Towards a splitter theorem for internally 4-connected binary matroids VII ☆

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A R T I C L E   I N F O

A B S T R A C T

Let M be a 3-connected binary matroid; M is internally 4-connected if one side of every 3-separation is a triangle or a triad, and M is $(4, 4, S)$-connected if one side of every 3-separation is a triangle, a triad, or a 4-element fan. Assume M is internally 4-connected and that neither M nor its dual is a cubic Möbius or planar ladder or a certain coextension thereof. Let N be an internally 4-connected proper minor of M. Our aim is to show that M has a proper internally 4-connected minor with an N-minor that can be obtained from M either by removing at most four elements, or by removing elements in an easily described way from a special substructure of M. When this aim cannot be met, the earlier papers in this series showed that, up to duality, M has a good bowtie, that is, a pair, $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$, of disjoint triangles and a cocircuit, $\{x_2, x_3, x_4, x_5\}$, where $M \backslash x_3$ has an N-minor and is $(4, 4, S)$-connected. We also showed that, when M has a good bowtie, either $M \backslash x_3, x_6$ has an N-minor and $M \backslash x_6$ is $(4, 4, S)$-connected; or $M \backslash x_3/x_2$ has an N-minor and is $(4, 4, S)$-connected. In this paper, we show that, when $M \backslash x_3, x_6$ has no N-minor, M has an internally 4-connected proper minor with an N-minor that can be obtained from M by removing at most three elements, or by removing elements in a well-described way from a special substructure of M. This

is a final step towards obtaining a splitter theorem for the class of internally 4-connected binary matroids.

1. Introduction

Seymour's Splitter Theorem [11] proved that if N is a 3-connected proper minor of a 3-connected matroid M, then M has a proper 3-connected minor M' with an N-minor such that $|E(M) - E(M')| \leq 2$. Furthermore, such an M' can be found with $|E(M)| - |E(M')| = 1$ unless $r(M) \geq 3$ and M is a wheel or a whirl. This result has been extremely useful in inductive and constructive arguments for 3-connected matroids. In [6], we prove a corresponding theorem for internally 4-connected binary matroids. In this paper, we prove one of the final steps needed to get that theorem. Specifically, we show here that if M and N are internally 4-connected binary matroids, and M has a proper N-minor, then M has a proper minor M' such that M' is internally 4-connected with an N-minor, and M' can be produced from M by a small number of simple operations.

Any unexplained matroid terminology used here will follow [10]. The only 3-separations allowed in an internally 4-connected matroid have a triangle or a triad on one side. A 3-connected matroid M is $(4, 4, S)$-*connected* if, for every 3-separation (X, Y) of M, one of X and Y is a triangle, a triad, or a 4-element fan, that is, a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad.

To provide a context for the main theorem of this paper, we briefly describe our progress towards obtaining the desired splitter theorem. Johnson and Thomas [7] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geq 3$, let G_{n+2} be the *biwheel* with $n+2$ vertices, that is, G_{n+2} consists of an n-cycle $v_1, v_2, \ldots, v_n, v_1$, the *rim*, and two additional vertices, u and w, both of which are adjacent to every v_i. Thus the dual of G_{n+2} is a cubic planar ladder. Let M be the cycle matroid of G_{2n+2} for some $n \geq 3$ and let N be the cycle matroid of the graph that is obtained by proceeding around the rim of G_{2n+2} and alternately deleting the edges from the rim vertex to u and to w. Both M and N are internally 4-connected but there is no internally 4-connected proper minor of M that has a proper N-minor. We can modify M slightly and still see the same phenomenon. Let G_{n+2}^+ be obtained from G_{n+2} by adding a new edge z joining the hubs u and w. Let Δ_{n+1} be the binary matroid that is obtained from $M(G_{n+2}^+)$ by deleting the element $v_{n-1}v_n$ and adding the third element on the line spanned by wv_n and uv_{n-1}. This new element is also on the line spanned by uv_n and wv_{n-1}. For $r \geq 3$, Mayhew, Royle, and Whittle [9] call Δ_r the *rank-r triangular Möbius matroid* and note that $\Delta_r \backslash z$ is the dual of the cycle matroid of a cubic Möbius ladder. The following is the main result of [3, Theorem 1.2].
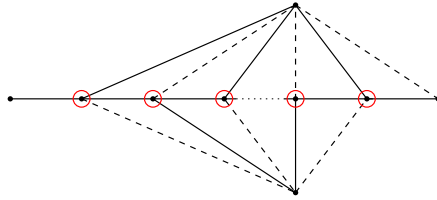
Fig. 1. All the elements shown are distinct. There are at least three dashed elements; and all dashed elements are deleted.

Theorem 1.1. *Let M be an internally 4-connected binary matroid with an internally 4-connected proper minor N such that $|E(M)| \geq 15$ and $|E(N)| \geq 6$. Then*

(i) *M has a proper minor M' such that $|E(M) - E(M')| \leq 3$ and M' is internally 4-connected with an N-minor; or*

(ii) *for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has a triangle T that contains an element e such that $M_0 \backslash e$ is $(4, 4, S)$-connected with an N_0-minor; or*

(iii) *M or M^* is isomorphic to $M(G^+_{r+1})$, $M(G_{r+1})$, Δ_r, or $\Delta_r \backslash z$ for some $r \geq 5$.*

That theorem led us to consider those matroids for which the second outcome in the theorem holds. In order to state the next result, we need to define some special structures. Let M be an internally 4-connected binary matroid and N be an internally 4-connected proper minor of M. Suppose M has disjoint triangles T_1 and T_2 and a 4-cocircuit D^* contained in their union. We call this structure a *bowtie* and denote it by (T_1, T_2, D^*). If D^* has an element d such that $M \backslash d$ has an N-minor and $M \backslash d$ is $(4, 4, S)$-connected, then (T_1, T_2, D^*) is a *good bowtie*. Motivated by (ii) of the last theorem, we aim to discover more about the structure of M when it has a triangle containing an element e such that $M \backslash e$ is $(4, 4, S)$-connected with an N-minor. One possible outcome here is that M has a good bowtie. Indeed, as the next result shows, if that outcome or its dual does not arise, we get a small number of easily described alternatives. We shall need two more definitions. A *terrahawk* is the graph that is obtained from a cube by adjoining a new vertex and adding edges from the new vertex to each of the four vertices that bound some fixed face of the cube. Fig. 1 shows a modified graph diagram, which we will use to keep track of some of the circuits and cocircuits in M, even though M need not be graphic. Each of the cycles in such a graph diagram corresponds to a circuit of M while a circled vertex indicates a known cocircuit of M. We refer to the structure in Fig. 1 as an *open rotor chain* noting that all of the elements in the figure are distinct and, for some $n \geq 3$, there are n dashed edges. The figure may suggest that n must be even but we impose no such restriction. We will refer to deleting the dashed elements from Fig. 1 as *trimming an open rotor chain*.

We need to define another special structure. An *augmented 4-wheel* consists of a 4-wheel restriction of M with triangles $\{z_2, x_1, y_2\}$, $\{y_2, x_3, z_3\}$, $\{z_3, y_3, x_2\}$, $\{x_2, y_1, z_2\}$ along with two additional distinct elements z_1 and z_4 such that M has $\{x_1, y_1, z_1, z_2\}$,
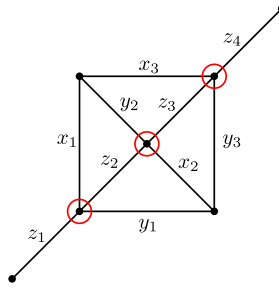
Fig. 2. An augmented 4-wheel.

$\{x_2, y_2, z_2, z_3\}$, and $\{x_3, y_3, z_3, z_4\}$ as cocircuits. We call $\{x_2, y_2, z_2, z_3\}$ the *central cocircuit* of the augmented 4-wheel. A diagrammatic representation of an augmented 4-wheel is shown in Fig. 2.

The following is [5, Corollary 1.4].

Theorem 1.2. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 6$. Suppose that M has a triangle T containing an element e for which $M \backslash e$ is $(4, 4, S)$-connected with an N-minor. Then one of the following holds.*

(i) *M has an internally 4-connected minor M' that has an N-minor such that $1 \leq |E(M) - E(M')| \leq 3$; or $|E(M) - E(M')| = 4$ and, for some (M_1, M_2) in $\{(M, M'), (M^*, (M')^*)\}$, the matroid M_2 is obtained from M_1 by deleting the central cocircuit of an augmented 4-wheel; or*
(ii) *M or M^* has a good bowtie; or*
(iii) *M is the cycle matroid of a terrahawk; or*
(iv) *for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 contains an open rotor chain that can be trimmed to obtain an internally 4-connected matroid with an N_0-minor.*

Note that there is a small error in [5, Theorem 1.1] since it requires at least five elements to be removed when trimming an open rotor chain. But, as the proof there makes clear, trimming exactly four elements is a possibility. Trimming exactly three elements is also possible but that is included under (i) of [5, Theorem 1.1].

This theorem leads us to consider a good bowtie $(\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_4, x_5\})$ in an internally 4-connected binary matroid M where $M \backslash x_3$ is $(4, 4, S)$-connected with an N-minor. In $M \backslash x_3$, we see that $\{x_5, x_4, x_2\}$ is a triad and $\{x_6, x_5, x_4\}$ is a triangle, so $\{x_6, x_5, x_4, x_2\}$ is a 4-element fan. By [4, Lemma 2.5], which is included below as Lemma 3.1, either
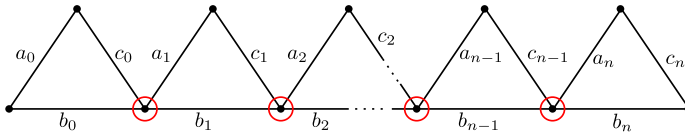
(i) $M \backslash x_3, x_6$ has an N-minor; or

Fig. 3. A string of bowties. All elements are distinct except that a_0 may be the same as c_n.

(ii) $M\backslash x_3, x_6$ does not have an N-minor, but $M\backslash x_3/x_2$ is $(4, 4, S)$-connected with an N-minor.

In [1], we considered the case when (i) holds and $M\backslash x_6$ is not $(4, 4, S)$-connected. In this paper, we focus on the case when (ii) holds. The next and final paper in this series will complete the work to obtain the splitter theorem by considering the case when $M\backslash x_3, x_6$ has an N-minor and $M\backslash x_6$ is $(4, 4, S)$-connected. Before stating the main result of [1], we define some structures that require special attention.

In a matroid M, a *string of bowties* is a sequence $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}$, $\{a_1, b_1, c_1\}, \{b_1, c_1, a_2, b_2\}, \ldots, \{a_n, b_n, c_n\}$ with $n \geq 1$ such that

(i) $\{a_i, b_i, c_i\}$ is a triangle for all i in $\{0, 1, \ldots, n\}$;
(ii) $\{b_j, c_j, a_{j+1}, b_{j+1}\}$ is a cocircuit for all j in $\{0, 1, \ldots, n-1\}$; and
(iii) the elements $a_0, b_0, c_0, a_1, b_1, c_1, \ldots, a_n, b_n$, and c_n are distinct except that a_0 and c_n may be equal.

The reader should note that this differs slightly from the definition we gave in [2] in that here we allow a_0 and c_n to be equal instead of requiring all of the elements to be distinct. Fig. 3 illustrates a string of bowties, but this diagram may obscure the potential complexity of such a string. Evidently $M\backslash c_0$ has $\{c_1, b_1, a_1, b_0\}$ as a 4-fan. Indeed, $M\backslash c_0, c_1, \ldots, c_i$ has a 4-fan for all i in $\{0, 1, \ldots, n-1\}$. We shall say that the matroid $M\backslash c_0, c_1, \ldots, c_n$ has been obtained from M by *trimming a string of bowties*. This operation plays a prominent role in our main theorem, and is the underlying operation in trimming an open rotor chain. Before stating this theorem, we introduce the other operations that incorporate this process of trimming a string of bowties. Such a string can attach to the rest of the matroid in a variety of ways. In most of these cases, the operation of trimming the string will produce an internally 4-connected minor of M with an N-minor. But, when the bowtie string is embedded in a modified quartic ladder in certain ways, we need to adjust the trimming process.

Consider the three configurations shown in Fig. 4 and Fig. 5 where the elements in each configuration are distinct except that d_2 may equal w_k. We refer to each of these configurations as an *enhanced quartic ladder*. Indeed, in each configuration, we can see a portion of a quartic ladder, which can be thought of as two interlocking bowtie strings, one pointing up and one pointing down. In each case, we focus on $M\backslash c_2, c_1, c_0, v_0, v_1, \ldots, v_k$ saying that this matroid has been obtained from M by an *enhanced-ladder move*.
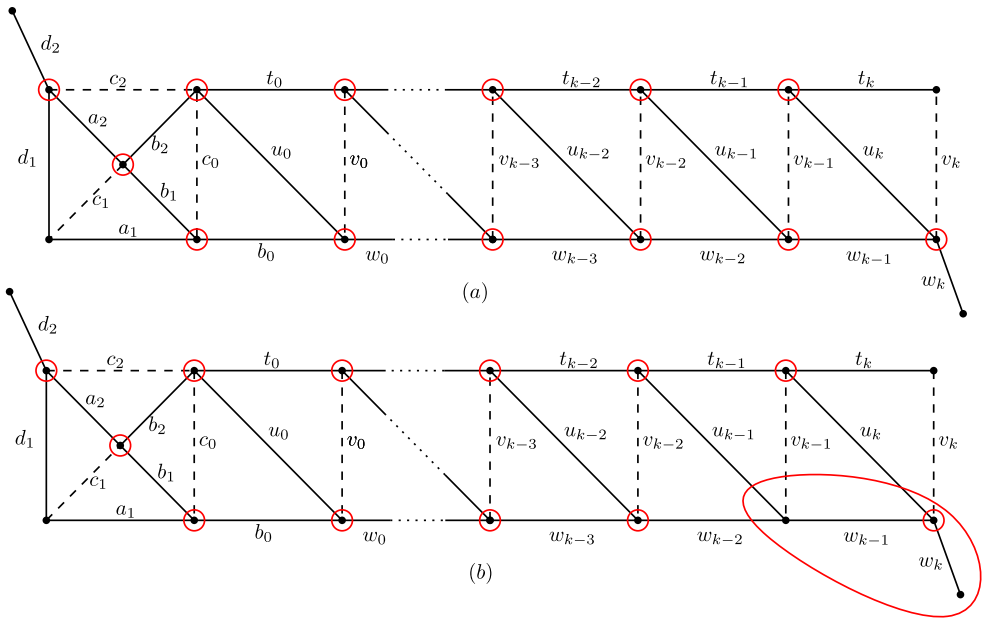
Fig. 4. In both (a) and (b), the elements shown are distinct, except that d_2 may be w_k. Furthermore, in (a), $k \geq 0$; and in (b), $k \geq 1$ and $\{w_{k-2}, u_{k-1}, v_{k-1}, u_k, v_k\}$ is a cocircuit.
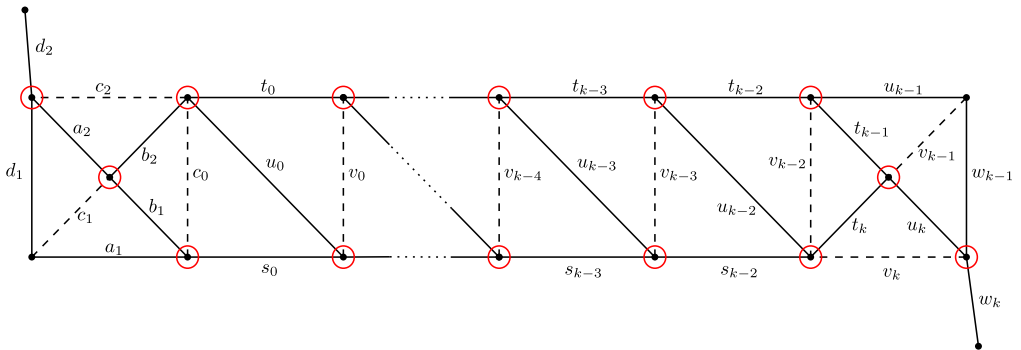
Fig. 5. In this configuration, $k \geq 2$ and the elements are all distinct except that d_2 may be w_k.

Suppose that $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \ldots, \{a_n, b_n, c_n\}$ is a bowtie string for some $n \geq 2$. Assume, in addition, that $\{b_n, c_n, a_0, b_0\}$ is a cocircuit. Then the string of bowties has wrapped around on itself as in Fig. 7, and we call the resulting structure a *ring of bowties*. We refer to each of the structures in Fig. 6 as a *ladder structure* and we refer to removing the dashed elements in Fig. 7 and Fig. 6 as *trimming a ring of bowties* and *trimming a ladder structure*, respectively.

In the case that trimming a string of bowties in M yields an internally 4-connected matroid with an N-minor, we are able to ensure that the string of bowties belongs to one
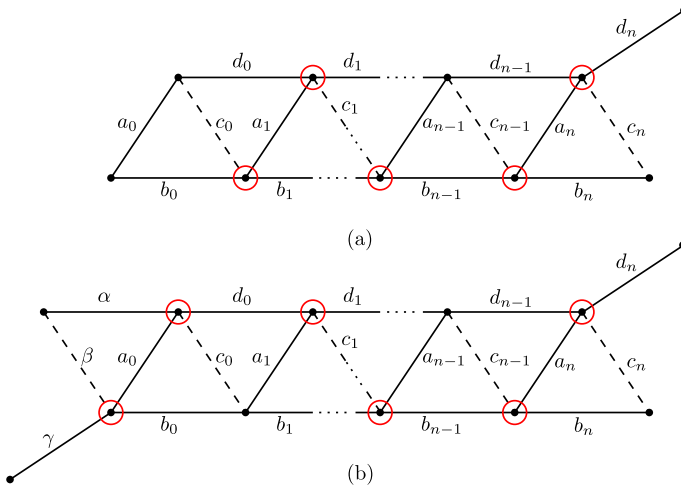
Fig. 6. In (a) and (b), $n \geq 2$ and the elements shown are distinct, with the exception that d_n may be the same as γ in (b). Either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit in (a) and (b). Either $\{b_0, c_0, a_1, b_1\}$ or $\{\beta, a_0, c_0, a_1, b_1\}$ is also a cocircuit in (b).
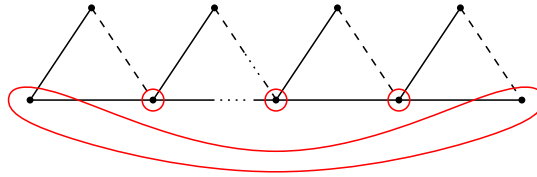
Fig. 7. A bowtie ring. All elements are distinct. The ring contains at least three triangles.

of the more highly structured objects we have described above. The following theorem is the main result of [1, Theorem 1.3].

Theorem 1.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Assume that M has a bowtie $(\{x_0, y_0, z_0\}, \{x_1, y_1, z_1\}, \{y_0, z_0, x_1, y_1\})$, where $M \backslash z_0$ is $(4, 4, S)$-connected, $M \backslash z_0, z_1$ has an N-minor, and $M \backslash z_1$ is not $(4, 4, S)$-connected. Then one of the following holds.*

(i) *M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N-minor; or*

(ii) *M contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N-minor; or*

(iii) *M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N-minor can be obtained by an enhanced-ladder move.*

In Theorem 1.3, not all of the moves that we perform on M to obtain an intermediate internally 4-connected binary matroid with an N-minor are bounded in size, but each
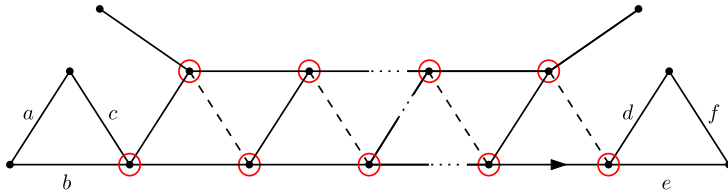
Fig. 8. All of the elements are distinct except that a may be f, or (a, b, c) may equal (d, e, f). There are at least three dashed elements.

unbounded move is highly structured. In this paper, we shall require one more such unbounded move. When M contains the structure in Fig. 8, where the elements are all distinct except that a may be f, or (a, b, c) may equal (d, e, f), we say that M contains an *open quartic ladder*. We will refer to deleting the dashed elements and contracting the arrow edge as a *mixed ladder move*. This is the only unbounded move that uses a contraction as well as a number of deletions. Note that both of the vertices of degree one in the diagram differ from the vertices closest to them.

The following theorem is the main result of this paper.

Theorem 1.4. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Let M have a bowtie $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$, where $M \backslash 4$ is $(4, 4, S)$-connected with an N-minor, and $M \backslash 1, 4$ has no N-minor. Then, for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, one of the following holds.*

(i) *M_0 has a proper internally 4-connected minor M' such that M' has an N_0-minor and either $|E(M)| - |E(M')| \leq 3$, or $|E(M) - E(M')| = 4$ and M' is obtained from M_0 by deleting the central cocircuit of an augmented 4-wheel; or*

(ii) *M_0 contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N_0-minor; or*

(iii) *M_0 contains an open quartic ladder and an internally 4-connected matroid with an N_0-minor can be obtained by a mixed ladder move; or*

(iv) *M_0 contains an enhanced quartic ladder from which an internally 4-connected minor of M_0 with an N_0-minor can be obtained by an enhanced-ladder move.*

An outline of the proof of this theorem is given in Section 5. That section separates the argument into three subcases and these cases are treated in the three subsequent sections. The results from those three sections are combined in Section 9 to complete the proof of the theorem. Before all of that, Section 2 gives some basic preliminaries while Sections 3 and 4 present some properties of, respectively, bowties and quasi rotors, and bowties and ladders.

The next corollary follows immediately by combining Theorem 1.4 with Theorem 1.3. This corollary provides the context for the final paper [6] in this series, which proves a splitter theorem for internally 4-connected binary matroids.

Corollary 1.5. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. If M has a bowtie $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$, where $M\backslash 4$ is $(4,4,S)$-connected with an N-minor, then either $M\backslash 1,4$ has an N-minor and $M\backslash 1$ is $(4,4,S)$-connected, or one of the following holds for some (M_0, N_0) in $\{(M,N),(M^*,N^*)\}$.*

(i) *M_0 has a proper internally 4-connected minor M' such that M' has an N_0-minor and either $|E(M)| - |E(M')| \leq 3$, or $|E(M) - E(M')| = 4$ and M' is obtained from M_0 by deleting the central cocircuit of an augmented 4-wheel; or*

(ii) *M_0 contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N_0-minor; or*

(iii) *M_0 contains an open quartic ladder and an internally 4-connected matroid with an N_0-minor can be obtained by a mixed ladder move; or*

(iv) *M_0 contains an enhanced quartic ladder from which an internally 4-connected minor of M_0 with an N_0-minor can be obtained by an enhanced-ladder move.*

2. Preliminaries

In this section, we give some basic definitions mainly relating to matroid connectivity. Let M and N be matroids. We shall sometimes write $N \preceq M$ to indicate that M has an N-minor, that is, a minor isomorphic to N. Now let E be the ground set of M and r be its rank function. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ. For a positive integer k, a subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ of E is a *k-separation* if $|X|, |E - X| \geq k$. If n is an integer exceeding one, a matroid is *n-connected* if it has no k-separations for all $k < n$. This definition [12] has the attractive property that a matroid is n-connected if and only if its dual is. Moreover, this matroid definition of n-connectivity is relatively compatible with the graph notion of n-connectivity when n is 2 or 3. For example, when G is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if G is a 3-connected simple graph. But the link between n-connectivity for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geq 3$, neither $M(K_{r+1})$ nor $PG(r - 1, 2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity in which certain 3-separations are allowed.

A matroid is *internally 4-connected* if it is 3-connected and, whenever (X, Y) is a 3-separation, either $|X| = 3$ or $|Y| = 3$. Equivalently, a 3-connected matroid M is internally 4-connected if and only if, for every 3-separation (X, Y) of M, either X or Y is a triangle or a triad of M. A graph G without isolated vertices is *internally 4-connected* if $M(G)$ is internally 4-connected.

Let M be a matroid. A subset S of $E(M)$ is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. For convenience, we will often refer to the fan ordering as the fan. We will be mainly concerned with 4-element and 5-element fans. By convention, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term *4-fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms *5-fan* and *5-cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let $(s_1, s_2, \ldots, s_n)$ be a fan ordering of a fan S. When M is 3-connected having at least five elements and $n \geq 4$, every fan ordering of S has its first and last elements in $\{s_1, s_n\}$. We call these elements the *ends* of the fan while the elements of $S - \{s_1, s_n\}$ are called the *internal elements* of the fan. When (s_1, s_2, s_3, s_4) is a 4-fan, our convention is that $\{s_1, s_2, s_3\}$ is a triangle, and we call s_1 the *guts element* of the fan and s_4 the *coguts element* of the fan since $s_1 \in \mathrm{cl}(\{s_2, s_3, s_4\})$ and $s_4 \in \mathrm{cl}^*(\{s_1, s_2, s_3\})$.

In a matroid M, a set U is *fully closed* if it is closed in both M and M^*. The *full closure* $\mathrm{fcl}(Z)$ of a set Z in M is the intersection of all fully closed sets containing Z. Let (X, Y) be a partition of $E(M)$. If (X, Y) is k-separating in M for some positive integer k, and y is an element of Y that is also in $\mathrm{cl}(X)$ or $\mathrm{cl}^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is k-separating, and we say that we have *moved* y into X. More generally, $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ is k-separating in M. Let n be an integer exceeding one. If M is n-connected, an n-separation (U, V) of M is *sequential* if $\mathrm{fcl}(U)$ or $\mathrm{fcl}(V)$ is $E(M)$. In particular, when $\mathrm{fcl}(U) = E(M)$, there is an ordering $(v_1, v_2, \ldots, v_m)$ of the elements of V such that $U \cup \{v_m, v_{m-1}, \ldots, v_i\}$ is n-separating for all i in $\{1, 2, \ldots, m\}$. When this occurs, the set V is called *sequential*. Moreover, if $n \leq m$, then $\{v_1, v_2, \ldots, v_n\}$ is a circuit or a cocircuit of M. A 3-connected matroid is *sequentially 4-connected* if all of its 3-separations are sequential. It is straightforward to check that, when M is binary, a sequential set with 3, 4, or 5 elements is a fan. Let (X, Y) be a 3-separation of a 3-connected binary matroid M. We shall frequently be interested in 3-separations that indicate that M is, for example, not internally 4-connected. We call (X, Y) or X a *(4, 3)-violator* if $|Y| \geq |X| \geq 4$. Similarly, (X, Y) is a *(4, 4, S)-violator* if, for each Z in $\{X, Y\}$, either $|Z| \geq 5$, or Z is non-sequential. We also say that (X, Y) is a *(4, 5, S, +)-violator* if, for each $Z \in \{X, Y\}$, either $|Z| \geq 6$, or Z is non-sequential, or Z is a 5-cofan. A binary matroid that has no *(4, 4, S)-violator* is *(4, 4, S)-connected*, as we defined in the introduction, and it is *(4, 5, S, +)-connected* if it has no *(4, 5, S, +)*-violator.

Next we note another special structure from [13], which has arisen frequently in our work towards the desired splitter theorem. In an internally 4-connected binary matroid M, we call $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ a *quasi rotor* with *central triangle* $\{4, 5, 6\}$ and *central element* 5 if $\{1, 2, 3\}, \{4, 5, 6\}$, and $\{7, 8, 9\}$ are disjoint triangles in M such that $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits and $\{3, 5, 7\}$ is a triangle (see Fig. 9). The next section is dedicated to results concerning bowties and quasi rotors.
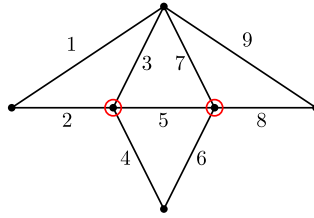
Fig. 9. A quasi rotor, where $\{2,3,4,5\}$ and $\{5,6,7,8\}$ are cocircuits.
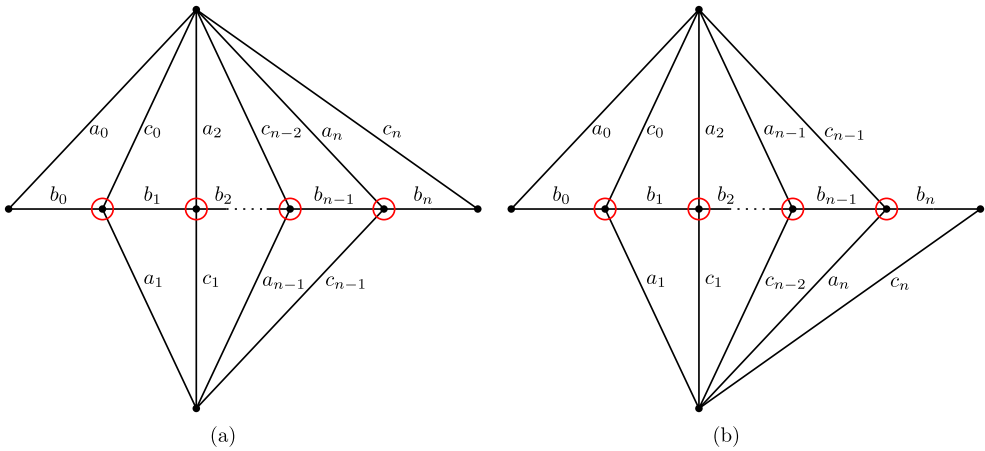

Fig. 10. Right-maximal rotor chain configurations. In the case that n is even, the rotor chain is depicted on the left. If n is odd, then the rotor chain has the form on the right.

