JOURNAL OF COMBINATORIAL THEORY, Series B **66**, 334–360 (1996)
ARTICLE NO. 0026

Unavoidable Minors of Large 3-Connected Binary Matroids

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Received July 11, 1994

We show that, for every integer n greater than two, there is a number N such that every 3-connected binary matroid with at least N elements has a minor that is isomorphic to the cycle matroid of $K_{3,n}$, its dual, the cycle matroid of the wheel with n spokes, or the vector matroid of the binary matrix $(I_n \mid J_n - I_n)$, where J_n is the $n \times n$ matrix of all ones.

1. INTRODUCTION

The general theme of Ramsey theory may be stated as follows: A class $\mathscr{A}$ of objects contains a subclass $\mathscr{B}$ of "more structured" objects so that every sufficiently large element of $\mathscr{A}$ "dominates" a large element of $\mathscr{B}$. The classical result of this type is the following.

(1.1) THEOREM. *There is a function R (the Ramsey function) such that, whenever the edges of a complete graph K on at least $R(x, y)$ vertices are colored with y colors, there is an induced subgraph of K on x vertices all of whose edges have the same color.*

The following is another well-known graph result of this type.

(1.2) THEOREM. *For every integer n greater than one, there is an integer $N(n)$ such that every 2-connected simple graph with more than $N(n)$ vertices contains a subdivision of the circuit on n edges or a subdivision of $K_{2,n}$.*

This result has been generalized to matroids by Lovász, Schrijver, and Seymour (see [6]). They proved the following result, which was strengthened in [8].

(1.3) THEOREM. *Let n be an integer greater than one. If M is a connected matroid with more than 4^n elements, then M contains a circuit or cocircuit with more than n elements.*

0095-8956/96 $18.00

Two of the authors, in collaboration with Thomas, proved analogs of (1.2) for 3- and 4-connected graphs [5]. The following is the result for 3-connected graphs. Let W_n denote a wheel with n spokes. Observe that, for every integer n greater than two, each of W_n and $K_{3,n}$ is 3-connected.

(1.4) THEOREM. *For every integer n greater than two, there is an integer $N(n)$ such that every 3-connected graph with more than $N(n)$ vertices contains a minor isomorphic to W_n or $K_{3,n}$.*

The next theorem, the main result of the paper, generalizes (1.4) to binary matroids. For an integer n greater than one, let S_n denote the matroid whose binary representation is of the form $(I_n \mid J_n - I_n)$, where I_n is the rank-n identity matrix and J_n is the matrix with n rows and n columns, all of whose entries equal 1. It is easy to show that S_n is 3-connected for every integer n greater than two.

(1.5) THEOREM. *For every integer n greater than two, there is an integer $N(n)$ such that every 3-connected binary matroid with more than $N(n)$ elements contains a minor isomorphic to one of $M(K_{3,n})$, $M^*(K_{3,n})$, $M(W_n)$, and S_n.*

Both Theorems 1.4 and 1.5 are existence results and we believe that the bounds on N that are obtained in their proofs are far from best-possible.

It is not surprising that (1.4) and (1.5) feature wheels, which arise frequently as minors of 3-connected binary matroids. Indeed, it is impossible to have a 3-connected binary matroid on more than three elements without a minor isomorphic to $M(W_3)$. One of the authors [7] investigated the class of 3-connected binary matroids with no minor isomorphic to $M(W_4)$. He proved that even though the members of this class may be arbitrarily large, every such member is a minor of some S_n. An even more striking property of wheels is stated in the following consequence of (1.4).

(1.6) COROLLARY. *Let n be an integer greater than two, and let $\mathscr{G}$ be a class of 3-connected simple planar graphs none of which has a W_n-minor. Then $\mathscr{G}$ contains only finitely many pairwise nonisomorphic elements.*

Similarly, Theorem 1.5 can be reformulated as follows.

(1.7) COROLLARY. *Let n be an integer greater than two, and let $\mathscr{M}$ be a class of 3-connected binary matroids none of which has a minor isomorphic to $M(K_{3,n})$, $M^*(K_{3,n})$, $M(W_n)$, or S_n. Then $\mathscr{M}$ has only finitely many pairwise nonisomorphic elements.*

The main tools in proving (1.5) are Ramsey-type results on matrices which we state and prove in Section 2. We believe that these results are interesting in their own right. Section 3 contains a result showing that it is sufficient to prove (1.5) for matroids that have a spanning circuit. The result of Section 4 relates 3-connectivity of matroids representable over finite fields to a property of matrix representations of such matroids. Section 5 contains the proof of (1.5). We note that in proving (1.5) we do not invoke (1.4), and hence the proof presented here may be viewed as an alternative proof of (1.4).

For all $n \geqslant 4$, the matroid S_n has the Fano matroid as a minor. This observation, together with (1.5), clearly implies the following.

