CHORDAL MATROIDS ARISING FROM GENERALIZED PARALLEL CONNECTIONS

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Abstract. A graph is chordal if every cycle of length at least four has a chord. In 1961, Dirac characterized chordal graphs as those graphs that can be built from complete graphs by repeated clique-sums. Generalizing this, we consider the class of simple $GF(q)$-representable matroids that can be built from projective geometries over $GF(q)$ by repeated generalized parallel connections across projective geometries. We show that this class of matroids is closed under induced minors. We characterize the class by its forbidden induced minors; the case when $q = 2$ is distinctive.

1. Introduction

The notation and terminology in this paper will follow [7] for graphs and [12] for matroids. Unless stated otherwise, all graphs and matroids considered here are simple. Thus every contraction of a set from a matroid is immediately followed by the simplification of the resulting matroid. A *chord* of a cycle C in a graph G is an edge e of G that is not in C such that both vertices of e are vertices of C. A graph is *chordal* if every cycle of length at least four has a chord. Such graphs were called *rigid circuit graphs* by Dirac [8] and *triangulated graphs* by Berge [2]. Let G_1 and G_2 be graphs and $V(G_1) \cap V(G_2) = V$, say. Assume that $G_1[V]$ is a complete graph H and $G_2[V]$ has edge set $E(H)$. The *clique-sum* of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Loosely speaking, the clique-sum is obtained by gluing G_1 and G_2 together across the clique H. While there are several characterizations of chordal graphs (see, for example, [13]), we choose to focus on the following one of Dirac [8].

Theorem 1.1. *A graph G is chordal if and only if G can be constructed from complete graphs by repeated clique-sums.*

Let M_1 and M_2 be matroids whose ground sets intersect in a set T such that T is a modular flat of M_1, and $M_1|T = M_2|T = N$. The

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generalized parallel connection of M_1 and M_2 across N is the matroid with ground set $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. We denote this matroid by $P_N(M_1, M_2)$ or $P_T(M_1, M_2)$. Note that T may be empty, in which case, $P_T(M_1, M_2) = M_1 \oplus M_2$.

For a prime power q, we will denote the projective geometry $PG(r-1, q)$ by P_r when context makes the field clear. Let $\mathcal{M}_q$ be the class of matroids that can be built from projective geometries over $GF(q)$ by a sequence of generalized parallel connections across projective geometries over $GF(q)$. A matroid M is $GF(q)$*-chordal* if M is a member of $\mathcal{M}_q$. By [5], each member of $\mathcal{M}_q$ is $GF(q)$-representable.

An *induced minor of a graph* G is a graph H that can be obtained from G by a sequence of vertex deletions and edge contractions. Similarly, an *induced minor of a matroid* M is a matroid N that can be obtained from M by a sequence of restrictions to flats and contractions, where each such contraction is followed by a simplification. Evidently, the class of chordal graphs is closed under vertex deletions, that is, it is closed under taking induced subgraphs. The analogous property for the class $\mathcal{M}_q$ is highlighted by the following result.

Theorem 1.2. *For all q, the class $\mathcal{M}_q$ is closed under taking induced minors.*

Our main results are the following characterizations of the forbidden induced minors for the class $\mathcal{M}_q$, first for $q = 2$ and then for $q > 2$.

Theorem 1.3. *The set of forbidden induced minors for the class $\mathcal{M}_2$ is $\{M(K_4), U_{3,4}\}$.*

Theorem 1.4. *For each $q > 2$, the set of forbidden induced minors for the class $\mathcal{M}_q$ is $\{U_{2,k} : 2 < k \leq q\} \cup \{U_{3,q+2}\}$.*

Several different notions of chordal matroids have been given over the last forty years [1, 3, 6, 10, 17]. Each of these papers, with the exception of [6, 10], focuses primarily on binary matroids. Following Cordovil, Forge, and Klein [6], we define a simple or non-simple matroid M to be *chordal* if, for each circuit C that has at least four elements, there are circuits C_1 and C_2 and an element e such that $C_1 \cap C_2 = \{e\}$ and $C = (C_1 \cup C_2) - e$.