For all non-negative integers i, it will be convenient to adopt the convention throughout the paper of using T_i and D_i to denote, respectively, a triangle $\{a_i, b_i, c_i\}$ and a cocircuit $\{b_i, c_i, a_{i+1}, b_{i+1}\}$. Let M have $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ as a quasi rotor. Now T_2 may also be the central triangle of a quasi rotor. In fact, we may have a structure like one of the two depicted in Fig. 10. If $T_0, D_0, T_1, D_1, \ldots, T_n$ is a string of bowties in M, for some $n \geq 2$, and M has the additional structure that $\{c_i, b_{i+1}, a_{i+2}\}$ is a triangle for all i in $\{0, 1, \ldots, n-2\}$, then we say that $((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_n, b_n, c_n))$ is a *rotor chain*. Clearly, deleting a_0 from a rotor chain gives an open rotor chain. Note that every three consecutive triangles within a rotor chain have the structure of a quasi rotor; that is, for all i in $\{0, 1, \ldots, n-2\}$, the sequence $(T_i, T_{i+1}, T_{i+2}, D_i, D_{i+1}, \{c_i, b_{i+1}, a_{i+2}\})$ is a quasi rotor. Zhou [13] considered a similar structure that he called a *double fan of length $n-1$*; it consists of all of the elements in the rotor chain except for a_0, b_0, b_n, and c_n.

If a rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_n, b_n, c_n))$ cannot be extended to a rotor chain of the form $((a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_{n+1}, b_{n+1}, c_{n+1}))$, then we call it a *right-maximal rotor chain*.

In the introduction, we defined a string of bowties. We say that such a string $T_0, D_0, T_1, D_1, \ldots, T_n$ is a *right-maximal bowtie string* in M if M has no triangle $\{u, v, w\}$

such that $T_0, D_0, T_1, D_1, \ldots, T_n, \{x, c_n, u, v\}, \{u, v, w\}$ is a bowtie string for some x in $\{a_n, b_n\}$.

For each positive integer $n \geq 3$, let M_n be the binary matroid that is obtained from a wheel of rank n by adding a single element γ such that if B is the basis of $M(\mathcal{W}_n)$ consisting of the set of spokes of the wheel, then the fundamental circuit $C(\gamma, B)$ is $B \cup \gamma$. Observe that $M_3 \cong F_7$ and $M_4 \cong M^*(K_{3,3})$. Assume that the spokes of $M(\mathcal{W}_n)$, in cyclic order, are $x_1, x_2, \ldots, x_n$ and that $\{x_i, y_i, x_{i+1}\}$ is a triangle of $M(\mathcal{W}_n)$ for all i in $\{1, 2, \ldots, n\}$ where we interpret all subscripts modulo n. Then, for all i in $\{1, 2, \ldots, n\}$, the set $\{y_{i-1}, x_i, y_i\}$ is a triad of $M(\mathcal{W}_n)$ and $\{\gamma, y_{i-1}, x_i, y_i\}$ is a cocircuit of M_n. It is straightforward to check that M_n is internally 4-connected. Kingan and Lemos [8] denote M_n by F_{2n+1}. When n is odd, which is the case that will be of most interest to us here, M_n^* is isomorphic to what Mayhew, Royle, and Whittle [9] call the *rank-$(n+1)$ triadic Möbius matroid*, Υ_{n+1}.

3. Some results for bowties and quasi rotors

In this section, we gather together a number of results that will be needed to prove the main theorem beginning with Lemma 2.5 from [4] and Lemmas 4.1 and 4.2 from [1]. The second of these will often be used implicitly without reference.

Lemma 3.1. *Let M and N be internally 4-connected binary matroids and $\{e, f, g\}$ be a triangle of M such that $N \preceq M \backslash e$ and $M \backslash e$ is $(4, 4, S)$-connected. Suppose $|E(N)| \geq 7$ and $M \backslash e$ has $(1, 2, 3, 4)$ as a 4-fan. Then either*

(i) $N \preceq M \backslash e \backslash 1$; *or*
(ii) $N \preceq M \backslash e / 4$ *and $M \backslash e / 4$ is $(4, 4, S)$-connected.*

Lemma 3.2. *Let N be an internally 4-connected matroid having at least seven elements and M be a binary matroid with an N-minor. If (s_1, s_2, s_3, s_4) is a 4-fan in M, then $M \backslash s_1$ or M / s_4 has an N-minor. If $(s_1, s_2, s_3, s_4, s_5)$ is a 5-fan in M, then either $M \backslash s_1, s_5$ has an N-minor, or both $M \backslash s_1 / s_2$ and $M \backslash s_5 / s_4$ have N-minors.*

Lemma 3.3. *Let M be an internally 4-connected matroid having at least ten elements. If $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ is a bowtie in M, then $\{2, 3, 4, 5\}$ is the unique 4-cocircuit of M that meets both $\{1, 2, 3\}$ and $\{4, 5, 6\}$.*

When dealing with bowtie structures, we will repeatedly use the following result [1, Lemma 4.3], a modification of [2, Lemma 6.3].

Lemma 3.4. *Let $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ be a bowtie in an internally 4-connected binary matroid M with $|E(M)| \geq 13$. Then $M \backslash 6$ is $(4, 4, S)$-connected unless $\{4, 5, 6\}$ is the central triangle of a quasi rotor $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{y, 6, 7, 8\},$*

$\{x, y, 7\})$ *for some* x *in* $\{2, 3\}$ *and some* y *in* $\{4, 5\}$. *In addition, when* $M \backslash 6$ *is* $(4, 4, S)$-*connected, one of the following holds.*

(i) $M \backslash 6$ *is internally 4-connected; or*
(ii) M *has a triangle* $\{7, 8, 9\}$ *disjoint from* $\{1, 2, 3, 4, 5, 6\}$ *such that* $(\{4, 5, 6\}, \{7, 8, 9\},$ $\{a, 6, 7, 8\})$ *is a bowtie for some* a *in* $\{4, 5\}$; *or*
(iii) *every* $(4, 3)$-*violator of* $M \backslash 6$ *is a 4-fan of the form* (u, v, w, x), *where* M *has a triangle* $\{u, v, w\}$ *and a cocircuit* $\{v, w, x, 6\}$ *for some* u *and* v *in* $\{2, 3\}$ *and* $\{4, 5\}$, *respectively, and* $|\{1, 2, 3, 4, 5, 6, w, x\}| = 8$; *or*
(iv) $M \backslash 1$ *is internally 4-connected and* M *has a triangle* $\{1, 7, 8\}$ *and a cocircuit* $\{a, 6, 7, 8\}$ *where* $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$ *and* $a \in \{4, 5\}$.

In Theorem 1.3, we dealt with the case when M has a bowtie $(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\},$ $\{b_0, c_0, a_1, b_1\})$ such that $M \backslash c_0$ is $(4, 4, S)$-connected with an N-minor and $M \backslash c_1$ has an N-minor but is not $(4, 4, S)$-connected. We will therefore use the following hypothesis in the next lemma, and throughout this paper.

Hypothesis VII. *If, for* $(M_1, N_1) \in \{(M, N), (M^*, N^*)\}$, *the matroid* M_1 *has a bowtie* $(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{b_0, c_0, a_1, b_1\})$, *where* $M_1 \backslash c_0$ *is* $(4, 4, S)$-*connected and* $M_1 \backslash c_0, c_1$ *has an* N_1-*minor, then* $M_1 \backslash c_1$ *is* $(4, 4, S)$-*connected.*

The next lemma is related to the previous lemma. We begin with the same structure in M, a bowtie, but we add the additional consideration of preserving an N-minor, and we eliminate one outcome by adding Hypothesis VII.

Lemma 3.5. *Let* $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ *be a bowtie in an internally 4-connected binary matroid* M *with* $|E(M)| \geq 13$. *Let* N *be an internally 4-connected minor of* M *having at least seven elements. Suppose that* $M \backslash 4$ *is* $(4, 4, S)$-*connected, that* $N \preceq M \backslash 1, 4$, *and that Hypothesis VII holds. Then* $M \backslash 1$ *is* $(4, 4, S)$-*connected with an* N-*minor and*

(i) $M \backslash 1$ *is internally 4-connected; or*
(ii) M *has a triangle* $\{7, 8, 9\}$ *such that* $(\{1, 2, 3\}, \{7, 8, 9\}, \{7, 8, 1, s\})$ *is a bowtie for some* s *in* $\{2, 3\}$ *and* $|\{1, 2, \ldots, 9\}| = 9$; *or*
(iii) *every* $(4, 3)$-*violator of* $M \backslash 1$ *is a 4-fan of the form* $(4, t, 7, 8)$, *for some* t *in* $\{2, 3\}$ *where* $|\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$; *or*
(iv) $M \backslash 6$ *is internally 4-connected with an* N-*minor.*

Proof. By Hypothesis VII, we may assume that $M \backslash 1$ is $(4, 4, S)$-connected. Now, if part (i) or (ii) of Lemma 3.4 holds, then (i) or (ii) of the current lemma holds. Moreover, if (iii) of Lemma 3.4 holds, then, since $M \backslash 4$ is $(4, 4, S)$-connected, part (iii) of the current lemma holds. Thus we may assume that (iv) of Lemma 3.4 holds. Then, by symmetry, we may assume that M has a triangle $\{6, 7, 8\}$ and a cocircuit $\{1, 3, 7, 8\}$ where

$|\{1,2,3,4,5,6,7,8\}| = 8$ and $M\backslash 6$ is internally 4-connected. We may also assume that $M\backslash 6$ has no N-minor otherwise (iv) holds.

Now $M\backslash 1, 4$ is 3-connected having $(6,7,8,3)$ as a 4-fan and $N \preceq M\backslash 1, 4$. As $N \npreceq M\backslash 6$, it follows, by Lemma 3.2, that $N \preceq M\backslash 1, 4/3$. But $M\backslash 1, 4/3 \cong M\backslash 2, 4/3 \cong M\backslash 2, 4/5 \cong M\backslash 2/5\backslash 6$. Hence $N \preceq M\backslash 6$, a contradiction. □

The next two results [1, Lemmas 5.2 and 5.7] are helpful when dealing with bowtie strings.

Lemma 3.6. *Let* $T_0, D_0, T_1, D_1, \ldots, T_n$ *be a string of bowties in a matroid* M. *Then*

$$M\backslash c_0, c_1, \ldots, c_n/b_n \cong M\backslash a_0, a_1, \ldots, a_n/b_0$$
$$\cong M\backslash c_0, c_1, \ldots, c_{k-1}/b_k\backslash a_k, a_{k+1}, \ldots, a_n$$
$$\cong M\backslash c_0, c_1, \ldots, c_{k-1}/b_{k-1}\backslash a_k, a_{k+1}, \ldots, a_n$$

for all k *in* $\{1, 2, \ldots, n\}$.

Lemma 3.7. *Let* M *be a binary matroid with an internally 4-connected minor* N *where* $|E(N)| \geq 7$. *Let* $T_0, D_0, T_1, D_1, \ldots, T_n$ *be a string of bowties in* M. *Suppose* $M\backslash c_0, c_1$ *has an* N-minor but $M\backslash c_0, c_1/b_1$ *does not. Then* $M\backslash c_0, c_1, \ldots, c_n$ *has an* N-minor, but $M\backslash c_0, c_1, \ldots, c_i/b_i$ *has no* N-minor for all i in $\{1, 2, \ldots, n\}$, and $M\backslash c_0, c_1, \ldots, c_j/a_j$ *has no* N-minor for all j in $\{2, 3, \ldots, n\}$.

In the following result, we consider a short bowtie string.

Lemma 3.8. *Let* T_0, D_0, T_1, D_1, T_2 *be a string of bowties in an internally 4-connected binary matroid* M. *Suppose* $M\backslash c_1$ *is* $(4,4,S)$-connected. *Then* $M\backslash c_1/b_1$ *is* $(4,5,S,+)$-connected. *Moreover, either* $M\backslash c_1/b_1$ *is internally 4-connected, or* $M\backslash c_1/b_1$ *has a 4-fan and, whenever* $(\alpha, \beta, \gamma, \delta)$ *is such a 4-fan,* $a_1 \in \{\beta, \gamma, \delta\}$, *and* $\{\beta, \gamma, \delta, c_1\}$ *is a cocircuit of* M.

Proof. As $M\backslash c_1$ is $(4,4,S)$-connected having (c_2, b_2, a_2, b_1) as a 4-fan, $M\backslash c_1/b_1$ is 3-connected. Suppose $M\backslash c_1/b_1$ has a $(4,5,S,+)$-violator (U,V). Then, without loss of generality, $|T_2 \cap U| \geq 2$. It is not difficult to check that $(U \cup T_2 \cup b_1, V - T_2)$ is a $(4,4,S)$-violator of $M\backslash c_1$, a contradiction. We conclude that $M\backslash c_1/b_1$ is $(4,5,S,+)$-connected.

Suppose that $(\alpha, \beta, \gamma, \delta)$ is a 4-fan in $M\backslash c_1/b_1$ such that $a_1 \notin \{\beta, \gamma, \delta\}$. Orthogonality with T_1 implies that $\{\beta, \gamma, \delta, c_1\}$ is not a cocircuit of M. Hence $\{\beta, \gamma, \delta\}$ is a triad of M. As M is internally 4-connected, we deduce that $\{\alpha, \beta, \gamma, b_1\}$ is a circuit of M, so, by orthogonality, $\{\alpha, \beta, \gamma\}$ meets $\{b_0, c_0, a_1\}$ and $\{a_2, b_2\}$. Thus $\{\beta, \gamma, \delta\}$ meets a triangle of M, a contradiction. We conclude that $a_1 \in \{\beta, \gamma, \delta\}$. Since a_1 is in a triangle of M, we deduce that $\{c_1, \beta, \gamma, \delta\}$ is a 4-cocircuit of M. □

We continue on this theme with the following lemma.

Lemma 3.9. *Let M be an internally 4-connected binary matroid and suppose that M has T_0, D_0, T_1, D_1, T_2 as a string of bowties and that $M\backslash c_1$ is $(4,4,S)$-connected. Then*

(i) $M\backslash c_1/b_1$ *is internally 4-connected; or*

(ii) $M\backslash c_1/b_1$ *is $(4,5,S,+)$-connected and M has a triangle $\{1,2,3\}$ that avoids T_1 such that $\{2,3,a_1,c_1\}$ is a cocircuit; or*

(iii) $M\backslash c_1/b_1$ *is $(4,5,S,+)$-connected and M has elements d_0 and d_1 such that $\{d_0,d_1\}$ avoids $T_0 \cup T_1 \cup T_2$ where $\{d_0,a_1,c_1,d_1\}$ is a cocircuit, and $\{d_0,a_1,s\}$ or $\{d_1,c_1,t\}$ is a triangle for some s in $\{b_0,c_0\}$ or t in $\{a_2,b_2\}$.*

Proof. Suppose (i) does not hold. By Lemma 3.8, $M\backslash c_1/b_1$ has a 4-fan, $(1,2,3,4)$, where $a_1 \in \{2,3,4\}$, and $\{2,3,4,c_1\}$ is a cocircuit. Lemma 3.3 implies that $\{2,3,4\}$ avoids T_0 and T_2. Now M has $\{1,2,3\}$ or $\{1,2,3,b_1\}$ as a circuit. Suppose that $a_1 = 4$. If $\{1,2,3\}$ is a triangle, then (ii) holds, so we assume not. Then $\{1,2,3,b_1\}$ is a circuit. Now orthogonality implies that $\{1,2,3\}$ meets both $\{b_0,c_0\}$ and $\{a_2,b_2\}$, so $\{2,3\}$ meets T_0 or T_2, a contradiction. We deduce that $a_1 \neq 4$. Without loss of generality, $a_1 = 3$. If $\{1,2,a_1\}$ is a triangle in M, then orthogonality implies that $\{1,2\}$ meets $\{b_0,c_0\}$. Hence $1 \in \{b_0,c_0\}$ and, relabelling $(1,2,4)$ as (s,d_0,d_1), we see that (iii) holds. If $\{1,2,a_1,b_1\}$ is a circuit of M, then orthogonality with D_1 implies that $\{1,2\}$ meets $\{a_2,b_2\}$. Thus $1 \in \{a_2,b_2\}$ and $\{1,2,a_1,b_1\} \triangle T_1$ is $\{1,2,c_1\}$, a triangle, so, relabelling $(1,2,4)$ as (t,d_1,d_0), (iii) holds. $\square$

Next, we prove a stronger version of [2, Lemma 8.4].

Lemma 3.10. *Let M be an internally 4-connected binary matroid having T_0, D_0, T_1, D_1, T_2 as a string of bowties. Then*

(i) M/T_1 *is internally 4-connected; or*

(ii) T_1 *is the central triangle of a quasi rotor; or*

(iii) $M\backslash c_1/b_1$ *is internally 4-connected; or*

(iv) $M\backslash c_1/b_1$ *is $(4,5,S,+)$-connected and M has elements d_0 and d_1 such that $\{d_0,d_1\}$ avoids $T_0 \cup T_1 \cup T_2$, and $\{d_0,a_1,c_1,d_1\}$ is a cocircuit, and $\{d_0,a_1,s\}$ or $\{d_1,c_1,t\}$ is a triangle for some s in $\{b_0,c_0\}$ or t in $\{a_2,b_2\}$.*

Proof. Assume the lemma does not hold. By Lemma 3.4, since T_1 is not the central triangle of a quasi rotor, $M\backslash c_1$ is $(4,4,S)$-connected. By Lemma 3.9, M has a triangle $\{1,2,3\}$ avoiding T_1 such that $\{2,3,a_1,c_1\}$ is a cocircuit. Lemma 3.3 implies that $\{1,2,3\}$ avoids D_0 and D_1, and that T_0 and T_2 avoid $\{2,3\}$. Then (i) or (ii) holds by [2, Lemma 8.3], a contradiction. $\square$

To conclude this section, we recall [1, Lemma 4.5], which is useful for dealing with quasi rotors.
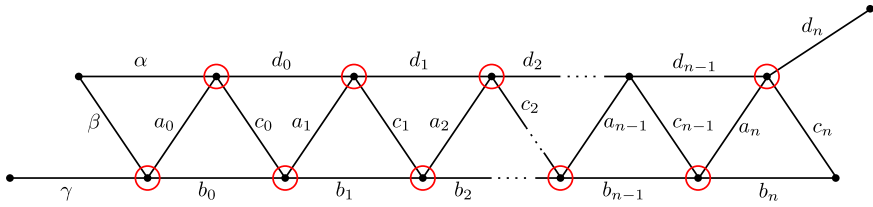
Fig. 11. $n \geq 1$ and M has either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit, where $d_{-1} = \alpha$ if $n = 1$.

Lemma 3.11. *Let M be an internally 4-connected binary matroid having $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ as a quasi rotor and having at least thirteen elements. Let N be an internally 4-connected matroid containing at least seven elements such that M/e has an N-minor for some e in $\{3, 5, 7\}$. Then one of $M\backslash 1$, $M\backslash 9$, $M\backslash 1/2$, $M\backslash 9/8$, or $M\backslash 3, 4/5$ is internally 4-connected with an N-minor.*

4. Bowties and ladders

In this section, we consider how bowties can interact with ladders. We begin with a lemma that builds from the configuration in Fig. 11.

Lemma 4.1. *Let M be an internally 4-connected binary matroid that has at least thirteen elements. Assume that M contains the configuration shown in Fig. 11 where $n \geq 1$, all the elements shown are distinct except that d_n and γ may be equal, and, in addition to the cocircuits shown, exactly one of $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit of M. Assume also that M is not isomorphic to the cycle matroid of a quartic Möbius ladder and that $M\backslash c_n$ is $(4, 4, S)$-connected. Then $M\backslash c_0, c_1, \ldots, c_n, \beta$ is $(4, 4, S)$-connected and if it has a $(4, 3)$-violator, then one side of that $(4, 3)$-violator is a 4-fan F where either*

(i) *F is a 4-fan in $M\backslash c_n$ with b_n as its coguts element; or*
(ii) *F is a 4-fan in $M\backslash \beta$ with α as its coguts element.*

Proof. First we show the following.

4.1.1. *When $n = 1$, neither $\{d_0, d_1\}$ nor $\{b_0, b_1\}$ is contained in a triangle of M. Moreover, none of a_1, b_0, b_1, d_0, nor d_1 is in a triangle of $M\backslash c_0, c_1$.*

If $\{d_{n-1}, d_n\}$ is in a triangle, then $M\backslash c_n$ has a 5-fan, a contradiction. If $\{b_0, b_1\}$ is in a triangle, then orthogonality implies that the triangle's third element is in $\{\beta, \gamma\}$, so $\lambda(\{\alpha, \beta, \gamma, d_0\} \cup T_0 \cup \{a_1, b_1\}) \leq 2$, a contradiction. By [1, Lemma 6.1], a_1 is not in a triangle of $M\backslash c_0, c_1$. If b_0 or b_1 is in a triangle of $M\backslash c_0, c_1$, then orthogonality implies that this triangle contains $\{b_0, b_1\}$, a contradiction. Similarly, if d_0 or d_1 is in a triangle of

$M \backslash c_0, c_1$, then orthogonality implies that this triangle contains $\{d_0, d_1\}$, a contradiction. We conclude that 4.1.1 holds.

Next we note that

4.1.2. $M \backslash c_0, c_1, \ldots, c_n$ *is* $(4, 4, S)$-*connected.*

This follows immediately from [1, Lemma 6.5] when $n > 1$. Moreover, it holds when $n = 1$ by combining 4.1.1 with [1, Lemma 6.1].

The matroid $M \backslash c_0, c_1, \ldots, c_n$ has $(\beta, \alpha, a_0, d_0)$ or $(\beta, \alpha, a_0, a_1)$ as a 4-fan where the latter can only arise when $n = 1$. Thus $M \backslash c_0, c_1, \ldots, c_n, \beta$ is 3-connected. Next we observe that $M \backslash c_0, c_1, \ldots, c_n, \beta$ is sequentially 4-connected. To see this, note that if $M \backslash c_0, c_1, \ldots, c_n, \beta$ has a non-sequential 3-separation (U, V), then, as $\{a_0, \alpha\}$ is in a triad, we may assume that this triad is contained in U. Thus $(U \cup \beta, V)$ is a non-sequential 3-separation of $M \backslash c_0, c_1, \ldots, c_n$, a contradiction.

Now suppose $M \backslash c_0, c_1, \ldots, c_n, \beta$ has a 4-fan (w_1, w_2, w_3, w_4). Then M has a cocircuit C^* such that $\{w_2, w_3, w_4\} \subsetneq C^* \subseteq \{w_2, w_3, w_4, \beta, c_0, c_1, \ldots, c_n\}$.

4.1.3. *If* (w_1, w_2, w_3, w_4) *is a 4-fan of* $M \backslash c_0, c_1, \ldots, c_n$, *then* $w_4 = b_n$ *and* $\{b_n, c_n, w_2, w_3\}$ *is a cocircuit of* M, *so* (w_1, w_2, w_3, w_4) *is a 4-fan of* $M \backslash c_n$.

Suppose that this fails. If $n > 1$, then, by (iii) of [1, Lemma 6.5], $w_4 = d_0$ and $a_0 \in \{w_2, w_3\}$. Moreover, (w_1, w_2, w_3, w_4) is a 4-fan of $M \backslash c_0$. Thus $\{w_2, w_3, c_0, d_0\}$ is a 4-cocircuit of M containing $\{a_0, c_0, d_0\}$, so it is the 4-cocircuit $\{\alpha, a_0, c_0, d_0\}$. Thus $\{w_1, w_2, w_3\}$ contains $\{\alpha, a_0\}$ and so is $\{\alpha, a_0, \beta\}$, a contradiction. We conclude that 4.1.3 holds if $n > 1$.

Now let $n = 1$. By 4.1.1, neither $\{b_0, b_1\}$ nor $\{d_0, d_1\}$ is contained in a triangle of M. It follows by [1, Lemma 6.1] that $w_4 = b_1$. Now 4.1.3 holds if $\{w_2, w_3, b_1, c_1\}$ is a cocircuit, so we assume that $\{w_2, w_3, b_1, c_0\}$ or $\{w_2, w_3, b_1, c_0, c_1\}$ is a cocircuit. Orthogonality implies that $\{w_2, w_3\}$ meets $\{d_0, a_1\}$, a contradiction to 4.1.1. Thus 4.1.3 holds.

We may now assume that (w_1, w_2, w_3, w_4) is not a 4-fan of $M \backslash c_0, c_1, \ldots, c_n$. Then $\beta \in C^*$. Thus $\{\alpha, a_0\}$ meets $\{w_2, w_3, w_4\}$. Next we show that

4.1.4. $a_0 \notin \{w_1, w_2, w_3, w_4\}$.

First we show that $a_0 \notin \{w_1, w_2, w_3\}$. Assume the contrary. Let $n = 1$. Then, by orthogonality, $\{w_1, w_2, w_3\}$ meets $\{\alpha, d_0\}$ or $\{\alpha, a_1\}$. Thus d_0 or a_1 is in a triangle of $M \backslash c_0, c_1$, a contradiction to 4.1.1. Hence we may assume that $n \geq 2$. Then, by [1, Lemma 6.3], the triangle $\{w_1, w_2, w_3\}$ of $M \backslash c_0, c_1, \ldots, c_n$ meets $\{a_0, b_0, d_0, a_1, b_1, d_1, \ldots, a_n, b_n, d_n\}$ in $\{a_0\}$ or $\{a_0, d_{n-1}, d_n\}$. By orthogonality, $\{a_0, d_{n-1}, d_n\}$ is not a triangle. Thus $\{w_1, w_2, w_3\}$ avoids $\{b_0, d_0, a_1, b_1, d_1, \ldots, a_n, b_n, d_n\}$. By orthogonality between $\{w_1, w_2, w_3\}$ and both $\{\beta, \gamma, a_0, b_0\}$ and $\{\alpha, a_0, c_0, d_0\}$, we find that $\{w_1, w_2, w_3\} = \{a_0, \alpha, \gamma\}$. But $\{a_0, \alpha, \beta\}$ is a triangle, a contradiction. Hence $a_0 \notin \{w_1, w_2, w_3\}$.

Suppose now that $a_0 = w_4$. Orthogonality between C^* and the circuit $\{a_0, b_0, d_0, a_1\}$ implies that $\{w_2, w_3\}$ meets $\{b_0, d_0, a_1\}$. Thus, by [1, Lemma 6.3], $n = 1$. But now we have a contradiction to 4.1.1. Thus 4.1.4 holds.

We now know that $\alpha \in \{w_2, w_3, w_4\}$. Suppose $\alpha \in \{w_2, w_3\}$. If $\{\alpha, a_0, c_0, d_0\}$ is a cocircuit, then, by orthogonality, $\{a_0, d_0\}$ meets $\{w_1, w_2, w_3\}$. By [1, Lemma 6.3] and 4.1.4, $n = 1$ and $d_0 \in \{w_1, w_2, w_3\}$, a contradiction to 4.1.1. We deduce that $\{\alpha, a_0, c_0, a_1, c_1\}$ is a cocircuit of M, so $n = 1$. Then orthogonality implies that $\{w_1, w_2, w_3\}$ meets $\{a_0, a_1\}$. Thus, by 4.1.4 and 4.1.1, we have a contradiction. Hence $\alpha = w_4$.

Now suppose that $c_i \in C^*$ for some i in $\{0, 1, \ldots, n\}$. Then $\{w_2, w_3\}$ meets $\{a_i, b_i\}$. Thus, by [1, Lemma 6.3], if $n \geq 2$, then $i = 0$ and $a_0 \in \{w_2, w_3\}$, a contradiction to 4.1.4. Moreover, if $n = 1$, then one of a_0, b_0, a_1, or b_1 is in $\{w_2, w_3\}$, a contradiction to 4.1.4 or 4.1.1. We conclude that C^* avoids $\{c_0, c_1, \ldots, c_n\}$, so $C^* = \{w_2, w_3, \alpha, \beta\}$, and (w_1, w_2, w_3, α) is a 4-fan of $M \backslash \beta$. $\quad\square$

Beginning with the next lemma and for the rest of the paper, we shall start abbreviating how we refer to the following four outcomes in the main theorem.