(1.8) COROLLARY. *For every integer n greater than two, there is an integer N such that every* 3*-connected regular matroid with more than N elements contains a minor isomorphic to one of* $M(K_{3,n})$, $M^*(K_{3,n})$, *and* $M(W_n)$.

The last corollary can also be derived independently of (1.5) by using (1.4) and Seymour's decomposition of regular matroids [9]. However, the details of this argument seem to require almost as much effort as we needed to prove the more general result (1.5).

In the remainder of this section, we establish some terminology and notation. We shall assume familiarity with basic matroid theory and follow [6] for notation.

A matrix having all of its entries in a set F will be called an *F-matrix*. If F is a set containing zero, $A = (a_{i,j})$ is an F-matrix, and l is a column of A, then $s_A(l)$, or simply $s(l)$, will denote the set of rows k of A for which $a_{k,l} \neq 0$. The rank function of a matroid M will be denoted r_M, or simply r if the matroid M can be inferred from the context. A matroid M is *hamiltonian* if M has a circuit with $r(M)+1$ elements. Suppose M is a hamiltonian matroid of rank at least one and corank at least two that is representable over a finite field F. It is well known that M can be represented by an F-matrix of the form:

$$\begin{pmatrix} & 1 & \\ & 1 & \\ I_r & \vdots & A \\ & 1 & \end{pmatrix}.$$

Such a matrix will be called a *normal F-representation* of M. Observe that if M is the cycle matroid of a graph G, then the columns outside A correspond to a hamiltonian cycle C of G, and the columns of A correspond to the chords of C in G. For this reason, the matrix A in a normal representation of any hamiltonian matroid will be called a *chordal matrix*. The set $\{1, 2, ..., n\}$ will be written as $[n]$, and $[0]$ will denote the empty set.

2. Ramsey-Type Theorems for Matrices

Throughout this section, F will be a set containing 0 together with exactly $q-1$ other elements for some integer $q \geqslant 2$. We remark that to prove the main result of this paper, we need only consider $F = \{0, 1\}$. However, allowing F to have more than two elements does not substantially increase the difficulty of the proofs of the results of this section, and hence we present these results in full generality.

Suppose A is an F-matrix. A matrix B obtained from A by deleting rows and columns is a *submatrix* of A. A matrix that is obtained from a submatrix of A by permuting its rows is a *row-permuted submatrix* of A. In this section, we prove several results that identify unavoidable row-permuted submatrices of large F-matrices whose columns are "sufficiently diverse." The major results of this section will be used in Section 5.

We begin with a definition and two lemmas. Let n and p be nonnegative integers not both zero. An F-matrix $A = (a_{i,j})$ is *$[n, p]$-semidiagonal* if A has exactly $n+p$ columns and at least n rows, and, for every row $i \in [n] - \{n+p\}$, we have $a_{i,i} \neq a_{i,i+1}$ and $a_{i,i+1} = a_{i,j}$ for all $j \in [n+p] - [i]$.

(2.1) Lemma. *For an integer n greater than one, let $g_1(n, q) = 3(q+1)^n$. Let C be an F-matrix with at least $g_1(n, q)$ columns no two of which are identical. Then there is an $[n, 0]$-semidiagonal matrix D obtained from C by deleting columns and permuting rows.*

Proof. Without loss of generality, we may assume that C has exactly $g_1(n, q)$ columns. We shall inductively construct a sequence of matrices $C = C_0, C_1, ..., C_n = D$ where, for each $m \in \{0\} \cup [n]$, the matrix C_m is $[m, g_1(n-m, q)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows.

Trivially, C_0, which equals C, is $[0, g_1(n, q)]$-semidiagonal. Now suppose that m is an integer in $[n]$ and that $C_{m-1} = (c_{i,j})$ is $[m-1, g_1(n-m+1, q)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows. Since $g_1(x, q) \geqslant 3$ for all nonnegative integers x, the matrix C_{m-1} has at least $m+2$ columns. As the columns m and $m+1$ of C_{m-1} are not identical, yet they agree in the first $m-1$ rows, there is a row i of C_{m-1} such that $i \geqslant m$ and $c_{i,m} \neq c_{i,m+1}$. Let $J = [m-1+g_1(n-m+1, q)] - [m+1]$. Consider the entries of the form $c_{i,t}$ for $t \in J$. The cardinality of J is at least

$$\begin{aligned} m-1+g_1(n-m+1, q)-(m+1) &= g_1(n-m+1, q)-2 \\ &= 3(q+1)^{n-m+1}-2 \\ &= 3(q+1)^{n-m}(q+1)-2 \\ &> g_1(n-m, q)\, q. \end{aligned}$$