Cordovil, Forge, and Klein [6] and Mayhew and Probert [10] study the relation between chordal graphs and supersolvable matroids. Such matroids were originally introduced by Stanley [15]. A rank-r matroid is *supersolvable* if there is a chain of modular flats $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r$ in M where the rank of F_i is i for each i in $[r]$. The class $\mathcal{M}_q$ is given in [10] as an example of a class of matroids whose members are supersolvable and form a saturated class of matroids, where a matroid M is *saturated* if, for every modular flat F, the restriction $M|F$ has no two disjoint cocircuits.

2. Preliminaries

Before beginning the discussions of forbidden induced minors for the class $\mathcal{M}_q$, we show that this class is closed under taking induced minors. We shall use the following well-known property of generalized parallel connections (see, for example, [12, 11.23]).

Lemma 2.1. *For every flat F of $P_N(M_1, M_2)$,*

$$r(F) = r(F \cap E(M_1)) + r(F \cap E(M_2)) - r(F \cap E(N)).$$

Lemma 2.2. *The class $\mathcal{M}_q$ of $GF(q)$-chordal matroids is closed under induced restrictions.*

Proof. It is enough to show that the class $\mathcal{M}_q$ is closed under restricting to a hyperplane. Let M be a minimum-rank matroid in $\mathcal{M}_q$ such that $M|H \notin \mathcal{M}_q$ for some hyperplane H of M. Then M is not a projective geometry, so $M = P_N(M_1, M_2)$ where M_1 and M_2 are in $\mathcal{M}_q$, and N is a projective geometry over $GF(q)$. Let $H_i = H \cap E(M_i)$ for each i in $\{1, 2\}$ and let $H_N = H \cap E(N)$. Then, since H is a both flat of a generalized parallel connection and a hyperplane of M,

$$r(M_1) + r(M_2) - r(N) - 1 = r(H) = r(H_1) + r(H_2) - r(H_N). \quad (2.1)$$

Suppose H contains all of $E(N)$. Then $r(H_N) = r(N)$, so, from (2.1), we have

$$r(M_1) + r(M_2) - 1 = r(H_1) + r(H_2).$$

This implies that, for some i and j such that $\{i, j\} = \{1, 2\}$, the hyperplane H contains $E(M_i)$, and H_j is a hyperplane of M_j. It follows by the minimality of M that $M|H$ is in $\mathcal{M}_q$, a contradiction.

We now assume that H does not contain $E(N)$. Hence H does not contain $E(M_1)$ or $E(M_2)$. By [5], $E(M_1)$ is a modular flat of $P_N(M_1, M_2)$, so

$$r(H) = r(H \cup E(M_1)) + r(H_1) - r(M_1). \quad (2.2)$$

As $E(M_1) \not\subseteq H$, we deduce that $r(H \cup E(M_1)) > r(H)$, so $r(H \cup E(M_1)) = r(H) + 1$. It follows from (2.2) that $r(H_1) = r(M_1) - 1$. By symmetry, $r(H_2) = r(M_2) - 1$. Hence H_i is a hyperplane of M_i for each i in $\{1, 2\}$. Therefore, by (2.1) and (2.2),

$$r(M_1) + r(M_2) - r(N) - 1 = r(M_1) - 1 + r(M_2) - 1 - r(H_N).$$

Thus $r(H_N) = r(N)-1$. Hence H_N is a hyperplane of N, so $M|H_N$ is a projective geometry. By [5] (see also [12, Proposition 11.4.15]), $M|H = P_{M|H_N}(M|H_1, M|H_2)$, so $M|H$ is a generalized parallel connection of members of $\mathcal{M}_q$ along a projective geometry and is therefore a member of $\mathcal{M}_q$, a contradiction. □

In the next result, where we show that $\mathcal{M}_q$ is closed under contractions, we shall make repeated use of the fact that every contraction is followed by a simplification. We observe that a consequence of this result is that $\mathcal{M}_q$ is closed under parallel minors.