  (i) M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N-minor; or
 (ii) M contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N-minor;
(iii) M contains an open quartic ladder from which an internally 4-connected minor of M with an N-minor can be obtained by a mixed ladder move;
(iv) M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N-minor can be obtained by an enhanced-ladder move.

When (i) or (iv) holds, we say, respectively, that M has a *quick win* or an *enhanced-ladder win*. When trimming an open rotor chain, a ladder structure, or a ring of bowties in M produces an internally 4-connected matroid with an N-minor, we say, respectively, that M has an *open-rotor-chain win*, a *ladder win*, or a *bowtie-ring win*. When (iii) holds, we say that M has a *mixed ladder win*.

5. An outline of the proof of the main theorem

Since the proof of the main theorem is long, we give an outline of it in this section. By hypothesis, M and N are internally 4-connected binary matroids and M has a bowtie $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ where $M \backslash 4$ is $(4, 4, S)$-connected with an N-minor, and $M \backslash 1, 4$ has no N-minor. We may assume that $M \backslash 6$ is $(4, 4, S)$-connected otherwise the theorem holds by Theorem 1.3. The one result in this section, Lemma 5.1, shows that either we get a quick win, or M contains one of configurations (A), (B), and C in Fig. 13. In Section 6, we treat the case when M contains configuration (C) noting first that, by Lemma 3.1, $M \backslash 4/5$ is $(4, 4, S)$-connected with an N-minor and with $(a, b, c, 6)$ as a 4-fan.
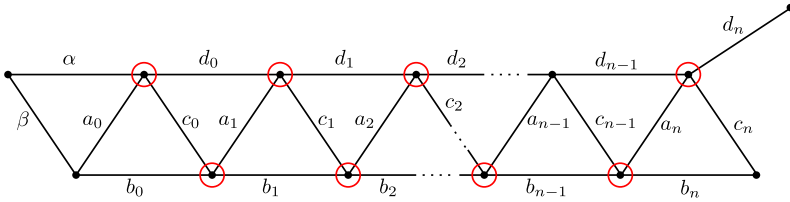
Fig. 12. Either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit, where $d_{-1} = \alpha$.
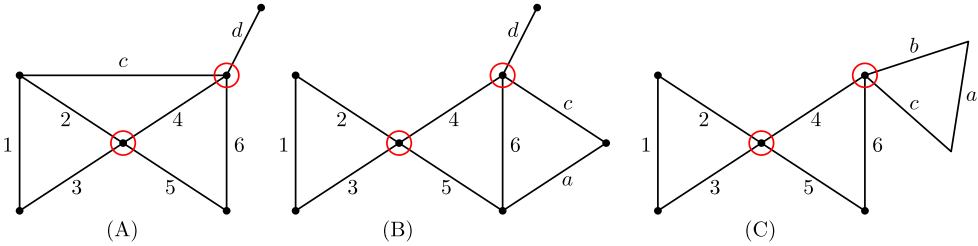

Fig. 13. In each structure, we view the labels on 2 and 3 as being interchangeable. The elements in each part are distinct except that a may equal 1 in (B) and (C).

Thus $M\backslash 4/5, 6$ or $M\backslash 4/5\backslash a$ has an N-minor. These two possibilities are dealt with in Lemmas 6.1 and 6.4, respectively.

In Section 7, we deal with the case when M contains configuration (A). First we prove a technical lemma detailing the possible structures surrounding a right-maximal bowtie chain in M that is also a right-maximal bowtie chain in M', a minor of M. Then we show in Lemma 7.3 that we obtain our result.

The results of Sections 6 and 7 mean that we can assume that M contains neither of configurations (C) or (A). It remains to consider when M contains configuration (B) from Fig. 13. This is done in Section 8. Finally, in Section 9, the parts already proved are combined to complete the proof of the main theorem.

We now show that M does indeed contain one of the three structures in Fig. 13.

Lemma 5.1. *Let* $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ *be a bowtie in an internally 4-connected binary matroid* M *with* $|E(M)| \geq 13$. *Let* N *be an internally 4-connected binary matroid having at least seven elements. Suppose that* $M\backslash 4$ *is* $(4, 4, S)$-*connected with an* N-*minor and that* $N \npreceq M\backslash 1, 4$. *Then either* M *has an internally 4-connected minor* M' *with an* N-*minor such that* $1 \leq |E(M) - E(M')| \leq 2$, *or, up to switching the labels on the elements* 2 *and* 3, *the matroid* M *contains one of the configurations shown in Fig. 13, the deletion* $M\backslash 6$ *is* $(4, 4, S)$-*connected, and* $\{4, 5, 6\}$ *is the only triangle in* M *containing* 5. *Moreover, in each of (A), (B), and (C), the elements shown are distinct except that, in (B) and (C), it is possible that* $a = 1$.

Proof. Since $N \npreceq M\backslash 1, 4$, it follows by Lemma 3.1 that $M\backslash 4/5$ is $(4, 4, S)$-connected with an N-minor. Thus $M\backslash 6/5$ is $(4, 4, S)$-connected with an N-minor. We may assume

that $M\backslash 6/5$ is not internally 4-connected otherwise the lemma holds. If 5 is in a triangle T other than $\{4, 5, 6\}$, then $M\backslash 4/5$ has $T - 5$ as a circuit, a contradiction. Thus $\{4, 5, 6\}$ is the only triangle in M containing 5.

With a view to using Lemma 3.4, we now consider $M\backslash 6$. First suppose that $\{4, 5, 6\}$ is the central triangle of a quasi rotor $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{y, 6, 7, 8\}, \{x, y, 7\})$ for some x in $\{2, 3\}$ and some $y \in \{4, 5\}$. Since $M\backslash 4$ is $(4, 4, S)$-connected, we deduce that $y = 4$. Then $M\backslash 4/5$ has $(x, 6, 7, 8, 9)$ as a 5-fan, a contradiction. We conclude that $\{4, 5, 6\}$ is not the central triangle of such a quasi rotor. Then Lemma 3.4 implies that either $M\backslash 6$ is internally 4-connected, and the lemma holds; or $M\backslash 6$ is $(4, 4, S)$-connected but not internally 4-connected. Moreover, in the latter case, one of the following holds.

(i) M has $\{a, b, c\}$ as a triangle and $\{b, c, d, 6\}$ as a cocircuit for some a in $\{2, 3\}$ and b in $\{4, 5\}$, where $|\{1, 2, \ldots, 6, c, d\}| = 8$; or

(ii) M contains the structure in Fig. 13(C) where the elements are all distinct except that a may be the same as 1; or

(iii) M has a triangle $\{7, 8, 9\}$ and a cocircuit $\{5, 6, 7, 8\}$ where the elements are all distinct except that 1 may be the same as 9.

To see this, observe first that (i) above occurs when (iii) of Lemma 3.4 holds. On the other hand, outcomes (ii) and (iv) of Lemma 3.4 have been combined into (ii) and (iii) above with the separation between the latter being determined by the relative placement of the elements 4 and 5.

If (i) holds, then we know that $b = 4$, otherwise $\{a, 5, c\}$ is a triangle in M that contains 5 but is not $\{4, 5, 6\}$; a contradiction. Thus, up to switching the labels on 2 and 3, if (i) holds, then M contains the structure in Fig. 13(A) where all of the elements shown are distinct.

Since (ii) yields (C) in Fig. 13, we may now assume that (iii) holds. Recall that $M\backslash 6/5$ is $(4, 4, S)$-connected but not internally 4-connected. Thus $M\backslash 6/5$ has a 4-fan (a, b, c, d). Lemma 3.8 implies that $4 \in \{b, c, d\}$, and $\{b, c, d, 6\}$ is a cocircuit of M. By symmetry between b and c, we may assume that $4 = b$ or $4 = d$.

Assume first that $4 = b$. Then M has $\{4, c, d, 6\}$ as a cocircuit, and Lemma 3.3 implies that $\{c, d\}$ avoids $\{1, 2, 3\}$ so $|\{1, 2, \ldots, 6, c, d\}| = 8$, and M has $\{a, 4, c\}$ or $\{5, a, 4, c\}$ as a circuit. In the first case, orthogonality with $\{2, 3, 4, 5\}$ implies that $a \in \{2, 3\}$, so, up to switching the labels of 2 and 3, the structure (A) in Fig. 13 occurs, where all of the elements are distinct. If $\{5, a, 4, c\}$ is a circuit of M, then M also has $\{a, c, 6\}$ as a circuit, so M contains the structure in Fig. 13(B) where all of the elements are distinct except that a may be a repeated element. Certainly $a \notin \{5, 6, b, c, d\}$. Hence, by orthogonality, either a is distinct from the other elements in (B), or $a = 1$.

We may now assume that $4 = d$. Then $\{b, c, 4, 6\}$ is a cocircuit of M. Lemma 3.3 implies that $\{b, c\}$ avoids $\{1, 2, 3, 7, 8, 9\}$. Either $\{a, b, c\}$ or $\{a, b, c, 5\}$ is a circuit of M. In the first case, (C) in Fig. 13 occurs and orthogonality implies that all of the elements

are distinct except that a may be 1. Now suppose that M has $\{a, b, c, 5\}$ as a circuit. Orthogonality between this circuit and $\{2, 3, 4, 5\}$ implies that $\{a, b, c\}$ meets $\{2, 3\}$. By symmetry between 2 and 3, we may assume that $3 \in \{a, b, c\}$. As $\{b, c\}$ avoids $\{1, 2, 3, 7, 8, 9\}$, we deduce that $3 = a$. In (iii), we know that $\{5, 6, 7, 8\}$ is a cocircuit. Orthogonality between this cocircuit and the circuit $\{3, b, c, 5\}$ implies that $\{3, b, c\}$ meets $\{6, 7, 8\}$, a contradiction. □

We reiterate here that, in each of configurations (A), (B), and (C) in Fig. 13, while the labels of $1, 4, 5$, and 6 are fixed, we allow the labels on 2 and 3 to be interchanged without regarding the resulting structures as being different.

6. Configuration (C)

In this section, we treat the case when M contains the configuration in Fig. 13(C). We arrived at this configuration by assuming that $N \npreceq M \backslash 1, 4$. Thus, by Lemma 3.1, $M \backslash 4/5$ is $(4, 4, S)$-connected with an N-minor. Since $M \backslash 4/5$ has $(a, b, c, 6)$ as a 4-fan, by Lemma 3.2, $N \preceq M \backslash 4/5/6$ or $N \preceq M \backslash 4/5 \backslash a$. The next lemma deals with the first of these cases.

Lemma 6.1. *Let M and N be binary internally 4-connected matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Suppose that M contains structure (C) in Fig. 13, where $M \backslash 4$ is $(4, 4, S)$-connected with an N-minor and $N \npreceq M \backslash 1, 4$. If $M \backslash 4/5, 6$ has an N-minor, then M has a quick win.*

Proof. We observe that $M \backslash 4/5, 6 \cong M/4/5/6$. Now apply Lemma 3.10. If (i) or (iii) of that lemma holds, then the required result is immediate. If (ii) holds, then, as M/e has an N-minor for all e in $\{4, 5, 6\}$, it follows by [1, Lemma 4.5] that the lemma holds. Finally, suppose that (iv) holds. Then either M has a triangle containing 5 and a member of $\{2, 3\}$; or M has a triangle containing 6 and a member of $\{b, c\}$. In each case, we obtain a contradiction to the fact that $M \backslash 4$ is $(4, 4, S)$-connected. □

The next lemma concerns the structure in Fig. 14, which arises in Lemma 6.3.

Lemma 6.2. *Suppose M is an internally 4-connected binary matroid. If M contains the structure in Fig. 14, where all of the elements are distinct, then M is the cycle matroid of a 16-element quartic planar ladder having $\{a_1, c_0, d_0\}$ and $\{a_0, c_3, d_3\}$ as triangles.*

Proof. Clearly $|E(M)| \geq 16$. Moreover, $T_0 \cup T_1 \cup T_2 \cup T_3 \cup \{d_1, d_2\}$ has rank at most seven and contains at least five cocircuits, none of which is the symmetric difference of any others. Thus this 14-element set is 3-separating, so $|E(M)| \leq 17$, and has rank equal to seven. Suppose $|E(M)| = 17$. Then M has $\{d_0, d_3, e\}$ as a triad for some element e that is not shown in Fig. 14. Then the symmetric difference of all of the vertex cocircuits and $\{d_0, d_1, e\}$ is $\{e, a_0, c_0\}$, so M is not internally 4-connected, a contradiction. We conclude
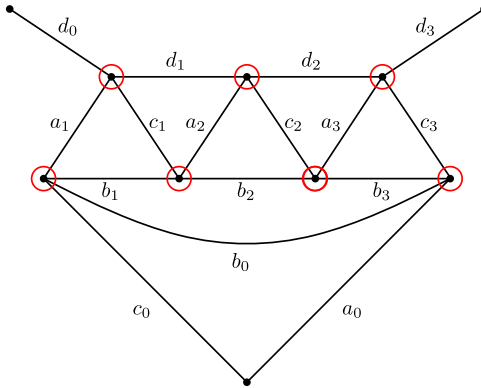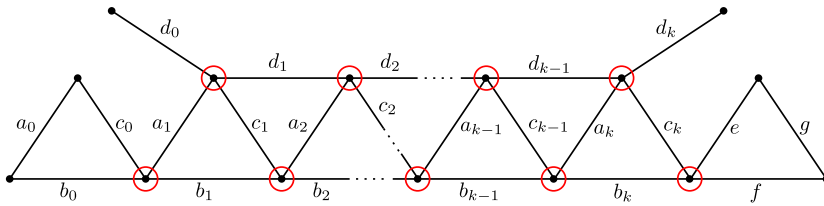
Fig. 14. All of the elements are distinct.

Fig. 15. All of the elements are distinct except a_0 may be g, or T_0 may be $\{e, f, g\}$.

that $|E(M)| = 16$. As $r(M) = 7$, we see that $\{a_0, c_0, a_1, b_1, a_2, b_2, c_3\}$ is a basis B of M. By orthogonality with the vertex cocircuits in Fig. 14, we deduce that the fundamental circuit $C(d_0, B)$ for d_0 is $\{d_0, a_1, c_0\}$. Similarly, $C(d_3, B) = \{d_3, a_0, c_3\}$. The lemma now follows because M is binary and so is determined by its fundamental circuits with respect to B. $\square$

Next we deal with a structure that leads to a mixed ladder win.

Lemma 6.3. *Let M be an internally 4-connected binary matroid having at least fifteen elements. Suppose that M contains the structure in Fig. 15, where $k \geq 2$ and all of the elements are distinct except that a_0 may be g, or T_0 may be $\{e, f, g\}$, or d_0 may be d_k.*

- (i) *The set $\{b_0, c_0\} \neq \{e, f\}$ and $d_0 \neq d_k$.*
- (ii) *If $M \backslash c_i$ is $(4, 4, S)$-connected for all i in $\{1, 2, \ldots, k\}$, then*
    - (a) *$\{d_0, a_1\}$ is contained in a triangle; or*
    - (b) *$\{c_k, d_k\}$ is contained in a triangle; or*
    - (c) *$M \backslash c_1, c_2, \ldots, c_k / b_k$ is internally 4-connected.*

Proof. Let $X = T_1 \cup T_2 \cup \cdots \cup T_k \cup \{d_1, d_2, \ldots, d_{k-1}\}$. Then $E(M) - X$ contains $T_0 \cup d_0$. To establish (i), it suffices to show that $\lambda(X) \leq 2$ if $\{b_0, c_0\} = \{e, f\}$ or if $d_0 = d_k$. In the first case, $D_0 \triangle \{b_k, c_k, e, f\} = \{a_1, b_1, a_k, c_k\}$, which is a cocircuit contained in X.

In the second case, $\{d_0, a_1, c_1, d_1\} \bigtriangleup \{d_{k-1}, a_k, c_k, d_k\}$ is also a cocircuit contained in X. In each case, one easily checks that $\lambda(X) \leq 2$, so (i) holds.

Next we note that, by [1, Lemma 5.3], $M \backslash c_1, c_2, \ldots, c_k$ is 3-connected unless this matroid has a 1- or 2-element cocircuit D^* that contains a_j or b_j for some j in $\{3, 4, \ldots, k\}$. Consider the exceptional case. Then M has a cocircuit C^* such that $D^* \subsetneqq C^* \subseteq D^* \cup \{c_1, c_2, \ldots, c_k\}$. For $i \neq j$, orthogonality implies that $c_i \in C^*$ if and only if $|D^* \cap \{a_i, b_i\}| = 1$. As C^* meets T_j, it must have at least four elements. Hence $C^* = \{x, c_i, y, c_j\}$ for some x in $\{a_i, b_i\}$ and y in $\{a_j, b_j\}$. Let s be the smaller of i and j. Then $\{i, j\} = \{s, s+1\}$ and $a_{s+1} \in C^*$ otherwise we get a contradiction to orthogonality between C^* and $\{c_s, d_s, a_{s+1}\}$. Thus C^* is a 4-cocircuit meeting T_s and T_{s+1} and containing $\{a_{s+1}, c_{s+1}\}$. This contradiction to Lemma 3.3 implies that $M \backslash c_1, c_2, \ldots, c_k$ is indeed 3-connected.

Clearly $M \backslash c_1, c_2, \ldots, c_k$ has b_k as the coguts element of a 4-fan. The proof of the next assertion occupies most of the rest of the proof of the lemma finishing just before 6.3.5.

6.3.1. *If $M \backslash c_1, c_2, \ldots, c_k$ is $(4, 4, S)$-connected and has b_k as the coguts element of every 4-fan, then part (b) of the lemma holds.*

Certainly $M \backslash c_1, c_2, \ldots, c_k / b_k$ is 3-connected. Suppose (U, V) is a non-sequential 3-separation of this matroid. Without loss of generality, $\{e, f, g\} \subseteq U$, so $(U \cup b_k, V)$ is a non-sequential 3-separation of $M \backslash c_1, c_2, \ldots, c_k$, a contradiction. Thus $M \backslash c_1, c_2, \ldots, c_k / b_k$ is sequentially 4-connected.

Let $(\alpha, \beta, \gamma, \delta)$ be a 4-fan in $M \backslash c_1, c_2, \ldots, c_k / b_k$. By the hypothesis, $(\alpha, \beta, \gamma, \delta)$ is not a 4-fan in $M \backslash c_1, c_2, \ldots, c_k$, so $\{\alpha, \beta, \gamma, b_k\}$ is a circuit, C. Next we observe that

6.3.2. *C contains $\{b_{k-1}, b_k\}$, avoids $\{a_{k-1}, a_k, c_{k-1}, c_k\}$, and meets both $\{b_{k-2}, c_{k-2}\}$ and $\{e, f\}$.*

The last part of 6.3.2 is an immediate consequence of orthogonality. Suppose $a_k \in C$. Then orthogonality with the cocircuits $\{d_{k-1}, a_k, c_k, d_k\}$ and $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ implies that the last element in C is d_k, so C contains $\{a_k, d_k, b_k\}$. The symmetric difference of C with T_k is a triangle containing $\{c_k, d_k\}$. Thus (ii)(b) holds, a contradiction. We may now assume that $a_k \notin C$. Then $b_{k-1} \in C$ and orthogonality implies that $\{b_{k-2}, c_{k-2}, a_{k-1}\}$ meets C. Moreover, orthogonality with $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ implies that a_{k-1} is not in C. Thus 6.3.2 holds.

Suppose that $k \geq 3$. Then, by orthogonality, $c_{k-2} \notin C$. Moreover, without loss of generality, we may assume that $e \in C$. Thus $C = \{e, b_{k-2}, b_{k-1}, b_k\}$. Then, by orthogonality between C and $\{b_{k-3}, c_{k-3}, a_{k-2}, b_{k-2}\}$, we deduce that $e \in \{b_{k-3}, c_{k-3}\}$. Thus $k = 3$ and $T_0 = \{e, f, g\}$. Without loss of generality, $e = b_0$. Since (i) holds, $c_0 = g$, and M contains the structure in Fig. 14 with all the elements in that figure being distinct. Then Lemma 6.2 implies that (ii)(a) holds.

We may now assume that $k = 2$. Recall that $C = \{\alpha, \beta, \gamma, b_2\}$ where $(\alpha, \beta, \gamma, \delta)$ is a 4-fan of $M \backslash c_1, c_2/b_2$. Now M has a cocircuit C^* such that $\{\beta, \gamma, \delta\} \subseteq C^* \subseteq \{\beta, \gamma, \delta, c_1, c_2\}$. Since $\{\beta, \gamma\}$ meets $\{b_1, b_0, c_0\}$, it follows that $|C^*| \neq 3$.

Next we show the following.

6.3.3. *If $\{e, f, g\} = T_0$, then $C = \{b_0, b_1, b_2, y\}$ for some element y that is not in $T_0 \cup T_1 \cup T_2 \cup \{d_0, d_1, d_2\}$. If $\{e, f, g\} \neq T_0$, then, without loss of generality, $e \in C$ and C is $\{b_0, b_1, b_2, e\}$ or $\{c_0, b_1, b_2, e\}$.*

Suppose first that $\{e, f, g\} = T_0$. By (a), we may assume that $(a_0, b_0, c_0) = (e, f, g)$. If M has a triangle that meets T_0, T_1, and T_2, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Thus we may assume that M has no such triangle. Now C contains $\{b_1, b_2\}$ and meets both $\{b_0, c_0\}$ and $\{a_0, b_0\}$. Thus C meets each of T_0, T_1, and T_2. If C contains two elements of one of these triangles, say T_i, then $C \triangle T_i$ is a triangle that meets each of T_0, T_1, and T_2, a contradiction. Thus $C = \{b_0, b_1, b_2, y\}$ for some element y that avoids $T_0 \cup T_1 \cup T_2$. By orthogonality, y also avoids $\{d_0, d_1, d_2\}$. Thus the first part of 6.3.3 holds. The second part is an immediate consequence of 6.3.2.

6.3.4. *If $\{e, f, g\} = T_0$, then $\{\beta, \gamma\}$ is $\{b_1, y\}$; otherwise $\{\beta, \gamma\}$ is $\{b_1, e\}$.*

To see this, note that, by 6.3.3, $\{\alpha, \beta, \gamma\}$ is one of $\{b_0, b_1, y\}$, $\{b_0, b_1, e\}$, or $\{c_0, b_1, e\}$. To prove 6.3.4, it suffices to show that T_0 avoids $\{\beta, \gamma\}$. Assume the contrary. Then orthogonality between C^* and T_0 implies that $\delta \in T_0$. Suppose $c_1 \in C^*$. Then orthogonality implies that $\{\beta, \gamma\} - T_0 = \{b_1\}$ and $c_2 \notin C^*$. Thus C^* is a 4-cocircuit that meets both T_0 and T_1 but is not $\{b_0, c_0, a_1, b_1\}$, a contradiction to Lemma 3.3. Thus $c_1 \notin C^*$. Hence $c_2 \in C^*$ and we contradict orthogonality with T_2. We conclude that 6.3.4 holds.

We now know that $b_1 \in C^*$. As $|C^* \cap T_1|$ is even, either $c_1 \in C^*$, or $c_1 \notin C^*$ and C^* is $\{b_1, y, a_1, c_2\}$ or $\{b_1, e, a_1, c_2\}$. Thus $c_1 \in C^*$ otherwise orthogonality between C^* and the circuit $\{c_1, b_2, c_2, d_1\}$ gives a contradiction. Suppose $c_2 \in C^*$. Then, by orthogonality, $\delta = a_2$, so C^* is a 5-cocircuit containing $\{b_1, c_1, a_2\}$. The symmetric difference of this cocircuit with the cocircuit $\{b_1, c_1, a_2, b_2\}$ is a triad that contains c_2, a contradiction. Hence $c_2 \notin C^*$. Thus C^* is $\{b_1, y, \delta, c_1\}$ or $\{b_1, e, \delta, c_1\}$. By orthogonality, with the circuit $\{c_1, c_2, d_1, b_2\}$, we deduce that $\delta = d_1$. If $\{e, f, g\} \neq \{a_0, b_0, c_0\}$, then we have a contradiction to orthogonality between C^* and $\{e, f, g\}$. If $\{e, f, g\} = \{a_0, b_0, c_0\}$, then $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_1, y\}) \leq 2$, a contradiction as $|E(M)| \geq 15$. This completes the proof of 6.3.1.

Now assume that part (ii) of the lemma fails. Next we show the following.

6.3.5. $k \geq 3$

Assume that $k = 2$. Then applying Lemma 6.1 of [1], we see that part (i) of that lemma does not hold. Moreover, part (v) of that lemma does not hold by 6.3.1. If (ii) of [1, Lemma 6.1] holds, that is, $\{d_1, d_2\}$ is in a triangle of M, then this triangle together with

c_1 and a_2 forms a 5-fan in $M \backslash c_2$, a contradiction. If (iv) of [1, Lemma 6.1] holds, that is, a_1 is in a triangle that avoids $\{b_1, c_1, d_1\}$, then orthogonality with $\{d_0, a_1, c_1, d_1\}$ implies that this triangle contains d_0, so (ii)(a) of the current lemma holds, a contradiction. Thus (iii) of [1, Lemma 6.1] holds, that is, $\{b_1, b_2\}$ is in a triangle T of M. By orthogonality, T meets both $\{b_0, c_0\}$ and $\{e, f\}$. Thus $T_0 = \{e, f, g\}$, so $\{b_1, b_2, b_0\}$ or $\{b_1, b_2, c_0\}$ is a triangle. Then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. We conclude that 6.3.5 holds.

We now apply Lemma 6.5 of [1] to the configuration induced by $T_1 \cup T_2 \cup \cdots \cup T_k \cup \{d_1, d_2, \ldots, d_k\}$. Neither (i) nor (ii) of that lemma holds, and if (iv) holds, then $\{d_k, c_k\}$ is in a triangle, that is, (ii)(b) of the current lemma holds, a contradiction. We deduce that (iii) of [1, Lemma 6.5] holds. Thus $M \backslash c_1, c_2, \ldots, c_k$ is $(4, 4, S)$-connected and every $(4, 3)$-violator of it is a 4-fan (u_1, u_2, u_3, u_4) where either $u_4 = d_1$ and a_1 is in $\{u_2, u_3\}$; or $u_4 = b_k$. Suppose that (u_1, u_2, a_1, d_1) is a 4-fan in $M \backslash c_1, c_2, \ldots, c_k$. By orthogonality, $\{u_1, u_2\}$ meets $\{d_0, d_1\}$. Hence $\{d_0, a_1\}$ is contained in a triangle and (ii)(a) holds, a contradiction. We conclude that every 4-fan of $M \backslash c_1, c_2, \ldots, c_k$ has b_k as its coguts element. This contradiction to 6.3.1 completes the proof of the lemma. □

We now consider the case when M contains the configuration in Fig. 13(C), but $N \not\preceq M \backslash 1, 4$ and $N \preceq M \backslash 4/5 \backslash a$.

Lemma 6.4. *Let M and N be binary internally 4-connected matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose that Hypothesis VII holds and that M contains structure (C) in Fig. 13, where $M \backslash 6/5 \backslash a$ has an N-minor, and $M \backslash 4$ and $M \backslash 6$ are $(4, 4, S)$-connected. If $M \backslash 4, 1$ has no N-minor, then*

(i) *M has a quick win; or*
(ii) *M has an open-rotor-chain win or a ladder win; or*
(iii) *M has a mixed ladder win; or*
(iv) *M has an enhanced-ladder win.*

Proof. Suppose that M has no quick win. As $M \backslash 6/5 \backslash a \cong M \backslash 4/5 \backslash a$, each of these matroids has an N-minor.

6.4.1. *Neither $M \backslash 4, a/c$ nor $M \backslash 4, a/b$ has an N-minor.*

Assume that 6.4.1 fails. Note that $M \backslash 4, a/b \cong M \backslash 4, c/b \cong M \backslash 4, c/6$ and, by symmetry, $M \backslash 4, a/c \cong M \backslash 4, b/6$. Since $M \backslash 4, a/c$ or $M \backslash 4, a/b$ has an N-minor, we deduce that $M \backslash 4/6$ has an N-minor. As $M \backslash 4/6$ has $(1, 2, 3, 5)$ as a 4-fan, but $M \backslash 4, 1$ has no N-minor, it follows that $M \backslash 4/6/5$ has an N-minor. Therefore, by Lemma 6.1, we conclude that 6.4.1 holds.

