lemma 6. *Let G be a graph and let uv be an edge e of G. Then $\overline{G/e}$ is the graph obtained by adding a vertex w with neighbourhood $N_{\overline{G}}(u) \cap N_{\overline{G}}(v)$ to the graph $\overline{G} - \{u, v\}$.*

Cographs are also called complement-reducible graphs due to the following recursive-generation result [4]. The operation of taking the complement of a graph is called **complementation**.

Proposition 7. *A graph G is a cograph if and only if G can be generated from K_1 using complementation and 0-sum.*

Next, we show that, for $k \geqslant 2$, the class of k-cographs can be generated similarly.

Proposition 8. *For all positive integers k, a graph G is a k-cograph if and only if G can be generated from K_1 using complementation and the operation of t-sum for all t with $0 \leqslant t < k$.*

Proof. Let G be a k-cograph. If $|V(G)| \leqslant 2$, the result holds. We proceed via induction on the number of vertices of G. Assume that the result holds for all k-cographs of order less than $|V(G)|$. Since G is a k-cograph, G or $\overline{G}$ is not k-connected. Without loss of generality, we may assume that G is not k-connected. Therefore, for some $t < k$, we can write G as a t-sum of two induced subgraphs G_1 and G_2 of G. By Lemma 4, G_1 and G_2 are k-cographs and the result follows by induction.

Conversely, let G be a graph that can be generated from K_1 using complementation and t-sums. Since K_1 is a k-cograph, the result follows by Lemmas 4 and 5. $\qquad\square$

The following recursive-generation result for cographs is due to Royle [12]. It uses the concept of **join** of two disjoint graphs G and H, which is the graph $G \bigtriangledown H$ that is obtained from the union of G and H by joining every vertex of G to every vertex of H.

Proposition 9. *Let $\mathcal{C}$ be the class of graphs defined as follows:*

(i) *K_1 is in $\mathcal{C}$;*

(ii) *if G and H are in $\mathcal{C}$, then so is $G \oplus_0 H$; and*

(iii) *if G and H are in $\mathcal{C}$, then so is $G \bigtriangledown H$.*

Then $\mathcal{C}$ is the class of cographs.

For graphs G and H such that $V(G) \cap V(H) = T$ and $G[T] = H[T]$, suppose that $|T| = t$. We generalize the join operation letting $G \bigtriangledown_t H$ be the graph that is obtained from the union of G and H by joining every vertex of $V(G) - V(H)$ to every vertex of $V(H) - V(G)$. Note that $G \bigtriangledown_t H$ is the graph $\overline{\overline{G} \oplus_t \overline{H}}$.

The next result generalizes Proposition 9 to k-cographs.

Proposition 10. *For $k \geqslant 1$, let $\mathcal{C}$ be the class of graphs defined as follows:*

(i) *K_1 is in $\mathcal{C}$;*

(ii) *if G and H are in $\mathcal{C}$, then so is $G \oplus_t H$ for all t with $0 \leqslant t < k$; and*

(iii) *if G and H are in $\mathcal{C}$, then so is $G \bigtriangledown_t H$ for all t with $0 \leqslant t < k$.*

Then $\mathcal{C}$ is the class of k-cographs.

Proof. Since $G \bigtriangledown_t H$ can be written in terms of t-sum and complementation, every graph in $\mathcal{C}$ is a k-cograph. Conversely, let G be a k-cograph. If $|V(G)| = 1$, then $G \in \mathcal{C}$. We proceed by induction on $|V(G)|$. Let $|V(G)| = n \geqslant 2$ and assume that $H \in \mathcal{C}$ when H is a k-cograph with $|H| < n$. By Proposition 8, G or $\overline{G}$ is a t-sum of two smaller k-cographs. If G is the graph that can be decomposed as a t-sum, then the result follows by induction. Therefore we may assume that $\overline{G}$ is $G_1 \oplus_t G_2$ for two smaller k-cographs G_1 and G_2. Observe that $G = \overline{G_1} \bigtriangledown_t \overline{G_2}$. By Lemma 4, $\overline{G_1}$ and $\overline{G_2}$ are k-cographs and so are in $\mathcal{C}$ by induction. Therefore G is in $\mathcal{C}$. $\qquad \square$

Next we show that the class of 2-cographs is closed under contractions.

Proposition 11. *Let G be a 2-cograph and e be an edge of G. Then G/e is a 2-cograph.*

Proof. Assume to the contrary that G/e is not a 2-cograph. Then there is an induced subgraph H of G/e such that both H and $\overline{H}$ are 2-connected. Let $e = uv$ and let w denote the vertex in G/e obtained by identifying u and v. We may assume that w is a vertex of H, otherwise H is an induced subgraph of G, a contradiction. We assert that the subgraph H' of G induced on the vertex set $(V(H) \cup \{u, v\}) - \{w\}$ is 2-connected, as

is its complement $\overline{H'}$. To see this, note that, since H is 2-connected, H' is 2-connected unless one of u and v, say u, is a leaf of H'. In the exceptional case, we have $H' - u \cong H$, so G has an induced subgraph for which both it and its complement are 2-connected, a contradiction. We deduce that H' is 2-connected.

By Lemma 6, the neighbours of w in $\overline{H}$ are the common neighbours of u and v in $\overline{H'}$. Thus the degrees of u and v in $\overline{H'}$ each equal at least the degree of w in $\overline{H}$. Moreover, $\overline{H'} - u$ has a spanning subgraph isomorphic to $\overline{H}$ and is therefore 2-connected. Since u has degree at least two in $\overline{H'}$, it follows that $\overline{H'}$ is 2-connected, a contradiction. $\square$

We show next that, for all $k \geqslant 3$, a contraction of a k-cograph need not be a k-cograph. We use the following construction for the proof. Start with a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and a copy G' of G with vertex set $\{v'_1, v'_2, \ldots, v'_n\}$. Take the disjoint union of G and G', and add all the edges joining v_i to v'_i. The resulting graph, $G \square K_2$, is the Cartesian product of G and K_2.

Proposition 12. *For $k \geqslant 3$, the class of k-cographs is not closed under contraction.*

Proof. Let $G_2 = C_5$. For all $m \geqslant 3$, let $G_m = G_{m-1} \square K_2$. One can easily check that G_m is an m-connected, m-regular graph whose complement is also m-connected.

Let G'_m be a graph having an edge $e = v_1 v_2$ such that $G'_m / e = G_m$, both v_1 and v_2 have degree less than m, and v_1 and v_2 have no common neighbours. Note that every proper induced subgraph of G'_m has a vertex of degree less than m and so G'_m is a k-cograph. However, G'_m / e is not a k-cograph as it equals G_m. $\square$

By Lemma 4 and Proposition 11, the class of 2-cographs is closed under taking induced minors. In the rest of the paper, we will focus our attention on 2-cographs. The next lemma is straightforward.

Lemma 13. *All graphs having at most four vertices are 2-cographs.*

Note that a graph G is a 2-cograph if and only if G or $\overline{G}$ can be decomposed as a 0-sum or a 1-sum of two smaller 2-cographs. For an input graph G and, for t in $\{0, 1\}$, the recognition algorithm in Figure 5 attempts to decompose G as a t-sum of graphs having at most four vertices using complementation. Since such graphs are 2-cographs by Lemma 13 and we can compute the blocks of a graph in polynomial time [15, 4.1.23], the algorithm recognizes 2-cographs in polynomial time. Since 2-cographs do not have induced subgraphs isomorphic to odd cycles of length at least five or their complements, it follows by the Strong Perfect Graph Theorem [3] that all 2-cographs are perfect. However, this inclusion is proper. For example, the well-known **domino** graph obtained from a 6-cycle by adding a chord to create two 4-cycles is a perfect graph that is not a 2-cograph. We let C_6^+ denote the domino.

Akiyama and Harary [1, Corollary 1a] claimed that a 2-connected graph G has a 2-connected complement if and only if the red and green degrees of every vertex of G are at least two and G has no spanning complete bipartite subgraph. However, this result is not true. The graphs in Figure 6 are complements of each other. The first graph in the

Require: Input a simple graph G

    Set $H \leftarrow G$, BlocksList $\leftarrow [G]$

    if $|V(H)| \leqslant 4$ **then**

        remove H from BlocksList

        if BlocksList is empty **then**

            return G is a 2-cograph and exit the algorithm

        else

            update H to be an element of BlocksList

    if some K in $\{H, \overline{H}\}$ can be decomposed into 2-connected blocks **then**

        remove H from BlocksList

        add all the blocks of K to BlocksList

        update H to be an element of BlocksList

    else

        return G is not a 2-cograph and exit the algorithm

Figure 5: Algorithm for recognizing a 2-cograph.

figure satisfies the hypotheses of [1, Corollary 1a] but its complement, $C_4 \oplus_1 C_4$, is not 2-connected.

We can repair Akiyama and Harary's result as follows.

Proposition 14. *If G is a 2-connected graph, then $\overline{G}$ is a 2-connected graph if and only if G has no complete bipartite subgraph using at least $|V(G)| - 1$ vertices.*

Proof. Note that if $\overline{G}$ is not 2-connected, then G has a spanning complete bipartite subgraph or a complete bipartite subgraph on $|V(G)| - 1$ vertices. The converse is immediate. $\qquad\square$
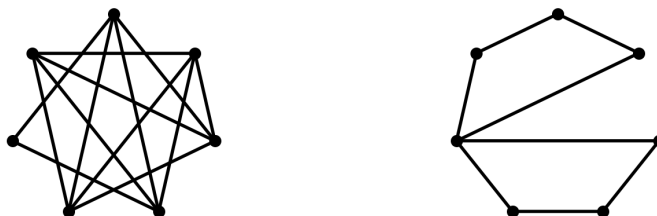

Figure 6: A counterexample to a result of Akiyama and Harary.

3 Induced-minor-minimal non-2-cographs

We noted in Section 2 that 2-cographs are closed under induced minors. In this section, we consider those non-2-cographs for which every proper induced minor is a 2-cograph. We call these graphs **induced-minor-minimal non-2-cographs**. The goal of this section is to characterize such graphs. We begin by showing that there are infinitely many of them.

Theorem 2, whose proof appears at the end of this section, specifies all of the infinite families of such graphs.

Because the proof of Theorem 2 is long, we now outline its key steps. A 2-connected graph H is **critically 2-connected** if $H - v$ is not 2-connected for all vertices v of H. In Lemma 24, we show that if G is a non-2-cograph for which every single-vertex deletion is a 2-cograph, then either G or $\overline{G}$ is critically 2-connected, or both G and $\overline{G}$ have vertex connectivity two. Propositions 26 and 28 identify the induced-minor-minimal non-2-cographs G for which, respectively, G is critically 2-connected, or $\overline{G}$ is critically 2-connected. We are then able to focus on 2-cuts in an induced-minor-minimal non-2-cograph G. Lemmas 29-36 are a sequence of incremental results whose aim is to determine the structure of G. Corollary 37 summarizes the information determined about this structure to that point. Outcome (iii) of that corollary is that G has connectivity two, and, for every 2-cut $\{g_1, g_2\}$ of G, there are exactly two components in $G - \{g_1, g_2\}$ and one of these has a single vertex. The rest of the proof of Theorem 2 deals with this case. Much of the focus there and indeed throughout the section is on the sets V_g and V_r of green-degree-two vertices and red-degree-two vertices in G.