Thus, by the pigeon-hole principle, there is a subset J' of J and an element α of F such that $|J'| = g_1(n-m, q)$ and $c_{i,t} = \alpha$ for all $t \in J'$. Moreover, since $c_{i,m} \neq c_{i,m+1}$, there is an $m' \in \{m, m+1\}$ such that $c_{i,m'} \neq \alpha$. Construct C_m from C_{m-1} by deleting all columns except those in $[m-1] \cup \{m'\} \cup J'$ and swapping rows i and m. Clearly, C_m is $[m, g_1(n-m, q)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows. By induction, the lemma follows. ∎

Suppose α, β, and γ are elements of F that are not all equal. A square F-matrix $A = (a_{i,j})$ is (α, β, γ)-*diagonal* if

$$a_{i,j} = \begin{cases} \alpha, & \text{if} \quad i < j; \\ \beta, & \text{if} \quad i = j; \\ \gamma, & \text{if} \quad i > j. \end{cases}$$

Suppose a matrix B has m rows and n columns. A submatrix C of B is *principal* if there is a subset I of $[m] \cap [n]$ such that C is obtained from B by deleting all rows except those in I, and deleting all columns except those in I.

(2.2) LEMMA. *For an integer n greater than one, let $g_2(n, q) = R(n, q)\, q^2$, where R is the Ramsey function. Suppose $D = (d_{i,j})$ is a $[g_2(n, q), p]$-semidiagonal matrix for some nonnegative integer p. Then D has a principal submatrix E that has n columns and is (α, β, γ)-diagonal for some $\alpha \neq \beta$.*

Proof. Let $m = g_2(n, q)$ and consider $d_{1,1}, d_{2,2}, ..., d_{m,m}$. Since $m > (R(n, q)\, q - 1)\, q$, by the pigeon-hole principle, there is a subset J of $[m]$ that has $R(n, q)\, q$ elements and such that all $d_{i,i}$ for $i \in J$ are identical. Construct a principal submatrix $D' = (d'_{i,j})$ of D by deleting all columns except those in J, and deleting all rows except those in J. Observe that all elements on the main diagonal of D' are identical.

Now consider all entries of D' of the form $d'_{i,i+1}$. The number of entries of this form is $R(n, q)\, q - 1 > (R(n, q) - 1)(q - 1)$. As each of them differs from the entries on the main diagonal of D', they may only take on values from $F - \{d'_{1,1}\}$. By the pigeon-hole principle, there is a subset J' of $[R(n, q)\, q - 1]$ that has $R(n, q)$ elements and for which all entries $d'_{i,j}$, for $i \in J'$ and $j > i$, are identical. Construct a matrix $D'' = (d''_{i,j})$ from D' by deleting all columns except those in J' and by deleting all rows except those in J'. Observe that D'' is a principal submatrix of D' and, hence, also of D.

Now consider a q-coloring of the edges of a complete graph K on the vertex set $[R(n, q)]$ in which the edge (i, j), where $i > j$, is colored by $d''_{i,j}$. From the definition of $R(n, q)$, there is a subset J'' of $[R(n, q)]$ such that the subgraph of K induced by J'' has all its edges colored identically.

Construct E from D'' by deleting all columns except those in J'' and by deleting all rows except those in J''. It is straightforward to verify that E satisfies the conclusion of (2.2). ∎

The following result describes the unavoidable row-permuted submatrices of large F-matrices no two columns of which are identical. It is the first of the main results of this section and it is a slight strengthening of (3.2) from [2].

(2.3) THEOREM. *There is a function g_3 with the following property: If n is an integer greater than one and A is an F-matrix with at least $g_3(n, q)$ columns no two of which are identical, then A contains a row-permuted submatrix B that has n columns and is (α, β, γ)-diagonal for some $\alpha \neq \beta$.*

Proof. Let g_1 denote the function from (2.1), let g_2 denote the function from (2.2), and let $g_3(n, q) = g_1(g_2(n, q), q)$. The conclusion follows immediately from (2.1) and (2.2). ∎

We remark that the theorems of this section can be stated in the language of bipartite graphs instead of the language of matrices. We shall give an example of this by, essentially, restating the special case of (2.3) when $F = \{0, 1\}$ as a theorem on unavoidable induced subgraphs of large bipartite graphs. The general version of (2.3) can be restated in terms of unavoidable monochromatic induced subgraphs of edge-colored complete bipartite graphs. We leave it to the reader to translate the theorems of this section, including the general version of (2.3), into the language of bipartite graphs.

(2.4) COROLLARY. *For every positive integer n, there is an integer N with the following property: Suppose G is a bipartite graph with a bipartition U and V such that $|U| = m \geqslant N$ and the neighbor sets of the vertices in U are all distinct. Then U and V have n-element subsets U' and V', respectively, such that the subgraph G' of G induced by $U' \cup V'$ satisfies one of the following:*

(i) *G' is a matching;*

(ii) *G' is the bipartite complement of a matching; or*

(iii) *the vertices of U' and V' can be labelled $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ so that the neighbor set of u_i is $\{v_j : j \in [i]\}$ for all $i \in [n]$.*

We return to matrices. The next is an easy lemma.