Next we relabel letting $(5, 6, 4) = (a_0, b_0, c_0)$ and $(b, c, a) = (a_1, b_1, c_1)$. Take $T_0, D_0, T_1, D_1, \ldots, T_n$ to be a right-maximal bowtie string in M. Now $M \backslash c_0, c_1$ has an N-minor but none of $M \backslash c_0, c_1/b_1, M \backslash c_0, c_1/a_1$, and $M \backslash c_0, 1$ has an N-minor.

Suppose that $\{a_0, b_0, x, c_n\}$ is not a cocircuit for all x in $\{a_n, b_n\}$. Now, by Lemma 5.1, a_0 is in a unique triangle of M. Therefore, by [1, Lemma 10.1], the lemma holds.

We may now assume that $\{a_0, b_0, x, c_n\}$ is a cocircuit for some x in $\{a_n, b_n\}$. Then $a_0 \neq c_n$, so all of the elements in the bowtie string are distinct. Up to relabelling a_n and b_n, we may assume that $x = b_n$. Lemma 3.3 implies that $n > 1$. If $n = 2$, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Therefore $n \geq 3$. By Lemma 3.7 and the observations at the end of the second-last paragraph, we have that

6.4.2. $M \backslash c_0, c_1, \ldots, c_n$ *has an N-minor, but $M \backslash c_0, c_1, \ldots, c_i / a_i$ has no N-minor for all i in $\{1, 2, \ldots, n\}$.*

Next we show that

6.4.3. $\{1, 2, 3\}$ *avoids* $\{c_0, c_1, \ldots, c_n\} \cup \{a_n, b_n\}$.

Suppose first that $\{1, 2, 3\}$ meets $\{c_0, c_1, \ldots, c_n\}$. Then $\{2, 3\}$ meets the last set, since $M \backslash c_0, 1$ has no N-minor. Up to switching the labels on 2 and 3, we may assume that $3 = c_i$ for some i in $\{0, 1, \ldots, n\}$. Then 2 is in a 1- or 2-cocircuit of $M \backslash c_0, c_i$, so $M \backslash c_0, c_i / 2$ has an N-minor. Hence so does $M \backslash c_0, /2 \backslash 1$, a contradiction. We deduce that $\{1, 2, 3\}$ avoids $\{c_0, c_1, \ldots, c_n\}$.

Now suppose that $\{1, 2, 3\}$ meets $\{a_n, b_n\}$. As $\{1, 2, 3\} \neq T_n$, orthogonality between $\{1, 2, 3\}$ and D_{n-1} implies that $b_{n-1} \in \{1, 2, 3\}$. If $n = 2$, then $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. If $n \geq 3$, then orthogonality with D_{n-2} implies, since $\{1, 2, 3\} \neq T_{n-1}$, that $b_{n-2} \in \{1, 2, 3\}$. Then orthogonality between $\{1, 2, 3\}$ and D_{n-3} gives a contradiction. Thus 6.4.3 holds.

Evidently $(1, 2, 3, a_0)$ is a 4-fan in $M \backslash c_0, c_1, \ldots, c_n$. Since deleting 1 from the last matroid destroys all N-minors, Lemma 3.2 implies that $M \backslash c_0, c_1, \ldots, c_n / a_0$ has an N-minor. Now

$$
\begin{aligned}
M \backslash c_0, c_1, \ldots, c_{n-1}, c_n / a_0 &\cong M \backslash b_0, c_1, \ldots, c_{n-1}, c_n / a_0 \\
&\cong M \backslash b_0, c_1, \ldots, c_{n-1}, c_n / b_n \\
&\cong M \backslash b_0, c_1, \ldots, c_{n-1}, a_n / b_n \\
&\cong M \backslash b_0, c_1, \ldots, c_{n-1}, a_n / b_{n-1} \\
&\vdots \\
&\cong M \backslash b_0, c_1, a_2 \ldots, a_{n-1}, a_n / b_1 \\
&\cong M \backslash b_0, a_1, \ldots, a_{n-1}, a_n / c_0.
\end{aligned}
$$

Therefore $M / b_i \backslash c_i$ has an N-minor for all i in $\{1, 2, \ldots, n\}$. Hypothesis VII implies that $M \backslash c_1$ is $(4, 4, S)$-connected and, indeed, that $M \backslash c_i$ is $(4, 4, S)$-connected for all i in $\{1, 2, \ldots, n\}$.

Consider $M/b_n \backslash c_n$, which has an N-minor. Lemma 3.9 implies that $M/b_n \backslash c_n$ is $(4, 5, S, +)$-connected and either

(I) M has a triangle $\{x, y, z\}$ such that $\{y, z, a_n, c_n\}$ is a cocircuit; or
(II) M has elements d_{n-1} and d_n such that $\{d_{n-1}, d_n\}$ avoids $T_{n-1} \cup T_n \cup T_0$ where $\{d_{n-1}, a_n, c_n, d_n\}$ is a cocircuit, and $\{d_{n-1}, a_n, s\}$ or $\{d_n, c_n, t\}$ is a triangle for some s in $\{b_{n-1}, c_{n-1}\}$ or t in $\{a_0, b_0\}$.

6.4.4. *Part (I) does not hold.*

Suppose that (I) does hold. Since we have a right-maximal bowtie string, we know that $\{x, y, z\}$ meets $T_0 \cup T_1 \cup \cdots \cup T_n$. By Lemma 5.1, T_0 is the only triangle containing a_0. Thus, by [1, Lemma 5.4], $\{x, y, z\} = T_i$ for some i in $\{0, 1, \ldots, n-2\}$. Moreover, by Lemma 3.3, $i \neq 0$. If $c_i \in \{y, z\}$, then $M \backslash c_0, c_1, \ldots, c_n$ has a_n in a cocircuit of size at most two, so we can contract a_n from the last matroid keeping an N-minor, a contradiction to 6.4.2. Therefore $c_i = x$, so $\{a_i, b_i\} = \{y, z\}$. Now $D_{i-1} \triangle \{y, z, a_n, c_n\}$ is $\{b_{i-1}, c_{i-1}, a_n, c_n\}$, which must be a cocircuit. Again a_n is in a cocircuit in $M \backslash c_0, c_1, \ldots, c_n$ of size at most two, so contracting a_n from the last matroid retains an N-minor, a contradiction to 6.4.2. We conclude that 6.4.4 holds.

We may now assume that (II) holds. Next we show the following.

6.4.5. *M has no triangle containing $\{d_n, c_n\}$.*

Suppose M has a triangle T containing $\{d_n, c_n\}$. By orthogonality with the cocircuit $\{b_n, c_n, a_0, b_0\}$, we deduce that a_0 or b_0 is in T. As T_0 is the only triangle containing a_0, it follows that $T = \{d_n, c_n, b_0\}$. Orthogonality implies that $d_n \in \{c_0, a_1, b_1\}$ and hence that $\{d_{n-1}, d_n\} \subseteq T_1$. Then (I) holds so we have a contradiction to 6.4.4 that completes the proof of 6.4.5.

We now know that $\{d_{n-1}, a_n, s\}$ is a triangle for some s in $\{b_{n-1}, c_{n-1}\}$. If $s = b_{n-1}$, then orthogonality implies that $d_{n-1} \in \{b_{n-2}, c_{n-2}\}$. Hence orthogonality implies that $\{d_{n-1}, d_n\} \subseteq T_{n-2}$, and $\lambda(T_{n-2} \cup T_{n-1} \cup T_n) \leq 2$, a contradiction. Thus $s = c_{n-1}$. By assumption, $\{d_{n-1}, d_n\}$ avoids $T_{n-1} \cup T_n \cup T_0$. If $\{d_{n-1}, d_n\}$ meets $T_0 \cup T_1 \cup \cdots \cup T_n$, then, by orthogonality between $\{d_{n-1}, a_n, b_n, d_n\}$ and each T_i, we see that $\{d_{n-1}, d_n\} \subseteq T_i$ for some $i \notin \{n-1, n, 0\}$, and (I) holds, a contradiction. We deduce that the elements of $T_0 \cup T_1 \cup \cdots \cup T_n \cup \{d_{n-1}, d_n\}$ are distinct.

By taking $j = n - 1$, we see that M contains the structure in Fig. 16 where all of the elements shown are distinct except those with the same labels. Next we show the following.

6.4.6. *Suppose M contains the structure in Fig. 16 for some j with $1 \leq j \leq n-1$ where all of the elements are distinct except those with the same label. Then either M has a mixed ladder win, or there is an element d_{j-1} that is not in $T_0 \cup T_1 \cup \cdots \cup T_n \cup \{d_j, d_{j+1}, \ldots, d_n\}$ such that $\{c_{j-1}, d_{j-1}, a_j\}$ is a triangle and $\{d_{j-1}, a_j, c_j, d_j\}$ is a cocircuit.*
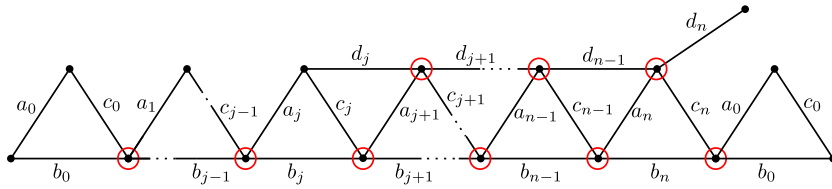
Fig. 16. All of the elements are distinct except those with the same label.

We apply Lemma 3.9 to the bowtie string $T_{j-1}, D_{j-1}, T_j, D_j, T_{j+1}$, noting that $M/b_j \backslash c_j$ has an N-minor. As M has no quick win, outcome (i) of that lemma does not hold. Thus (ii) or (iii) of Lemma 3.9 holds, so $\{a_j, c_j\}$ is contained in a 4-cocircuit D^*. Lemma 3.3 implies that D^* avoids T_{j+1}. By orthogonality with the circuit $\{c_j, d_j, a_{j+1}\}$, we see that D^* contains d_j. Let d_{j-1} be the fourth element of D^*. The structure induced on $T_{j-1} \cup T_j \cup \cdots \cup T_n \cup T_0 \cup \{d_{j-1}, d_j, \ldots, d_n\}$ has the form of the one shown in Fig. 15.

By orthogonality using the cocircuit $\{d_{j-1}, a_j, c_j, d_j\}$ and the triangles in Fig. 16, we see that d_{j-1} avoids $T_0 \cup T_1 \cup \cdots \cup T_n$, and d_{j-1} avoids $\{d_j, d_{j+1}, \ldots, d_{n-1}\}$. We now apply Lemma 6.3 to the structure on $T_{j-1} \cup T_j \cup \cdots \cup T_n \cup T_0 \cup \{d_{j-1}, d_j, \ldots, d_n\}$. If (ii)(c) of that lemma holds, then M has a mixed ladder win. Part (ii)(b) does not hold by 6.4.5, so part (ii)(a) holds; that is, $\{d_{j-1}, a_j\}$ is contained in a triangle T. By orthogonality between T and the cocircuits $\{b_{j-1}, c_{j-1}, a_j, b_j\}$ and $\{b_{j-2}, c_{j-2}, a_{j-1}, b_{j-1}\}$, we deduce that $T = \{d_{j-1}, a_j, c_{j-1}\}$. We conclude that 6.4.6 holds.

By repeatedly applying 6.4.6, we find that either M has a mixed ladder win or M has $\{c_0, d_0, a_1\}$ as a triangle. Hence we may assume the latter. But recall that we began with a triangle $\{1, 2, 3\}$ and a cocircuit that, after relabelling, became $\{2, 3, a_0, c_0\}$. Now the elements in $\{1, 2, 3, a_0, b_0, c_0, a_1, b_1, c_1\}$ are distinct except that 1 and c_1 may be equal. By orthogonality between $\{c_0, d_0, a_1\}$ and $\{2, 3, a_0, c_0\}$, we see that $\{d_0, a_1\}$ meets $\{2, 3, b_0\}$ so $d_0 \in \{2, 3\}$. Suppose $1 = c_1$. Then the circuit $\{1, 2, 3\}$ is $\{d_0, c_1, a_2\}$ or $\{d_0, c_1, b_2\}$. The first possibility contradicts the fact that $\{d_1, c_1, a_2\}$ is a circuit; the second violates orthogonality. We deduce that $1 \neq c_1$, so $\{1, 2, 3\}$ avoids $\{a_0, b_0, c_0, a_1, b_1, c_1\}$. Now orthogonality between $\{1, 2, 3\}$ and $\{d_0, a_1, c_1, d_1\}$ implies that $\{d_0, d_1\} \subseteq \{1, 2, 3\}$. Then $\lambda(\{1, 2, 3\} \cup T_0 \cup T_1) \leq 2$, a contradiction. $\square$

The results in this section enable us to conclude that M does not contain the structure in Fig. 13(C).

7. Configuration (A)

In this section, we deal with the case when M contains configuration (A) from Fig. 13, where $M\backslash 4/5$ has an N-minor and $M\backslash 4, 1$ has no N-minor. We begin with a straightforward lemma that will aid our efforts in this case.

Lemma 7.1. *Suppose that* $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ *is a bowtie in an internally 4-connected binary matroid* M *and that* M *has* $\{2, 4, 7\}$ *as a triangle. Let* N *be an*

internally 4*-connected matroid with at least seven elements such that N is a minor of $M/\{x, y\}$ for some pair $\{x, y\}$ of elements of $\{4, 5, 6\}$. Then $N \preceq M \backslash 1, 4$.*

Proof. As $\{4, 5, 6\}$ is a triangle of M, clearly $N \preceq M/\{4, 5, 6\}$, so $N \preceq M/5, 6 \backslash 4$. Since $M/5, 6 \backslash 4$ has $\{2, 7\}$ as a circuit, it follows that $N \preceq M \backslash 4, 2$, so $N \preceq M \backslash 4, 2/3$. Hence $N \preceq M \backslash 1, 4$. □

In the following lemma, we consider a matroid that is not necessarily internally 4-connected, or even 3-connected. In the case that a binary matroid has $\{a, b, c\}$ as a disjoint union of circuits and $\{b, c, d\}$ as a disjoint union of cocircuits, we say that (a, b, c, d) is a *loose* 4*-fan*. If (a, b, c, d) and (e, d, c, b) are loose 4-fans, then we say that (a, b, c, d, e) is a *loose* 5*-fan* in M and a *loose* 5*-cofan* in M^*. It is easy to see, by modifying the proof of Lemma 3.2, that if M has a loose 4-fan and has an internally 4-connected minor N having at least seven elements, then either deleting the first element or contracting the last element of the loose 4-fan retains the N-minor. We will use this fact within the proof of the next lemma.

Lemma 7.2. *Let M and N be internally* 4*-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Suppose that Hypothesis VII holds and that M is not the cycle matroid of a terrahawk or a quartic Möbius ladder and is not the dual of a triadic Möbius matroid. Let $M' = M \backslash X/Y$ for some sets X and Y, and let M have $T_0, D_0, T_1, D_1, \ldots, T_n$ as a right-maximal bowtie string that is also a bowtie string in M'. Suppose that $M' \backslash c_0, c_1, \ldots, c_n$ has an N-minor. Then one of the following holds.*

   (i) *M has a quick win; or*
  (ii) *M has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or*
 (iii) *M has $\{a_0, b_0, z, c_n\}$ as a 4-cocircuit for some z in $\{a_n, b_n\}$; or*
  (iv) *the structure in Fig. 12 is contained in M, up to switching the labels on a_n and b_n, and either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit, where $d_{-1} = \alpha$; or*
   (v) *$M' \backslash c_0, c_1/b_1$ has an N-minor, or $n = 1$ and $M' \backslash c_0, c_1/a_1$ has an N-minor; or*
  (vi) *$n = 1$ and M contains one of the structures in Fig. 17 or Fig. 18, where all of the elements are distinct and the labels a_1 and b_1 are viewed as being interchangeable. Moreover, if M contains the structure in Fig. 18, then $d_1 \in Y$, and either $d_0 \in X$ or $M' \backslash c_0, c_1, d_0$ has an N-minor; or*
 (vii) *deleting the central cocircuit of some augmented 4-wheel in M gives an internally 4-connected matroid with an N-minor.*

Furthermore, if neither (iii) nor (v) holds, then M has no triangle T_{n+1} such that $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a 4-cocircuit for any x in $\{a_n, b_n\}$.
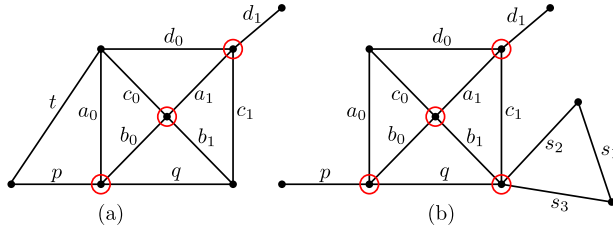
Fig. 17. The elements in both structures are all distinct, and we view the labels on a_1 and b_1 as being interchangeable.
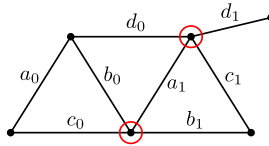
Fig. 18. The elements in this structure are all distinct, and we view the labels on a_1 and b_1 as being interchangeable. Furthermore, $d_1 \in Y$, and either $d_0 \in X$ or $M' \backslash c_0, c_1, d_0$ has an N-minor.

Proof. Suppose that neither (iii) nor (v) holds. By Lemma 3.6, we get the following.

7.2.1. *If* $i \in \{1, 2, \ldots, n\}$, *then* $M' \backslash c_0, c_1, \ldots, c_i / b_i$ *has no N-minor. If* $j \in \{1, 2, \ldots, n-1\}$, *then* $M' \backslash c_0, c_1, \ldots, c_j / a_{j+1}$ *has no N-minor.*

7.2.2. *If M has a triangle T_{n+1} where $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a cocircuit for some x in $\{a_n, b_n\}$, then $a_0 \neq c_n$.*

To show this, suppose that $a_0 = c_n$. Then $n \geq 2$. Without loss of generality, we assume that $x = b_n$. By orthogonality with T_0, the cocircuit D_n meets $\{b_0, c_0\}$. Up to switching the labels on a_{n+1} and b_{n+1}, we may assume that $b_{n+1} \in \{b_0, c_0\}$. Then orthogonality with D_0 implies that T_{n+1} meets $\{a_1, b_1\}$. As $a_0 \in D_n$, Lemma 3.3 implies that $a_{n+1} \notin T_1$. Hence $c_{n+1} \in \{a_1, b_1\}$.

Suppose $c_{n+1} = b_1$. Then orthogonality between T_{n+1} and D_1 implies that $a_{n+1} \in \{a_2, b_2\}$. Moreover, orthogonality with D_n implies that T_2 meets $\{b_n, a_0, b_{n+1}\}$. It follows that $n = 2$, so $c_2 = a_0$ and $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Thus $c_{n+1} = a_1$.

As the next step towards 7.2.2, we now show that

7.2.3. $\{b_{n+1}, c_{n+1}\} \subseteq E(M')$ *and* $a_{n+1} \in Y$.

Since $b_{n+1} \in \{b_0, c_0\}$ and $c_{n+1} = a_1$, we know that $\{b_{n+1}, c_{n+1}\} \subseteq E(M')$. If $a_{n+1} \in X$, then $M' \backslash c_0, c_1, \ldots, c_n$ has b_n in a 1- or 2-cocircuit, so $M' \backslash c_0, c_1, \ldots, c_n / b_n$ has an N-minor, a contradiction to 7.2.1. Suppose $a_{n+1} \in E(M')$. If $b_{n+1} = c_0$, then $M' \backslash c_0, c_1, \ldots, c_n$ has $\{b_n, a_{n+1}\}$ as a disjoint union of cocircuits and again we get a contradiction to 7.2.1. Thus $b_{n+1} = b_0$ and (a_1, b_0, a_{n+1}, b_n) is a loose 4-fan in $M' \backslash c_0, c_1, \ldots, c_n$. By 7.2.1, we see that $M' \backslash c_0, c_1, \ldots, c_n, a_1$ has an N-minor. The last
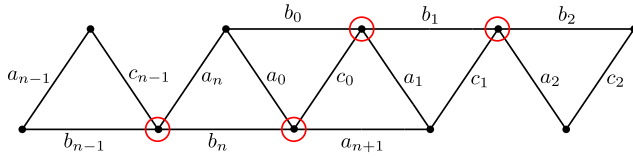
Fig. 19. From 7.2.4, M contains this structure.

matroid has $\{b_0, b_1\}$ as a disjoint union of cocircuits, a contradiction to 7.2.1. Hence 7.2.3 holds.

Still aiming to show 7.2.2, we show next that

7.2.4. $b_{n+1} = c_0$.

Suppose that $b_{n+1} \neq c_0$. Then $M' \backslash c_0, c_1, \ldots, c_n$ has $\{b_{n+1}, a_1\}$ as a disjoint union of circuits, so $M' \backslash c_0, c_1, \ldots, c_n, a_1$ has an N-minor. But the last matroid has $\{b_0, b_1\}$ as a disjoint union of cocircuits, a contradiction to 7.2.1. Thus 7.2.4 holds.

We now know that M contains the structure in Fig. 19 and that $M \backslash c_0, c_1, \ldots, c_{n-1}, c_n / b_n$ has an N-minor. By Lemma 3.6, $M \backslash c_i / b_i$ has an N-minor for all i in $\{0, 1, \ldots, n\}$.

As the next step towards 7.2.2, we now show the following.

7.2.5. *For each i in $\{1, 2, \ldots, n\}$, the matroid M has a 4-cocircuit containing $\{a_i, c_i\}$.*

Since M has no quick win, this follows by Lemma 3.8 and Hypothesis VII unless $M \backslash c_1$ is not $(4, 4, S)$-connected. Consider the exceptional case. By Lemma 3.4, M has a quasi rotor $(T_0, T_1, \{7, 8, 9\}, D_0, \{v, c_1, 7, 8\}, \{u, v, 7\})$ for some u in $\{b_0, c_0\}$ and v in $\{a_1, b_1\}$. Since M / b_1 has an N-minor, Lemma 3.11 implies that $b_1 \neq v$. Thus $v = a_1$. By orthogonality between $\{c_0, a_1, a_{n+1}\}$ and the cocircuit $\{a_1, c_1, 7, 8\}$, it follows that $\{7, 8\}$ meets $\{c_0, a_{n+1}\}$. Since the triangles T_0, T_1, and $\{7, 8, 9\}$ are disjoint, $c_0 \notin \{a_1, c_1, 7, 8\}$. Hence $a_{n+1} \in \{7, 8\}$. By orthogonality between $\{7, 8, 9\}$ and $\{a_0, c_0, b_n, a_{n+1}\}$, it follows that $b_n \in \{7, 8, 9\}$. Then orthogonality between $\{7, 8, 9\}$ and D_{n-1} implies that $\{7, 8, 9\}$ meets $\{b_{n-1}, c_{n-1}\}$. Then M has $(T_{n-1}, T_n, \{c_0, a_1, a_{n+1}\}, D_{n-1}, \{a_0, c_0, b_n, a_{n+1}\}, \{7, 8, 9\})$ as a quasi rotor in which b_n is in two triangles. As M / b_n has an N-minor, Lemma 3.11 gives a contradiction. Hence 7.2.5 holds.

We continue the proof of 7.2.2 by showing that M contains the structure in Fig. 20. We will construct the left side of the figure first. Take $d_n = b_0$. Let $\ell = n$ if $\{c_{n-1}, a_n\}$ is not contained in a triangle of M. If $\{c_{n-1}, a_n\}$ is contained in a triangle, then take ℓ minimal in $\{1, 2, \ldots, n-1\}$ such that $\{c_i, a_{i+1}, d_i\}$ is a triangle of M for all i in $\{\ell, \ell+1, \ldots, n-1\}$, where d_i is an element in $E(M)$. By 7.2.5, M has a 4-cocircuit containing $\{a_n, a_0\}$. By orthogonality with T_0 and Lemma 3.3 applied to $(T_n, \{c_0, a_1, a_{n+1}\}, \{a_0, c_0, b_n, a_{n+1}\})$, this cocircuit contains b_0. If $\ell < n$, then orthogonality and Lemma 3.3 imply that the fourth element of this cocircuit is d_{n-1}. By repeated application of this argument, we see that M has $\{d_{i-1}, a_i, c_i, d_i\}$ as a cocircuit for all i in $\{\ell+1, \ell+2, \ldots, n-1\}$. Moreover,
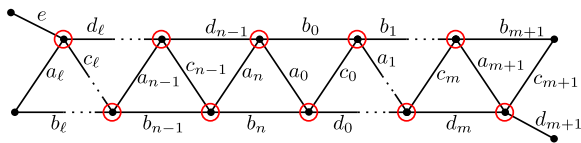
Fig. 20. The elements depicted are all distinct, $\ell \leq n$, where we let $d_n = b_0$, and $m \geq 0$. Furthermore, $m \leq \ell - 2$.

M has $\{e, a_\ell, c_\ell, d_\ell\}$ as a cocircuit for some element e where we note that this includes the case when $\ell = n$.

We continue to construct the structure in Fig. 20 by showing that

7.2.6. $\ell \geq 2$.

Suppose instead that $\ell = 1$. Then by orthogonality between $\{e, a_1, c_1, d_1\}$ and the triangle $\{c_0, a_1, a_{n+1}\}$ and Lemma 3.3, we deduce that $e = a_{n+1}$. By [1, Lemma 6.4], the elements in $T_1 \cup T_2 \cup \cdots \cup T_n \cup \{c_0, a_{n+1}\} \cup \{d_1, d_2, \ldots, d_{n-1}, b_0\}$ are distinct. Now it is easy to see that M is the cycle matroid of a quartic Möbius ladder or is the dual of a triadic Möbius ladder, a contradiction. Thus 7.2.6 holds.

We now consider the right side of Fig. 20. It is convenient to let $a_{n+1} = d_0$. Take m maximal in $\{0, 1, \ldots, \ell - 1\}$ such that, for all j in $\{0, 1, \ldots, m\}$, there is an element d_j such that $\{c_j, d_j, a_{j+1}\}$ is a triangle. By the definition of ℓ, we know that $\{c_{\ell-1}, a_\ell\}$ is not contained in a triangle of M. Hence

7.2.7. $m \leq \ell - 2$.

As before, by orthogonality and Lemma 3.3, we know that $\{d_{j-1}, a_j, c_j, d_j\}$ is a cocircuit for all j in $\{1, 2, \ldots, m\}$. Moreover, $\{d_m, a_m, c_m, d_{m+1}\}$ is a cocircuit for some element d_{m+1}. We deduce that M contains the structure in Fig. 20 where $\ell \leq n$, and $m \geq 0$.

Next we show that

7.2.8. *The elements in Fig. 20 are distinct.*

By [1, Lemma 6.4], since M is neither the dual of the triadic Möbius matroid nor the cycle matroid of a quartic Möbius ladder, the elements in Fig. 20 other than $\{e, c_{m+1}, d_{m+1}\}$ are distinct except that (b_m, b_{m+1}) may equal (a_ℓ, b_ℓ). As $\ell \neq m+1$, we deduce that the elements in Fig. 20 are distinct unless e, c_{m+1}, or d_{m+1} is equal to one of the other elements.

Suppose c_{m+1} is equal to another element in Fig. 20. Then $c_{m+1} \in \{e, d_0, d_1, \ldots, d_m, d_\ell, d_{\ell+1}, \ldots, d_{n-1}\}$. By orthogonality between T_{m+1} and the cocircuits in Fig. 20, it follows that $e \in \{a_{m+1}, b_{m+1}\}$ and $c_{m+1} = d_\ell$. Then orthogonality with the cocircuits in Fig. 20 implies that $(\ell, m) = (n, 0)$, and T_1 is $\{e, b_0, b_1\}$ or $\{e, b_0, a_1\}$. In either case,

$\lambda(T_n \cup \{c_0, d_0, a_1, e, b_0, b_1\}) \leq 2$, a contradiction. Thus c_{m+1} is not equal to any other element in Fig. 20.

By orthogonality between the triangles in Fig. 20 and the cocircuit $\{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$, we know that d_{m+1} avoids all of the triangles in that figure. Similarly, orthogonality between these triangles and the cocircuit $\{e, a_\ell, c_\ell, d_\ell\}$ implies that e avoids all these triangles. We deduce that 7.2.8 holds unless $d_{m+1} = e$. In the exceptional case, the set $\{e, a_\ell, c_\ell, d_\ell\} \triangle \{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$ contains a cocircuit, and the set Z of elements in Fig. 20 other than $\{e, d_{m+1}\}$ is 3-separating. Then Z meets $T_{\ell-1} \cup e$ otherwise we contradict the fact that M is internally 4-connected. But now we get a contradiction to orthogonality between $T_{\ell-1}$ and the cocircuits in the figure. Thus 7.2.8 holds.