Lemma 15. *Let G be the complement of a cycle C of length exceeding four. Then G is an induced-minor-minimal non-2-cograph.*

Proof. Certainly G is not a 2-cograph since both G and its complement are 2-connected. Moreover, by Lemma 4, $G - v$ is a 2-cograph for all vertices v of G because $\overline{G} - v$ is a path and is therefore a 2-cograph. It remains to show that G/e is a 2-cograph for all edges e of G. By Lemma 6, the complement of G/e is either a 0-sum of two paths and an isolated vertex, or a 0-sum of a path and K_2. This implies that the complement of G/e is a 2-cograph and, by Lemma 4, the result follows. $\square$

Note that the complements of cycles of length at least five are not the only induced-minor-minimal non-2-cographs. It can be checked that both C_6^+ and its complement are induced-minor-minimal non-2-cographs.

The following lemma is obtained by applying [11, Lemma 2.3] (see also [10, Lemma 4.3.10]) to the bond matroid of a 2-connected graph.

Lemma 16. *Let G be a 2-connected graph other than K_3 and let v be an arbitrary vertex of G. Then G has at least two edges incident with v each of whose contraction yields a 2-connected graph.*

An edge e of a 2-connected graph G is **contractible** if G/e is 2-connected. The following observation is immediate.

Lemma 17. *Let G be an induced-minor-minimal non-2-cograph. Then both G and $\overline{G}$ are 2-connected.*

In the rest of the section, we use the next two theorems of Chan about contractible edges in 2-connected graphs [2, Theorems 3.1, 3.3, and 3.5]. A component of a graph is **trivial** if it has just one vertex. In a 2-connected graph, a **2-cut** is a 2-element vertex cut.

Theorem 18. *Let G be a 2-connected graph that is not isomorphic to K_3. Suppose all the contractible edges of G meet a 3-element subset S of $V(G)$. Then either $G - S$ has no edges, or $G - S$ has exactly one non-trivial component and this component has at most three vertices.*

Theorem 19. *Let G be a 2-connected graph that is not isomorphic to K_3. Suppose all the contractible edges of G meet a subset S of $V(G)$ such that $|S| \geqslant 4$. Then $G - S$ has at most $|S| - 2$ non-trivial components and, between them, these components have at most $2|S| - 4$ vertices.*

We will also frequently use the following straightforward result.

Lemma 20. *Let G be a 2-connected graph. If G has a 2-cut $\{g_1, g_2\}$ such that each of g_1 and g_2 has red degree at least two and the components of $G - \{g_1, g_2\}$ can be partitioned into two sets each of which contains at least two vertices, then the red graph $\overline{G}$ is 2-connected.*

Lemma 21. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| \geqslant 6$ and let $wxyz$ be a path P of G such that both x and y have degree two in G. Then w and z are adjacent.*

Proof. Assume that w and z are not adjacent. By Lemma 17, G is 2-connected, so there is a path P' joining w and z such that P and P' are internally disjoint. This implies that G has C_5 as a proper induced minor. As C_5 is not a 2-cograph, this is a contradiction. $\square$

Lemma 22. *Let G be an induced-minor-minimal non-2-cograph. If G has two adjacent vertices of degree two, then $|V(G)| \leqslant 10$.*

Proof. Assume $|V(G)| \geqslant 11$. Let a and b be two vertices of G of degree two such that ab is a green edge. Let c be the green neighbour of a distinct from b, and let d be the green neighbour of b distinct from a. Then $c \neq d$, otherwise G is not 2-connected, contradicting Lemma 17. By Lemma 21, cd is a green edge. Observe that every vertex of $V(G) - \{a, b, c, d\}$ has red edges joining it to each of a and b. Thus $\overline{G} - \{c, d\}$ is 2-connected.

Suppose that both c and d have red degree at least three. Let w be a red neighbour of d such that $w \neq a$. It follows by Lemma 16 that w has a contractible green edge incident to it, say e, such that the other endpoint of e is not c. Then $\overline{G/e}$ is 2-connected, a contradiction.

Next suppose that both c and d have red degree two. First, we assume that c and d have the same red neighbour, say v, in $G - \{a, b\}$. Since v has green degree at least two, we have two green neighbours of v, say x and y. Note that x and y are in $V(G) - \{a, b, c, d\}$. Since x and y are adjacent to both c and d in the green graph, both the red and the green graphs induced on $\{a, b, c, d, v, x, y\}$ are 2-connected. This implies $|V(G)| \leqslant 7$, a contradiction. We may now assume that c and d have distinct red neighbours in $G - \{a, b\}$; call them v and w, respectively. Note that $vdcw$ is a green vw-path.

22.1. $G - \{a, b\}$ has no vw-path P internally disjoint from the path $vdcw$.

Assume that $G - \{a, b\}$ has such a path. Observe that the red graph and the green graph induced on the vertex set $V(P) \cup \{a, b, c, d\}$ are 2-connected and therefore, $V(G) = V(P) \cup \{a, b, c, d\}$. Now $|V(P)| \geqslant 7$ since $|V(G)| \geqslant 11$. Let e be an edge in the path P such that neither of the endpoints of e is in $\{v, w\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction. Thus 22.1 holds.

Let P_1 and P_2 be shortest vw-paths in $G - \{a, b, d\}$ and $G - \{a, b, c\}$, respectively. By 22.1, P_1 contains the vertex c and P_2 contains d. Note that $V(G) = V(P_1) \cup V(P_2) \cup \{a, b\}$. As $|V(G)| \geqslant 11$, we may assume that $P_1 - w$ has length at least three. Let e be an edge in $P_1 - w$ such that the endpoints of e are not in $\{c, v\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction.

Finally, without loss of generality, we may assume that c has red degree two and d has red degree at least three. Let v be the red neighbour of c distinct from b. Suppose that dv is red. Let x and y be two green neighbours of v, and let P be a shortest path from d to $\{v, x, y\}$ in $G - \{a, b, c\}$. Then, for $V' = \{a, b, c, d, v, x, y\} \cup V(P)$, the red and green graphs induced by V' are 2-connected, so $V' = V(G)$. As $|V(G)| \geqslant 11$, we may assume that P has length at least three. Let e be an edge in P such that the endpoints of e are not in $\{d, v, x, y\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction. Therefore, dv is green. Let w be a red neighbour of d in $G - \{a, b\}$. Let u be a green neighbour of v distinct from d. Observe that $u \neq w$, otherwise $|V(G)| \leqslant 6$ since both $G[\{a, b, c, d, v, w\}]$ and $\overline{G}[\{a, b, c, d, v, w\}]$ are 2-connected. Let P be a shortest path from w to $\{d, u, v\}$ in $G - \{a, b, c\}$. Then $V(G) = \{a, b, c, d, u, v, w\} \cup V(P)$, so we may assume that P has length at least three. Then, for an edge e of P having neither endpoint in $\{d, u, v, w\}$, both G/e and $\overline{G/e}$ are 2-connected, a contradiction. $\qquad\square$

The next lemma shows that if a path of an induced-minor-minimal non-2-cograph G has three consecutive vertices of degree two, then $G \cong C_5$.

Lemma 23. *Let G be an induced-minor-minimal non-2-cograph such that G has a path P of length exceeding three and all the internal vertices of P are of degree two. Then $G \cong C_5$.*

Proof. Let u and v be vertices of P such that the subpath P_{uv} of P joining u and v has length four. Since G is 2-connected, there is a uv-path P' such that P_{uv} and P' are internally disjoint. Assume that P' is a shortest such path. Then contracting all but one edge in P' and deleting all the vertices not in $V(P_{uv})$, we obtain C_5. Since G cannot have C_5 as a proper induced minor, $G \cong C_5$. $\qquad\square$

Lemma 24. *If G is a non-2-cograph such that $G - v$ is a 2-cograph for all vertices v of G, then G or $\overline{G}$ is critically 2-connected, or both G and $\overline{G}$ have vertex connectivity two.*

Proof. Certainly, G and $\overline{G}$ are 2-connected and, for all vertices v of G, either $G - v$ or $\overline{G} - v$ is not 2-connected. Observe that if neither G nor $\overline{G}$ is critically 2-connected, then G has vertices v and v_c such that $G - v$ and $\overline{G} - v_c$ are 2-connected. It follows that $G - v_c$ and $\overline{G} - v$ are not 2-connected, so both G and $\overline{G}$ have vertex connectivity two. $\qquad\square$

Next we find those induced-minor-minimal non-2-cographs G such that G or $\overline{G}$ is critically 2-connected. We will use the following result of Nebesky [9].

Lemma 25. *Let G be a critically 2-connected graph such that $|V(G)| \geqslant 6$. Then G has at least two distinct paths of length exceeding two such that the internal vertices of these paths have degree two in G.*

Proposition 26. *Let G be an induced-minor-minimal non-2-cograph such that G is critically 2-connected. Then G is isomorphic to C_5 or C_6^+.*

Proof. By Lemmas 13 and 15, it follows that C_5 is the unique induced-minor-minimal non-2-cograph with at most five vertices, so we may assume that $|V(G)| \geqslant 6$. Thus, by Lemma 25, G has two distinct paths P_1 and P_2 of length exceeding two such that their internal vertices have degree two. Since G is not isomorphic to C_5, by Lemma 23, we may assume that both P_1 and P_2 have length three. Lemma 21 implies that, for each i, the endpoints of P_i are adjacent. We deduce that G has C_6^+ as an induced minor. As C_6^+ is an induced-minor-minimal non-2-cograph, we deduce that $G \cong C_6^+$. $\square$

Lemma 27. *A graph G is an induced-minor-minimal non-2-cograph for which the graph $G[V_r]$ induced on V_r has at least two disjoint red edges if and only if $\overline{G}$ is a cycle with at least five vertices, or $\overline{G}$ is isomorphic to H_1, H_2, or F_m for some $m \geqslant 0$ where H_1, H_2, and F_m are shown in Figure 1.*

Proof. First we observe that if $\overline{G}$ is a cycle with $|V(\overline{G})| \geqslant 5$ or if $\overline{G}$ is isomorphic to H_1, H_2, or F_m, then $G[V_r]$ has at least two disjoint red edges. Moreover, by Lemma 15, if $\overline{G}$ is a cycle with $|V(\overline{G})| \geqslant 5$, then G is an induced-minor-minimal non-2-cograph. It is straightforward to check that if $\overline{G}$ is isomorphic to H_1 or H_2, then G is an induced-minor-minimal non-2-cograph. Finally, we show that, for all $m \geqslant 0$, the complement of F_m is an induced-minor-minimal non-2-cograph. Since $F_0 \cong C_6^+$ and the complement of the latter is an induced-minor-minimal non-2-cograph, we may assume that $m > 0$. As both F_m and $\overline{F_m}$ are 2-connected, the graph $\overline{F_m}$ is not a 2-cograph. We show that every proper induced minor H of $\overline{F_m}$ is a 2-cograph. First assume that H is an induced subgraph of $\overline{F_m}$. Deleting the vertex x from F_m leaves a path, which is a 2-cograph. Thus we may assume that x is a vertex of H. Once a vertex distinct from x is deleted from F_m, if we were to find a non-2-cograph, it must be contained in one of the blocks of the vertex deletion. Each block B of a vertex deletion of F_m that has at least three vertices must have x as a vertex. Moreover, B has x adjacent to all but at most one other vertex, so its complement is not 2-connected. It is now straightforward to see that H is a 2-cograph. For an edge uv of $\overline{F_m}$, it follows by Lemma 6 that the complement of $\overline{F_m}/uv$ is either an induced subgraph of F_m or a 1-sum of an induced subgraph of F_m with K_2 or K_3. Thus $\overline{F_m}/uv$ is a 2-cograph and so $\overline{F_m}$ is an induced-minor-minimal non-2-cograph.