Lemma 2.3. *The class $\mathcal{M}_q$ of $GF(q)$-chordal matroids is closed under taking contractions.*

Proof. Let M be a $GF(q)$-chordal matroid. We argue by induction on $|E(M)|$ that M/e is a $GF(q)$-chordal matroid for all e in $E(M)$. The result is certainly true if $|E(M)| \leq 2$. Suppose the result holds when $|E(M)| < k$ and let $|E(M)| = k$. The result holds if M is a projective geometry, so we may assume that $M = P_N(M_1, M_2)$, where M_1 and M_2 are in $\mathcal{M}_q$, and N is a projective geometry over $GF(q)$. Suppose that $e \in E(M_1) - E(N)$. Then M_1/e is in $\mathcal{M}_q$ by the induction hypothesis and M_1/e has N as a restriction. As $M/e = P_N(M_1/e, M_2)$, we deduce that M/e is in $\mathcal{M}_q$. We may now assume that $e \in E(N)$. This implies that e is in both $E(M_1)$ and $E(M_2)$, and, by the induction hypothesis, M_1/e and M_2/e are in $\mathcal{M}_q$ and contain the projective geometry N/e as a restriction. By [5], $M/e = P_{N/e}(M_1/e, M_2/e)$, and we again get that M/e is in $\mathcal{M}_q$. □

Proof of Theorem 1.2. This is an immediate consequence of combining Lemmas 2.2 and 2.3. □

3. Characterizing $GF(q)$-chordal matroids

In this section, we prove Theorems 1.3 and 1.4. An *induced-minor-minimal non-$GF(q)$-chordal matroid* is a $GF(q)$-representable matroid that is not a $GF(q)$-chordal matroid such that every proper induced minor of M is a $GF(q)$-chordal matroid. When $q = 2$, the only rank-2 binary matroids are $U_{2,2}$ and $U_{2,3}$ both of which are $GF(2)$-chordal. Clearly, neither $M(K_4)$ nor $U_{3,4}$ is $GF(2)$-chordal. It follows that each is an induced-minor-minimal non-$GF(2)$-chordal matroid. When $q > 2$, the matroids in $\{U_{2,3}, U_{2,4}, \dots, U_{2,q}\}$ are induced-minor-minimal non-$GF(q)$-chordal matroids.

For the remainder of this paper, it will be convenient to view a $GF(q)$-representable matroid M of rank r as a restriction of P_r by coloring the elements of $E(M)$ *green* and coloring the other elements *red*. Note that when we contract an element e from M, we can obtain M/e as follows. Take a hyperplane H of P_r that avoids e. Then project from e onto H. Because e is green, an element z of H is green in the contraction precisely when there are at least two green points on the line $\mathrm{cl}_{P_r}(\{e,z\})$ of P_r.

For a positive integer k, a partition (X,Y) of the ground set of a matroid M is a *vertical k-separation* of M if $r(X)+r(Y)-r(M) \leq k-1$ and $\min\{r(X), r(Y)\} \geq k$. This vertical k-separation is *exact* if $r(X)+r(Y)-r(M) = k-1$.

Lemma 3.1. *Let (X,Y) be a vertical 2-separation in a matroid M such that each of $M|\mathrm{cl}(X)$ and $M|\mathrm{cl}(Y)$ is in $\mathcal{M}_q$. Then either*

(i) $|\mathrm{cl}(X) \cap \mathrm{cl}(Y)| = 1$ *and* $M \in \mathcal{M}_q$*; or*

(ii) $|\mathrm{cl}(X) \cap \mathrm{cl}(Y)| = 0$ *and M has $U_{3,4}$ and $U_{2,3}$ as induced minors.*