Continuing the proof of 7.2.2, we show next that

7.2.9. *M/d_{m+1} has an N-minor.*

Recall that $M \backslash c_0, c_1, \ldots, c_{n-1}, c_n/b_n$ has an N-minor. By Lemma 3.6, $M \backslash a_1, a_2, \ldots, a_{m+1}, c_{m+1}, c_{m+2}, \ldots, c_{n-1}, c_n/d_m$ has an N-minor. The last matroid is isomorphic to $M \backslash a_1, a_2, \ldots, a_{m+1}, c_{m+1}, c_{m+2}, \ldots, c_{n-1}, a_0/d_{m+1}$. Hence 7.2.9 holds.

Next we show that

7.2.10. *d_{m+1} is in no triangle of M.*

Suppose M has d_{m+1} in a triangle, T. By orthogonality between T and the 4-cocircuits in Fig. 20, we see that $\{c_{m+1}, d_{m+1}\} \subseteq T$. By the maximality of m, we know that $a_{m+2} \notin T$. Thus orthogonality with D_{m+1} implies that $b_{m+2} \in T$, a contradiction to orthogonality with D_{m+2}. Thus 7.2.10 holds.

As M has no quick win, we know that M/d_{m+1} is not internally 4-connected. We complete the proof of 7.2.2 by giving a contradiction to this fact.

Suppose (U, V) is a non-sequential 2- or 3-separation of M/d_{m+1}. By [5, Lemma 3.3], we may assume that $T_m \cup T_{m+1} \cup d_m \subseteq U$. Then $(U \cup d_{m+1}, V)$ is a non-sequential 2- or 3-separation of M, a contradiction. By 7.2.10, M/d_{m+1} has no sequential 2-separation. It follows that M/d_{m+1} is 3-connected having a 4-fan, (u, v, w, x). Hence M has $\{v, w, x\}$ as a cocircuit and $\{d_{m+1}, u, v, w\}$ as a circuit. By orthogonality with the cocircuit $\{d_m, a_{m+1}, c_{m+1}, d_{m+1}\}$, the set $\{u, v, w\}$ meets $\{d_m, a_{m+1}, c_{m+1}\}$. If $\{u, v, w\}$ meets D_m or D_{m+1}, then orthogonality implies that two elements of $\{u, v, w\}$ is in one of these cocircuits. Then $\{v, w, x\}$ meets T_m, T_{m+1}, or T_{m+2}, a contradiction. Thus $\{u, v, w\}$ avoids $\{a_{m+1}, c_{m+1}\}$, so $d_m \in \{u, v, w\}$. Either $\{d_{m-1}, a_m, c_m, d_m\}$ is a cocircuit, or $m = 0$ and $\{b_n, a_0, c_0, d_0\}$ is a cocircuit. Hence $\{u, v, w\}$ contains two elements from one of these cocircuits, so the triad $\{v, w, x\}$ meets a triangle in Fig. 20, a contradiction. Thus M/d_{m+1} has no 4-fan, a contradiction. We conclude that $a_0 \neq c_n$. Thus 7.2.2 holds.

Next we show that

7.2.11. *M does not have a triangle T_{n+1} such that $(T_n, T_{n+1}, \{x, c_n, a_{n+1}, b_{n+1}\})$ is a bowtie for some x in $\{a_n, b_n\}$.*

Suppose M has such a bowtie. Then [1, Lemma 5.4] implies that $T_{n+1} = T_j$ for some j in $\{0, 1, \ldots, n-2\}$, so $n \geq 2$. If $c_j \in \{a_{n+1}, b_{n+1}\}$, then $M'\backslash c_0, c_1, \ldots, c_n$ has x in a 1- or 2-cocircuit, so $M'\backslash c_0, c_1, \ldots, c_n/x$ has an N-minor, a contradiction to 7.2.1. Thus $c_j = c_{n+1}$ so $\{a_j, b_j\} = \{a_{n+1}, b_{n+1}\}$. If $j = 0$, then (iii) holds, a contradiction. Thus $j \geq 1$, and $D_{j-1} \triangle \{x, c_n, a_{n+1}, b_{n+1}\}$ is $\{b_{j-1}, c_{j-1}, x, c_n\}$, a disjoint union of cocircuits in M'. Again $M'\backslash c_0, c_1, \ldots, c_n$ has x in a 1- or 2-cocircuit, a contradiction. Thus 7.2.11 holds, as does the last assertion of the lemma.

We now assume that the lemma fails. We show next that

7.2.12. $M\backslash c_n$ is $(4, 4, S)$-connected and every 4-fan in $M\backslash c_n$ has the form (u, v, d_{n-1}, d_n) for some u and v in $\{b_{n-1}, c_{n-1}\}$ and $\{a_n, b_n\}$, respectively, where $|T_{n-1} \cup T_n \cup \{d_{n-1}, d_n\}| = 8$.

As (i) does not hold, $M\backslash c_n$ is not internally 4-connected. Observe that, if T_n is the central triangle of a quasi rotor with a_n or b_n as its central element, then we have a contradiction to 7.2.11. Thus, applying Lemma 3.4 to the bowtie (T_{n-1}, T_n, D_{n-1}), we deduce using 7.2.11 that outcome (iii) of that lemma holds; that is, 7.2.12 holds.

Without loss of generality, we may now assume that $M\backslash c_n$ has a 4-fan of the form (u, a_n, d_{n-1}, d_n) for some u in $\{b_{n-1}, c_{n-1}\}$.

7.2.13. $u \neq b_{n-1}$.

To show this, we assume the contrary. Suppose that $n > 1$. Then orthogonality between $\{b_{n-1}, a_n, d_{n-1}\}$ and D_{n-2} implies that $d_{n-1} \in \{b_{n-2}, c_{n-2}\}$. Then orthogonality between $\{a_n, d_{n-1}, d_n, c_n\}$ and T_{n-2} implies that $\{a_n, c_n, d_n\}$ meets T_{n-2}. If $a_{n-2} = c_n$, then $n = 2$ and, since $d_1 \in \{b_0, c_0\}$, the triangle T_0 is in the closure of $T_1 \cup T_2$, so $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Thus $a_{n-2} \neq c_n$, so $d_n \in T_{n-2}$, a contradiction to 7.2.11. We deduce that $n = 1$ and M contains the structure in Fig. 18.

Suppose $\{d_0, d_1\}$ avoids Y. If $\{d_0, d_1\}$ meets X, then $M'\backslash c_0, c_1$ has a_1 in a 1- or 2-cocircuit, so (v) holds, a contradiction. Thus $\{d_0, d_1\}$ avoids X, and $(b_1, b_0, a_1, d_0, d_1)$ is a loose 5-cofan of $M'\backslash c_0, c_1$, so (v) holds, a contradiction. We deduce that $\{d_0, d_1\}$ meets Y. If $d_0 \in Y$, then $\{b_0, a_1\}$ is a disjoint union of circuits in $M'\backslash c_0, c_1$, so $M'\backslash c_0, c_1, b_0$ has an N-minor. As this matroid has $\{a_1, b_1\}$ as a disjoint union of cocircuits, (v) holds, a contradiction. Thus $d_0 \notin Y$, so $d_1 \in Y$. If $d_0 \in X$, then (vi) holds, a contradiction. Thus $d_0 \notin X$, so (d_0, b_0, a_1, b_1) is a loose 4-fan in $M'\backslash c_0, c_1$, and (v) or (vi) holds, a contradiction. We conclude that 7.2.13 holds.

We show next that

7.2.14. M contains the configuration in Fig. 21 where the elements a_n and b_n are viewed as being interchangeable. Moreover, the elements in this figure are distinct with the possible exception that (a_0, b_0) may equal (c_n, d_n) when $n > 1$. Also if $n > 1$, then $d_{n-1} \notin Y$.
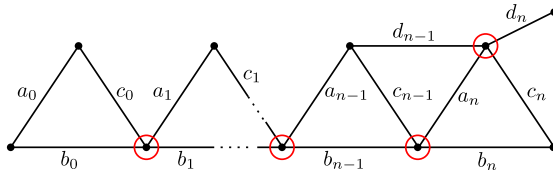
Fig. 21. We view the labels a_n and b_n as being interchangeable. These elements are all distinct with the possible exception that (a_0, b_0) may be (c_n, d_n) in the case that $n > 1$. Furthermore, if $n > 1$, then $d_{n-1} \notin Y$.

By 7.2.13, $u = c_{n-1}$. Thus, M contains the configuration in Fig. 21. Moreover, by 7.2.12, when $n = 1$, the elements in the figure are all distinct. Hence we may suppose that $n \geq 2$. By 7.2.11, the pair $\{d_{n-1}, d_n\}$ is not contained in a triangle of M. Hence, by orthogonality, $\{d_{n-1}, a_n, c_n, d_n\}$ avoids $T_1 \cup T_2 \cup \cdots \cup T_{n-2}$. Now $d_{n-1} \notin Y$ otherwise $(b_{n-2}, a_{n-1}, b_{n-1}, a_n, b_n)$ is a loose 5-cofan in $M' \backslash c_0, c_1, \ldots, c_n$, so $M' \backslash c_0, c_1, \ldots, c_n / b_n$ has an N-minor, a contradiction to 7.2.1.

Suppose T_0 meets $\{d_{n-1}, d_n\}$. By orthogonality with $\{d_{n-1}, a_n, c_n, d_n\}$, it follows that $a_0 = c_n$, and $\{b_0, c_0\}$ meets $\{d_{n-1}, d_n\}$. Orthogonality between D_0 and $\{c_{n-1}, d_{n-1}, a_n\}$ implies that $d_{n-1} \notin \{b_0, c_0\}$. Hence $d_n \in \{b_0, c_0\}$. Suppose $d_n = c_0$. As $d_{n-1} \in E(M') \cup X$, we see that a_n is in a 1- or 2-cocircuit of $M' \backslash c_0, c_1, \ldots, c_n$, a contradiction to 7.2.1. We conclude that 7.2.14 holds.

7.2.15. $n \geq 2$.

As a first step towards proving this, we now show the following.

7.2.16. *If $n = 1$, then M has $\{b_0, b_1, q\}$ as a triangle for some element q that is not in $T_0 \cup T_1 \cup \{d_0, d_1\}$.*

As $n = 1$, 7.2.14 implies that the elements in Fig. 21 are distinct. We now apply [1, Lemma 6.1] assuming first that (v) of that lemma does not hold, that is, that M has no triangle containing $\{b_0, b_1\}$. As the current lemma fails, it follows using 7.2.11 that none of (i), (ii), or (iv) of [1, Lemma 6.1] holds. Thus (iii) of that lemma holds, so $M \backslash c_0, c_1$ has a 4-fan of the form $(1, 2, 3, b_1)$. By 7.2.11, M does not have $\{2, 3, b_1, c_1\}$ as a cocircuit. Thus M has $\{2, 3, b_1, c_0, c_1\}$ or $\{2, 3, b_1, c_0\}$ as a cocircuit. Since $T_0 \cup T_1 \cup d_0$ is not 3-separating, this set contains no cocircuit of M other than D_0. By orthogonality, $\{2, 3\}$ meets both $\{a_0, b_0\}$ and $\{d_0, a_1\}$. Hence $\{2, 3\} = \{b_0, a_1\}$, so a_1 is in a triangle of $M \backslash c_0, c_1$, a contradiction to [1, Lemma 6.1]. We conclude that (v) of [1, Lemma 6.1] holds, that is, M has a triangle of the form $\{b_0, b_1, q\}$. Clearly $q \neq a_0$, and orthogonality with the cocircuits D_0 and $\{d_0, a_1, c_1, d_1\}$ implies that $q \notin T_0 \cup T_1 \cup \{d_0, d_1\}$. Thus 7.2.16 holds.

We continue our proof of 7.2.15 by showing the following.

7.2.17. *Either $q \in X$, or $M' \backslash c_0, c_1, q$ has an N-minor.*
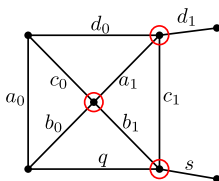
Fig. 22. We view the labels a_1 and b_1 as being interchangeable. The elements shown are distinct.

To prove this, suppose first that $q \in Y$. Then $\{b_0, b_1\}$ is a disjoint union of circuits in $M' \backslash c_0, c_1$, so $M' \backslash c_0, c_1, b_0$ has an N-minor. As this matroid has $\{a_1, b_1\}$ as a disjoint union of cocircuits, (v) holds, a contradiction. Suppose $q \in E(M')$. Then (q, b_0, b_1, a_1) is a loose 4-fan in $M' \backslash c_0, c_1$, and, as (v) does not hold, $M' \backslash c_0, c_1, q$ has an N-minor. Thus 7.2.17 holds.

Consider $M \backslash q$. Since this matroid is not internally 4-connected, by applying Lemma 3.4 to the bowtie $(\{d_0, c_0, a_1\}, \{b_0, b_1, q\}, \{a_1, c_0, b_0, b_1\})$, we deduce that $\{q, b_0\}$ or $\{q, b_1\}$ is contained in a 4-cocircuit of M. By orthogonality with the circuits T_0 and T_1, it follows using Lemma 3.3 that $\{q, a_0, b_0\}$ or $\{q, b_1, c_1\}$ is contained in a 4-cocircuit of M.

Next we show that

7.2.18. *M has no 4-cocircuit containing $\{q, b_1, c_1\}$.*

Suppose that $\{q, b_1, c_1, s\}$ is a 4-cocircuit in M, for some element s. By orthogonality between this cocircuit and T_0, T_1, and $\{c_0, d_0, a_1\}$, we see that either s is a new element, or $s = d_1$. The latter gives the contradiction that $\lambda(T_0 \cup T_1 \cup \{d_0, q, s\}) \leq 2$. Thus s is a new element and M contains the structure in Fig. 22. Since $M \backslash c_0, c_1, q$ has $\{b_1, s\}$ as a disjoint union of cocircuits, $M \backslash c_0, c_1, q/b_1$ has an N-minor. This matroid is isomorphic to $M \backslash a_0, a_1, q/b_0$ by Lemma 3.6. By Lemma 3.4, as M has no quick win, a_0 is in a 4-cocircuit D of M. By orthogonality, D meets $\{b_0, c_0\}$. Lemma 3.3 implies that D avoids T_1, so orthogonality with $\{c_0, d_0, a_1\}$ and $\{b_0, b_1, q\}$ implies that D contains $\{a_0, c_0, d_0\}$ or $\{q, a_0, b_0\}$. If the fourth element of D is d_1 or s, then the symmetric difference of D with the 4-cocircuit in Fig. 22 containing d_1 or s, respectively, is a new cocircuit contained in $T_0 \cup T_1 \cup \{d_0, q\}$. Thus the last set is 3-separating, a contradiction. Using D together with the structure in Fig. 22, we see that M contains a good augmented 4-wheel. Since $M \backslash c_0, c_1, q/b_1$ has an N-minor and is isomorphic to $M \backslash c_0, a_1, b_0/b_1$ and hence to $M \backslash c_0, a_1, b_0, b_1$, we see that M contains an augmented 4-wheel such that removing the central cocircuit preserves an N-minor. Then [5, Theorem 4.1] implies that (i) or (vii) holds, a contradiction. Thus 7.2.18 holds.

We now continue the proof of 7.2.15 knowing that $\{a_0, b_0, q, p\}$ is a cocircuit in M, for some element p. By orthogonality between this cocircuit and the circuits T_0, T_1, and $\{c_0, d_0, a_1\}$, we see that either p is a new element, or $p = d_1$. If $p = d_1$, then $\lambda(T_0 \cup T_1 \cup \{d_0, q, d_1\}) \leq 2$, a contradiction. We now apply [1, Lemma 6.2]. Clearly (i) of that lemma does not hold. Also 7.2.11 and 7.2.18 imply that neither (ii) nor

(v) holds. If (iv) holds, then M has a triangle $\{a_0, p, t\}$ for some element t which is new unless it equals s. Orthogonality excludes the exceptional case. Thus M contains structure (a) in Fig. 17, and (vi) of the current lemma holds, a contradiction. We deduce that (iii) of [1, Lemma 6.2] holds, that is, M has a triangle $\{s_1, s_2, s_3\}$ and a cocircuit $\{q, c_1, b_1, s_2, s_3\}$ where $\{s_1, s_2, s_3\}$ avoids $\{b_0, c_0, q, a_1, b_1, c_1\}$. Orthogonality and 7.2.11 imply that $\{s_1, s_2, s_3\}$ avoids $\{d_0, d_1\}$. If $\{s_1, s_2, s_3\}$ meets $\{p, a_0\}$, then orthogonality implies that $\{p, a_0\} \subseteq \{s_1, s_2, s_3\}$, and $\lambda(\{b_0, c_0\} \cup T_1 \cup \{s_1, s_2, s_3, q\}) \leq 2$, a contradiction. We conclude that $\{s_1, s_2, s_3\}$ avoids p as well as all of the elements in Fig. 22. Thus M contains structure (a) in Fig. 17, and (vi) holds, a contradiction. Therefore 7.2.15 holds.

Take m minimal such that, for all i in $\{m, m+1, \ldots, n-1\}$, the matroid M has $\{c_i, d_i, a_{i+1}\}$ as a triangle and has $\{d_i, a_{i+1}, c_{i+1}, d_{i+1}\}$ or $\{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$ as a cocircuit, where the 5-cocircuit option is only possible when $i = n - 2$.

7.2.19. *The elements in* $T_m \cup T_{m+1} \cup \cdots \cup T_n \cup \{d_m, d_{m+1}, \ldots, d_n\}$ *are all distinct and* $d_j \notin Y$ *if* $\max\{1, m\} \leq j \leq n - 1$.

To see this, suppose first that the elements in this set are not all distinct. By 7.2.11, $\{d_{n-1}, d_n\}$ is not contained in a triangle. By [1, Lemma 6.4], we deduce that $(a_m, b_m) = (c_n, d_n)$, and M has $\{b_n, c_n, c_m, d_m\}$ as a cocircuit. Then $m \leq n - 2$. But $\{c_m, d_m, a_{m+1}\}$ is a triangle of M, so we get a contradiction to 7.2.11. Thus the elements in $T_m \cup T_{m+1} \cup \cdots \cup T_n \cup \{d_m, d_{m+1}, \ldots, d_n\}$ are all distinct.

Suppose that $d_j \in Y$ for some j with $\max\{1, m\} \leq j \leq n - 1$. Then $(b_{j-1}, a_j, b_j, a_{j+1}, b_{j+1})$ is a loose 5-cofan in $M' \backslash c_0, c_1, \ldots, c_n$. Then $M' \backslash c_0, c_1, \ldots, c_n / b_{j+1}$ has an N-minor, a contradiction to 7.2.1. Thus 7.2.19 holds.

Next we show that

7.2.20. $m = n - 1$.

Suppose $m < n - 1$. We consider $M \backslash c_m, c_{m+1}, \ldots, c_n$. By [1, Lemma 6.5] and 7.2.11, since M has no ladder win, $M \backslash c_m, c_{m+1}, \ldots, c_n$ is $(4, 4, S)$-connected and every 4-fan of this matroid is either a 4-fan of $M \backslash c_n$ with b_n as its coguts element or a 4-fan of $M \backslash c_m$ with d_m as its coguts element and with a_m as an interior element. By 7.2.11, $M \backslash c_n$ does not have a 4-fan with b_n as its coguts element. Therefore $M \backslash c_m$ has a 4-fan of the form $(\beta, \alpha, a_m, d_m)$, so $\{\alpha, a_m, d_m, c_m\}$ is a cocircuit of M. As (iv) does not occur, we know that $m \geq 1$. Orthogonality implies that $\{\alpha, \beta\}$ meets $\{b_{m-1}, c_{m-1}, b_m\}$. As $\{\alpha, \beta, a_m\} \neq T_m$, we know that $b_m \notin \{\alpha, \beta\}$. Furthermore, Lemma 3.3 implies that $\alpha \notin T_{m-1}$. Hence $\beta \in \{b_{m-1}, c_{m-1}\}$. By the minimality of m, we deduce that $\beta = b_{m-1}$.

Suppose $m > 1$. Then orthogonality between $\{b_{m-1}, \alpha, a_m\}$ and D_{m-2}, implies that $\alpha \in \{b_{m-2}, c_{m-2}, a_{m-1}\}$. As $\alpha \notin T_{m-1}$, the cocircuit $\{\alpha, a_m, c_m, d_m\}$ meets T_{m-2}. Orthogonality implies that this 4-cocircuit is contained in $T_{m-2} \cup T_m$. Hence $\lambda(T_{m-2} \cup T_{m-1} \cup T_m) \leq 2$, a contradiction. We conclude that $m = 1$.

By 7.2.19, $d_1 \notin Y$. Thus $d_1 \in X \cup E(M')$. If $d_1 \in X$, then a_2 is in a 1- or 2-cocircuit of $M'\backslash c_0, c_1, \ldots, c_n$, so $M'\backslash c_0, c_1/a_2$ has an N-minor, a contradiction to 7.2.1. Hence $d_1 \in E(M')$. If $\alpha \in E(M')$, then $(b_1, b_0, a_1, \alpha, d_1)$ is a loose 5-cofan in $M'\backslash c_0, c_1$, so $M'\backslash c_0, c_1/b_1$ has an N-minor, a contradiction to 7.2.1. Thus $\alpha \notin E(M')$. Then $\alpha \in Y$ or $\alpha \in X$ so, in $M'\backslash c_0, c_1, \ldots, c_n$, either $\{b_0, a_1\}$ is a disjoint union of circuits, or $\{a_1, d_1\}$ is a disjoint union of cocircuits. The former implies that $M'\backslash c_0, c_1, \ldots, c_n, a_1$ has an N-minor. As the last matroid also has $\{b_0, b_1\}$ as a disjoint union of cocircuits, we get a contradiction to 7.2.1. Therefore $\{a_1, d_1\}$ is a disjoint union of cocircuits in $M'\backslash c_0, c_1, \ldots, c_n$, so $M'\backslash c_0, c_1/d_1$ has an N-minor. As the last matroid has $(b_0, a_1, b_1, a_2, b_2)$ as a loose 5-cofan, $M'\backslash c_0, c_1/b_2$ has an N-minor. This contradiction to 7.2.1 completes the proof of 7.2.20.

We now apply [1, Lemma 6.1] to $M\backslash c_{n-1}, c_n$. By assumption and 7.2.11, neither (i) nor (ii) of that lemma holds.

We show next that

7.2.21. *b_n is not the coguts element of any 4-fan of $M\backslash c_{n-1}, c_n$.*

Suppose $(\alpha, \beta, \gamma, b_n)$ is a 4-fan in $M\backslash c_{n-1}, c_n$. By 7.2.11, we know that $\{\beta, \gamma, b_n, c_n\}$ is not a cocircuit of M. Hence either $\{\beta, \gamma, b_n, c_{n-1}, c_n\}$ or $\{\beta, \gamma, b_n, c_{n-1}\}$ is a cocircuit D of M. By orthogonality between D and the triangles T_{n-1} and $\{c_{n-1}, d_{n-1}, a_n\}$, we see that $\{\beta, \gamma\} \subseteq \{a_{n-1}, b_{n-1}, d_{n-1}, a_n\}$. Now $T_{n-1} \cup T_n \cup d_{n-1}$ is not 3-separating in M, and so $D = D_{n-1}$. Thus $\{\beta, \gamma\} = \{b_{n-1}, a_n\}$, so a_n is in a triangle of $M\backslash c_{n-1}, c_n$, a contradiction to [1, Lemma 6.1]. Hence 7.2.21 holds.

By 7.2.21, part (v) of [1, Lemma 6.1] does not hold. Now suppose that (iv) of [1, Lemma 6.1] holds. Then M has a triangle $\{\alpha, \beta, a_{n-1}\}$ that differs from T_{n-1}, and M has $\{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit. By the minimality of m, we deduce that $\beta \neq c_{n-2}$. By 7.2.15, $n \geq 2$. Orthogonality between $\{\alpha, \beta, a_{n-1}\}$ and D_{n-2} implies that $\{\alpha, \beta\}$ meets $\{b_{n-2}, c_{n-2}, b_{n-1}\}$, so $b_{n-2} = \beta$, or $\alpha \in \{b_{n-2}, c_{n-2}\}$.

7.2.22. *$b_{n-2} \neq \beta$.*

Suppose that $b_{n-2} = \beta$. We begin by locating the element α. Suppose first that $\alpha \in Y$. Then $\{b_{n-2}, a_{n-1}\}$ is a disjoint union of circuits in $M'\backslash c_0, c_1, \ldots, c_n$, so $M'\backslash c_0, c_1, \ldots, c_n, a_{n-1}$ has an N-minor. The last matroid has $\{b_{n-2}, b_{n-1}\}$ as a disjoint union of cocircuits, so we get a contradiction to 7.2.1. Thus $\alpha \notin Y$.

Suppose next that $\alpha \in X$. Then, since $\{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit in M, either $M'\backslash c_0, c_1, \ldots, c_n/d_{n-1}$ or $M'\backslash c_0, c_1, \ldots, c_n/a_n$ has an N-minor. The second option contradicts 7.2.1. If $M'\backslash c_0, c_1, \ldots, c_n/d_{n-1}$ has an N-minor, then, as the last matroid has $(b_{n-2}, a_{n-1}, b_{n-1}, a_n, b_n)$ as a loose 5-cofan, $M'\backslash c_0, c_1, \ldots, c_n/d_{n-1}/b_n$ has an N-minor. This contradiction to 7.2.1 establishes that $\alpha \notin X$.

We now know that $\alpha \in E(M')$. Then $(b_{n-1}, b_{n-2}, a_{n-1}, \alpha, x)$ is a loose 5-cofan in $M'\backslash c_0, c_1, \ldots, c_n$ for some x in $\{d_{n-1}, a_n\}$, so $M'\backslash c_0, c_1, \ldots, c_n/b_{n-1}$ has an N-minor. This contradiction to 7.2.1 completes the proof of 7.2.22.

From the above, if (iv) of [1, Lemma 6.1] holds, then $\alpha \in \{b_{n-2}, c_{n-2}\}$. Thus, by Lemma 3.3, $\{\alpha, a_{n-1}, c_{n-1}, d_{n-1}\}$ is not a cocircuit of M, so $\{\alpha, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Orthogonality with T_{n-2} implies that $n = 2$ and $a_0 = c_2$. By orthogonality with $\{d_1, a_2, c_2, d_2\}$, the triangle T_0 meets $\{d_1, d_2\}$. Thus T_0 is $\{\alpha, c_2, d_1\}$ or $\{\alpha, c_2, d_2\}$, so $\lambda(T_1 \cup T_2 \cup \{\alpha, d_1, d_2\}) \leq 2$, a contradiction. We conclude that (iv) of [1, Lemma 6.1] does not hold.

It now follows that (iii) of [1, Lemma 6.1] holds, that is, $\{b_{n-1}, b_n\}$ is contained in a triangle. Recall that $n \geq 2$ by 7.2.15. Orthogonality with D_{n-2} implies that the third element of this triangle is in $\{b_{n-2}, c_{n-2}\}$. If $\{b_{n-2}, b_{n-1}, b_n\}$ is a triangle, then $M' \backslash c_0, c_1, \ldots, c_n$ has $(a_n, b_n, b_{n-1}, b_{n-2}, a_{n-1})$ as a loose 5-cofan, so $M' \backslash c_0, c_1, \ldots, c_n / a_n$ has an N-minor, a contradiction to 7.2.1. We conclude that

7.2.23. $\{c_{n-2}, b_{n-1}, b_n\}$ *is a triangle of* M.

Next we show that

7.2.24. $\{c_n, d_n\}$ *is not contained in a triangle.*

Suppose $\{c_n, d_n\}$ is contained in a triangle T. Then $(T, \{a_n, d_{n-1}, c_{n-1}\}, \{d_n, c_n, a_n, d_{n-1}\})$ is a bowtie of M. By 7.2.12, $M \backslash c_n$ is $(4, 4, S)$-connected and $M \backslash c_n, c_{n-1}$ has an N-minor. Thus, by applying Hypothesis VII to the last bowtie, we get that $M \backslash c_{n-1}$ is $(4, 4, S)$-connected. This is a contradiction as the last matroid has a 5-fan. Hence 7.2.24 holds.

Consider the rotor chain $((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), (c_{n-2}, b_{n-2}, a_{n-2}))$. If this rotor chain can be extended to the right by adjoining (x, y, z), then $\{b_{n-2}, a_{n-2}, x, y\}$ is a cocircuit and $\{a_{n-1}, b_{n-2}, x\}$ is a circuit. By orthogonality, $x \in \{b_{n-3}, c_{n-3}\}$. Indeed, by orthogonality with D_{n-4}, it follows that $x = c_{n-3}$ unless $n = 3$. Thus $\{x, y\} = \{b_{n-3}, c_{n-3}\}$, so $z = a_{n-3}$. Continuing in this way, we see that a right-maximal bowtie chain that begins as above has one of the following forms, where we note that we have (b_0, c_0, a_0) rather than (c_0, b_0, a_0) in (b).