Conversely, assume that G is an induced-minor-minimal non-2-cograph for which $G[V_r]$ has u_1u_2 and v_1v_2 as two disjoint red edges. Since C_5 is the unique induced-minor-minimal non-2-cograph with five vertices, we may assume that $|V(\overline{G})| \geqslant 6$, that $\overline{G}$ is not a cycle, and that no F_m for $m \geqslant 0$ is isomorphic to $\overline{G}$. Next we show the following.

27.1. In $\overline{G}$, no u_i is adjacent to any v_j.

Note that if we have a red edge connecting $\{u_1, u_2\}$ to $\{v_1, v_2\}$, then $\overline{G}$ has a path P of length three such that all the vertices of P have red degree two. Let Q be a shortest path in $\overline{G} \backslash E(P)$ joining the endpoints of P. Then $\overline{G}$ has as an induced subgraph a cycle with edge set $E(P) \cup E(Q)$. This cycle has at least five edges, a contradiction. Thus 27.1 holds.

In $\overline{G}$, let x and y be the neighbours of u_1 and u_2, respectively, other than u_2 and u_1; and let w and z be the neighbours of v_1 and v_2, respectively, other than v_2 and v_1. Because $\overline{G}$ is 2-connected, it has a cycle C containing $u_1 u_2$ and $v_1 v_2$. We show next that C is Hamiltonian. Assume it is not. Certainly $\overline{G}[V(C)]$ is 2-connected. Consider $G[V(C)]$. In it, u_1 and u_2 are adjacent to every vertex not in $\{x, u_1, u_2, y\}$, and v_1 and v_2 are adjacent to every vertex not in $\{w, v_1, v_2, z\}$. In addition, u_1 and v_1 are adjacent to y and it follows by symmetry that $G[V(C)]$ is 2-connected. The minimality of G implies that $V(G) = V(C)$. Thus C is indeed Hamiltonian.

Assume that C consists of the path $x u_1 u_2 y$, a path P_{yz} from y to z, the path $z v_2 v_1 w$, and a path P_{wx} from w to x. Now x and y must be distinct. Likewise, w and z are distinct. If $x = w$ and $y = z$, then $\overline{G}$ is either C_6 or C_6^+. As $C_6^+ = F_0$, this is a contradiction. Thus $x \neq w$ or $y \neq z$.

The graph $\overline{G} - \{u_1, u_2\}$ is connected. Take a shortest path P in this graph from x to y. This path P must be a single edge otherwise $\overline{G}$ has an induced cycle of length at least five consisting of the union of P and the path $x u_1 u_2 y$. By Lemma 15, the complement of this induced cycle is an induced-minor-minimal non-2-cograph, so G is this complement, a contradiction.

By symmetry, we may assume that $\overline{G}$ has xy and wz as edges. Assume that $x = w$ but $y \neq z$. Because the only cycles of $\overline{G}$ containing $u_1 u_2$ and $v_1 v_2$ are Hamiltonian, the path P_{yz} in C is a shortest path from y to z in $\overline{G} - x$. Let $P_{yz} = y_0 y_1 \ldots y_m$ where $y = y_0$ and $z = y_m$. For each i in $\{1, 2, \ldots, m-1\}$, the only possible neighbour of y_i in $\overline{G}$ other than y_{i-1} and y_{i+1} is x. We argue by induction on i that y_i is adjacent to x. Suppose y_1 is not adjacent to x. If y_2 is adjacent to x, then $\overline{G}$ has C_6^+ as an induced subgraph, a contradiction. Thus y_2 is not adjacent to x. As y_m is adjacent to x, for some $j \geqslant 3$, the vertex y_j is adjacent to x, but none of $y_{j-1}, y_{j-2}, \ldots, y_2, y_1$ is adjacent to x. Then $\overline{G}$ has a cycle of length at least five as an induced subgraph, a contradiction. We conclude that y_1 is adjacent to x. Assume that all of $y_1, y_2, \ldots, y_t$ are adjacent to x but y_{t+1} is not. If y_{t+2} is not adjacent to x, then $\overline{G}$ contains an induced cycle of length at least five, a contradiction. Thus y_{t+2} is adjacent to x and $\overline{G}$ has F_t as a proper induced subgraph, a contradiction. We conclude that y_{t+1} is adjacent to x. Hence, by induction, y_i is adjacent to x for all i in $\{1, 2, \ldots, m-1\}$. Thus $\overline{G} \cong F_m$, a contradiction.

It remains to consider the case when $x \neq w$ and $y \neq z$. If xz and wy are both green, then $G - \{v_1, v_2\}$ and its complement are both 2-connected, a contradiction. Suppose both xz and wy are red. Then $\overline{G}$ has a cycle using $u_1 u_2$ and $v_1 v_2$ and having exactly eight vertices. Thus $|V(\overline{G})| = 8$. If both xw and yz are green, then $\overline{G} - \{u_1, u_2\} \cong C_6^+$, a contradiction. Thus $\overline{G}$ is isomorphic to either H_1 or H_2. Now assume that xz is red and wy is green. If both xw and yz are red, then $|V(G)| = 8$ and $\overline{G}$ is isomorphic to H_2.

If xw is green, then, using the paths P_{yz} and P_{wz} in $\overline{G}$, we see that $G - \{v_1, v_2\}$ and its complement are both 2-connected. Thus we may assume that xw is red. Likewise, yz is red otherwise $G - \{u_1, u_2\}$ and its complement are both 2-connected, a contradiction. Hence $\overline{G}$ is isomorphic to H_2. $\qquad\square$

The following is a straightforward consequence of Lemmas 25 and 27.

Proposition 28. *A graph G is an induced-minor-minimal non-2-cograph for which $\overline{G}$ is critically 2-connected if and only if $\overline{G}$ is a cycle with at least five vertices, or $\overline{G}$ is isomorphic to H_1, H_2, or F_m for some $m \geqslant 0$.*

The next three lemmas show that the number of vertices of an induced-minor-minimal non 2-cograph is bounded above given some conditions on the sizes of components after the removal of a green 2-cut and on the red degrees of the vertices in that cut.

Lemma 29. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that each of g_1 and g_2 has red degree exceeding two and the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leqslant 8$.*

Proof. Assume that $|V(G)| > 8$. Without loss of generality, let $|V(A)| \geqslant 4$. Suppose A contains no red neighbours of g_1 or g_2. Then all vertices in A are incident to both g_1 and g_2 via a green edge. Let v be any vertex in A. Note that both $G - v$ and $\overline{G} - v$ are 2-connected, a contradiction. Therefore, we may assume that A has a red neighbour, say a_1, of g_1. Lemma 16 implies that we can find a contractible green edge, say e, of G incident to a_1 such that the other endpoint of e is in A. By Lemma 20, $\overline{G/e}$ is 2-connected, a contradiction. $\qquad\square$

Lemma 30. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the red degree of g_1 is two and the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs A and B such that $|V(A)| \geqslant |V(B)| \geqslant 2$. Suppose that A contains exactly one red neighbour v of g_1, and either g_2 has no red neighbours in $A - v$, or g_2 has red degree greater than two. If all of the contractible edges of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to a vertex in $\{g_1, g_2, v\}$, then $|V(A)| \leqslant 4$.*

Proof. Assume that $|V(A)| > 4$. Let G_A be the subgraph of G induced by $V(A) \cup \{g_1, g_2\}$, and let Q denote the vertex set $\{g_1, g_2, v\}$. By colouring the edge $g_1 g_2$ green if necessary, we may assume that G_A is 2-connected. Since the contractible edges of G_A must meet Q, by Theorem 18, either $G_A - Q$ has no edges, or $G_A - Q$ has one non-trivial component and this component has at most three vertices. First suppose that $G_A - Q$ is edgeless. Let $\Gamma = V(G_A) - Q$. Next we show the following.

30.1. There is no vertex γ in Γ such that $G_A - \gamma$ is 2-connected.

If such a vertex exists, then $G - \gamma$ is 2-connected. Moreover, by Lemma 20, $\overline{G} - \gamma$ is 2-connected, a contradiction. Thus 30.1 holds.

30.2. The edge vg_2 is red.

Suppose vg_2 is green. Let α be a neighbour of v in Γ. Then $g_1 \alpha v g_2 g_1$ is a cycle of G_A. Because $G_A - Q$ is edgeless and G_A is 2-connected, every vertex in $\Gamma - \alpha$ is adjacent to at least two members of $\{g_1, g_2, v\}$. Thus $G_A - \gamma$ is 2-connected for all γ in $\Gamma - \alpha$, a contradiction to 30.1. Thus 30.2 holds.

Observe that v and g_2 have a common neighbour β in Γ otherwise, as $G_A - Q$ is edgeless, g_1 is a cut vertex of G_A. By 30.2, v has a neighbour α in $\Gamma - \beta$. Since $g_1 \alpha v \beta g_2 g_1$ is a cycle and all vertices in $\Gamma - \{\alpha, \beta\}$ are adjacent to at least two vertices in $\{g_1, g_2, v\}$, we deduce that $G_A - \gamma$ is 2-connected for all γ in $\Gamma - \{\alpha, \beta\}$, a contradiction.

We may now assume that $G_A - Q$ has one non-trivial component, say C_A, and a set I_A of isolated vertices. Moreover, $|V(C_A)| \leqslant 3$. Then I_A is non-empty since $|V(A)| > 4$. Let $\alpha\beta$ be an edge in C_A. Note that $\alpha\beta$ is not contractible in G_A, so $\{\alpha, \beta\}$ is a 2-cut of G_A and, therefore, of G. Since $|V(B)| \geqslant 2$ and I_A is non-empty, each of α and β has red degree at least three in G. Therefore, by Lemma 29, as $|V(G)| = |V(A)| + 2 + |V(B)| > 8$, there is a vertex t of G whose only green neighbours are α and β. Since g_1 is adjacent to all vertices in $I_A \cup V(C_A)$, it follows that $t = v$. This implies that all vertices in I_A are adjacent only to g_1 and g_2. Taking w in I_A, we see that $G_A - w$ is 2-connected, a contradiction to 30.1 $\qquad \square$

Lemma 31. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs, A and B, each having at least two vertices. If the red degree of g_1 is two and that of g_2 is greater than two such that one red neighbour of g_1 is in A and the other is in B, then $|V(G)| \leqslant 10$.*

Proof. Without loss of generality, assume $|V(A)| \geqslant |V(B)|$. Let G_A be the subgraph of G induced by $V(A) \cup \{g_1, g_2\}$. Note that G_A is 2-connected since $g_1 g_2$ is green. Denote the red neighbour of g_1 in A by v and let $Q = \{g_1, g_2, v\}$. Observe that if we have a contractible edge e of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ such that neither of the endpoints of e is in Q, then, by Lemma 20, both G/e and $\overline{G/e}$ are 2-connected, a contradiction. Therefore, we may assume that all contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ meet Q. Thus, by Lemma 30, $|V(A)| \leqslant 4$, so $|V(G)| \leqslant 10$. $\qquad \square$

Lemma 32. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Suppose that, for each i in $\{1, 2\}$, if g_i has red degree two, then g_i has no red neighbour in B. Then $|V(B)| = 2$.*

Proof. Suppose $|V(B)| \geqslant 3$. If all vertices in B are green neighbours of both g_1 and g_2, then $G - z$ is 2-connected for all z in $V(B)$. But, by Lemma 20, $\overline{G} - z$ is also 2-connected, a contradiction. Thus B has a red neighbour, say b, of g_1. Note that g_1 has red degree greater than two. Now, by Lemma 16, we can find a contractible edge, say e, of G incident to b such that the other endpoint of e is in $V(B)$. By Lemma 20, $\overline{G/e}$ is 2-connected, a contradiction. $\qquad \square$

Lemma 33. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the red degree of g_1 is two and the components of $G - \{g_1, g_2\}$ can be partitioned into*

subgraphs A and B such that $|V(A)| \geqslant |V(B)| \geqslant 2$. Suppose that one of the following holds.