Proof. Observe that if $|\mathrm{cl}(X) \cap \mathrm{cl}(Y)| = 1$, then M is the parallel connection of $M|\mathrm{cl}(X)$ and $M|\mathrm{cl}(Y)$, so $M \in \mathcal{M}_q$. Now suppose that $\mathrm{cl}(X) \cap \mathrm{cl}(Y) = \emptyset$. Let C and D be circuits of M each of which meets both X and Y such that $|C \cap X|$ is a minimum and $|D \cap Y|$ is a minimum. Because M can be written as a 2-sum with basepoint b of matroids with ground sets $X \cup b$ and $Y \cup b$, it follows that $(C \cap X) \cup (D \cap Y)$ is a circuit of M. Clearly both $M|\mathrm{cl}(C \cap X)$ and $M|\mathrm{cl}(D \cap Y)$ are in $\mathcal{M}_q$. Let X_1 and Y_1 be subsets of $C \cap X$ and $D \cap Y$, respectively, such that $|X_1| = |C \cap X| - 2$ and $|Y_1| = |D \cap Y| - 2$. Then each of $(M|\mathrm{cl}(C \cap X))/X_1$ and $(M|\mathrm{cl}(D \cap Y))/Y_1$ is a rank-2 matroid in $\mathcal{M}_q$. Moreover, in M/X_1, there is no element x of X that is in the closure of Y otherwise $(X_1 \cup x) \cup (D \cap Y)$ contains a circuit of M that contains $x \cup (D \cap Y)$ and violates the choice of C. Hence the rank-2 matroid $(M|\mathrm{cl}(C \cap X))/X_1$, which is in $\mathcal{M}_q$, is not isomorphic to $U_{2,q+1}$. Thus it is isomorphic to $U_{2,2}$. By symmetry, $(M|\mathrm{cl}(D \cap Y))/Y_1 \cong U_{2,2}$. Hence $(M|\mathrm{cl}((C \cap X) \cup (D \cap Y)))/(X_1 \cup Y_1) \cong U_{3,4}$. Thus both $U_{3,4}$ and $U_{2,3}$ are induced minors of M. □

In the next result, we denote by $P_{r+1} \backslash P_{r-i}$ the matroid that is obtained from P_{r+1} by deleting the elements of a rank-$(r-i)$ flat. Clearly this matroid does not depend on the choice of the rank-$(r-i)$ flat.

Lemma 3.2. *Let M be a binary rank-$(r+1)$ matroid. Then $M/e \cong P_r$ for all e in $E(M)$ if and only if $M \cong P_{r+1} \backslash P_{r-i}$ for some i in $\{0,1,\dots,r\}$.*

Proof. If $M \cong P_{r+1} \backslash P_{r-i}$ for some i in $\{0,1,\dots,r\}$, then, by, for example, [12, Corollary 6.2.6], $M/e \cong P_r$ for all e in $E(M)$.

Now suppose $E(M) \cong P_r$ for each e in $E(M)$. We may assume that $|E(P_{r+1}) - E(M)| \geq 2$ otherwise the result certainly holds. Let

x and y be distinct elements of $E(P_{r+1}) - E(M)$. Then the third element z on the line of P_{r+1} that is spanned by $\{x, y\}$ must also be in $E(P_{r+1}) - E(M)$ otherwise M/z is not isomorphic to P_r. It follows that, for a basis X of $P_{r+1} \backslash E(M)$, by [11, Theorem 1], $\mathrm{cl}_{P_{r+1}}(X) = E(P_{r+1}) - E(M)$. Since M has rank $r+1$, we deduce that $\mathrm{cl}_{P_{r+1}}(X)$ is a projective geometry of rank at most r. The lemma follows. □

We omit the straightforward proof of the next result.

Lemma 3.3. *For $r \geq 3$, if M is the binary matroid $P_r \backslash P_{r-i}$ for some i in the set $\{1, 2, \ldots, r-1\}$, then M has a flat isomorphic to either $M(K_4)$ or $U_{3,4}$.*

For all $q > 2$, the only rank-2 members of $\mathcal{M}_q$ are $U_{2,2}$ and $U_{2,q+1}$. Natural obstructions to membership of $\mathcal{M}_q$ are the lines that contain more than two but fewer than $q+1$ points, that is, the collection $\{U_{2,i} : 3 \leq i \leq q\}$. In rank three, the only matroid that is not in $\mathcal{M}_q$ and has no member of $\{U_{2,i} : 3 \leq i \leq q\}$ as an induced minor is $U_{3,q+2}$. By Bose [4], we note that $U_{3,q+2}$ is representable over $GF(q)$ if and only if q is even. Therefore, the collection $\mathcal{N} = \{U_{2,3}, U_{2,4}, \ldots, U_{2,q}, U_{3,q+2}\}$ is contained in the collection of forbidden induced minors for the class $\mathcal{M}_q$. The next result highlights some structure in matroids that have members of $\mathcal{N}$ as induced minors.