(a) $((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}), \ldots, (c_\ell, b_\ell, a_\ell))$ for some ℓ, which may be negative; or
(b) $((c_n, a_n, b_n),\ (c_{n-1}, b_{n-1}, a_{n-1}),\ \ldots,\ (c_1, b_1, a_1),\ (b_0, c_0, a_0),\ (c_{-1}, b_{-1}, a_{-1}),\ \ldots, (c_\ell, b_\ell, a_\ell))$ for some $\ell \leq 0$.

To eliminate the second possibility, assume that it arises. Recall that $M \backslash c_0, c_1, \ldots, c_n$ has an N-minor. The last matroid has (b_0, b_1, a_2, b_2) as a 4-fan. Then $M \backslash c_0, c_1, \ldots, c_n \backslash b_0$ or $M \backslash c_0, c_1, \ldots, c_n / b_2$ has an N-minor. In both cases, M / b_1 has an N-minor. To see this, observe in the second case that $M \backslash c_0, c_1, \ldots, c_n / b_2 \cong M \backslash c_0, c_1, c_3, c_4 \ldots, c_n \backslash a_2 / b_2 \cong M \backslash c_0, c_1, c_3, c_4 \ldots, c_n \backslash a_2 / b_1$; in the first case, note that $M \backslash c_0, c_1, \ldots, c_n \backslash b_0$ has b_1 in a cocircuit of size at most two. Thus M / b_1 does indeed have an N-minor, and it follows by Lemma 3.11 that M has a quick win, a contradiction. We conclude that (a) holds.
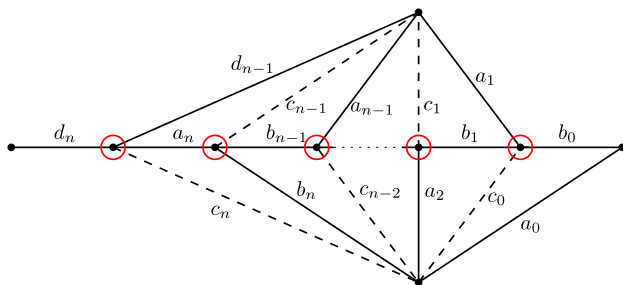
Fig. 23. These elements are all distinct.

If $\ell > 0$, then $M \backslash c_0, c_1, \ldots, c_\ell$ has an N-minor. Now suppose that $\ell \leq 0$. Let k be a non-negative integer such that $-k \geq \ell$. We argue by induction on k that $M \backslash c_0, c_1, \ldots,$ $c_0, c_{-1}, \ldots, c_{-k}$ has an N-minor. This is certainly true if $k = 0$. Hence we may assume that $k \geq 1$. Now $M \backslash c_n, c_{n-1}, \ldots, c_0, c_{-1}, \ldots, c_{-k+1}$ has $(c_{-k}, b_{-k+1}, a_{-k+2}, b_{-k+2})$ as a loose 4-fan. By Lemma 3.11, M/b_{-k+2} does not have an N-minor. Thus $M \backslash c_0, c_1, \ldots, c_{-k}$ has an N-minor.

When $\ell \neq 0$, we will now adjust the notation to make $\ell = 0$ irrespective of whether it was originally positive or negative. This will change n but we will continue to use the same symbol for this quantity noting that we still know that its value is at least two and that $M \backslash c_n, c_{n-1}, \ldots, c_0$ has an N-minor.

We show next that the elements in the rotor chain $((c_n, a_n, b_n), (c_{n-1}, b_{n-1}, a_{n-1}),$ $\ldots, (c_0, b_0, a_0))$ are distinct. Suppose not. Then $a_0 = c_n$. Orthogonality between T_0 and $\{d_{n-1}, a_n, c_n, d_n\}$ implies that T_0 meets $\{d_{n-1}, d_n\}$. By 7.2.24, we see that $d_{n-1} \in T_0$ but $d_{n-1} \neq a_0$. Then $\{c_{n-1}, d_{n-1}, a_n\}$ meets D_0 in a single element, a contradiction to orthogonality. We conclude that the elements in the rotor chain are distinct.

Suppose $\{d_{n-1}, d_n\}$ meets a triangle in this rotor chain. Then orthogonality with the cocircuit $\{d_{n-1}, a_n, c_n, d_n\}$ implies that $\{d_{n-1}, d_n\}$ is contained in this triangle, a contradiction to 7.2.11. Thus M contains the structure in Fig. 23 where all of the elements shown are distinct.

Since $N \preceq M \backslash c_1, c_0$ and $M \backslash c_1$ is not $(4, 4, S)$-connected, it follows by Hypothesis VII that $M \backslash c_0$ is not $(4, 4, S)$-connected. Now M has a bowtie of the form $(T, \{z, b_1, c_0\}, D_1)$ where z is b_2 when $n = 2$, and z is c_2 otherwise. Applying Lemma 3.4 to this bowtie gives that M has a quasi rotor having $\{z, b_1, c_0\}$ as its central triangle. Moreover, $M \backslash c_0$ has a 5-fan (p, q, s, t, u) whose elements are contained in this quasi rotor. Then $\{q, s, t, c_0\}$ is a cocircuit D of M. By orthogonality with $\{z, b_1, c_0\}$, the cocircuit D meets either T_1 or T_2. If D meets T_2, then orthogonality implies that it is contained in $T_0 \cup T_2$, and $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Thus D meets T_1. Lemma 3.3 implies that $D = D_0$. Since $D_0 \cap D_1 = \{b_1\}$, it follows that b_1 is the central element of the quasi rotor. Thus $\{b_0, a_1, g\}$ is a triangle for some element g. By orthogonality with the cocircuits shown in Fig. 23, we know that g is a new element.

Now $M \backslash c_0, c_1, \ldots, c_n$ has an N-minor and has (g, b_0, a_1, b_1) as a loose 4-fan. Since, by Lemma 3.11, M/b_1 has no N-minor, we deduce that
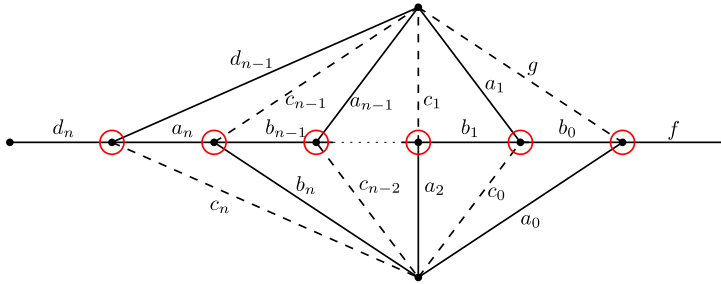
Fig. 24. These elements are all distinct except that f may be d_n. No set in $\{\{f, g\}, \{c_n, d_n\}, \{d_{n-1}, d_n\}\}$ is contained in a triangle. Deleting the dashed elements preserves an N-minor.

7.2.25. $M \backslash c_0, c_1, \ldots, c_n, g$ *has an N-minor.*

Now M has $(\{z, b_1, c_0\}, \{b_0, a_1, g\}, D_0)$ as a bowtie where z is b_2 when $n = 2$ and is a_2 otherwise. As $M \backslash g$ is not internally 4-connected, Lemma 3.4 implies that M has a 4-cocircuit of the form $\{v, w, x, g\}$. By orthogonality with $\{g, b_0, a_1\}, T_0$, and T_1, we know that $\{v, w, x\}$ contains two elements of T_0 or two elements of T_1. By Lemma 3.3, $\{v, w, x\}$ avoids $\{z, b_1, c_0\}$, so $\{v, w, x, g\}$ contains $\{a_0, b_0, g\}$ or $\{a_1, c_1, g\}$. If the latter holds, then, since $M \backslash c_0, c_1, g$ has an N-minor, we deduce that $M \backslash c_0, c_1, g / a_1$ has an N-minor. This gives a contradiction to Lemma 3.11 because a_1 is in two triangles of a quasi rotor of M. Thus $\{v, w, x, g\} = \{a_0, b_0, f, g\}$ for some element f. By orthogonality with the triangles in Fig. 23, we deduce that either $f = d_n$, or f differs from all the elements in Fig. 23.

We show next that

7.2.26. $\{f, g\}$ *is not contained in a triangle of M.*

Suppose $\{e, f, g\}$ is a triangle of M. Since we constructed a right-maximal rotor chain, we know that $\{e, f, g\}$ meets the set of elements in Fig. 23. But g avoids this set of elements, and so does f unless $f = d_n$. By orthogonality with the cocircuits shown in Fig. 23, it follows that $f = d_n$ and $e \in \{d_{n-1}, a_n, c_n\}$. Furthermore, orthogonality implies that $e \notin D_{n-1}$, so $e \neq a_n$. By 7.2.11, we deduce that $e = c_n$, so $\{c_n, d_n\}$ is contained in a triangle, a contradiction to 7.2.24. Thus 7.2.26 holds.

Now M contains the structure in Fig. 24. Moreover, by 7.2.11, 7.2.24, and 6.3.3, none of $\{d_{n-1}, d_n\}, \{c_n, d_n\}$, and $\{f, g\}$ is contained in a triangle. Next we show that

7.2.27. *neither M / a_1 nor M / b_0 has an N-minor, and $\{f, g, a_0, b_0\}$ is the only 4-cocircuit in M containing a_0.*

Suppose M / b_0 has an N-minor. Then, since M / b_0 has $\{a_0, c_0\}$ and $\{g, a_1\}$ as circuits, $M / b_0 \backslash a_1, c_0$ has an N-minor. This matroid is isomorphic to $M / b_1 \backslash a_1, c_0$, so M / b_1 has an N-minor, a contradiction to Lemma 3.11. Thus M / b_0 has no N-minor. Moreover, an immediate consequence of Lemma 3.11 is that M / a_1 has no N-minor.

Now suppose that a_0 is in a 4-cocircuit D other than $\{f, g, a_0, b_0\}$. Lemma 3.3 implies that D avoids T_1. Orthogonality with T_0 implies that D meets $\{b_0, c_0\}$, so orthogonality with $\{g, b_0, a_1\}$ and either $\{a_2, b_1, c_0\}$ or $\{b_2, b_1, c_0\}$ implies that D contains $\{a_0, b_0, g\}$ or $\{a_0, c_0, z\}$, for some element z in $\{a_2, b_2\}$. Since D is not $\{f, g, a_0, b_0\}$, the latter holds. Then orthogonality implies that it is contained in $T_0 \cup T_2$, and $\lambda(T_0 \cup T_1 \cup T_2) \leq 2$, a contradiction. Thus 7.2.27 holds.

7.2.28. $M \backslash c_0, c_1, \ldots, c_n, g$ *is sequentially* 4-*connected.*

To see this, first observe that, by [1, Lemma 7.1], $M \backslash c_0, c_1, \ldots, c_n$ is sequentially 4-connected. The last matroid has g in a triangle, so either it has g in a triad, or $M \backslash c_0, c_1, \ldots, c_n, g$ is 3-connected. The former implies, by orthogonality with $\{g, b_0, a_1\}$, that $\{b_0, g\}$ or $\{a_1, g\}$ is contained in a triad of $M \backslash c_0, c_1, \ldots, c_n$. Then 7.2.25 implies that $M \backslash g, c_0, c_1, \ldots, c_n / x$ has an N-minor for some $x \in \{b_0, a_1\}$, a contradiction to 7.2.27. Thus $M \backslash c_0, c_1, \ldots, c_n, g$ is 3-connected. Suppose this matroid has a non-sequential 3-separation. Without loss of generality, we may assume that the triad $\{b_0, a_1, b_1\}$ is contained on one side of the 3-separation, and we can add g to that side to get a non-sequential 3-separation of $M \backslash c_0, c_1, \ldots, c_n$, a contradiction. Thus 7.2.28 holds.

Next we show that

7.2.29. M *has an element h such that $\{a_0, f, h\}$ is a triangle and $M \backslash g, h$ has an N-minor.*

Since M has no open-rotor-chain win, $M \backslash c_0, c_1, \ldots, c_n, g$ has a 4-fan $(1, 2, 3, 4)$, so M has a cocircuit C^* such that $\{2, 3, 4\} \subsetneq C^* \subseteq \{2, 3, 4, c_0, c_1, \ldots, c_n, g\}$. By orthogonality with the cocircuits in Fig. 24, we deduce that $\{1, 2, 3\}$ can only meet the set of elements in that figure if it contains $\{d_{n-1}, d_n\}$, $\{c_n, d_n\}$, $\{f, g\}$, or $\{a_0, f\}$. The first three possibilities have been excluded, so either $\{1, 2, 3\}$ avoids the set of elements in Fig. 24, or $\{a_0, f\} \subseteq \{1, 2, 3\}$. Suppose the latter occurs. Then M has a triangle of the form $\{a_0, f, h\}$, so $M \backslash g$ has (h, f, a_0, b_0) as a 4-fan. By 7.2.27, $M \backslash g / b_0$ does not have an N-minor. Thus $M \backslash g, h$ has an N-minor, so 7.2.29 holds. We may now assume that $\{1, 2, 3\}$ avoids the set of elements in Fig. 24. Suppose $c_i \in C^*$ for some i in $\{0, 1, \ldots, n-1\}$. Then, since c_i is in two triangles in Fig. 24, orthogonality with these triangles implies that $\{2, 3, 4\}$ contains two elements in this figure. Hence $\{2, 3\}$ contains an element in the figure, a contradiction. Thus C^* avoids $\{c_0, c_1, \ldots, c_{n-1}\}$, so $C^* \subseteq \{2, 3, 4, c_n, g\}$. Furthermore, orthogonality with $\{g, b_0, a_1\}$ and T_n implies that either $C^* = \{2, 3, 4, g\}$ and $4 \in \{b_0, a_1\}$, or $C^* = \{2, 3, 4, c_n\}$ and $4 \in \{a_n, b_n\}$. Thus C^* meets one of the triangles $T_0, T_1, \{c_{n-1}, d_{n-1}, a_n\}$, or $\{c_{n-2}, b_{n-1}, b_n\}$ in a single element, a contradiction to orthogonality. Thus 7.2.29 holds.

By orthogonality with the cocircuits in Fig. 24, either h differs from all the elements in that figure, or $d_n = f$ and $h \in \{c_n, d_{n-1}\}$. By 7.2.11 and 7.2.24, we deduce that h is a new element.

We now show that

7.2.30. $M \backslash h$ *is* $(4, 4, S)$-*connected, and every* 4-*fan of this matroid has* f *as its coguts element.*

Let (z, y, x, w) be a 4-fan in $M \backslash h$. Then $\{w, x, y, h\}$ is a cocircuit of M, and orthogonality with $\{a_0, f, h\}$ implies that $\{w, x, y\}$ meets $\{a_0, f\}$ in a single element. By 7.2.27, $a_0 \notin \{w, x, y\}$, so $f \in \{w, x, y\}$. Suppose $f \in \{x, y, z\}$. Then orthogonality with $\{f, g, a_0, b_0\}$ implies that $\{x, y, z\}$ meets $\{g, a_0, b_0\}$. By 7.2.26, $g \notin \{x, y, z\}$. As $\{x, y, z\}$ does not contain $\{e_0, f\}$, it must contain $\{f, b_0\}$. By orthogonality with D_0, the triangle $\{x, y, z\}$ is $\{f, b_0, b_1\}$, a contradiction to orthogonality with D_1. Thus $f = w$, that is, f is the coguts element of every 4-fan of $M \backslash h$. Thus $M \backslash h$ has no 5-fan. It follows by Lemma 3.4 that $M \backslash h$ is $(4, 4, S)$-connected. Thus 7.2.30 holds.

Since M has no quick win, $M \backslash h$ has a 4-fan (z_0, y_0, x_0, f). Thus $(\{a_0, f, h\}, \{x_0, y_0, z_0\}, \{f, h, x_0, y_0\})$ is a bowtie.

7.2.31. $M \backslash h / f$ *has an* N-*minor.*

To show this, we assume the contrary. Then $M \backslash h, z_0$ has an N-minor. Extend the bowtie $(\{a_0, f, h\}, \{x_0, y_0, z_0\}, \{f, h, x_0, y_0\})$ to a right-maximal bowtie string $\{a_0, f, h\}, \{f, h, x_0, y_0\}, \{x_0, y_0, z_0\}, \ldots, \{x_k, y_k, z_k\}$. Now $M \backslash h, z_0 / y_0 \cong M \backslash h, x_0 / y_0 \cong M \backslash h, x_0 / f$. Thus $M \backslash h, z_0 / y_0$ has no N-minor. Therefore, by Lemma 3.7, $M \backslash h, z_0, z_1, \ldots, z_k$ has an N-minor.

To complete the proof of 7.2.31, we aim to apply [1, Lemma 10.1], but first we need to show that

7.2.32. $(\{x_k, y_k, z_k\}, \{a_0, f, h\}, \{\gamma, z_k, a_0, f\})$ *is not a bowtie for all* γ *in* $\{x_k, y_k\}$.

Assume that M contains such a bowtie. Then, by 7.2.27, $\{\gamma, z_k\} = \{g, b_0\}$. By Lemma 3.3, $k > 0$. Suppose $z_k = b_0$. Then $M \backslash b_0$ has an N-minor. This matroid has (c_1, a_1, b_1, c_0, z) as a 5-fan where $z = a_2$ unless $n = 2$, in which case $z = b_2$. By Lemma 3.2, $M \backslash b_0, c_1, z$ or $M \backslash b_0, c_1 / a_1$ has an N-minor. The former implies that M / b_1 has an N-minor, so in both cases we get a contradiction to Lemma 3.11. We conclude that $z_k \neq b_0$, so $z_k = g$. Now $M \backslash h, z_0, z_1, \ldots, z_k$ has an N-minor, and, by 7.2.30, $M \backslash h$ is $(4, 4, S)$-connected. By Hypothesis VII, it follows that $M \backslash z_0$ is $(4, 4, S)$-connected, that $M \backslash z_i$ is $(4, 4, S)$-connected for all i in $\{1, 2, \ldots, k - 1\}$, and that $M \backslash g$ is $(4, 4, S)$-connected. But $M \backslash g$ has a 5-fan, a contradiction. Thus 7.2.32 holds.

We can now apply [1, Lemma 10.1]. None of (i), (ii), or (v) of that lemma holds. If (iii) holds, then a_0 is in a 4-cocircuit with h, a contradiction to 7.2.27. Thus (iv) holds, so $M \backslash h, z_0 / y_0$ or $M \backslash h, z_0 / x_0$ has an N-minor. Lemma 3.6 implies that $M \backslash h / f$ has an N-minor, a contradiction. We conclude that 7.2.31 holds.

Since M has no quick win, we know that $M \backslash h / f$ is not internally 4-connected. We now apply Lemma 3.8 to the bowtie string $\{a_1, g, b_0\}, \{b_0, g, a_0, f\}, \{a_0, f, h\}, \{f, h, x_0, y_0\}$ to obtain that M has $\{a_0, h\}$ contained in a 4-cocircuit, a contradiction to 7.2.27. □
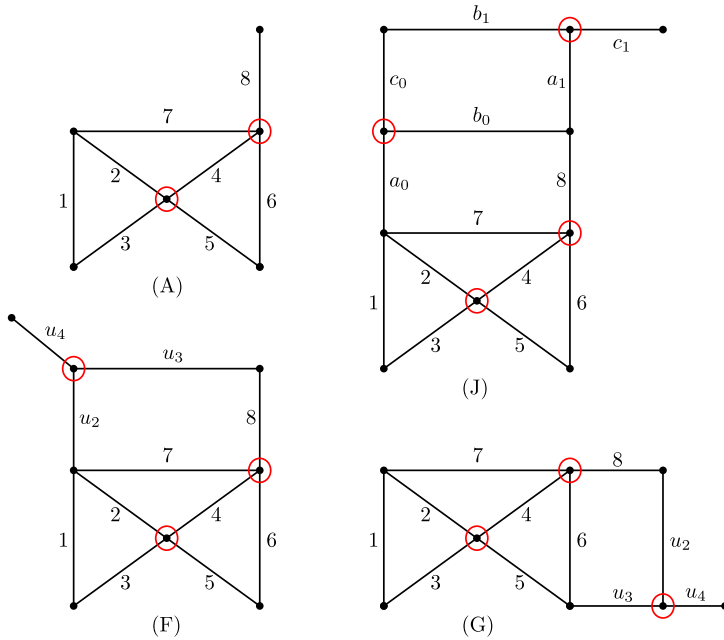
Fig. 25. Some possible configurations in M.

Lemma 7.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Suppose Hypothesis VII holds. Suppose that M contains configuration (A) labelled as in Fig. 25 where $M\backslash 4$ is $(4, 4, S)$-connected with an N-minor, but $N \not\preceq M\backslash 1, 4$. Then*

(i) *M has a quick win; or*

(ii) *M^* has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or*

(iii) *deleting the central cocircuit of an augmented 4-wheel in M^* gives an internally 4-connected matroid with an N^*-minor.*

Proof. Assume the lemma does not hold. First we show that

7.3.1. *$M\backslash 6/8$ and $M/8$ are 3-connected, and $M/5\backslash 6/8$ is 3-connected with an N-minor.*

Clearly $M/5\backslash 4 \cong M/5\backslash 6$. As $N \not\preceq M\backslash 1, 4$, it follows by Lemma 3.1 that $M/5\backslash 6$ is $(4, 4, S)$-connected having an N-minor. Since $M/5\backslash 6$ has $(2, 4, 7, 8)$ as a 4-fan, both $M/5\backslash 6/8$ and $M/5\backslash 6\backslash 2$ are 3-connected, and at least one of these matroids has an N-minor. As $M/5\backslash 6, 2 \cong M/5\backslash 4, 2 \cong M\backslash 4, 2/3 \cong M/3\backslash 4, 1$, and $N \not\preceq M\backslash 1, 4$, we deduce that $N \preceq M/5\backslash 6/8$.

By Lemma 5.1, we know that $M\backslash 6$ is $(4, 4, S)$-connected. As $M\backslash 6$ has $(2, 4, 7, 8)$ as a 4-fan, $M\backslash 6/8$ is 3-connected. Thus $M/8$ is 3-connected unless M has a triangle

containing $\{6, 8\}$. In the exceptional case, $M\backslash 4/5$ has a 5-fan containing $\{2, 6, 7, 8\}$. This contradiction completes the proof of 7.3.1.

Next we note that

7.3.2. *$M/8$ is sequentially 4-connected.*

Suppose that $M/8$ has a non-sequential 3-separation (U, V). Then, by [5, Lemma 3.3], we may assume that $\{1, 2, \ldots, 7\} \subseteq U$. Thus we can adjoin 8 to U to obtain a non-sequential 3-separation of M, a contradiction. Thus 7.3.2 holds.

Next we show that

7.3.3. *M contains one of the configurations (F) and (G) in Fig. 25, where all of the elements shown are distinct, and every 4-fan in $M/8$ has its guts element in $\{6, 7\}$.*

To see this, note that, since $N \preceq M/8$ but M has no quick win, $M/8$ has a 4-fan (u_1, u_2, u_3, u_4). Thus M has $\{u_1, u_2, u_3, 8\}$ as a circuit and has $\{u_2, u_3, u_4\}$ as a triad. By orthogonality, $\{u_1, u_2, u_3\}$ meets $\{4, 6, 7\}$, so $u_1 \in \{4, 6, 7\}$. If $u_1 = 4$, then, by orthogonality, 2, 3, or 5 is in $\{u_2, u_3, u_4\}$, so M has a 4-fan, a contradiction. Thus $u_1 \in \{6, 7\}$. By construction, $8 \notin \{u_2, u_3, u_4\}$. As $\{u_2, u_3, u_4\}$ is a triad, this set also avoids $\{1, 2, \ldots, 7\}$. Hence 7.3.3 holds.

Next we show the following.

7.3.4. *If M contains configuration (G), then $M\backslash 6/8$ is sequentially 4-connected.*

By 7.3.1, we know that $M\backslash 6/8$ is 3-connected. Let (U, V) be a non-sequential 3-separation of $M\backslash 6/8$. Then we may assume that $\{u_2, u_3, u_4\} \subseteq U$. Thus $(U \cup 6, V)$ is a non-sequential 3-separation of $M/8$, a contradiction to 7.3.2. Hence 7.3.4 holds.

7.3.5. *If M contains configuration (G), then M contains configuration (F).*

To see this, observe, since $M\backslash 6/8$ is sequentially 4-connected with an N-minor, but (i) does not hold, $M\backslash 6/8$ has a 4-fan (v_1, v_2, v_3, v_4). Suppose first that $\{v_2, v_3, v_4\}$ is a triad of M. Then the 4-fan is a fan in $M/8$, and 7.3.3 implies that $v_1 = 7$. Hence M contains configuration (F) where v_i replaces u_i for each i in $\{1, 2, 3\}$.

We may now assume that $\{v_2, v_3, v_4, 6\}$ is a cocircuit of M. Lemma 3.3 implies that $\{v_2, v_3, v_4\}$ avoids $\{1, 2, 3\}$. Orthogonality implies that $\{4, 5\}$ meets $\{v_2, v_3, v_4\}$. Suppose $4 \in \{v_2, v_3, v_4\}$. Then, by orthogonality, $\{2, 7\}$ meets $\{v_2, v_3, v_4\}$. Thus $\{v_2, v_3, v_4\}$ contains $\{4, 7\}$, so $\{6, v_2, v_3, v_4\} = \{6, 4, 7, 8\}$, a contradiction. We conclude that $4 \notin \{v_2, v_3, v_4\}$. Thus $5 \in \{v_2, v_3, v_4\}$.

By Lemma 5.1, we know that $\{4, 5, 6\}$ is the only triangle containing 5. Suppose that $5 \in \{v_2, v_3\}$. Then, without loss of generality, $\{v_1, v_2, 5, 8\}$ is a circuit. In this case, orthogonality with $\{2, 3, 4, 5\}$ and $\{4, 6, 7, 8\}$ implies that either $4 \in \{v_1, v_2\}$, or $\{v_1, v_2\} = \{7, z\}$, for some z in $\{2, 3\}$. If $4 \in \{v_1, v_2\}$, then $\{v_1, v_2, 5, 8\} \triangle \{4, 5, 8, u_2, u_3\}$

is a triangle meeting $\{u_2, u_3\}$, a contradiction. Thus $\{v_1, v_2\} = \{7, z\}$, for some $z \in \{2, 3\}$, and $\lambda(\{1, 2, \ldots, 8\}) \leq 2$, a contradiction. We conclude that $5 \notin \{v_2, v_3\}$, so $5 = v_4$. Moreover, by orthogonality between $\{6, 8, u_2, u_3\}$ and $\{v_2, v_3, 5, 6\}$, we deduce that $\{u_2, u_3\}$ meets $\{v_2, v_3\}$. Thus, we may assume that $v_3 = u_3$. As u_3 is not in a triangle of M, the set $\{v_1, v_2, u_3, 8\}$ is a circuit. By orthogonality, $\{4, 7\}$ and $\{u_2, u_4\}$ meet $\{v_1, v_2\}$. Hence $v_2 \in \{4, 7, u_2, u_4\}$. By orthogonality between $\{v_2, v_3, 5, 6\}$ and the circuits $\{2, 4, 7\}$ and $\{6, 8, u_2, u_3\}$, we know that $v_2 \notin \{4, 7, u_2\}$. Hence $v_2 = u_4$. Then $\{u_4, u_3, 5, 6\} \triangle \{u_2, u_3, u_4\}$, which is $\{u_2, 5, 6\}$, is a triad, a contradiction. We conclude that 7.3.5 holds.

We may now assume that M contains configuration (F). We relabel (u_2, u_3, u_4) as (a_0, b_0, c_0), for reasons that will become clear later. We show next that

7.3.6. $M/5\backslash 6\backslash 8/c_0$ *has an N-minor but $M/5\backslash 6\backslash 8\backslash 7$ does not.*

As $M/5\backslash 6\backslash 8$ has an N-minor and has $(7, a_0, b_0, c_0)$ as a 4-fan, we deduce that $N \preceq M/5\backslash 6\backslash 8\backslash 7$ or $N \preceq M/5\backslash 6\backslash 8/c_0$. In the first case, as $M/5\backslash 6\backslash 8\backslash 7 \cong M/5\backslash 6, 7/4$, we see that $N \preceq M/5, 4$. Thus, by Lemma 7.1, we deduce that $N \preceq M\backslash 4, 1$, a contradiction. We conclude that 7.3.6 holds.

From 7.3.6, $N \preceq M/c_0$. Since c_0 is in a triad of M, we see that M/c_0 is 3-connected.