(2.5) LEMMA. *Let n be a positive integer, let α, β, γ be elements of F such that $\alpha \neq \gamma$, and suppose that C is an (α, β, γ)-diagonal matrix with $2n$*

columns. Then C contains submatrices C' and C'', each with n columns, such that C' is (α, γ, γ)-diagonal, and C'' is (α, α, γ)-diagonal.

Proof. Construct C' from C by deleting all odd-numbered rows and all even-numbered columns. The construction of C'' is very similar. ∎

By imposing a stronger condition than merely that the columns be distinct, we are able to sharpen the conclusion of Theorem 2.3. A column j of an F-matrix $A = (a_{i,j})$ *dominates* a column k if the columns j and k are identical or if there is an element α in $F - \{0\}$ such that $a_{i,j} = \alpha$ whenever $a_{i,k} \neq 0$.

(2.6) THEOREM. *There is a function g_4 with the following property: If n is a positive integer and A is an F-matrix with at least $g_4(n, q)$ columns such that no column of A dominates another, then A contains a row-permuted submatrix B that has n columns and satisfies one of the following conditions:*

(i) *B has n rows and is an (α, β, γ)-diagonal matrix where $\alpha \neq \beta$, and $\alpha = 0$ if and only if $\gamma = 0$.*

(ii) *B has $n+1$ rows the first n of which form an $(\alpha, \alpha, 0)$-diagonal matrix and the last of which has all its entries equal to β for some $\beta \in F - \{0, \alpha\}$.*

(iii) *B has $n+1$ rows the first n of which form a $(0, \alpha, \alpha)$-diagonal matrix and the last of which has all its entries equal to β for some $\beta \in F - \{0, \alpha\}$.*

(iv) *B has $2n$ rows the first n of which form a $(0, \alpha, \alpha)$-diagonal matrix and the last n of which form an $(\alpha, \alpha, 0)$-diagonal matrix.*

Proof. We begin by proving the following.

(1) *Let m be an integer greater than n, let $p = g_3(m, q)$, and let $h(m, q) = R(p, 2) + 1$, where g_3 is the function from (2.3) and R is the Ramsey function. Let $E = (e_{i,j})$ be an F-matrix with $h(m, q)$ columns none of which dominates another. Suppose that the first $h(m, q)$ rows of E form a $(0, \alpha, \alpha)$-diagonal matrix for some $\alpha \in F - \{0\}$, and that all row-permuted submatrices B of E fail all of* (i)–(iii). *Then there is a row i_E of E, and a subset J_E of $[h(m, q)] - \{1\}$ with m elements such that $d_{i_E, 1} \neq \alpha$ and $d_{i_E, j} = \alpha$ for all $j \in J_E$.*

Let I denote the set of rows i of E for which $e_{i,1} \neq \alpha$. Since column 1 of E does not dominate column 2, the set I is nonempty. Let E' denote the matrix obtained from E by deleting all rows except those in I. Consider the complete graph K on the vertex set $[h(m, q)] - \{1\}$. Color the edge (j', j'') with color 0 if the columns j' and j'' of E' differ, and with color 1 otherwise. Then K contains an induced subgraph K' on $g_3(m, q)$ vertices all of whose edges are colored the same.

Suppose first that all of the edges of K' are colored 0. Upon applying (2.3) to the submatrix of E' obtained by deleting those columns not in K', we conclude that E' contains a row-permuted submatrix E'' that has m columns and is (β, γ, δ)-diagonal with $\beta \neq \gamma$. Since $m > n$, if $\beta \notin \{0, \alpha\}$ or $\delta \notin \{0, \alpha\}$, then the matrix obtained from E by deleting all columns except those corresponding to the columns of E'' has a row-permuted submatrix B that satisfies (iii); a contradiction. If $\beta = \delta$, then E'' has a row-permuted submatrix B that satisfies (i); again, a contradiction. Hence $\{\beta, \delta\} = \{0, \alpha\}$. Now let i_E be the row of E that corresponds to the first or the last row of E'' depending on whether $\beta = \alpha$ or $\delta = \alpha$, respectively, and let J_E be the set of columns of E that correspond to the columns of E''.

Now we may assume that all of the edges of K' are colored 1. Let E'' be the matrix obtained from E' by deleting the columns not in K'. Since the column 1 of E dominates no other column of E, it follows that no column of E'' has all entries equal to zero. Since all columns of E'' are identical, E'' has a row consisting of $g_3(m, q)$ identical nonzero entries equal to γ. Since $n < m \leqslant g_3(m, q)$, and since (iii) fails for all row-permuted submatrices B of E, it follows that $\gamma = \alpha$. This proves (1).