Lemma 3.4. *For some $q > 2$, let M be a $GF(q)$-representable matroid having rank at least three. Suppose that $M \not\cong P_r$ but that $M/e \cong P_{r-1}$ for all e in $E(M)$. Then M has a member of $\mathcal{N}$ as an induced minor.*

Proof. Suppose M has no member of $\mathcal{N}$ as an induced minor. Suppose $r(M) = 3$. Since the contraction of any element would result in a $(q+1)$-point line, $E(M)$ must have at least $q+2$ elements. Moreover, since M is not $U_{3,q+2}$, we deduce that M contains a triangle $\{p_1, p_2, p_3\}$. This triangle must be contained in a full $(q+1)$-point line of M, otherwise M would contain a member of $\mathcal{N}$ as an induced restriction. Label this line $\{p_1, p_2, \ldots, p_{q+1}\}$. Since $|E(M)| \geq q+2$, we may choose an element e in $E(M) - \{p_1, p_2, \ldots, p_{q+1}\}$. If e is unique, then M/p_1 is isomorphic to $U_{2,2}$, a contradiction. Without loss of generality, suppose there is a third point on the line $\mathrm{cl}(\{e, p_1\})$. Then this line is a full $(q+1)$-point line. Therefore, there are two full lines meeting at p_1. If this were the entire matroid, then M/p_1 consists of only two points. Hence there is an additional element f in M not on either of the lines $\mathrm{cl}(\{e, p_1\})$ or $\{p_1, p_2, \ldots, p_{q+1}\}$. Each point of the line $\mathrm{cl}(\{e, p_1\})$ together with f defines a line that meets the line $\{p_1, p_2, \ldots, p_{q+1}\}$ in a distinct point and so each of these lines is also full. This gives that every line is full except possibly the line $\text{cl}(\{p_1, f\})$. If any of the points on the line $\text{cl}(\{p_1, f\})$ is absent, then, for some i in $[q+1]$, the line $\text{cl}(\{e, p_i\})$ contains only q points, a contradiction. Therefore, every line of M is full and M must be a projective geometry, a contradiction. Thus the result holds when $r(M) = 3$.

Now suppose the result holds for $r(M) < k$ and let $r(M) = k \geq 4$. Let e be an element of M. Since $M/e \cong P_{r-1}$, each of the lines of P_r that contains e must contain a second point of M. There must be such a line, say L, that contains exactly two points of M otherwise every such line has exactly $q+1$ points and $M \cong P_r$, a contradiction. Let $L = \{e, f\}$. Take a point g in $E(M) - L$ and consider the plane $Q = \text{cl}_M(\{e, f, g\})$. Since $M/h \cong P_r$ for every point h of this plane, it follows that $Q/h \cong U_{2,q+1}$. It follows by the induction assumption that $Q \cong P_3$. Hence L is a $(q+1)$-point line, a contradiction. We conclude, by induction, that the lemma holds. □

Lemma 3.5. *Let M be a $GF(q)$-representable matroid and let (X, Y) be an exact vertical k-separation of M. Suppose that both $M|\text{cl}_M(X)$ and $M|\text{cl}_M(Y)$ are in $\mathcal{M}_q$. Then either $\text{cl}_M(X) \cap \text{cl}_M(Y)$ is a projective geometry of rank $k-1$, or M has a member of $\mathcal{N}$ as an induced minor.*

Proof. This is immediate if $k = 1$ and follows by Lemma 3.1 when $k = 2$, so we may assume that $k \geq 3$. Let M be an induced-minor-minimal counterexample and let $r(M) = r$. Suppose first that $\text{cl}_M(X) \cap \text{cl}_M(Y)$ is empty. Then in the green-red coloring of P_r corresponding to M, all of the elements of $\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y)$ are red. Take e in X and suppose that $r(X) > k$. Then M/e has an exact vertical k-separation (X', Y') corresponding to $(X - e, Y)$. By the minimality of M, we deduce that $\text{cl}_{M/e}(X') \cap \text{cl}_{M/e}(Y')$ is a projective geometry of rank $k-1$.