7.3.7. M/c_0 *is sequentially 4-connected.*

To show this, suppose that (U, V) is a non-sequential 3-separation of M/c_0. Then, by [5, Lemma 3.3], we may assume that $\{1, 2, \ldots, 7\} \subseteq U$. Hence we may also assume that $8 \in U$. Now if a_0 or b_0 is in U, then we may assume that both are in U, in which case we can adjoin c_0 to U to get a non-sequential 3-separation of M, a contradiction. Hence $\{a_0, b_0\} \subseteq V$ and we can adjoin c_0 to V to get a non-sequential 3-separation of M, a contradiction. Thus 7.3.7 holds.

Next we observe the following.

7.3.8. *If a_0 or b_0 is the guts element of a 4-fan of M/c_0, then, up to switching the labels on a_0 and b_0, the matroid M contains structure (J) in Fig. 25 where all of the elements shown are distinct.*

We need only check that the elements are distinct. Clearly $\{a_1, b_1, c_1\}$ avoids $\{1, 2, \ldots, 7, b_0, c_0\}$. If $\{a_1, b_1, c_1\}$ meets $\{a_0, 8\}$, then it contains this set. Now $a_0 \notin \{a_1, b_1\}$ as the binary matroid M does not have a 3-element circuit–cocircuit intersection. Thus $a_0 = c_1$. Then $8 \in \{a_1, b_1\}$ and we get a contradiction to orthogonality. Hence 7.3.8 holds.

We now show that

7.3.9. M *contains structure (J) in Fig. 25 where all of the elements shown are distinct and the labels on a_0 and b_0 may be interchanged.*

As M has no quick win, by 7.3.7, M/c_0 has a 4-fan $(\alpha, \beta, \gamma, \delta)$. Thus M has $\{\alpha, \beta, \gamma, c_0\}$ as a circuit. By orthogonality, $\{a_0, b_0\}$ meets $\{\alpha, \beta, \gamma\}$. If $\alpha \in \{a_0, b_0\}$, then the result follows by 7.3.8. Thus we may assume that $\gamma = b_0$. Then $\{\beta, \gamma, \delta\}$ contains exactly two elements of $\{a_0, b_0, 7, 8\}$. Now $7 \notin \{\beta, \gamma, \delta\}$, and $\{\beta, \gamma, \delta\}$ does not contain $\{a_0, b_0\}$, so $8 \in \{\beta, \delta\}$.

As the next step towards proving 7.3.9, we now show that

7.3.10. *M does not have* $\{2, 5, a_0, c_0\}$ *or* $\{6, 8, b_0, c_0\}$ *as a circuit.*

Since M has $\{a_0, b_0, 2, 5, 6, 8\}$ as a circuit and $\{a_0, b_0, 2, 5, 6, 8\} \triangle \{2, 5, a_0, c_0\} = \{6, 8, b_0, c_0\}$, it suffices to prove that $\{2, 5, a_0, c_0\}$ is not a circuit. Assume otherwise. By 7.3.6, we know that $M/5, 8, c_0 \backslash 6$ has an N-minor. Hence so does $M/5, 8, c_0 \backslash 6, 2$. But $M/5, 8, c_0 \backslash 6, 2 \cong M/5, 8, c_0 \backslash 4, 2 \cong M \backslash 4, 2/3, 8, c_0 \cong M/3, 8, c_0 \backslash 4, 1$, so $N \preceq M \backslash 1, 4$, a contradiction. Hence 7.3.10 holds.

Recall that $8 \in \{\beta, \delta\}$. Suppose that $8 = \beta$. Then, by orthogonality between $\{\alpha, 8, b_0, c_0\}$ and the cocircuits $\{4, 6, 7, 8\}$ and $\{2, 3, 4, 5\}$, it follows that $\alpha \in \{6, 7\}$. As $\{7, 8, a_0, b_0\}$ is also a circuit, $\alpha \neq 7$, so $\alpha = 6$ and we have a contradiction to 7.3.10. We deduce that $8 \neq \beta$, so $8 = \delta$. Hence M/c_0 has $(\alpha, \beta, b_0, 8)$ as a 4-fan.

7.3.11. *M has* $\{\alpha, \beta, b_0, c_0\}$ *as a circuit and has* $\{\beta, b_0, 8\}$ *and* $\{a_0, b_0, c_0\}$ *as triads, these being its only triads containing* b_0.

The first part of this is immediate. By orthogonality, a triad containing b_0 must contain a_0 or 8, so the last part also holds.

By 7.3.6, we see that $N \preceq M/c_0, 8$. Still aiming at obtaining 7.3.9, we show next that

7.3.12. $M/c_0, 8$ *is 3-connected.*

Assume the contrary. As M/c_0 is 3-connected having $(\alpha, \beta, b_0, 8)$ as a 4-fan, we deduce that M/c_0 has $\{8, b_0\}$ or $\{8, \beta\}$ in a triangle. Suppose $\{8, \beta\}$ is in a triangle of M/c_0. Then $\{8, \beta, c_0\}$ is contained in a 4-circuit of M, which, by orthogonality, must be $\{8, \beta, c_0, a_0\}$ or $\{8, \beta, c_0, b_0\}$. But M has $\{\alpha, \beta, b_0, c_0\}$ as a circuit. As $\alpha \neq 8$, it follows that $\{8, \beta, c_0, a_0\}$ is a circuit. By taking the symmetric difference of the last two circuits, we deduce that $\{\alpha, 8, a_0, b_0\}$ is a circuit, so $\alpha = 7$. Then $\{7, \beta, b_0, c_0\}$ is a circuit, so $\beta \in \{4, 6\}$, a contradiction to orthogonality. We conclude that $\{8, \beta\}$ is not in a triangle of M/c_0. Thus M/c_0 has $\{8, b_0\}$ contained in a triangle, so M has $\{8, b_0, c_0\}$ contained in a 4-circuit. By orthogonality, this 4-circuit is $\{8, b_0, c_0, 6\}$, a contradiction to 7.3.10. We conclude that 7.3.12 holds.

By 7.3.7, M/c_0 is sequentially 4-connected. It follows that $M/c_0, 8$ is sequentially 4-connected for if (U, V) is a non-sequential 3-separation of the last matroid, then we may assume that $\{\alpha, \beta, b_0\} \subseteq U$, so $(U \cup 8, V)$ is a non-sequential 3-separation of M/c_0, a contradiction.

Since M has no quick win, $M/c_0, 8$ has a 4-fan (s_1, s_2, s_3, s_4). Thus M has a circuit C such that $\{s_1, s_2, s_3\} \subsetneq C \subseteq \{s_1, s_2, s_3, c_0, 8\}$, and $\{s_2, s_3, s_4\}$ avoids $\{1, 2, \ldots, 7, 8, c_0\}$. Suppose $c_0 \notin C$. Then $C = \{s_1, s_2, s_3, 8\}$. By orthogonality, $s_1 \in \{4, 6, 7\}$. But $s_1 \neq 4$ otherwise $\{s_2, s_3\}$ meets $\{2, 3, 5\}$. Thus $s_1 \in \{6, 7\}$. Now M has $\{\beta, b_0, 8\}$ as a triad, so $\{s_2, s_3\}$ meets $\{\beta, b_0\}$ in a single element. The triad $\{s_2, s_3, s_4\}$ avoids $\{c_0, 8\}$ so differs from both $\{a_0, b_0, c_0\}$ and $\{b_0, \beta, 8\}$. Hence, by 7.3.11, $b_0 \notin \{s_2, s_3, s_4\}$, so $\beta \in \{s_2, s_3\}$. Thus, by orthogonality between $\{s_2, s_3, s_4\}$ and $\{\alpha, \beta, b_0, c_0\}$, we deduce that $\alpha \in \{s_2, s_3, s_4\}$. It follows that M/c_0 has a 4-fan having b_0 as its guts element and $\{s_2, s_3, s_4\}$ as its triad, a contradiction to 7.3.8. We conclude that $c_0 \in C$.

Suppose $C = \{s_1, s_2, s_3, c_0\}$. By orthogonality and 7.3.8, $\{s_2, s_3\}$ meets $\{a_0, b_0\}$. As $\{s_2, s_3, s_4\}$ is not $\{a_0, b_0, c_0\}$, it follows by orthogonality that $\{s_2, s_3, s_4\}$ meets $\{7, 8\}$, a contradiction. We deduce that $C \neq \{s_1, s_2, s_3, c_0\}$, so $C = \{s_1, s_2, s_3, c_0, 8\}$. Then, by orthogonality, $s_1 \in \{4, 6, 7\}$, and $\{s_2, s_3\}$ meets $\{a_0, b_0\}$. By 7.3.11, $b_0 \notin \{s_2, s_3, s_4\}$ so $a_0 \in \{s_2, s_3\}$. Thus, by orthogonality between $\{s_2, s_3, s_4\}$ and $\{a_0, b_0, 7, 8\}$, we obtain a contradiction. We conclude that 7.3.9 holds.

We may now assume that M^* has (T_0, T_1, D_0) as a bowtie. In M^*, take a right-maximal bowtie string $T_0, D_0, T_1, D_2, \ldots, T_n$ noting that n may equal 1. Next we show the following.

7.3.13. *The elements in $\{1, 2, \ldots, 8\} \cup T_0 \cup T_1 \cup \cdots \cup T_n$ are distinct.*

The triangles in this string are triads in M, so the elements in the string avoid $\{1, 2, \ldots, 7\}$. Therefore 7.3.13 holds unless either $a_0 = c_n$, or 8 is in the bowtie string. Suppose $8 \in T_i$. Then, since $8 \notin T_0$, orthogonality between T_i and $\{7, 8, a_0, b_0\}$ implies that $a_0 = c_i$ and $i = n$. Then $8 \in \{a_n, b_n\}$. By orthogonality, $\{b_{n-1}, c_{n-1}, a_n, b_n\}$ meets $\{4, 6, 7\}$, so a triangle in M meets a triangle in M^*, a contradiction. We conclude that 8 is not in the bowtie string. By orthogonality between T_n and the cocircuit $\{7, 8, a_0, b_0\}$ in M^*, we see that $c_n \neq a_0$. Thus 7.3.13 holds.

We want to apply Lemma 7.2 to M^*. Since M^* contains both triangles and triads, it is not isomorphic to the cycle matroid of a quartic Möbius ladder. Moreover, if M^* is isomorphic to the cycle matroid of a terrahawk, then so is M, and therefore M has a second triangle containing 5, a contradiction to Lemma 5.1. Let $X = \{5, 8\}$, let $Y = \{6\}$, and let $M' = M^* \backslash X / Y$. In M', each T_i is a disjoint union of circuits while each D_j is a disjoint union of cocircuits. Since M' is 3-connected, it follows that $T_0, D_0, T_1, D_2, \ldots, T_n$ is a bowtie string in M'.

7.3.14. $M' \backslash c_0, c_1, \ldots, c_n$ *has an N^*-minor and* $M' \backslash c_0, c_1, \ldots, c_i / e$ *has no N^*-minor for all e in $\{a_i, b_i\}$ where $i \in \{1, 2, \ldots, n\}$.*

To see this, observe first that, by 7.3.6, $M' \backslash c_0$ has an N^*-minor but $M'/7$ does not. It follows from this that $M' \backslash a_0 / b_0$ has no N^*-minor since $M' \backslash a_0$ has $\{b_0, 7\}$ as a cocircuit, so $M' \backslash a_0 / b_0 \cong M' \backslash a_0 / 7$. Suppose $M' \backslash c_0, c_1, \ldots, c_i$ has an N^*-minor

for some i in $\{0, 1, \ldots, n-1\}$. As this matroid has $(c_{i+1}, a_{i+1}, b_{i+1}, b_i)$ as a 4-fan, either $M' \backslash c_0, c_1, \ldots, c_{i+1}$ has an N^*-minor, or $M' \backslash c_0, c_1, \ldots, c_i / b_i$ has an N^*-minor. By Lemma 3.6, the latter implies that $M' \backslash a_0, a_1, \ldots, a_i / b_0$ has an N^*-minor. Hence so does $M' \backslash a_0 / b_0$, a contradiction. It follows by induction that $M' \backslash c_0, c_1, \ldots, c_n$ has an N^*-minor. Furthermore, if $M' \backslash c_0, c_1, \ldots, c_i / a_i$ has an N^*-minor for some i in $\{1, 2, \ldots, n\}$, then Lemma 3.6 implies that $M' \backslash a_0, a_1, \ldots, a_{i-1}, b_i / b_0$ has an N^*-minor, so $M' \backslash a_0 / b_0$ has an N^*-minor, a contradiction. Thus 7.3.14 holds.

Clearly none of (i), (ii), and (vii) of Lemma 7.2 holds in M^*.

7.3.15. *Neither (iii) nor (v) of Lemma 7.2 holds in M^*.*

To see this, observe that, by 7.3.14, it follows that (v) of Lemma 7.2 does not hold. Suppose (iii) of Lemma 7.2 holds in M^*. Then $\{a_0, b_0, z, c_n\}$ is a 4-circuit of M for some z in $\{a_n, b_n\}$. Taking the symmetric difference of this circuit with $\{7, 8, a_0, b_0\}$, we see that $\{7, 8, z, c_n\}$ is a circuit of M. But $M/5\backslash6/8, c_n$ has an N-minor and has 7 in a 2-circuit. Hence $M/5\backslash6/8, c_n\backslash7$ has an N-minor, a contradiction to 7.3.6. We deduce that 7.3.15 holds.

Suppose M^* contains the structure in Fig. 18 where $d_1 \in Y$ and either $d_0 \in X$ or $M' \backslash c_0, c_1, d_0$ has an N^*-minor. Then $d_1 = 6$, and $\{d_0, a_1, c_1, 6\}$ is a circuit in M. By orthogonality with $\{4, 6, 7, 8\}$, we know that $d_0 \in \{4, 7, 8\}$. But $\{b_0, d_0, a_1\}$ is a triad of M, so it avoids $\{4, 7\}$. Thus $d_0 = 8$. The symmetric difference $\{4, 5, 6\} \triangle \{6, 8, a_1, c_1\}$ is $\{4, 5, 8, a_1, c_1\}$, which must be a circuit in M. Now $M/5\backslash6/8/c_0/c_1$ has $\{4, a_1\}$ as a circuit. Hence $M/5\backslash6/8/c_0, c_1\backslash a_1$ has an N-minor and (v) of Lemma 7.2 holds, a contradiction. Thus M^* does not contain the structure in Fig. 18.

Suppose M^* contains the structure in Fig. 17(a). Then M has $\{p, t, a_0\}$, $\{a_0, b_0, c_0\}$, and $\{b_0, b_1, q\}$ as distinct cocircuits. By orthogonality with the circuit $\{7, 8, a_0, b_0\}$, each of q and $\{p, t\}$ meets $\{7, 8\}$ in a unique element. As 7 is in no triad of M, we deduce that $q = 8$ and $8 \in \{p, t\}$. Hence $q \in \{p, t\}$, a contradiction. Suppose next that M^* contains the structure in Fig. 17(b). Then $\{b_0, b_1, q\}$ and $\{s_1, s_2, s_3\}$ are disjoint cocircuits of M and $\{b_1, c_1, q, s_2, s_3\}$ is a circuit of M. By orthogonality with the circuit $\{7, 8, a_0, b_0\}$, it follows that $q = 8$. By orthogonality with the cocircuit $\{4, 6, 7, 8\}$, we see that $\{b_1, c_1, s_2, s_3\}$ meets $\{4, 6, 7\}$, a contradiction. We conclude that M^* contains neither of the structures in Fig. 17.

By Lemma 7.2, it follows that M^* contains the structure in Fig. 12 and $\{d_{n-1}, d_n\}$ is not contained in a triangle of M^*. By [1, Lemma 6.4], as $a_0 \neq c_n$ and $\{d_{n-1}, d_n\}$ is not contained in a triangle, we know that the elements in this figure are all distinct with the possible exception that α and β may be repeated elements. Thus $\{\alpha, \beta, a_0\}$ is a triangle of M^* distinct from T_0, and $\{\alpha, a_0, c_0, d_0\}$ or $\{\alpha, a_0, c_0, a_1, c_1\}$ is a cocircuit of M^* for some elements α, β, and d_0. Furthermore, $\{c_0, d_0, a_1\}$ is a triangle of M^*. Since M^* has $\{7, 8, a_0, b_0\}$ as a cocircuit, orthogonality implies that $\{\alpha, \beta\}$ meets $\{7, 8\}$. Clearly 7 avoids $\{\alpha, \beta\}$, so $8 \in \{\alpha, \beta\}$.

7.3.16. $\beta = 8$.

To show this, suppose that $\alpha = 8$. Then orthogonality implies that $\{4, 6, 7\}$ meets $\{a_0, c_0, d_0\}$ or $\{a_0, c_0, a_1, c_1\}$. Thus M^* is not internally 4-connected, a contradiction. Thus 7.3.16 holds.

We relabel 7 as γ to see that M^* contains the structure in Fig. 11, where the elements are all distinct with the possible exception that α, β, and γ may be repeated elements.

In preparation for applying Lemma 4.1, we now show the following.

7.3.17. *The elements in Fig. 11 are distinct except that γ and d_n may be equal.*

As γ is in a triad of M^*, we know that γ avoids all of the other elements in Fig. 11 with the possible exception of d_n. By orthogonality between the triangles in this figure and the cocircuits $\{\beta, \gamma, a_0, b_0\}$ and $\{\alpha, a_0, c_0, d_0\}$, we deduce that β and α avoid all of the elements with the possible exception of d_n. Thus the elements are all distinct except that d_n may be in $\{\alpha, \beta, \gamma\}$. But orthogonality between $\{d_{n-1}, a_n, c_n, d_n\}$ and $\{\alpha, \beta, a_0\}$ implies that $d_n \notin \{\alpha, \beta\}$. Thus 7.3.17 holds.

By 7.3.15 and Lemma 7.2, M^* has no triangle T_{n+1} such that $\{x, c_n, a_{n+1}, b_{n+1}\}$ is a 4-cocircuit for any x in $\{a_n, b_n\}$. Then Lemma 3.4 implies that $M^* \backslash c_n$ is $(4, 4, S)$-connected.

By 7.3.14, $M^* \backslash \beta, c_0, c_1, \ldots, c_n$ has an N-minor. By 7.3.17, we are now in a position to apply Lemma 4.1 to get that $M^* \backslash \beta, c_0, c_1, \ldots, c_n$ is $(4, 4, S)$-connected and every 4-fan of this matroid is either a 4-fan in $M^* \backslash c_n$ with b_n as its coguts element, or is a 4-fan in $M^* \backslash \beta$ with α as its coguts element. Let (u, v, w, x) be such a 4-fan. Suppose first that $x = b_n$ and this 4-fan is a 4-fan in $M^* \backslash c_n$. Then $\{v, w, b_n, c_n\}$ is a 4-cocircuit of M^* and, taking $T_{n+1} = \{u, v, w\}$, we get a contradiction to the previous paragraph. It follows that (u, v, w, α) is a 4-fan in $M^* \backslash \beta$. By 7.3.16, we know that $\beta = 8$. Thus (α, w, v, u) is a 4-fan in $M/8$, a contradiction to 7.3.3. We conclude that $M^* \backslash \beta, c_0, c_1, \ldots, c_n$ has no 4-fans and so is internally 4-connected. Thus (ii) holds, a contradiction. $\square$

This completes our analysis of the case when M contains configuration (A) in Fig. 13.

8. Configuration (B)

In this section, we deal with the case when M contains configuration (B) in Fig. 13. The results from the last two sections mean that if we find that M contains configuration (C) or (A) from Fig. 13, then we are guaranteed to get one of the desired outcomes from the main theorem.

Lemma 8.1. *Suppose M and N are internally 4-connected binary matroids, $|E(M)| \geq 13$ and $|E(N)| \geq 7$, and M contains structure (B) in Fig. 13 where all of the elements are distinct except that 1 may be a. Suppose $M \backslash 4$ is $(4, 4, S)$-connected with an N-minor but $M \backslash 1, 4$ does not have an N-minor. Then*
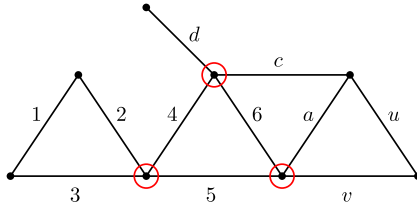
Fig. 26. All of the elements are distinct except that u may be 1.

(i) *M has a triangle $\{7, 8, 9\}$ where $\{4, 6, 7, 8\}$ is a cocircuit and the elements in $\{1, 2, \ldots, 9\}$ are distinct except that 1 may be 9; or*

(ii) *M contains the configuration in Fig. 26, where all of the elements are distinct except that u may be 1, and $M \backslash 6$ is $(4, 4, S)$-connected with an N-minor; or*

(iii) *$M \backslash 6$ is internally 4-connected with an N-minor.*

Proof. Suppose that the lemma fails. As $N \npreceq M \backslash 1, 4$, it follows by Lemma 3.1 that $N \preceq M \backslash 4/5$, and $M \backslash 4/5$ is $(4, 4, S)$-connected. Since $M/5 \backslash 4 \cong M/5 \backslash 6$, we deduce that $N \preceq M \backslash 6$.

8.1.1. *$M \backslash 6$ is $(4, 4, S)$-connected with an N-minor.*

Assume that 8.1.1 fails. Then, by Lemma 3.4, $\{4, 5, 6\}$ is the central triangle of a quasi rotor. Moreover, since $M \backslash 4$ is $(4, 4, S)$-connected, the central element of this quasi rotor is 4, and Lemma 3.4 specifies that this quasi rotor is $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{4, 6, 7, 8\}, \{x, 4, 7\})$ for some x in $\{2, 3\}$, so (i) holds, a contradiction. We conclude that 8.1.1 holds.

Since $M \backslash 6$ is not internally 4-connected, it has a 4-fan (u, v, w, x). Thus M has $\{v, w, x, 6\}$ as a cocircuit. Hence $\{v, w, x\}$ meets $\{4, 5\}$ and $\{a, c\}$. Clearly $\{u, v, w\} \neq \{4, 5, 6\}$, so Lemma 5.1 implies that $5 \notin \{u, v, w\}$. Suppose $4 \in \{u, v, w\}$. Then orthogonality between $\{u, v, w\}$ and $\{4, 6, c, d\}$ implies that $\{c, d\}$ meets $\{u, v, w\}$. If $\{4, c\}$ is in a triangle, then the symmetric difference of this triangle with $\{4, 5, a, c\}$ is a triangle other than $\{4, 5, 6\}$ that contains 5, a contradiction to Lemma 5.1. Thus $\{4, d\} \subseteq \{u, v, w\}$, so $M \backslash 6/5$ has a 5-fan, a contradiction. We conclude that $4 \notin \{u, v, w\}$, so $x \in \{4, 5\}$ and, without loss of generality, w is in $\{a, c\}$.

8.1.2. *Either $\{1, 2, 3\}$ avoids $\{u, v, w\}$, or $\{1, 2, 3\} \cap \{u, v, w\} = \{1\} = \{u\}$.*

To see this, observe that, as $(\{u, v, w\}, \{4, 5, 6\}, \{v, w, x, 6\})$ is a bowtie, Lemma 3.3 implies that the cocircuit $\{2, 3, 4, 5\}$ avoids $\{u, v, w\}$. Furthermore, Lemma 3.3 applied to the bowtie $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ implies that $\{v, w, x, 6\}$ avoids $\{1, 2, 3\}$. It follows that 8.1.2 holds.

By 8.1.2, if $x = 4$, then (i) of the lemma holds, a contradiction. Thus we may assume that $x = 5$. Suppose $w = c$. Then $\{c, d\}$ is not in a triangle, as (i) does not hold, so orthogonality implies that $4 \in \{u, v\}$, which we have already shown does not occur. We conclude that $a = w$. Then $\{v, a, 5, 6\}$ is a cocircuit of M, so M contains the configuration in Fig. 26.

By 8.1.2, since $a = w$, we see that $a \neq 1$. Thus the elements in Fig. 13(B) are distinct. Lemma 3.3 implies that $\{u, v\}$ avoids $\{2, 3, 4, 5\}$ and $\{4, 6, c, d\}$. Hence the elements in Fig. 26 are distinct with the possible exception that $1 \in \{u, v\}$. By 8.1.2, $1 \neq v$. Thus (ii) holds. □

If (i) of the last lemma holds, then, possibly after a minor relabelling, we see that M contains structure (C) from Fig. 13. Since this case has already been treated, it remains for us to consider the structure in Fig. 26.

Lemma 8.2. *Suppose M and N are internally 4-connected binary matroids, $|E(M)| \geq 16$ and $|E(N)| \geq 7$, and M contains the structure in Fig. 26 where all of the elements are distinct except that u may be 1. Suppose Hypothesis VII holds. If $M \backslash 6$ is $(4, 4, S)$-connected with an N-minor but $M \backslash 6, u$ has no N-minor, then*

  (i) *M has a quick win; or*
 (ii) *M^* has an open-rotor-chain win, a ladder win, or an enhanced-ladder win; or*
(iii) *deleting the central cocircuit of an augmented 4-wheel in M^* gives an internally 4-connected matroid with an N^*-minor.*

Proof. As $M \backslash 6, u$ has no N-minor, we relabel the elements $1, 2, 3, 4, 5, 6, a, v, u, c$, and d in Fig. 26, as $x, y, z, 6, 5, 4, 2, 3, 1, c$, and d. We then restrict our attention to the configuration in M that is the same as that in Fig. 13(A). Now $M \backslash 1, 4$ has no N-minor. Lemma 7.3 implies that the result holds. □

We may now assume that M contains the structure in Fig. 26 and $M \backslash 6, u$ has an N-minor. Recall that $M \backslash 4/5$ has an N-minor. As this matroid has $(a, 6, c, d)$ as a 4-fan, we may delete a or contract d keeping an N-minor. In the next lemma, we consider the second possibility.

Lemma 8.3. *Suppose M is an internally 4-connected binary matroid containing at least thirteen elements, and M has the structure in Fig. 26 where all of the elements are distinct except that u may be 1. Suppose that both $M \backslash 4$ and $M \backslash 6$ are $(4, 4, S)$-connected. Then*

  (i) *M/d is $(4, 4, S)$-connected and every $(4, 3)$-violator of M/d is a 4-fan with c as its guts element; or*
 (ii) *M contains the structure in Fig. 13(A).*

Proof. Assume that (ii) does not hold. First we show that

8.3.1. *M has no triangle containing d.*

Assume that M has a triangle T containing d. Then, by orthogonality, T contains c, 6, or 4. In the first two cases, we obtain the contradiction that $M\backslash 4$ is not $(4, 4, S)$-connected. We conclude that $4 \in T$. Then, by orthogonality and symmetry, we may assume that $2 \in T$, so (ii) holds. This contradiction implies that 8.3.1 holds.

8.3.2. *M/d is sequentially 4-connected.*

By 8.3.1, M/d is 3-connected. Suppose that M/d has a non-sequential 3-separation (U, V). Then, by [5, Lemma 3.3], we may assume that $\{4, 5, 6, a, u, v, c\} \subseteq U$. Then $(U \cup d, V)$ is a non-sequential 3-separation of M, a contradiction. Thus 8.3.2 holds.

Now suppose that M/d has a 4-fan $(\alpha, \beta, \gamma, \delta)$. Then M has $\{\alpha, \beta, \gamma, d\}$ as a circuit. By orthogonality, $\{\alpha, \beta, \gamma\}$ meets $\{4, 6, c\}$. Since M has no 4-fan, $\{\beta, \gamma\}$ avoids the triangles of M. Thus $\alpha \in \{4, 6, c\}$. By orthogonality between $\{\alpha, \beta, \gamma, d\}$ and the cocircuits $\{2, 3, 4, 5\}$ and $\{5, 6, a, v\}$, we see that $\alpha \notin \{4, 6\}$. Thus $\alpha = c$.

We now know that every 4-fan in M/d has c as its guts element. It follows easily that M has no 5-fan and no 5-cofan, so M/d is $(4, 4, S)$-connected as required. $\square$

We have not eliminated the case that M contains the structure in Fig. 26 and M/d has an N-minor, but we have built up more structure, which will assist us in our later analysis.