Let g_2 be the function from (2.2), and let $g = g_2(2n, q)$. We define the sequence of integers $k_g, k_{g-1}, ..., k_0$ by letting $k_g = n + 1$ and, inductively, $k_{l-1} = h(k_l, q)$ for $l \in [g]$. Finally, let $g_4(n, q) = g_3(2k_0, q)$, where g_3 is the function from (2.3). Suppose that all row-permuted submatrices B of A fail all of (i)–(iii). Clearly, no two columns of A are identical, and hence, by (2.3), A contains a row-permuted submatrix C with $2k_0$ columns that is (γ, β, α)-diagonal with $\gamma \neq \beta$. Since (i) fails with $B = C$, exactly one of α and γ is zero. We shall assume that $\gamma = 0$; the proof in the other case is very similar. Now apply (2.5) to C to obtain a matrix C' that has k_0 columns and is $(0, \alpha, \alpha)$-diagonal. Let D be a matrix obtained from A by deleting columns and permuting rows so that the first k_0 rows of D form C'.

We shall define a sequence of matrices $D_0, D_1, ..., D_g$ such that D_l, for $l \in [g]$, has the form

$$\begin{pmatrix} D'_l \\ D''_l \\ D'''_l \end{pmatrix}$$

where D'_l is $(0, \alpha, \alpha)$-diagonal and $D''_l = (d''_{i,j})$ is $[l, k_l]$-semidiagonal with $d''_{i,j} = \alpha$ whenever $i < j$. Let $D_0 = D$. Suppose l is a nonnegative integer smaller than g and suppose D_l has been defined so that the submatrix E_l obtained from D_l by deleting the first l rows and deleting the first l columns satisfies the hypotheses of (1) with $m=k_{l+1}$. Let i_{E_l} and a subset J_{E_l} of $[k_l]$ be as in the conclusion of (1). Note that $|J_{E_l}|=k_{l+1}$. Let $K_{E_l}=[l+k_l]-([l]\cup\{l+j\colon j\in\{1\}\cup J_{E_l}\})$. Now construct the matrix D_{l+1} from D_l by deleting all columns in K_{E_l}, deleting all rows in K_{E_l}, and swapping two rows so that the row of D_{l+1} corresponding to the row i_{E_l} of E_l is at position $k_{l+1}+l+1$ in D_{l+1}. It is clear that if $l+1<g$, then the matrix E_{l+1} obtained from D_{l+1} by deleting the first $l+1$ rows and the first $l+1$ columns satisfies the hypotheses of (1) with $m=k_{l+2}$.

Now consider the matrix E obtained from D_g by removing the first g rows. It is clear that E is $[g_2(2n, q), n+1]$-semidiagonal. Therefore, by (2.2), it has a principal submatrix E' that has $2n$ rows and is $(\beta, \delta, \varepsilon)$-diagonal for some $\beta \neq \delta$. It is clear from the construction that $\beta=\alpha$. Since (i) fails with $B=E'$, we conclude that $\varepsilon=0$. Upon applying (2.5) to E' , we obtain a matrix E'' that has n columns and is $(\alpha, \alpha, 0)$-diagonal. Let I and J be, respectively, the sets of rows and columns of E deleted in the process of obtaining E''. Construct B from D_g by deleting all rows in I, and deleting all columns in J. It is clear that, after permuting the rows of B if necessary, B satisfies (iv). The result follows. ▌

In the next result, we study a binary relation on the columns of an F-matrix that is even stronger than nondomination. Two columns j and k of an F-matrix $A=(a_{i,j})$ *cross* if neither of these columns dominates the other, and there is a row i of A for which both $a_{i,j}$ and $a_{i,k}$ are nonzero.

Suppose α and β are elements of $F-\{0\}$. An F-matrix $A=(a_{i,j})$ is (α, β)-*complete* if the number of rows of A is $\binom{n}{2}$, where n is the number of columns of A, and, for every two distinct columns j' and j'' of A, there is exactly one row i of A such that $a_{i,\min\{j',j''\}}$ and $a_{i,\max\{j',j''\}}$ are α and β, respectively, and $a_{i,j}=0$ for all $j\notin\{j',j''\}$.

(2.7) THEOREM. *There is a function g_5 with the following property: If n is an integer greater than one and A is an F-matrix with at least $g_5(n, q)$ columns such that every two columns of A cross, then A contains a row-permuted submatrix B that has n columns and satisfies one of the following conditions:*

(i) *B has n rows and is (α, β, γ)-diagonal with $\alpha\neq\beta$, $\alpha\neq 0$, and $\gamma\neq 0$.*

(ii) *B has $n+1$ rows the first n of which form an $(\alpha, \alpha, 0)$-diagonal matrix and the last of which has all its entries equal to β for some $\beta\in F-\{0, \alpha\}$.*

(iii) *B has $n+1$ rows the first n of which form a $(0, \alpha, \alpha)$-diagonal matrix and the last of which has all its entries equal to β for some $\beta\in F-\{0, \alpha\}$.*

(iv) *B has $2n$ rows the first n of which form a $(0, \alpha, \alpha)$-diagonal matrix and the last n of which form an $(\alpha, \alpha, 0)$-diagonal matrix.*

(v) *B has $n+1$ rows the first n which form a $(0, \alpha, 0)$-diagonal matrix and the last of which has all its entries equal to some nonzero β.*

(vi) *B is (α, β)-complete for some nonzero elements α and β of F.*

Note that condition (i) above is a strengthening of condition (i) of (2.6), and conditions (ii)–(iv) are exactly the same as in (2.6).