Since $k \geq 3$, there is a projective line L contained in $\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y)$. In the green-red coloring of P_r, every element of L is red. But every element of L is green in the coloring of P_r/e. Thus, in P_r, for each of the points $z_1, z_2, \ldots, z_{q+1}$ of L, there is a green point on the line $\text{cl}_{P_r}(\{e, z_i\})$ other than e. Thus, when $q = 2$, we see that the four green points in $\text{cl}_{P_r}(L \cup e)$ form a 4-circuit, a contradiction as $M|\text{cl}_M(X)$ is $GF(2)$-chordal. When $q > 2$, because all of the elements of L are red, the set of points in $\text{cl}_{P_r}(L \cup e)$ contains no line with more than q points. Since $M|\text{cl}_M(X)$ is in $\mathcal{M}_q$, it follows that each line in $\text{cl}_{P_r}(L \cup e)$ that contains at least two green points contains exactly two green points. It follows that M has $U_{3,q+2}$ as an induced restriction, a contradiction.

We may now assume that $r(X) = k = r(Y)$. Since (X, Y) is an exact k-separation, $r(X) + r(Y) - r(M) = k - 1$, so $r(M) = k + 1$. As M has at least $2k$ elements, it has an element f that is not a coloop. Then the construction of members of $\mathcal{M}_q$ implies that f is on a $(q+1)$-point green line of M. This line must meet $\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)$ so $\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)$ is not entirely red, a contradiction.

We conclude $\mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$ contains at least one point, say z. In M/z, there is an exact vertical $(k-1)$-separation (X'', Y'') corresponding to $(X - z, Y - z)$. We deduce, by the minimality of M, that $\mathrm{cl}_{P_r/z}(X'') \cap \mathrm{cl}_{P_r/z}(Y'')$ is a projective geometry of rank $k-2$. By Lemmas 3.3 and 3.4, we deduce that $M|\,(\mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y))$ must have, as an induced minor, a matroid that is not in $\mathcal{M}_q$. □

Corollary 3.6. *For all $k \geq 1$, an induced-minor-minimal non-$GF(q)$-chordal matroid has no vertical k-separations.*

A matroid M is *round* if M has no two disjoint cocircuits. Equivalently, M is round if there is no k for which M has a vertical k-separation (see, for example, [12, Lemma 8.6.2]). The following result is immediate from the constructive definition of $GF(q)$-chordal matroids.

Lemma 3.7. *A rank-r matroid M in $\mathcal{M}_q$ is round if and only if $M \cong P_r$.*

The following is a straightforward consequence of the definition.

Lemma 3.8. *A simple binary matroid is chordal if and only if it does not have $U_{3,4}$ as an induced minor.*

Lemma 3.9. *All $GF(2)$-chordal matroids are chordal matroids.*

Proof. Let M be a $GF(2)$-chordal matroid. Since the class of $GF(2)$-chordal matroids is closed under induced minors, $U_{3,4}$ is not an induced minor of M, so M is a chordal matroid. □

Since $M(K_4)$ is chordal but not $GF(2)$-chordal, it is clear that the class of binary matroids that are chordal properly contains the class of $GF(2)$-chordal matroids.

We now give a common proof of the two main results of the paper.

Proof of Theorems 1.3 and 1.4. Let M be an induced-minor-minimal non-$GF(q)$-chordal matroid. By Corollary 3.6, M is round. By Geelen, Gerards, and Whittle [9], M/e is also round for all e in $E(M)$. Thus by Lemma 3.7, $M/e \cong P_{r-1}$ where $r(M) = r$.