Lemma 8.4. *Suppose M and N are internally 4-connected binary matroids, $|E(M)| \geq 13$ and $|E(N)| \geq 7$, and M contains the structure in Fig. 26 where all of the elements are distinct except that u may be 1. Suppose that Hypothesis VII holds. Suppose that $M\backslash 4$ and $M\backslash 6$ are each $(4, 4, S)$-connected with an N-minor, and $M\backslash 1, 4$ has no N-minor. Then*

(i) *$\{4, 5, 6\}$ is the only triangle of M containing 4, and M contains the structure in Fig. 27 where all of the elements are distinct except that u_2 may be the same as 1, or $\{1, 2, 3\}$ may equal $\{a_2, u_2, v_2\}$; or*

(ii) *M has a quick win; or*

(iii) *M has a ladder win.*

Proof. We assume that neither (ii) nor (iii) holds. We show first that

8.4.1. *$\{4, 5, 6\}$ is the only triangle that meets $\{4, 5\}$.*

Lemma 5.1 implies that $\{4, 5, 6\}$ is the only triangle that contains 5. Let T be a triangle that contains 4 but differs from $\{4, 5, 6\}$. Then orthogonality implies that T
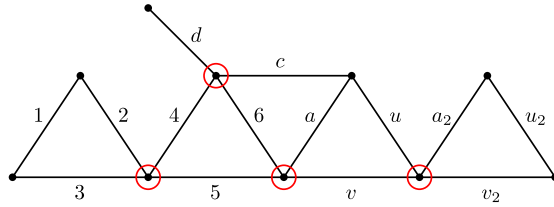
Fig. 27. All of the elements are distinct except that u_2 may be the same as 1, or $\{1, 2, 3\}$ may be $\{a_2, u_2, v_2\}$.

meets $\{2, 3, 5\}$ and $\{6, c, d\}$, so, up to switching the labels on 2 and 3, the triangle is $\{2, 4, d\}$ or $\{2, 4, c\}$, so $M\backslash 4/5$ has a 5-fan, a contradiction to Lemma 3.1. We deduce that 8.4.1 holds.

Lemma 8.2 implies that $M\backslash 6, u$ has an N-minor. From Lemma 3.5, we know that $M\backslash u$ is $(4, 4, S)$-connected and, as $M\backslash 4$ is not internally 4-connected, either

(I) M has a triangle $\{a_2, u_2, v_2\}$ and a cocircuit $\{x, u, a_2, v_2\}$ where $x \in \{a, v\}$ and $|\{4, 5, 6, a, u, v, a_2, u_2, v_2\}| = 9$, or

(II) every $(4, 3)$-violator of $M\backslash u$ is a 4-fan of the form $(6, y_2, y_3, y_4)$ where $y_2 \in \{a, v\}$.

Suppose (II) holds. Since $y_2 \in \{a, v\}$, orthogonality implies that (y_1, y_2, y_3) is $(6, v, d)$ or $(6, a, c)$. If $\{6, v, d\}$ is a triangle, then $\lambda(\{4, 5, 6, a, v, u, c, d\}) \leq 2$, a contradiction. Thus $y_2 = a$ and $y_3 = c$. We now consider $M\backslash 6, u$. By [1, Lemma 6.1], since $M\backslash 6, u$ is not internally 4-connected, one of the following occurs: $\{c, y_4\}$ is contained in a triangle; or $\{5, v\}$ is contained in a triangle; or M has a triangle that contains 4 but avoids $\{5, 6, a, c, u, v\}$; or $M\backslash 6, u$ is $(4, 4, S)$-connected and v is the coguts element of every 4-fan in it. If $\{c, y_4\}$ is contained in a triangle, then orthogonality implies that the third element of this triangle is in $\{4, 6, d\}$ and so $\lambda(\{4, 5, 6, a, u, v, c, d, y_4\}) \leq 2$, a contradiction. By 8.4.1, $\{5, v\}$ is not contained in a triangle and M has no triangle that contains 4 but avoids $\{5, 6\}$. We deduce that $M\backslash 6, u$ has a 4-fan of the form (z_1, z_2, z_3, v). As every 4-fan of $M\backslash u$ contains 6, we see that (z_1, z_2, z_3, v) is not a 4-fan of $M\backslash u$. Hence $\{6, z_2, z_3, v\}$ or $\{6, u, z_2, z_3, v\}$ is a cocircuit of M. Then orthogonality implies that $\{z_2, z_3\}$ meets $\{4, 5\}$, a contradiction to 8.4.1. We conclude that (II) does not hold. Therefore (I) holds.

If the triangle $\{a_2, u_2, v_2\}$ meets $\{c, d\}$, then orthogonality with the cocircuit $\{4, 6, c, d\}$ implies that $\{c, d\} \subseteq \{a_2, u_2, v_2\}$, so $M\backslash 4$ has a 5-fan, a contradiction. Thus the elements in $\{4, 5, 6, a, u, v, a_2, u_2, v_2, c, d\}$ are distinct. Now consider the cocircuit $\{x, u, a_2, v_2\}$ recalling that $x \in \{a, v\}$. If $x = a$, then orthogonality implies that $\{u, a_2, v_2\}$ meets $\{6, c\}$, a contradiction. Thus $x = v$. By hypothesis, $\{2, 3\}$ avoids $\{4, 5, 6, a, c, d, u, v\}$. Suppose $\{2, 3\}$ meets $\{a_2, u_2, v_2\}$. Then orthogonality with $\{2, 3, 4, 5\}$ implies that $\{2, 3\} \subseteq \{a_2, u_2, v_2\}$, so $\{1, 2, 3\} = \{a_2, u_2, v_2\}$, and (i) holds. Now suppose that $\{2, 3\}$ avoids $\{a_2, u_2, v_2\}$. Then the elements in $\{2, 3, 4, 5, 6, a, u, v, a_2, u_2, v_2, c, d\}$ are distinct. Finally, orthogonality implies that 1 can only be in the last set if it equals u_2. Thus (i) holds. $\square$

When M contains the configuration in Fig. 27 and $M\backslash 4$ is $(4,4,S)$-connected with an N-minor but $M\backslash 1, 4$ does not have an N-minor, we know, by Lemma 3.1, that $M\backslash 6/5$ is $(4,4,S)$-connected with an N-minor. Since $M\backslash 6/5$ has $(a, 4, c, d)$ as a 4-fan, it follows that either

  (i) $N \preceq M\backslash 6/5\backslash a$; or
 (ii) $N \preceq M\backslash 6/5/d$.

As we showed in Lemma 8.3, we are able to find a new triad in the case that (ii) holds. In the following lemma, we dispense with the case that (i) holds.

Lemma 8.5. *Suppose M and N are internally 4-connected binary matroids, $|E(M)| \geq 15$ and $|E(N)| \geq 7$, and M contains the structure in Fig. 27 where all of the elements are distinct except that u_2 may be 1, or $\{1, 2, 3\}$ may equal $\{a_2, u_2, v_2\}$. Suppose that Hypothesis VII holds and that $\{4, 5, 6\}$ is the only triangle that contains 4. Suppose that $M\backslash 4$ and $M\backslash 6$ are each $(4, 4, S)$-connected with an N-minor and $M\backslash 6/5\backslash a$ has an N-minor but $M\backslash 1, 4$ has no N-minor. Then*

  (i) *M has a quick win; or*
 (ii) *M has a ladder win; or*
(iii) *M has a mixed ladder win.*

Proof. Assume that the lemma does not hold. We relabel the elements $4, 5, 6, a, v, u, d$, and c in Fig. 27 as $a_0, v_0, u_0, a_1, v_1, u_1, t_0$, and t_1, respectively. Since $M\backslash u_0/v_0\backslash a_1$ is isomorphic to $M\backslash u_0, u_1/v_1$, the second matroid has an N-minor. We take a right-maximal bowtie string $\{a_0, v_0, u_0\}, \{v_0, u_0, a_1, v_1\}, \{a_1, v_1, u_1\}, \{v_1, u_1, a_2, v_2\}, \ldots,$ $\{a_n, v_n, u_n\}$. Let X be the set of elements in this bowtie string.

8.5.1. *Suppose M contains the structure in Fig. 28, where $i \geq 2$, all of the elements are distinct, and $\{t_0, t_1, \ldots, t_{i-1}\}$ avoids X. Suppose $M\backslash u_0, u_1, \ldots, u_{i-1}/v_{i-1}$ has an N-minor. Then M has an element t_i that is not in $\{t_0, t_1, \ldots, t_{i-1}\} \cup X$ such that $\{t_{i-1}, a_{i-1}, u_{i-1}, t_i\}$ is a cocircuit, $\{u_{i-1}, t_i, a_i\}$ is a triangle, and $M\backslash u_0, u_1, \ldots, u_i/v_i$ has an N-minor. Moreover, if $\{1, 2, 3\}$ meets $(\cup_{j=0}^i \{a_j, u_j, v_j\}) \cup \{t_0, t_1, \ldots, t_i\}$, then either $1 = u_i$, or $\{1, 2, 3\} = \{a_i, u_i, v_i\}$.*

Since $M/v_{i-1}\backslash u_{i-1}$ has an N-minor, Lemma 3.8 implies that $\{a_{i-1}, u_{i-1}\}$ is contained in a 4-cocircuit. Orthogonality implies that this cocircuit meets $\{u_{i-2}, t_{i-1}\}$, and Lemma 3.3 implies that it avoids $\{a_{i-2}, u_{i-2}, v_{i-2}\}$, hence it contains t_{i-1}. Let t_i be the fourth element in this cocircuit. Orthogonality implies that t_i avoids the triangles in Fig. 28 and also avoids X. Thus t_i is a new element unless $t_i = t_0$. In the exceptional
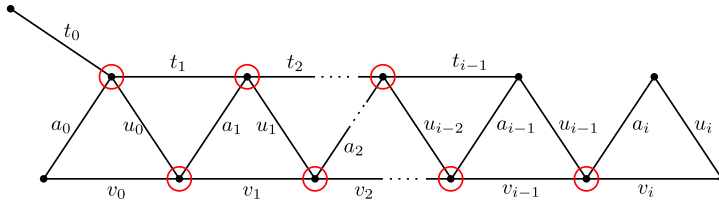
Fig. 28. All of the elements shown are distinct.

case, $\{t_0, a_0, u_0, t_1\} \triangle \{t_{i-1}, a_{i-1}, u_{i-1}, t_0\}$, which equals $\{a_0, u_0, t_1, t_{i-1}, a_{i-1}, u_{i-1}\}$, is a cocircuit. Hence the elements in Fig. 28, excluding $\{t_0, a_i, u_i, v_i\}$, comprise a 3-separating set in M, a contradiction. Thus $t_i \neq t_0$.

Next we establish the last part of 8.5.1. Suppose that $\{1, 2, 3\}$ meets $(\cup_{j=0}^{i} \{a_j, u_j, v_j\}) \cup \{t_0, t_1, \ldots, t_i\}$. If $\{1, 2, 3\}$ avoids X, then orthogonality with the cocircuits in Fig. 28 implies that $\{1, 2, 3\}$ contains $\{t_0, t_1, \ldots, t_i\}$. Hence $i = 2$, and $\lambda(\{a_0, u_0, v_0, a_1, u_1, v_1, t_0, t_1, t_2\}) \leq 2$, a contradiction. We deduce that $\{1, 2, 3\}$ meets X. By [1, Lemma 5.4], we see that $1 = u_i$, as desired, or $\{1, 2, 3\} = \{a_k, u_k, v_k\}$ for some k in $\{2, 3, \ldots, i\}$. We assume the latter. By orthogonality with the cocircuit $\{2, 3, a_0, v_0\}$, we see that $\{2, 3\}$ avoids $\{u_{k-1}, t_k, a_k\}$, so $1 = a_k$ and $\{2, 3\} = \{u_k, v_k\}$. Now orthogonality with $\{2, 3, a_0, v_0\}$ implies that $\{u_k, t_{k+1}, a_{k+1}\}$ is not a triangle. Hence $k = i$. Thus the last part of 8.5.1 holds.

We can now apply Lemma 6.3 to our structure noting that, by assumption, (ii)(c) of that lemma cannot hold. Since $\{a_0, u_0, v_0\}$ is the unique triangle of M containing a_0, it follows that $\{a_0, t_0\}$ is not in a triangle of M. Thus M has a triangle containing $\{u_{i-1}, t_i\}$. By orthogonality, this triangle meets $\{v_{i-1}, a_i, v_i\}$. If $i < n$, then orthogonality with $\{v_i, u_i, a_{i+1}, v_{i+1}\}$ implies that the third element of this triangle is a_i. If $i = n$, then, up to switching the labels on a_n and v_n, we may assume that the third element of this triangle is a_i.

To complete the proof of 8.5.1, it remains only to show that $M \backslash u_0, u_1, \ldots, u_i / v_i$ has an N-minor. Suppose not. Since $M \backslash u_0, u_1, \ldots, u_{i-1} / v_{i-1}$ has an N-minor and has $(a_i, t_i, a_{i-1}, t_{i-1})$ as a 4-fan, we know that $M \backslash u_0, u_1, \ldots, u_{i-1} / v_{i-1} \backslash a_i$ or $M \backslash u_0, u_1, \ldots, u_{i-1} / v_{i-1} / t_{i-1}$ has an N-minor. Since the first matroid is isomorphic to $M \backslash u_0, u_1, \ldots, u_{i-1} / v_i \backslash u_i$ by Lemma 3.6, we may assume that the second matroid has an N-minor. Now Lemma 3.6 implies that $M \backslash u_0, u_1, \ldots, u_{i-1} / v_{i-1} / t_{i-1}$ is isomorphic to $M \backslash a_0, a_1, \ldots, a_{i-1} / v_0 / t_{i-1}$. Applying Lemma 3.6 again, this time focusing on the bowtie string at the top of the diagram, we get that the last matroid is isomorphic to $M \backslash a_0, u_0, u_1, \ldots, u_{i-2} / v_0 / t_1$. Thus $M \backslash a_0, u_0$ has an N-minor. As this matroid has $\{t_0, t_1\}$ as a cocircuit, we deduce that M / t_0 has an N-minor. Because $\{a_0, u_0, v_0\}$ is the unique triangle containing a_0, it follows that M does not contain the structure in Fig. 13(A). Thus Lemma 8.3 implies that M / t_0 has a 4-fan of the form $(t_1, \beta, \gamma, \delta)$. Hence $\{t_0, t_1, \beta, \gamma\}$ is a circuit of M, and $\{\beta, \gamma, \delta\}$ is a triad of M. Orthogonality with $\{t_1, a_1, u_1, t_2\}$ implies that $\{\beta, \gamma\}$ meets a triangle of M, a contradiction. Thus 8.5.1 holds.
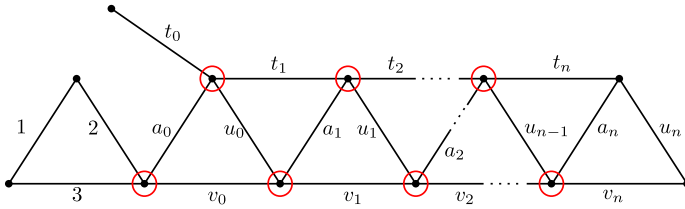
Fig. 29. $n \geq 2$ and all of the elements shown are distinct except that 1 may be the same as u_n.

By repeatedly applying 8.5.1 on our bowtie string, we deduce that M contains the structure in Fig. 29 and $M\backslash u_0, u_1, \ldots, u_n/v_n$ has an N-minor. By Lemma 3.6, we see that

8.5.2. $M\backslash u_0, u_1, \ldots, u_n/v_n \cong M\backslash a_0, a_1, \ldots, a_n/v_0$.

We also get from 8.5.1 that $\{1, 2, 3\}$ avoids the other elements in Fig. 29 except that 1 may be u_n, or $\{1, 2, 3\}$ may be $\{a_n, u_n, v_n\}$. If $\{1, 2, 3\} = \{a_n, u_n, v_n\}$, then orthogonality implies that $\{2, 3\} = \{u_n, v_n\}$, so $1 = a_n$. By 8.5.2, we deduce, since $a_0 = 4$, that $M\backslash 1, 4$ has an N-minor, a contradiction. We conclude that all the elements in Fig. 29 are distinct, except that 1 may be u_n.

Next we show that

8.5.3. $M\backslash u_n$ *has no 4-fan with* v_n *as its coguts element.*

Suppose that $M\backslash u_n$ has $(7, 8, 9, v_n)$ as a 4-fan. Then $(\{a_n, u_n, v_n\}, \{7, 8, 9\}, \{8, 9, v_n, u_n\})$ is a bowtie, and [1, Lemma 5.4] implies that $\{7, 8, 9\} = \{a_j, u_j, v_j\}$ for some j in $\{0, 1, \ldots, n-2\}$. Then $\lambda(X \cup \{t_1, t_2, \ldots, t_n\}) \leq 2$. Since $\{1, 2, 3, t_0\}$ avoids $X \cup \{t_1, t_2, \ldots, t_n\}$, to avoid a contradiction, we must have that $1 = u_n$. Hence $\{1, 2, 3\}$ meets $\{8, 9, v_n, u_n\}$ in a single element, a contradiction. Thus 8.5.3 holds.

Next we show that

8.5.4. $\{a_n, u_n\}$ *is contained in a 4-cocircuit.*

Since $M\backslash u_0, u_1, \ldots, u_n/v_n \cong M\backslash a_0, a_1, \ldots, a_n/v_0$ by 8.5.2, we deduce that the second matroid has an N-minor. Thus both $M\backslash u_n$ and $M\backslash a_{n-1}$ have N-minors. Moreover, by Hypothesis VII, $M\backslash u_n$ is $(4, 4, S)$-connected. As M has no quick win, Lemma 3.4 and 8.5.3 imply that 8.5.4 holds unless v_n is in a triangle T with u_{n-1} or v_{n-1}. In the exceptional case, orthogonality with the vertex cocircuits in Fig. 29 implies that T is $\{v_n, u_{n-1}, t_n\}$, a contradiction. Thus 8.5.4 holds.

Orthogonality implies that the 4-cocircuit containing $\{a_n, u_n\}$ meets $\{u_{n-1}, t_n\}$, and Lemma 3.3 implies that the cocircuit avoids $\{a_{n-1}, u_{n-1}, v_{n-1}\}$. Hence it contains t_n. Let t_{n+1} be the fourth element in this cocircuit. Orthogonality with the triangles in Fig. 29 implies that t_{n+1} avoids all of the elements in the figure with the possible exception of t_0. Now we apply [1, Lemma 6.5] and conclude that one of the following holds:

$M \backslash u_0, u_1, \ldots, u_n$ is internally 4-connected; or $M \backslash u_0, u_1, \ldots, u_n$ is $(4, 4, S)$-connected and every 4-fan in this matroid is also a 4-fan of $M \backslash u_n$ with v_n as its coguts element or is a 4-fan of $M \backslash u_0$ with a_0 in its triangle; or M is a quartic Möbius ladder with a_0 in two triangles. Since $\{a_0, u_0, v_0\}$ is the only triangle of M containing a_0, we deduce that either M has a ladder win, a contradiction; or $M \backslash u_n$ has a 4-fan with v_n as its coguts element, which contradicts 8.5.3. We conclude that the lemma holds. $\square$

We continue to consider the case when M contains the structure in Fig. 27, $M \backslash 4$ is $(4, 4, S)$-connected with an N-minor, $M \backslash 1, 4$ does not have an N-minor, and $M \backslash 6/5$ is $(4, 4, S)$-connected with an N-minor. Now $M \backslash 6/5$ has $(a, 4, c, d)$ as a 4-fan, and the preceding lemma dealt with the case when $M \backslash 6/5 \backslash a$ has an N-minor. Our final lemma deals with the case when the last matroid does not have an N-minor.

Lemma 8.6. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 16$ and $|E(N)| \geq 7$. Suppose that M contains the structure in Fig. 27, where the elements are all distinct except that u_2 may be 1, or $\{a_2, u_2, v_2\}$ may be $\{1, 2, 3\}$. Suppose that $M \backslash 4$ and $M \backslash 6$ are $(4, 4, S)$-connected having N-minors, and that $M \backslash 1, 4$ does not have an N-minor. Suppose that Hypothesis VII holds, that $\{4, 5, 6\}$ is the only triangle containing 4, and that $M \backslash 6/5 \backslash a$ does not have an N-minor. Then*

(i) *M has a quick win; or*
(ii) *M or M^* has an open-rotor-chain win, a bowtie-ring win, or a ladder win; or*
(iii) *M or M^* has an enhanced-ladder win; or*
(iv) *deleting the central cocircuit of an augmented 4-wheel in M^* gives an internally 4-connected matroid with an N^*-minor.*

Proof. Assume that the lemma fails. Then, by Lemma 8.2, $N \preceq M \backslash 6, u$. As before, we relabel $4, 5, 6, a, u$, and v as a_0, v_0, u_0, a_1, u_1, and v_1, respectively. Let $\{a_0, v_0, u_0\}$, $\{v_0, u_0, a_1, v_1\}$, $\{a_1, v_1, u_1\}$, $\{v_1, u_1, a_2, v_2\}, \ldots, \{a_n, v_n, u_n\}$ be a right-maximal bowtie string. We show next that

8.6.1. *$M \backslash u_0, u_1, \ldots, u_i/v_i$ has no N-minor for all i in $\{1, 2, \ldots, n\}$ and $M \backslash u_0, u_1, \ldots, u_j/a_j$ has no N-minor for all j in $\{2, 3, \ldots, n\}$, but $M \backslash u_0, u_1, \ldots, u_n$ has an N-minor.*

Since $M \backslash u_0, u_1/v_1 \cong M \backslash 6, a/5$, the first matroid does not have an N-minor. It follows by Lemma 3.7 that 8.6.1 holds.

By 8.6.1 and Hypothesis VII, it follows that $M \backslash u_i$ is $(4, 4, S)$-connected for all i in $\{0, 1, \ldots, n\}$. Establishing the next assertion will occupy most of the rest of the proof of Lemma 8.6.

8.6.2. *M has no bowtie of the form $(\{a_n, u_n, v_n\}, \{a_0, v_0, u_0\}, \{x, u_n, a_0, v_0\})$ with x in $\{a_n, v_n\}$.*

Suppose that M does have such a bowtie. Then $a_0 \neq u_n$. By possibly interchanging the labels on a_n and v_n, we may assume that $x = v_n$. Next we show the following.

8.6.3. *Either* $\{2, 3\} = \{u_n, v_n\}$; *or* $\{u_n, v_n, 2, 3\}$ *is a cocircuit of* M.

To see this, observe that both $\{u_n, v_n, a_0, v_0\}$ and $\{2, 3, a_0, v_0\}$ are cocircuits of M. Taking their symmetric difference, we immediately get 8.6.3.

We now eliminate the first possibility in 8.6.3.

8.6.4. $\{2, 3\} \neq \{u_n, v_n\}$.

Suppose $\{2, 3\} = \{u_n, v_n\}$. Then $a_n = 1$ and M has $(\{a_0, u_0, v_0\}, \{u_0, v_0, a_1, v_1\},$ $\{a_1, u_1, v_1\}, \ldots, \{a_n, u_n, v_n\}, \{u_n, v_n, a_0, v_0\})$ as a ring of bowties. We now apply [1, Lemma 5.5] noting that, since M does not have a bowtie-ring win, part (i) of that lemma does not hold. Moreover, by 8.6.1, part (iii) of that lemma does not hold. Thus part (ii) of that lemma holds, that is, $M \backslash u_0, u_1, \ldots, u_n$ is sequentially 4-connected but not internally 4-connected, and every 4-fan of it has the form $(\alpha, \beta, \gamma, \delta)$ where $\{\alpha, \beta, \gamma\}$ avoids $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$, and M has a cocircuit $\{\beta, \gamma, \delta, u_i\}$ for some i in $\{0, 1, \ldots, n\}$ and some δ in $\{a_i, v_i\}$.

With a view to applying Lemma 10.4 of [1], we show next that $i \neq 0$. Assume the contrary. Then M has $\{\beta, \gamma, \delta, u_0\}$ as a cocircuit and $\{u_0, c, a_1\}$ as a triangle where $\delta \in \{a_0, v_0\}$. By orthogonality, $c \in \{\beta, \gamma\}$. Then orthogonality between $\{\alpha, \beta, \gamma\}$ and $\{d, a_0, u_0, c\}$ implies that $d \in \{\alpha, \beta, \gamma\}$. Hence $M \backslash a_0$ has a 5-fan, a contradiction. We conclude that $i \neq 0$.

We now apply [1, Lemma 10.4] noting that we get a contradiction using 8.6.1 unless $i = 1$ and $\delta = a_i$. In the exceptional case, orthogonality between $\{u_0, c, a_1\}$ and $\{a_1, u_1, \beta, \gamma\}$ implies that $\{u_0, c\}$ meets $\{\beta, \gamma\}$. By construction, $u_0 \notin \{\alpha, \beta, \gamma\}$. Hence $c \in \{\beta, \gamma\}$. By orthogonality with $\{a_0, v_0, c, d\}$, the triangle $\{\alpha, \beta, \gamma\}$ contains $\{c, d\}$, so $M \backslash 4$ has a 5-fan, a contradiction. Thus 8.6.4 holds.

By 8.6.3, we now know that M has $\{2, 3, u_n, v_n\}$ as a cocircuit. If $\{1, 2, 3\}$ avoids $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$, then we can adjoin $\{2, 3, u_n, v_n\}$ and $\{1, 2, 3\}$ to our right-maximal bowtie string to get a contradiction. Thus $\{1, 2, 3\}$ meets $\{a_0, u_0, v_0, a_1, u_1, v_1, \ldots, a_n, u_n, v_n\}$. By [1, Lemma 5.4], $\{1, 2, 3\} = \{a_j, u_j, v_j\}$ for some j with $0 \leq j \leq n - 2$. Certainly $j \neq 0$. Moreover, $j \neq 1$ otherwise the cocircuit $\{2, 3, a_0, v_0\}$ contradicts Lemma 3.3. If $u_j \in \{2, 3\}$, then $M \backslash u_0, u_1, \ldots, u_n$ has v_n in a 2-cocircuit, a contradiction to 8.6.1. Thus $\{2, 3\} = \{a_j, v_j\}$ and $\{2, 3, u_n, v_n\} \triangle \{v_{j-1}, u_{j-1}, a_j, v_j\}$, which is $\{v_{j-1}, u_{j-1}, u_n, v_n\}$, is a cocircuit in M. Again $M \backslash u_0, u_1, \ldots, u_n$ has v_n in a 2-cocircuit, a contradiction. Thus 8.6.2 holds.

We can now apply [1, Lemma 10.1]. Since $n \geq 2$, we conclude that either $M \backslash u_0, u_1 / v_1$ has an N-minor, or M has a_0 in a triangle other than $\{a_0, u_0, v_0\}$. The former option gives a contradiction to 8.6.1, and the latter gives a contradiction to the assumptions of the lemma. $\quad \square$

9. The proof of the main theorem

In this section, as we shall see, it is quite straightforward to assemble the parts from earlier sections to complete the proof of the main result.

Proof of Theorem 1.4. Assume that the theorem fails. Theorem 1.3 implies that Hypothesis VII holds. Now M has a bowtie $(\{1,2,3\}, \{4,5,6\}, \{2,3,4,5\})$ where $M\backslash 4$ is $(4,4,S)$-connected with an N-minor and $M\backslash 1,4$ has no N-minor. Lemma 3.1 implies that $M\backslash 4/5$ is $(4,4,S)$-connected with an N-minor. By Lemma 5.1, we know that M contains (A), (B), or (C) in Fig. 13, that $M\backslash 6$ is $(4,4,S)$-connected, and that $\{4,5,6\}$ is the only triangle in M containing 5. Moreover, the elements in (A), (B), or (C) are all distinct except that a may equal 1 in (B) or (C).

Suppose that M contains the configuration in Fig. 13(C). Lemma 6.1 implies that $M\backslash 4/5, 6$ does not have an N-minor. Since $M\backslash 4$ is $(4,4,S)$-connected with an N-minor and has $(1,2,3,5)$ and $(a,b,c,6)$ as 4-fans, and $M\backslash 1,4$ does not have an N-minor, we deduce that $M\backslash 4/5\backslash a$ has an N-minor. Then Lemma 6.4 gives a contradiction. We conclude that M does not contain the configuration in Fig. 13(C).

If M contains the configuration in Fig. 13(A), then Lemma 7.3 gives a contradiction. Hence we may assume that M does not contain either of the configurations in Fig. 13(A) or Fig. 13(C). Thus M contains the configuration in Fig. 13(B). Lemma 8.1 implies that M contains the configuration in Fig. 26, where $M\backslash 6$ is $(4,4,S)$-connected and all of the elements are distinct except that 1 may be u or $\{1,2,3\}$ may equal $\{a_2, u_2, v_2\}$. By Lemma 8.4, M contains the configuration in Fig. 27 where $\{4,5,6\}$ is the only triangle containing 4, and all of the elements are distinct except that u_2 may be 1. If $M\backslash 6/5\backslash a$ has an N-minor, then Lemma 8.5 gives a contradiction. Thus we may assume that $N \not\preceq M\backslash 6/5\backslash a$. Now $M\backslash 4/5 \cong M\backslash 6/5$, so $M\backslash 6/5$ is $(4,4,S)$-connected with an N-minor. Using Lemma 8.6, we get a contradiction that completes the proof of the theorem. □

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