Proof. Let $m = g_4(n, q)$ and $\lambda_0 = R(n, (q-1)^2)$, where g_4 is the function from (2.6) and R is the Ramsey function. Let $\tau_0 = 11\lambda_0^2(\lambda_0+4)(\lambda_0+1)^2$. Let $g_5(n, q) = (q-1)(m-1)\,\tau_0$ and suppose $A = (a_{i,j})$ has $g_5(n, q)$ columns such that every two columns cross and every row-permuted submatrix B of A fails all of (i)–(iv). Let i_0 denote the row of A with the maximal number of nonzero elements, and let l denote the number of nonzero elements in this row. We shall divide the proof into two cases, depending on the value of l.

Suppose first that $l > (q-1)(m-1)$. By the pigeon-hole principle, there is an m-element set J of columns of A such that all $a_{i_0,j}$, for $j \in J$, equal the same element, say β, of $F - \{0\}$. Let A' be the matrix obtained from A by deleting all columns except those in J. Since the columns of A' pairwise cross, A' has a row-permuted submatrix C that satisfies one of conditions (i)–(iv) of (2.6). Since conditions (i)–(iv) of (2.7) fail with $B = C$, it follows from the note immediately preceding the proof that C is $(0, \alpha, 0)$-diagonal for some nonzero α. Note that all entries of row i_0 of A' equal β, and so this row was not used in constructing C. Let I denote the set of columns of A that were used in C. We can now obtain a matrix B that satisfies (v) by deleting all columns of A except those in I, deleting all rows except those in $I \cup \{i_0\}$, and finally permuting rows of the resulting matrix.

We may now assume that $l \leqslant (q-1)(m-1)$. Let τ denote the minimum k such that there is a k-element subset of the rows of A that meets $s(j)$ for all columns j of A. Since each row of A has at most l nonzero entries, it follows that $\tau \geqslant \tau_0$. Let λ be the maximum $k \geqslant 2$ for which there is a set J of k columns of A such that, for every two distinct elements j', j'' of J, there is a row $i(j', j'')$ for which $\{j \in J\colon a_{i(j',j''),j} \neq 0\} = \{j', j''\}$. Since every two columns of A cross, it follows from [3, (1.1)] that $\tau \leqslant 11\lambda^2(\lambda+4)(\lambda+1)^2$. Hence $\lambda \geqslant \lambda_0$ and A has a submatrix $C = (c_{i,j})$ with λ_0 columns such that, for every two columns j' and j'' of C, there is exactly one row $i(j', j'')$ in C for which $c_{i(j',j''),j'}$ and $c_{i(j',j''),j''}$ are the only nonzero entries in row $i(j', j'')$. Consider the complete graph K whose vertex set is the set of columns of C. In K, color each edge $\{j', j''\}$, where $j' < j''$, by the ordered pair $(c_{i(j',j''),j'}, c_{i(j',j''),j''})$. Since there are $(q-1)^2$ possible colors and $\lambda_0 = R(n, (q-1)^2)$, it follows that K contains an induced subgraph L with

n vertices all of whose edges have the same color. Construct a matrix B from C by deleting all columns and rows except those corresponding to the vertices and edges of L. Clearly, B satisfies (vi). ∎

Suppose n is an integer greater than two. We define three classes of F-matrices $\mathscr{B}'_n$, $\mathscr{B}''_n$, and $\mathscr{B}'''_n$ as follows. Typical matrices in these classes are illustrated in Fig. 1.

(1) Every matrix in $\mathscr{B}'_n$ can be obtained from an $(\alpha, \alpha, 0)$-diagonal matrix with n columns by replacing its first column by a column of the form $(\beta, \delta, \delta, ..., \delta)^{\mathrm{T}}$ for some $\beta \neq \delta$, and then adjoining, to the bottom of the matrix, a new row of the form $(\gamma, 0, 0, ..., 0)$ for some $\gamma \neq 0$.

(2) Every matrix in $\mathscr{B}''_n$ can be obtained from a $(0, \alpha, \alpha)$-diagonal matrix with n columns by deleting its last column, adjoining to the beginning of the matrix a new column of the form $(\delta, \delta, ..., \delta, \beta)^{\mathrm{T}}$, for some $\beta \neq \delta$, and then adjoining, to the bottom of the matrix, a new row of the form $(\gamma, 0, 0, ..., 0)$ for some $\gamma \neq 0$.