First, let $q = 2$. Then, by Lemma 3.2, $M \cong P_r \backslash P_{r-i}$ for some i in $\{1, 2, \ldots, r-1\}$. Moreover, since M is not a $GF(2)$-chordal matroid, $r \geq 3$. Then, by Lemma 3.3, M has $M(K_4)$ or $U_{3,4}$ as an induced restriction. As each of these matroids is an induced-minor-minimal non-$GF(2)$-chordal matroid, we deduce that M is isomorphic to $M(K_4)$ or $U_{3,4}$.

Now assume that $q > 2$. Since each member of $\mathcal{N}$ is an induced-minor-minimal non-$GF(q)$-chordal matroid, we may assume that M has no member of $\mathcal{N}$ as an induced minor. Thus $r(M) \geq 3$. By Lemma 3.4, we get a contradiction. □

4. Dirac's Other Characterization

In [8], another characterization is given of chordal graphs. In a graph G, a *vertex separator* is a set of vertices whose deletion produces a graph with more connected components than G.

Theorem 4.1. *A graph is chordal if and only if every minimal vertex separator induces a clique.*

It is shown in Lemma 3.5 that, if M is a $GF(q)$-chordal matroid, then, for every exact vertical k-separation (X,Y) of M, the restriction $M|(\text{cl}(X) \cap \text{cl}(Y)) \cong P_{k-1}$. However, the converse of this is not true. For example, the matroid $P_{U_{2,3}}(M(K_4), M(K_4))$ is not $GF(2)$-chordal, but the only exact vertical k-separation has $k = 3$ and has $U_{2,3}$ as the intersection of the closures of the two sides of the vertical 3-separation. A *divider* in a matroid is an exact vertical k-separation for some k. A divider (X,Y) is *minimal* if there is no vertical k'-separation (X',Y') such that $\text{cl}(X') \cap \text{cl}(Y') \subsetneqq \text{cl}(X) \cap \text{cl}(Y)$. Recall that, for sets X and Y in a matroid M, the *local connectivity* between X and Y, denoted $\sqcap(X,Y)$ or $\sqcap_M(X,Y)$, is given by $\sqcap(X,Y) = r(X) + r(Y) - r(X \cup Y)$. Let $\mathcal{N}_q$ be the class of $GF(q)$-representable matroids N such that, for every minimal divider (X,Y) of N, the matroid $N|(\text{cl}(X) \cap \text{cl}(Y))$ is a projective geometry of rank $\sqcap(X,Y)$. Since round matroids have no vertical k-separations, all $GF(q)$-round matroids are in $\mathcal{N}_q$.

Lemma 4.2. *Let (X,Y) be a minimal divider of a matroid N. Then* $\text{cl}(X) \cap \text{cl}(Y) = \text{cl}(X) \cap \text{cl}(Y - \text{cl}(X))$.

Proof. Suppose $y \in \text{cl}(X) \cap Y$. Then $r(Y) = r(Y - y)$; otherwise, y is a coloop of $N|Y$ and $(X \cup y, Y - y)$ is a divider with $\text{cl}(X \cup y) \cap \text{cl}(Y - y)$ properly contained in $\text{cl}(X) \cap \text{cl}(Y)$, a contradiction. We deduce that $\text{cl}(Y) = \text{cl}(Y - \text{cl}(X))$. □

The next theorem is an analog of Theorem 4.1.

Theorem 4.3. *A matroid M is in $\mathcal{N}_q$ if and only if M can be constructed from round $GF(q)$-representable matroids by a sequence of generalized parallel connections across projective geometries.*

Proof. Let $\mathcal{R}_q$ be the class of matroids that can be constructed from round $GF(q)$-representable matroids by a sequence of generalized parallel connections across projective geometries. It suffices to prove that a connected matroid $M \in \mathcal{N}_q$ if and only if it is in $\mathcal{R}_q$. Suppose $M \in \mathcal{N}_q$. If M has no dividers, then M is round, so $M \in \mathcal{R}_q$. Hence, we may assume M has a minimal divider (X, Y). Then $M|(\text{cl}(X) \cap \text{cl}(Y))$ is a projective geometry N of rank $\sqcap(X, Y)$, and $M = P_N(M|\text{cl}(X), M|\text{cl}(Y))$. Letting $M_X = M|\text{cl}(X)$, we see that M_X is either round or has a minimal divider (U, V), where $M_X|(\text{cl}(U) \cap \text{cl}(V))$ is a projective geometry, N', of rank $\sqcap_{M_X}(U, V)$, and M_X is equal to $P_{N'}(M_X|\text{cl}(U), M_X|\text{cl}(V))$. Continuing in this way, we see that every matroid in $\mathcal{N}_q$ can be obtained in the manner prescribed. Hence $\mathcal{N}_q \subseteq \mathcal{R}_q$.