  (i) A contains both the red neighbours $\{x, y\}$ of g_1, and g_2 has no red neighbour in $A - \{x, y\}$ if the red degree of g_2 is two; or

  (ii) g_2 has red degree two and A contains exactly one pair $\{x, y\}$ of distinct vertices such that x is a red neighbour of g_1, and y is a red neighbour of g_2.

If all contractible edges of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to a vertex in $\{g_1, g_2, x, y\}$, then $|V(A)| \leqslant 6$.

Proof. Assume that $|V(A)| > 6$ and so $|V(G)| > 10$. Let G_A be the graph induced on $V(A) \cup \{g_1, g_2\}$. Let $Q = \{g_1, g_2, x, y\}$. By colouring the edge $g_1 g_2$ green if necessary, we may assume that G_A is 2-connected. Note that all the contractible edges of G_A must meet Q, otherwise we have a contractible edge e of G such that $\overline{G/e}$ is 2-connected, a contradiction. By Theorem 19, $G_A - Q$ has at most two non-trivial components and, between them, these components have at most four vertices.

Let I_A and N_A be the sets of isolated and non-isolated vertices of $G_A - Q$, respectively. We note the following.

33.1. If two vertices i_1 and i_2 in I_A have the same green neighbourhood in G, then $\{i_1, i_2\}$ is a green 2-cut in G.

As $\overline{G} - \{g_1, g_2, i_1\}$ is a complete bipartite graph with each part having at least two vertices, it is 2-connected. Both g_1 and g_2 have at least two red neighbours in $\overline{G} - \{i_1\}$. Thus $\overline{G} - i_1$ is 2-connected. Therefore $G - i_1$ is not 2-connected. It follows that i_2 is a cut-vertex of $G - i_1$ and so $\{i_1, i_2\}$ is a green 2-cut. Thus 33.1 holds.

First suppose that N_A is empty. As $|V(A)| \geqslant 7$, we see that $|I_A| \geqslant 5$. Suppose that g_2 has red degree two. Then all vertices in I_A are adjacent to both g_1 and g_2 in G. Observe that if a vertex s in I_A has green neighbourhood $\{g_1, g_2\}$, then both $G - s$ and $\overline{G} - s$ are 2-connected, a contradiction. Since g_1 and g_2 have no red neighbours in I_A, the green neighbourhood of a vertex in I_A is $\{g_1, g_2, x\}, \{g_1, g_2, y\}$, or $\{g_1, g_2, x, y\}$. It follows that there are at least two pairs of vertices in I_A such that each vertex in a pair has the same green neighbourhood. Let $\{i_1, i_2\}$ be such a pair. By 33.1, $\{i_1, i_2\}$ is a green 2-cut. Since the red degrees of both i_1 and i_2 are greater than two, by Lemma 29, it follows that there is a vertex t of G that has green neighbourhood $\{i_1, i_2\}$. Note that t is either x or y. Since we have at least two such green 2-cuts, it follows that $g_2 x$ and $g_2 y$ are both red, and there is a red edge connecting $\{x, y\}$ to I_A. Observe that B has no red neighbour of g_1 or g_2. It now follows that, for each b in $V(B)$, both $G - b$ and $\overline{G} - b$ are 2-connected, a contradiction. Therefore g_2 has red degree at least three. By Lemma 32, $|V(B)| = 2$. Suppose there is no red edge connecting $\{x, y\}$ to I_A. Then the possible green neighbourhoods of the vertices in I_A are $\{x, y\}, \{x, y, g_1\}, \{x, y, g_2\}$, or $\{x, y, g_1, g_2\}$. Thus, by 33.1, I_A contains a green 2-cut $\{i_1, i_2\}$ of G. Then we get $|V(G)| \leqslant 8$ by applying Lemma 29 to the green 2-cut $\{i_1, i_2\}$. Therefore there is a red edge connecting $\{x, y\}$ to I_A. It follows that, for some b in $V(B)$, both $G - b$ and $\overline{G} - b$ are 2-connected, a contradiction.

We may now assume that $G_A - Q$ has at least one non-trivial component. Let C be such a component and let $\alpha\beta$ be an edge in C. Since $\alpha\beta$ is a non-contractible edge of G_A, we see that $\{\alpha, \beta\}$ is a green 2-cut of G_A and thus of G. Then $G_A - Q \neq C$ otherwise, by Theorem 19, $|V(A)| \leqslant 6$, a contradiction. Thus both α and β have red degree at least three in G. Therefore, by Lemma 29, G has a unique vertex t that has green neighbourhood $\{\alpha, \beta\}$. Since all vertices in G_A except x and y are adjacent to g_1 via a green edge, t is either x or y. As $\alpha\beta$ is an arbitrary green edge in $G_A - Q$, it follows that $G_A - Q$ has at most two edges and therefore has either one non-trivial component with at most three vertices, or has two non-trivial components each with two vertices.

Suppose that $G_A - Q$ has only one edge, $\alpha\beta$, and let t be the unique member of $\{x, y\}$ that has green neighbourhood $\{\alpha, \beta\}$. Then $|I_A| \geqslant 3$ and the green neighbourhood of every vertex in I_A is contained in $\{g_1, g_2, s\}$ where $\{t, s\} = \{x, y\}$. It is clear that if a vertex w in I_A has green neighbourhood $\{g_1, g_2\}$, then $G - w$ and $\overline{G} - w$ are 2-connected. It follows that the green neighbourhood of a vertex in I_A is either $\{g_1, s\}$ or $\{g_1, g_2, s\}$. As $|I_A| \geqslant 3$, it contains vertices i_1 and i_2 that have the same green neighbourhood. By 33.1, $\{i_1, i_2\}$ is a green 2-cut in G. As neither t nor s has $\{i_1, i_2\}$ as its green neighbourhood, Lemma 29 gives the contradiction that $|V(G)| \leqslant 8$. We now know that $G_A - Q$ has exactly two edges, so $3 \leqslant |N_A| \leqslant 4$. Observe that $I_A \neq \emptyset$ and all vertices in I_A have green neighbourhood equal to $\{g_1, g_2\}$ since x and y have their green neighbourhoods contained in N_A. Thus, for $w \in I_A$, both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. $\qquad\square$

Lemma 34. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that g_1 and g_2 are not adjacent in G and the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leqslant 10$.*

Proof. First suppose that both g_1 and g_2 have red degree two and that the red neighbour v of g_1 that is distinct from g_2 is in A, and the red neighbour u of g_2 distinct from g_1 is in B. We may assume that $|V(A)| \geqslant |V(B)|$. Observe that, if we can find a contractible edge e of G having both the endpoints in $V(A) - v$, then, by Lemma 20, $\overline{G/e}$ is 2-connected, a contradiction. This implies that all the contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, v\}$. By Lemma 30, $|V(A)| \leqslant 4$ and so $|V(G)| \leqslant 10$. Thus we may assume that both u and v are in A and all contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, u, v\}$. Note that $u \neq v$, otherwise $\overline{G}$ has a cut vertex. We get our result now by Lemmas 32 and 33. We may now assume that the red degree of g_2 exceeds two. By Lemma 29, we may further assume that the red degree of g_1 is two.

Let v be the red neighbour of g_1 other than g_2. We may assume that v is in A. By Lemma 32, $|V(B)| = 2$. Note that all the contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, v\}$. The result now follows by Lemma 30. $\qquad\square$

Lemma 29 can be modified as follows.

Lemma 35. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. If g_2 has red degree greater than two, then $|V(G)| \leqslant 10$.*

Proof. Assume that $|V(G)| \geqslant 11$. Then, by Lemma 29, the red degree of g_1 is two. Let x and y be the two red neighbours of g_1. Note that if x is in A and y is in B, then the result follows by Lemma 31. By Lemma 34, we may suppose that the edge $g_1 g_2$ is green and both x and y are in A.

The graph G_A induced on $V(A) \cup \{g_1, g_2\}$ is 2-connected. Let $Q = \{g_1, g_2, x, y\}$. Then every contractible edge e of G_A must meet Q otherwise, by Lemma 20, we obtain the contradiction that both G/e and $\overline{G/e}$ are 2-connected. The result now follows by Lemmas 32 and 33.

$\square$

We can generalize the above result by removing the condition on the red degrees of the vertices in the 2-cut at the cost of raising the bound on the number of vertices of G to 16.

Lemma 36. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leqslant 16$.*

Proof. Assume that $|V(G)| \geqslant 17$. By Lemmas 34 and 35, we may assume that the red degrees of both g_1 and g_2 are two and $g_1 g_2$ is green. We may further assume that $|V(A)| \geqslant |V(B)|$. The graph G_A induced on $V(A) \cup \{g_1, g_2\}$ is 2-connected. Let Q be the union of $\{g_1, g_2\}$, the set of red neighbours of g_1 in A, and the set of red neighbours of g_2 in A. Then every contractible edge e of G_A must meet Q, otherwise, by Lemma 20, we obtain a contradiction. Note that if $|Q| = 2$, then, by Lemma 32, $|V(A)| = 2$ and so $|V(G)| \leqslant 6$, a contradiction. By Theorems 18 and 19, $G_A - Q$ has at most four non-trivial components and between them, these components have at most eight vertices.

Let I_A and N_A be the sets of isolated and non-isolated vertices of $G_A - Q$, respectively. We note the following.

36.1. $|N_A| \leqslant 4$.

Assume that $|N_A| > 4$ and so $G_A - Q$ has at least three edges. Let $\alpha\beta$ be an edge of $G_A - Q$. Because $\alpha\beta$ is not a contractible edge of G_A, it follows that $\{\alpha, \beta\}$ is a green 2-cut of G. Observe that each of α and β has red degree at least three unless $|I_A|$ is empty, and $G_A - Q$ has one non-trivial component, and $|V(B)| = 2$. The exceptional case does not arise since it implies, as $V(G) = (V(A) - Q) \cup Q \cup V(B)$, that $|V(G)| \leqslant 8 + 6 + 2 = 16$, a contradiction. By Lemma 29, there is a vertex t that has green neighbourhood $\{\alpha, \beta\}$. Note that the only vertices that could have green neighbourhood $\{\alpha, \beta\}$ are the common red neighbours of g_1 and g_2. Since there are at most two such vertices and at least three edges in $G_A - Q$, each of which must have an associated such vertex, we have a contradiction. Thus 36.1 holds.