(3) Every matrix in $\mathscr{B}'''_n$ can be obtained by putting a $(0, \alpha, 0)$-diagonal matrix with n columns above a $(0, \beta, 0)$-diagonal matrix with n columns and then adjoining, to the beginning of the matrix, a new column in which the first n entries all equal some non-zero δ and the last n entries all equal some $\gamma \neq \delta$.

Observe that if $B \in \mathscr{B}'_n \cup \mathscr{B}''_n \cup \mathscr{B}'''_n$, then

(4) *two distinct columns j' and j'' of B cross if and only if* $1 \in \{j', j''\}$.

The last result of this section states that every sufficiently large matrix B satisfying (4) has a row-permuted submatrix in $\mathscr{B}'_n \cup \mathscr{B}''_n \cup \mathscr{B}'''_n$. More precisely, we have the following.

$$\begin{pmatrix} \beta & \alpha & \alpha & \cdots & \alpha \\ \delta & \alpha & \alpha & \cdots & \alpha \\ \delta & 0 & \alpha & \cdots & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta & 0 & 0 & \cdots & \alpha \\ \gamma & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} \delta & \alpha & 0 & \cdots & 0 \\ \delta & \alpha & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta & \alpha & \alpha & \cdots & \alpha \\ \beta & \alpha & \alpha & \cdots & \alpha \\ \gamma & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \begin{pmatrix} \delta & \alpha & 0 & \cdots & 0 \\ \delta & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ \delta & 0 & 0 & \cdots & \alpha \\ \gamma & \beta & 0 & \cdots & 0 \\ \gamma & 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & 0 & \cdots & \beta \end{pmatrix}$$

$\alpha \neq 0$; $\beta \neq \delta$; and $\gamma \neq 0$ — (1)

$\alpha \neq 0$; $\beta \neq \delta$; and $\gamma \neq 0$ — (2)

$0 \notin \{\alpha, \beta, \delta\}$ and $\delta \neq \gamma$ — (3)

FIGURE 1

(2.8) THEOREM. *There is a function g_6 with the following property: Suppose n is an integer greater than two and A is an F-matrix with at least $g_6(n, q)$ columns such that no two columns of A are identical, and two distinct columns j' and j'' cross if and only if $1 \in \{j', j''\}$. Then A has a row-permuted submatrix B that is in $\mathscr{B}'_n \cup \mathscr{B}''_n \cup \mathscr{B}'''_n$.*

Proof. Let g_3 be the function from (2.3). The following numbers will appear in the proof:

$$n_1 = g_3(nq, q)$$

$$n_2 = g_3(n_1 q, q)$$

$$g_6(n, q) = 4n^2 q^5 n_2.$$

Let A' denote the matrix obtained from A by removing the first column. Two columns j' and j'' of A' are *similar* if j' dominates j'', and j'' dominates j'. Clearly, similarity is an equivalence relation on the set of columns of A'. If two of the columns j' and j'' of A' are similar, then $s(j') = s(j'')$, all nonzero entries of j' are identical, and all nonzero entries of j'' are identical. Hence, it follows that each equivalence class of columns contains at most $q-1$ elements. Since the number of columns of A' is at least $4n^2q^5n_2$, which is greater than $(4n^2q^4n_2 - 1)(q-1)$, it follows that there are at least $4n^2q^4n_2$ equivalence classes of columns of A'. Let A'' be the matrix obtained from A' by deleting all but one column from each of the first $4n^2q^4n_2$ equivalence classes and by deleting all columns of all the remaining equivalence classes. Then the relation of domination on the columns of A'' is a partial order $\mathscr{D}$.

Suppose first that $\mathscr{D}$ has an antichain J with n_2 elements. Let $C = (c_{i,j})$ be the matrix obtained from A'' by deleting all columns except those in J. Observe that no column of C dominates another, and no two columns of C cross. Hence we conclude that

(1) *if j' and j'' are columns of C with $j' \neq j''$, then $s(j') \cap s(j'') = \varnothing$.*

Let C' be a submatrix of C obtained by deleting the rows except those in $s_A(1)$. Note that if a column of C' consists of all zeros, then the corresponding column of A fails to cross the first column of A; a contradiction. This fact and (1) imply that no two columns of C' are identical. Hence, upon applying (2.3) to C', we conclude that C' contains a row-permuted submatrix C'' that has $n_1 q$ columns and is $(\alpha_1, \alpha, \alpha_2)$-diagonal, where $\alpha_1 \neq \alpha$. But, as $n_1 q \geqslant n > 2$, it follows from (1) that $\alpha_1 = \alpha_2 = 0$. Hence C'' is $(0, \alpha, 0)$-diagonal.