We will prove that $\mathcal{R}_q \subseteq \mathcal{N}_q$ by induction on the number n of round matroids used to construct a connected member M of $\mathcal{R}_q$. If $n = 1$, then M is round and so M is in $\mathcal{N}_q$. Now suppose that the result holds when $n \leq t - 1$ and assume that M is constructed by using exactly t round matroids. Then $M \cong P_N(M_1, M_2)$, where M_2 is a round matroid and N is a projective geometry. Let (X, Y) be a minimal divider of M and let $F = \text{cl}(X) \cap \text{cl}(Y)$. We need to show that $M|F \cong P_k$ where $r(F) = k$. Let $X_N = X \cap E(N)$ and $Y_N = Y \cap E(N)$. Also let $X_i = (X \cap E(M_i)) - X_N$ and $Y_i = (Y \cap E(M_i)) - Y_N$ for each i in $\{1, 2\}$. Since N is a projective geometry, we may suppose that X_N spans Y_N. Therefore $\text{cl}(Y_N) \subseteq F$. As M_2 is round and has $(X_N \cup Y_N \cup X_2, Y_2)$ as a partition, either $X_N \cup X_2$ spans Y_2, or Y_2 spans M_2. In the latter case, Y spans $E(N)$, so $E(N) \subseteq F$. Now, $(E(M_1), E(M_2) - E(N))$ is a divider of M and $\text{cl}(E(M_1)) \cap \text{cl}(E(M_2) - E(N)) = E(N)$. Because (X, Y) is a minimal divider, we have $F = E(N)$, so $M|F \cong P_k$ where $r(F) = k$.

We deduce that $X_N \cup X_2$ spans Y_2. Then $\text{cl}(X)$ contains $E(M_2)$ and, by Lemma 4.2, we may assume that $Y \subseteq E(M_1) - E(N)$. Thus

$$F = \text{cl}(X) \cap \text{cl}(Y) = \text{cl}_{M_1}(X \cap E(M_1)) \cap \text{cl}_{M_1}(Y). \tag{4.1}$$

We show next that $(X \cap E(M_1), Y)$ is a minimal divider of M_1. It is a divider of M_1 unless $X \cap E(M_1)$ or Y spans M_1. In the exceptional case, as X does not span M, we see that $X \cap E(M_1)$ does not span M_1. Thus Y spans M_1, so $E(N) \subseteq F$. Hence $E(N) = F$, and $M|F \cong P_k$ as desired. Thus $(X \cap E(M_1), Y)$ is a divider of M_1. Suppose it is not minimal. Then, by (4.1), M_1 has a minimal divider (X_1, Y_1) such that $\text{cl}_{M_1}(X_1) \cap \text{cl}_{M_1}(Y_1) \subsetneqq F$. Now we may assume that $\text{cl}_{M_1}(X_1) \supseteq E(N)$, so $(\text{cl}_{M_1}(X_1), Y_1 - \text{cl}_{M_1}(X_1))$ is a minimal divider of M_1. It follows that $(E(M_2) \cup \text{cl}_{M_1}(X_1)) \cap \text{cl}_M(Y_1 - \text{cl}_{M_1}(X_1)) = \text{cl}_{M_1}(X_1) \cap \text{cl}_{M_1}(Y_1) \subsetneqq F$, a contradiction. Hence $(X \cap E(M_1), Y)$ is a minimal divider of M_1. By the induction assumption, $M_1|F \cong P_k$, so $M|F \cong P_k$, as desired. $\square$

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