Next we show the following.

36.2. $|I_A| \leqslant 4$.

Assume that $|I_A| \geqslant 5$. Note that all vertices in I_A are adjacent to both g_1 and g_2. Suppose I_A contains a vertex i such that all vertices in $Q - \{g_1, g_2\}$ have degree at least

two in $G - i$. Then both $G - i$ and $\overline{G}{-}i$ are 2-connected, a contradiction. It follows that, for every vertex i of I_A, there is a **special** green edge joining i to a vertex q of $Q - \{g_1, g_2\}$ such that q has green degree two. The set Q' of such vertices q is contained in $Q - \{g_1, g_2\}$. If a member q' of Q' is a common red neighbour of g_1 and g_2, then it meets at most two special green edges from I_A. If, instead, q' has a single red neighbour in $\{g_1, g_2\}$, then it has a single green neighbour in $\{g_1, g_2\}$ and so meets at most one special green edge. Thus the number of red edges from $\{g_1, g_2\}$ to Q' is an upper bound on the number of special green edges from I_A. Hence $|I_A| \leqslant 4$, a contradiction. Thus 36.2 holds.

36.3. $|V(B)| \geqslant 3$.

Suppose that $|V(B)| = 2$. Then, by 36.1 and 36.2, $|V(G)| \leqslant 4 + 4 + 6 + 2 = 16$, a contradiction. Thus 36.3 holds.

By 36.3, since $|V(B)| \neq 2$, Lemma 32 implies that B contains at least one red neighbour of $\{g_1, g_2\}$. Assume that B contains exactly one such red neighbour v. Let x and y be two green neighbours of v in $V(B) \cup \{g_1, g_2\}$. If $V(B) - \{v, x, y\}$ contains a vertex t, then $G - t$ and $\overline{G} - t$ are both 2-connected. It follows that $|V(B)| \leqslant 3$. Again by 36.1 and 36.2, we get $|V(G)| \leqslant 4 + 4 + 5 + 3 = 16$, a contradiction. Note that if A contains exactly one of the red neighbours of $\{g_1, g_2\}$, then, by Lemma 30, $|V(A)| \leqslant 4$, so $|V(G)| \leqslant 10$, a contradiction. We may now assume that the red neighbourhood of $\{g_1, g_2\}$ has size four, and each of A and B contains exactly two of those vertices. Then, by Lemma 33, $|V(A)| \leqslant 6$, so $|V(G)| \leqslant 14$, a contradiction. $\qquad\square$

The following corollary summarizes our results about the induced-minor-minimal non-2-cographs so far.

Corollary 37. *Let G be an induced-minor-minimal non-2-cograph. Then*

(i) $|V(G)| \leqslant 16$; *or*

(ii) $\overline{G}$ *is a cycle of length at least five; or*

(iii) G *has vertex connectivity two, and, for every 2-cut $\{g_1, g_2\}$ of G, the graph $G - \{g_1, g_2\}$ has exactly two components, and one component contains a single vertex.*

If an induced-minor-minimal non-2-cograph G satisfies (iii) of the above corollary, we say that G is an induced-minor-minimal non-2-cograph of *type (iii)*. The next lemma identifies several infinite families of such graphs.

Lemma 38. *Let G be a graph such that $\overline{G}$ is isomorphic to $L_m, M_m, M'_m, N_m, N'_m,$ or N''_m for some $m \geqslant 1$ where $L_m, M_m,$ and N_m are shown in Figures 2, 3, and 4, respectively, at the beginning of Section 2. Then G is an induced-minor-minimal non-2-cograph of type (iii).*

Proof. It is clear that G is not a 2-cograph as both G and $\overline{G}$ are 2-connected. Assume that H is an induced subgraph of G such that both H and $\overline{H}$ are 2-connected. It is clear that $v \in V(H)$ otherwise H or $\overline{H}$ is not 2-connected. Note that $V(H)$ also contains the

vertices x and y since x and y are the only green neighbours of v. It now follows that $V(H)$ contains the red neighbours of x and the red neighbours of y. It is now straightforward to see that $H = G$. Therefore every proper induced subgraph of G and of $\overline{G}$ is a 2-cograph. For an edge $\alpha\beta$ of G, it follows by Lemma 6 that the complement of $G/\alpha\beta$ is either a proper induced subgraph of $\overline{G}$ or a proper induced subgraph of $\overline{G}$ 1-summed with K_2 or K_3. Thus $G/\alpha\beta$ is a 2-cograph and so G is an induced minor-minimal non-2-cograph. Moreover, $\{x, y\}$ is its unique 2-cut and G is an induced-minor-minimal non-2-cograph of type (iii). $\qquad\square$

By a similar argument to that just given, we obtain the following.

Lemma 39. *For a non-negative integer j, the graph $\overline{F_j}$ is an induced-minor-minimal non-2-cograph of type (iii).*

In the rest of the section, we find all the other classes of induced-minor-minimal non-2-cographs of type (iii) thereby proving Theorem 2. Recall that, for a graph G, its sets of vertices of green-degree-two and of red-degree-two are denoted by V_g and V_r, respectively.

Lemma 40. *Let G be an induced-minor-minimal non-2-cograph of type (iii). Then $|V_g| \leqslant 3$ or $\overline{G}$ is of type (iii).*

Proof. Suppose that $\overline{G}$ is not of type (iii). By Lemma 24 and Proposition 26, we may assume that $\overline{G}$ has vertex connectivity two. Take a red 2-cut $\{r_1, r_2\}$ of G such that the components of $\overline{G} - \{r_1, r_2\}$ can be partitioned into subgraphs A and B, and $|V(A)| \geqslant |V(B)| \geqslant 2$. If $|V(B)| \geqslant 3$, then all vertices in $V(G) - \{r_1, r_2\}$ have green degree at least three and so $|V_g| \leqslant 2$. Now suppose that $V(B) = \{b_1, b_2\}$. Note that there is at most one vertex a in A that has green neighbourhood $\{b_1, b_2\}$ since G is of type (iii). One can now check that all vertices in $V(G) - \{r_1, r_2, a\}$ have green degree at least three, and so $|V_g| \leqslant 3$. $\qquad\square$

Lemma 41. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| > 10$. Suppose that $\overline{G}$ is not isomorphic to a cycle or to F_m for some $m \geqslant 0$. Then the graph induced on the vertex set V_g is a complete red graph and the graph induced on V_r has at most one red edge.*

Proof. By Lemma 22, the graph induced on V_g is a complete red graph. Assume that the graph induced on V_r has two red edges $e = u_1 u_2$ and $f = v_1 v_2$. Note that if e and f are disjoint, then, by Lemma 27, we obtain a contradiction. Therefore we may assume that $u_2 = v_1$. Let α and β be the respective neighbours of u_1 and v_2 in $\overline{G} - v_1$. Note that α and β are distinct otherwise we have a cut vertex in $\overline{G}$, a contradiction. Let P be a shortest $\alpha\beta$-path distinct from $\alpha u_1 u_2 v_2 \beta$. Then P avoids $\{u_1, u_2, v_2\}$ and the red graph induced on $V(P) \cup \{u_1, u_2, v_2\}$ is a cycle. It follows by the minimality of G that $V(G) = V(P) \cup \{u_1, u_2, v_2\}$ and so $\overline{G}$ is a cycle, a contradiction.

$\qquad\square$

In the following lemma, we note that either $|V_g|$ or $|V_r|$ is bounded.

Lemma 42. *Let G be an induced-minor-minimal non-2-cograph. Then either $|V_g|$ or $|V_r|$ is at most three, or $|V_g| = |V_r| = 4$.*

Proof. Note that there are at most $2|V_g|$ green edges and at most $2|V_r|$ red edges joining a vertex in V_g to a vertex in V_r. Since there are $|V_g||V_r|$ edges joining vertices in V_g to vertices in V_r, we have

$$2|V_g| + 2|V_r| \geqslant |V_g||V_r|.$$

This inequality is symmetric with respect to $|V_g|$ and $|V_r|$, so we may assume that $|V_g| \geqslant |V_r|$. Then $2 + 2\frac{|V_r|}{V_g} \geqslant |V_r|$. Thus $|V_r| \leqslant 4$. Moreover, if $|V_r| = 4$, then $|V_g| = 4$. $\square$

Next we note the following useful observation.

Lemma 43. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| > 10$. If all vertices of a subset S of $V(G) - (V_g \cup V_r)$ either have a red neighbour in V_r or a green neighbour in V_g, then*

43.1. $|S| \leqslant 2|V_g \cup V_r| - |V_g||V_r|$.

Moreover, when equality holds here, either each vertex in S has exactly one green neighbour in V_g or has exactly one red neighbour in V_r but not both. In particular, if $S = V(G) - (V_g \cup V_r)$, then

$$11 + |V_g||V_r| \leqslant 3|V_g| + 3|V_r|.$$

Proof. There are $|V_g||V_r|$ red or green edges joining a vertex in V_g to a vertex in V_r. There are at most $2|V_g|$ green such edges and at most $2|V_r|$ red such edges. Thus among the green edges meeting V_g and the red edges meeting V_r at most $2|V_g \cup V_r| - |V_g||V_r|$ have an endpoint in $V(G) - (V_g \cup V_r)$. Therefore, $|S| \leqslant 2|V_g \cup V_r| - |V_g||V_r|$ and it is clear that, when equality holds, each vertex in S satisfies the given condition. If $S = V(G) - (V_g \cup V_r)$, then it is clear that $11 + |V_g||V_r| \leqslant 3|V_g| + 3|V_r|$ since $|V(G)| \geqslant 11$. $\square$

Lemma 40 can be improved in the following way.

Lemma 44. *Let G be an induced-minor-minimal non-2-cograph of type (iii). Then $|V(G)| \leqslant 10$ or $|V_g| \leqslant 3$.*

Proof. By Lemma 40, it is enough to show that if $\overline{G}$ is of type (iii), then $|V(G)| \leqslant 10$ or $|V_g| \leqslant 3$. Suppose that $\overline{G}$ is of type (iii). Since every vertex of $V(G)$ is either in a red 2-cut or a green 2-cut, and both G and $\overline{G}$ are of type (iii), we have the following.

44.1. Every vertex in $V(G)$ either has a green neighbour in V_g or a red neighbour in V_r.

Since a vertex in V_g has no green neighbour in V_g by Lemma 22, it follows by 44.1 that $|V_g| \leqslant 2|V_r|$ since the number of red-degree-two neighbours of vertices in V_g is at least $|V_g|$ and at most $2|V_r|$. The following is an immediate consequence of Lemma 43 and 44.1.

44.2. $|V(G)| \leqslant |V_g| + |V_r| + 2|V_r \cup V_g| - |V_g||V_r| = 3|V_g| + 3|V_r| - |V_g||V_r|$.

Note that if $|V_g| = |V_r| = 4$, then $|V(G)| \leqslant 8$ and the result holds. Therefore, by Lemma 42, we may assume that $|V_r|$ is at most three. As $|V_g| \leqslant 2|V_r|$, by 44.2, checking the possibilities for $|V_r|$, we obtain that $|V(G)| \leqslant 10$. $\qquad\square$

In the next proof, we adopt the convention that, for a 2-cut $\{x, y\}$ of a graph H, the graphs A and B are disjoint subgraphs of $H - \{x, y\}$ with $V(A) \cup V(B) = V(H - \{x, y\})$ such that $|V(A)| \geqslant |V(B)|$, and $|V(B)|$ is maximal.

Lemma 45. *Let G be an induced-minor-minimal non-2-cograph such that G is of type (iii). Then $|V(G)| \leqslant 16$ or $|V_g| \leqslant 1$.*

Proof. By Lemma 44, we may assume that $|V_g| \leqslant 3$. The following observation is immediate.

45.1. Let $\{r_1, r_2\}$ be a red 2-cut of G. If $|V(B)| \geqslant 3$, then $\{r_1, r_2\} \subseteq V_g$.

Next we show the following.

45.2. There are at most two vertices outside of V_g that have neither a red neighbour in V_r nor a green neighbour in V_g.

Every vertex v of $V(G) - V_g$ is in a green 2-cut or a red 2-cut. In the first case, because G is of type (iii), v has a green neighbour in V_g. In the second case, let $\{v, r\}$ be a red 2-cut. By 45.1, we may assume that $|V(B)| \leqslant 2$. If $|V(B)| = 1$, then v has a red neighbour in V_r. Suppose $|V(B)| = 2$. If w is a vertex with green neighbourhood $V(B)$, then $V(B)$ is a green 2-cut. As G is of type (iii), w is unique. If $|V_g| = 3$, it follows that $\{v, r\} \subseteq V_g$, a contradiction, so 45.2 holds.

If $|V_g| \leqslant 1$, then the lemma holds, so we may assume $|V_g| = 2$. For the red 2-cut $\{v, r\}$, we know that $|V(B)| = 2$. Now each vertex u of $V(G) - V(B) - \{v, r\}$ has $V(B)$ in its green neighbourhood. Thus $\{u, r\}$ cannot be a red 2-cut with the same $V(B)$. Thus $\{v, r\}$ is the unique red 2-cut with the given $V(B)$. As $v \notin V_g$ and G is of type (iii), the set $V(B)$ is the green neighbourhood of exactly one vertex in V_g. Since $|V_g| = 2$, it follows that we have at most two red 2-cuts for which $|V(B)| = 2$. Moreover, each such red 2-cut contains a member of V_g. Now 45.2 follows immediately.

By 45.2 and Lemma 43, $|V(G)| \leqslant 3|V_g| + 3|V_r| - |V_g||V_r| + 2$. If $|V_g| = 3$, then $|V(G)| \leqslant 11$. Suppose $|V_g| = 2$. Then, by Lemma 41, $|V_r| \leqslant 2|V_g| + 2 + 2 = 8$, so $|V(G)| \leqslant 16$. $\qquad\square$

Proof of Theorem 2. We may assume that G is of type (iii) otherwise we have the result by Corollary 37. We may also assume that neither G nor $\overline{G}$ is critically 2-connected, otherwise the result follows by Proposition 26 or Proposition 28. It is now clear that V_g is non-empty. Therefore, by Lemma 45, $|V_g| = 1$ or $|V(G)| \leqslant 16$. If $|V(G)| \leqslant 16$, then we have our result. Therefore we may assume that $|V_g| = 1$. It now follows that G has a unique green 2-cut $\{x, y\}$. Thus every vertex not in $\{x, y\}$ is in a red 2-cut. As $\overline{G}$ is not critically 2-connected, we may assume that $\overline{G} - \{x\}$ is 2-connected. Note that $G - \{x, y\}$ has a non-trivial component A and a trivial component, say $\{v\}$.

32.1. There is no vertex t in A such that $\overline{G} - \{x, v, t\}$ is connected and each of x and y has at least two neighbours in $\overline{G} - \{t\}$.

Assume that this fails. Since $\overline{G} - \{x, v, t\}$ is connected and v is adjacent to all vertices of $\overline{G} - \{x, t\}$ except y, we conclude that $\overline{G} - \{x, t\}$ is 2-connected as $\overline{G} - x$ is 2-connected and y has at least two neighbours in $\overline{G} - \{x, t\}$. It now follows that $\overline{G} - \{t\}$ is 2-connected since x has at least two neighbours $\overline{G} - \{t\}$. This is a contradiction since t is in a red 2-cut.

32.2. $\overline{G}[A]$ is connected.

To show this, assume $\overline{G}[A]$ is disconnected. Because $\overline{G} - x$ is 2-connected, $\overline{G} - \{x, v\}$ is connected. Since $\overline{G}[A] = \overline{G} - \{x, v, y\}$, it follows that y has a neighbour in each component of $\overline{G}[A]$. As $\overline{G} - \{x, v\}$ is connected, there is a vertex t in A such that $\overline{G} - \{x, v, t\}$ is connected where, if possible, t is chosen from a component of $\overline{G}[A]$ with at least two vertices. By 32.1, t is a red neighbour of some z in $\{x, y\}$ such that z has degree two in $\overline{G}$. Suppose $z = y$. Then, as $\overline{G} - \{x, v, t\}$ is connected, y is adjacent to t and to each component of $\overline{G}[A]$, we deduce that $\{t\}$ is a component of $\overline{G}[A]$ and $|V(A)| = 2$. Thus $|V(G)| = 5$ and so, as G is a non-2-cograph, G is a 5-cycle, a contradiction. We deduce that $z = x$ and x has red degree two. Thus $\overline{G} - \{x, v\}$ has exactly two vertices t for which $\overline{G} - \{x, v, t\}$ is connected, and each such vertex is a red neighbour of x. It follows that $\overline{G} - \{x, v\}$ is a path and the leaves of this path are the neighbours of x in $\overline{G} - \{v\}$. Therefore $\overline{G} - \{v\}$ is a cycle, a contradiction.

Similar to 32.1, we have the following.

32.3. There is no vertex t in A such that $\overline{G} - \{y, v, t\}$ is connected and each of x and y has at least two neighbours in $\overline{G} - \{t\}$.

Assume that this fails. If x has at least two neighbours in $\overline{G} - \{y, t\}$, then the proof follows as in 32.1 by interchanging x and y. Therefore we may assume that x has exactly one neighbour in $\overline{G} - \{y, t\}$. Thus $\overline{G}[A] - \{t\}$ is connected and so $\overline{G} - \{x, y, t\}$ is 2-connected. Since each of x and y has at least two neighbours in $\overline{G} - \{t\}$, we conclude that $\overline{G} - \{t\}$ is 2-connected, a contradiction.

We call a vertex t of $\overline{G}[A]$ **deletable** if $\overline{G}[A] - \{t\}$ is connected. By combining 32.1 and 32.3, we obtain the following.

32.4. A deletable vertex t of $\overline{G}[A]$ is a neighbour in $\overline{G}$ of some z in $\{x, y\}$ where z has degree two in $\overline{G}$.

32.5. The number of deletable vertices in $\overline{G}[A]$ is in $\{2, 3, 4\}$.

To see this, first observe that, since $\overline{G}[A]$ is connected having at least two vertices, it has at least two deletable vertices. Now suppose that $\overline{G}[A]$ has at least five deletable vertices. Then there is such a vertex t so that, in $\overline{G} - \{t\}$, each of x and y has degree at least two. As $\overline{G} - \{x, v, t\}$ is connected, we have a contradiction to 32.1. Thus 32.5 holds.

The rest of the proof treats the three possibilities for the number of deletable vertices of $\overline{G}[A]$. First suppose that $\overline{G}[A]$ has exactly two deletable vertices s and t. Then $\overline{G}[A]$ is a path, which we may assume has at least five vertices. Let s' and t' be the respective neighbours of s and t in $\overline{G}[A]$. Note that if either x or y has red neighbourhood $\{s, t\}$, then we have an induced red cycle of size at least six, which is a contradiction. Thus, by 32.1 and 32.3, we may assume that both x and y have red degree two, and s is a red neighbour of x, and t is a red neighbour of y. If xy is red, then $\overline{G}$ has an induced cycle of

length at least seven, a contradiction. Thus both the red neighbours of x and y are in A. We show next that the respective red neighbourhoods of x and y are $\{s, s'\}$ and $\{t, t'\}$. To see this, let $\{s, w\}$ be the neighbourhood of x in $\overline{G}$ and suppose $w \neq s'$. If s' is not a red neighbour of y, then $\overline{G} - \{y, v, s'\}$ is connected and we get a contradiction to 32.1. Taking z to be a vertex of A not in $\{s, t, w, s'\}$, we see that $\overline{G} - \{x, v, z\}$ or $\overline{G} - \{y, v, z\}$ is connected and we get a contradiction to 32.1 or 32.3. We conclude that $\{s, s'\}$ is the red neighbourhood of x. By symmetry, $\{t, t'\}$ is the red neighbourhood of y. Thus $\overline{G}$ is isomorphic to L_m for some $m \geqslant 1$.

Next suppose that $\overline{G}[A]$ has exactly three deletable vertices, s, t, and u. Then $\overline{G}[A]$ has a spanning tree T having s, t, and u as its leaves. By 32.4, each vertex in $\{s, t, u\}$ is adjacent to a red-degree-2 vertex in $\{x, y\}$. Moreover, neither x nor y has red degree exceeding two, and xy is not red. Now $\overline{G}[A]$ is connected, so $\overline{G} - \{x, y\}$ is 2-connected. As xy is red, it follows that $\overline{G} - y$ is 2-connected. Recall that we already know that $\overline{G} - x$ is 2-connected. By symmetry, we may assume that the red neighbourhood of x is $\{s, t\}$, and so u is a red neighbour of y. Let u' be the red neighbour of u in T. Then the red neighbourhood of y is $\{u, u'\}$ otherwise $\overline{G} - \{x, v, u'\}$ is connected and we get a contradiction to 32.1. Similarly, the distance between s and t in T is two otherwise we get a contradiction to 32.3. As $\overline{G}[A]$ has exactly three deletable vertices, the only possible edge in $\overline{G}[A]$ that is not in T is st. Thus $\overline{G}$ is isomorphic to M_m or M'_m for some $m \geqslant 1$.

Finally, suppose that $\overline{G}[A]$ has four deletable vertices, s, t, u, and z. We may assume that the respective red neighbourhoods of x and y are $\{s, t\}$ and $\{u, z\}$. Again let T be a spanning tree of $\overline{G}[A]$ such that s, t, u, and z are leaves of T. Note that the distance between s and t, and u and z in T is two. Thus $\overline{G}$ is isomorphic to N_m, N'_m, or N''_m for some $m \geqslant 1$. □

We have now finished the proof of our first main result, Theorem 2.

4 Induced-minor-minimal non-2-cographs whose complements are also induced-minor-minimal non-2-cographs

In this section, we consider $\mathcal{G}$, the class of induced-minor-minimal non-2-cographs G such that $\overline{G}$ is also an induced-minor-minimal non-2-cograph. We show that all graphs in $\mathcal{G}$ have at most ten vertices. We give an exhaustive list of all these graphs in the appendix. We begin the section with the following immediate consequence of Lemma 22.

Corollary 33. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then the graph induced on the vertex set V_g is a complete red graph and the graph induced on V_r is a complete green graph.*

The next lemma shows that if the number of vertices of a graph G in $\mathcal{G}$ exceeds ten, then $V(G) - (V_g \cup V_r)$ is non-empty.

Lemma 34. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then $V(G) \neq V_g \cup V_r$.*

Proof. Assume that $V(G) = V_g \cup V_r$. There are $2|V_g|$ green edges and $2|V_r|$ red edges joining a vertex in V_g to vertex in V_r. Thus

34.1. $2|V_g| + 2|V_r| = |V_g||V_r|$.

We may assume that $|V_g| \leqslant |V_r|$. If $|V_g| = |V_r|$, then $4|V_r| = |V_r|^2$, so $|V_r| = 4$, a contradiction. Therefore $|V_g| \leqslant |V_r| - 1$ so, by 34.1, $|V_g||V_r| \leqslant 4|V_r| - 2$. Thus $|V_g| \leqslant 3$. If $|V_g| = 3$, then, by 34.1, $|V_r| = 6$, so $|V(G)| = 9$, a contradiction. If $|V_g| \leqslant 2$, then we contradict 34.1. $\qquad\square$

Next we note a useful observation about the vertices in $V(G) - (V_g \cup V_r)$.

Lemma 35. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then every vertex in $V(G) - (V_g \cup V_r)$ either has a green neighbour in V_g or a red neighbour in V_r.*

Proof. Since every vertex of G is in either a red 2-cut or a green 2-cut, the lemma follows by Lemma 35. $\qquad\square$

Lemma 36. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then neither V_g nor V_r is empty.*

Proof. It suffices to show that V_r is non-empty. Assume the contrary. By Lemma 35, every vertex outside V_g has a green neighbour in V_g. Thus, by Lemma 43, $11 \leqslant 3|V_g|$, so $|V_g| \geqslant 4$. Let $\{r_1, r_2\}$ be a red 2-cut T. Since V_r is empty, applying Lemma 35 to $\overline{G}$ gives that T is contained in V_g. Let v be a vertex in $V_g - T$ and let α and β be the two green neighbours of v. Consider the graph $\overline{G} - T$. Note that $\overline{G} - T$ is disconnected and v is incident to all the vertices in this graph except α and β. Let X be the component of $\overline{G} - T$ containing v. Since the red graph $\overline{G}$ has no degree-two vertices, $\overline{G} - T$ has exactly two components. The second component must have $\{\alpha, \beta\}$ as its vertex set.

Let w be a vertex in $V_g - T - v$. As w is in a different component of $\overline{G} - T$ from α and β, both $w\alpha$ and $w\beta$ are green edges. Since w has green degree two, it follows that $\{\alpha, \beta\}$ is the green neighbourhood of each vertex in $V_g - T$. By Lemma 35, each vertex in $V(G) - V_g - \{\alpha, \beta\}$ has a green neighbour in V_g. This neighbour is not in $V_g - T$, so it is in T. Thus $|V(G) - V_g - \{\alpha, \beta\}| \leqslant 4$. But $|V(G)| > 10$, so $|V_g - T| \geqslant 3$. Therefore $G - v$ and $\overline{G} - v$ are both 2-connected, a contradiction. We conclude that V_r is non-empty. $\qquad\square$

We are now ready to prove the second main result of the paper.

Proof of Theorem 3. Assume that $G \in \mathcal{G}$ and $|V(G)| > 10$. Without loss of generality, let $|V_g| \leqslant |V_r|$. By Lemma 35, every vertex in $V(G) - (V_g \cup V_r)$ either has a green neighbour in V_g or a red neighbour in V_r. By Lemmas 36 and 42, $1 \leqslant |V_g| \leqslant 4$. Suppose $|V_g| = 4$. Then, by Lemma 42, $|V_r| = 4$. Lemma 43 implies that $V(G) - (V_g \cup V_r)$ is empty. Therefore $|V(G)| = 8$, a contradiction.

Next we assume that $|V_g| = 3$. Then every vertex in V_r is a green neighbour of at least one vertex in V_g. Thus $|V_r| \leqslant 6$ as there are exactly six green edges incident to vertices in V_g. Then, by Lemma 43, as $|V_g| = 3$, we deduce that $11 \leqslant 3|V_g|$, a contradiction.

Now suppose that $|V_g| = 2$. Then, by Lemma 43, $|V_r| \geqslant 5$. Let $V_g = \{u, v\}$. Since there are only four green edges meeting V_g, there is a vertex w in V_r whose red neighbours

are u and v. Thus $\{u, v\}$ is a red 2-cut. Suppose that $V_r - \{w\}$ contains at least two vertices that are joined to both u and v by red edges. Then one can check that both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. Thus V_r has at most two vertices that are joined to both u and v by red edges. Therefore $|V_r| \leqslant 6$ since V_g meets only four green edges. Assume that $|V_r| = 6$. Then all the green neighbours of u and v are in V_r and are distinct. Since $|V(G)| \geqslant 11$, we see that $|V(G) - (V_g \cup V_r)| \geqslant 3$. Let $\{w, x\}$ be the vertices in V_r having both u and v as their red neighbours. All the vertices in $V_r - \{w, x\}$ have one red neighbour in V_g. Since $|V(G) - (V_g \cup V_r)| \geqslant 3$, Lemma 35 implies that each vertex in $V(G) - (V_g \cup V_r)$ has at most two red neighbours in $V_r - \{w, x\}$ and thus has at least two green neighbours in $V_r - \{w, x\}$. Thus $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. We may now assume that $|V_r| = 5$ and $|V(G) - (V_g \cup V_r)| \geqslant 4$. By Lemma 43, $|V(G) - (V_g \cup V_r)| = 4$. Thus, as equality holds in 43.1, every vertex in $V(G) - (V_g \cup V_r)$ has at most one red neighbour in $V_r - w$ and so has at least three green neighbours in $V_r - w$. Therefore we again have that both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction.

Finally, assume that $|V_g| = 1$. By Lemma 43, $|V_r| \geqslant 4$. Let $V_g = \{v\}$ and let $\alpha \in V_r$ be a red neighbour of v. First, we show that V_r does not contain a green 2-cut that contains α. Assume that $\{\alpha, \beta\}$ is a green 2-cut where $\{\alpha, \beta\} \subseteq V_r$. Then $G - \{\alpha, \beta\}$ has a component X that contains $V_r - \{\alpha, \beta\}$ and all but at most two vertices of $V(G) - \{\alpha, \beta\}$. Let Y be a component of $G - \{\alpha, \beta\}$ different from X. Then $|V(Y)| \leqslant 2$. Suppose $|V(Y)| = 1$. Then the vertex in Y must be in V_g, so it is v. This is a contradiction since αv is red. Thus $|V(Y)| = 2$ and $G - \{\alpha, \beta\}$ has exactly two components. Then $|V(X)| \geqslant 7$. Let x be a vertex in X such that x is not a red neighbour of α or β, and $X - \{x\}$ contains at least two vertices of $V_r - \{\alpha, \beta\}$. Since each vertex of $V_r - \{\alpha, \beta\}$ has its two red neighbours in Y and so is adjacent in G to every vertex of X, it follows that $G - x$ is 2-connected. Moreover, by Lemma 20, $\overline{G} - x$ is 2-connected, a contradiction. We conclude that V_r does not have a green 2-cut containing α.

Next, we show that no green 2-cut contains α. Assume that $\{\alpha, z\}$ is a green 2-cut. Then $z \notin V_r$. By Lemma 35, $G - \{\alpha, z\}$ has a single-vertex component Y. Since the vertex in Y has green degree two, $Y = \{v\}$. Thus αv is green, a contradiction. We conclude that deleting from G any red neighbour of v in V_r leaves a green graph that is still 2-connected.

To complete the proof of the theorem, we show that v has a red neighbour in V_r whose deletion from $\overline{G}$ leaves a 2-connected graph, thus arriving at a contradiction. Let β be a red neighbour of v in $V_r - \{\alpha\}$. If α and β have the same red neighbourhood, say $\{x, v\}$, then $\{x, v\}$ is a red 2-cut and we obtain a contradiction by applying Lemma 35 to $\overline{G}$. Thus α and β have distinct red neighbourhoods, $\{x, v\}$ and $\{y, v\}$, respectively. Note that if xv is red, then $\overline{G} - \alpha$ is 2-connected. Thus we may assume that both xv and yv are green. This implies γv is red for each γ in $V_r - \{\alpha, \beta\}$ since v has green degree two. Thus, for some fixed γ in $V_r - \{\alpha, \beta\}$, the other red neighbour, z, of γ is distinct from x and y. Since vz is red and γ has red degree two, we see that $\overline{G} - \gamma$ is 2-connected, a contradiction. $\qquad \square$

We have now finished the proof of our second main result, Theorem 3.

Require: $n = 6, 7, 8, 9$ or 10.

    Set FinalList $\leftarrow \emptyset$, $i \leftarrow 0$, $j \leftarrow 0$

    Generate all two connected graphs of order n using nauty geng nauty and store in an iterator L

    for g in L such that vertex connectivity of g and $\overline{g}$ is 2 **do**

        for v in $V(g)$ **do**

            $h = g \backslash v$

            if h is a 2-cograph **then**

                $i \leftarrow i + 1$

        for e in $E(g)$ **do**

            $h = g/e$

            if h is a 2-cograph **then**

                $j \leftarrow j + 1$

        if i equals $|V(g)|$ and j equals $|E(g)|$ **then**

            Add g to FinalList

    for g in FinalList **do**

        if FinalList does not contain $\overline{g}$ **then**

            remove g from FinalList

Figure 7: Finding graphs in $\mathcal{G}$ of order at most ten.

5 Appendix

We implemented the algorithm in Figure 7 using SageMathsage and provide a list of all graphs in $\mathcal{G}$ up to complementation. The graphs in this section are drawn using SageMath.

Graphs on six vertices. There are two graphs on six vertices in $\mathcal{G}$, namely, the graph shown in Figure 8 and its complement, which is the domino.
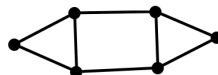
Figure 8: A 6-vertex graph in $\mathcal{G}$.

Graphs on seven vertices. There are sixteen graphs on seven vertices in $\mathcal{G}$, the graphs in Figure 9 and their complements.

Graphs on eight vertices. There are 87 graphs on eight vertices in $\mathcal{G}$, of which five are self-complementary. Figure 10 shows these self-complementary graphs. Figure 11 shows 41 non-self-complementary graphs that, together with their complements, are the remaining 8-vertex graphs in $\mathcal{G}$.

Graphs on nine vertices. There are 86 graphs on nine vertices in $\mathcal{G}$. These are the 43 graphs in Figure 12 and their complements.

Graphs on ten vertices. There are two graphs on ten vertices in $\mathcal{G}$, the graph in Figure 13 and its complement.
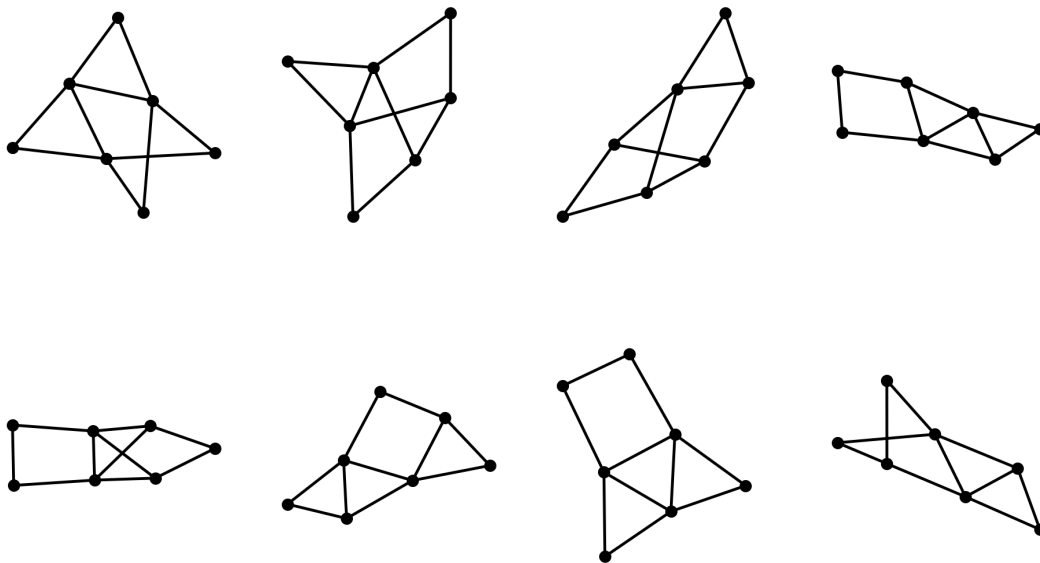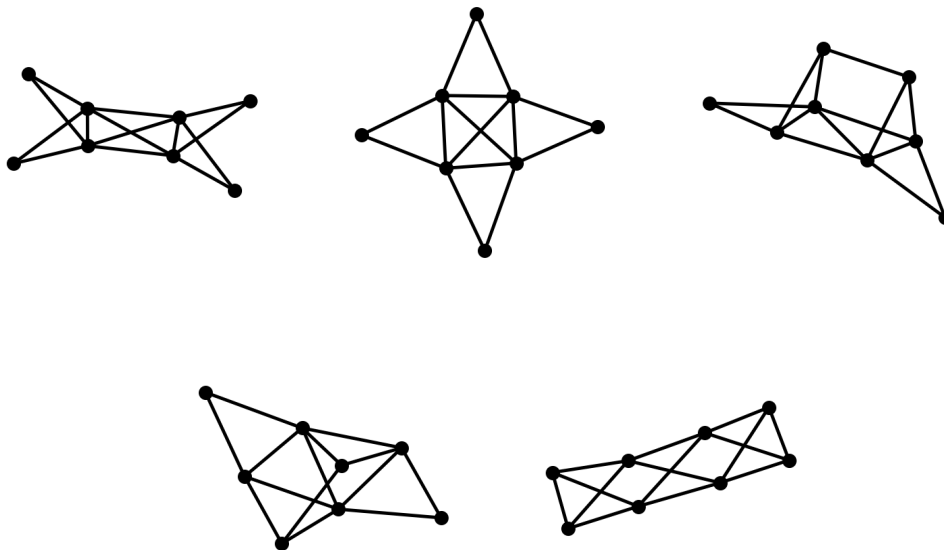
Figure 9: Graphs on seven vertices in $\mathcal{G}$.

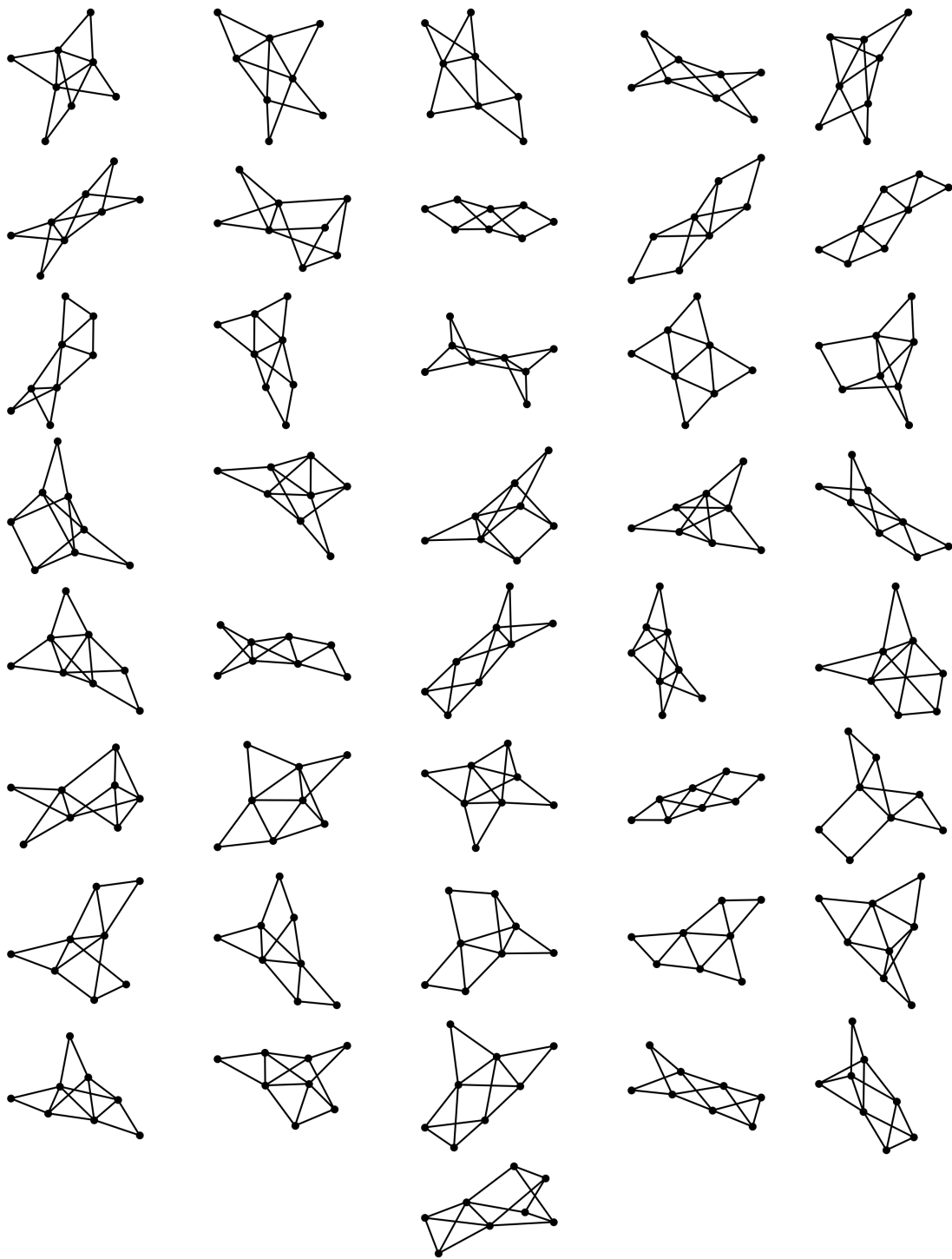Figure 10: Self-complementary graphs on eight vertices in $\mathcal{G}$.

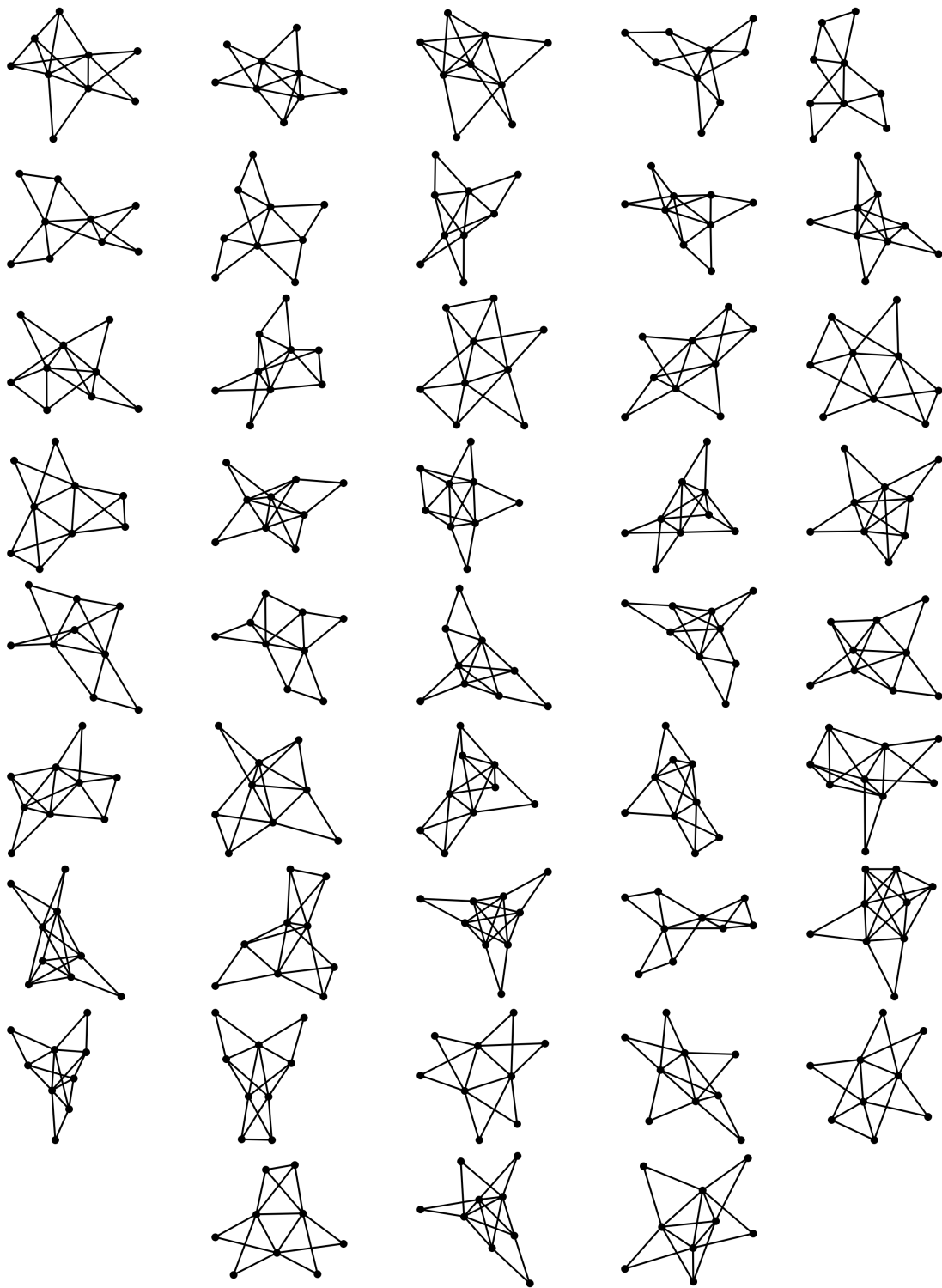Figure 11: Graphs on eight vertices in $\mathcal{G}$.
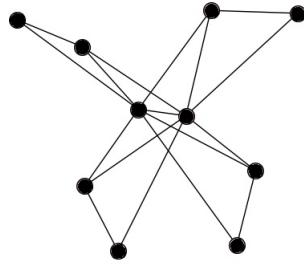
Figure 12: Graphs on nine vertices in $\mathcal{G}$.

Figure 13: A 10-vertex graph in $\mathcal{G}$.

Acknowledgement

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