Now consider the set K of rows of A that correspond to the rows of C''. Since $K \subseteq s_A(1)$ and $|K| = n_1 q > (n_1 - 1)(q-1)$, it follows that there is an n_1-element subset K' of K, and a δ in $F - \{0\}$ such that $a_{i,1} = \delta$ for all

$i \in K'$. Let C''' be the matrix obtained from C'' by deleting all columns that are not in K' and deleting all rows that are not in K'. It is clear that C''' is $(0, \alpha, 0)$-diagonal and, for every row of C''', the corresponding entry in the first column of A is δ.

Now let I be the set of rows i of A for which $a_{i,1} \neq \delta$. Let D be a matrix obtained from C by deleting all rows that are not in I and by deleting all columns except those corresponding to the columns of C'''. First, observe that if some column j of D contains only zeros, then the corresponding column of C is dominated by the first column of A; a contradiction. Using this fact and (1), we conclude that no two columns of D are identical. We apply (2.3) again and argue as before to conclude that D has a row-permuted submatrix D' that has nq columns and is $(0, \beta, 0)$-diagonal. Let L be the set of rows of A that correspond to the rows of D'. Since $L \subseteq F - \{\delta\}$ and $|L| = nq > (n-1)(q-1)$, it follows that there is a subset L' of L with n elements, and a γ in $F - \{0\}$ such that $a_{i,1} = \gamma$ for all $i \in L'$.

Construct the matrix B' from C by deleting all columns except those corresponding to the elements of L', and then adjoining the first column of A as the first column of B'. Now delete from B' all rows except those in $K' \cup L'$ to obtain a matrix B. It is clear that, after permuting the rows of B if necessary, we obtain a matrix in $\mathscr{B}_n'''$.

We may now assume that every antichain of $\mathscr{D}$ has fewer than n_2 elements. Then, by Dilworth's chain decomposition theorem, the smallest number of chains whose union is $\mathscr{D}$ is also smaller than n_2. As the number of elements of $\mathscr{D}$ is $4n^2q^4n_2$, and thus is greater than $4n^2q^4(n_2 - 1)$, the partial order $\mathscr{D}$ contains a chain J' of at least $4n^2q^4$ elements. It is a simple exercise to show that J' contains a subset J'' with $2nq^2$ elements for which one of the following holds:

(2) For all elements j' and j'' of J'' with $j' < j''$, column j'' dominates column j'.

(3) For all elements j' and j'' of J'' with $j' < j''$, column j' dominates column j''.

We shall assume that (2) holds; the proof when (3) holds is very similar. Let C be the matrix obtained from A'' by deleting all columns except those in J''. Since every column of C dominates all the preceding columns, but no two columns are similar, the sets $s(j)$, for $j \in [2nq^2]$, form an ascending sequence. It follows that, by permuting rows of C if necessary, we may assume that every column j of C satisfies $s(j) = [|s(j)|]$; that is, all the nonzero entries occupy the top portion of each column. Since, for all $j \in [2nq^2] - \{1\}$, column j of C dominates column $j-1$, there is an $\alpha_j \in F - \{0\}$ such that $c_{i,j} = \alpha_j$ for all $i \in s(j-1)$. Observe that the set $[2nq^2 - 1] - \{1\}$ has $2nq^2 - 2$ elements, and that $2nq^2 - 2 > (2nq - 1)(q - 1)$. Hence, by the pigeon-hole principle, there is a subset K of $[2nq^2 - 1] - \{1\}$ that has

nq members no two of which are consecutive and is such that, for some $\alpha \in F - \{0\}$, we have $\alpha_j = \alpha$ for all $j \in K$. Let $D = (d_{i,j})$ be the matrix obtained from C by deleting all columns except those in K, and then adjoining the first column of A as the first column of D.

Since no two consecutive columns of C appear in D, for each $j \in [nq+1] - \{1, 2\}$, there is a row i_j for which $d_{i_j, j-1} = 0$ and $d_{i_j, j} = \alpha$. Note that the number of elements of $[nq+1] - \{1, 2\}$ is $nq - 1$, which is greater than $(n-2)\,q$. Thus, by the pigeon-hole principle, there is a subset K' of $\{i_j\colon j \in [nq+1] - \{1, 2\}\}$ with $n-1$ elements and a $\delta \in F$ such that $d_{i,1} = \delta$ for all $i \in K'$. Observe that, as the first column of D crosses the first column of C, there are rows i' and i'' of D such that $\{i', i''\} \subseteq s_C(1)$ and $d_{i',1} \neq d_{i'',1}$. Let i_1 be an element from $\{i', i''\}$ for which $d_{i_1,1} \neq \delta$. Since the second column of D dominates, but differs from, the first column of C, it follows that $d_{i_1,2} = \alpha$. Also, note that the last column of C was not used in D, but it crosses the first column of D and dominates all other columns of D. Hence there is a row i''' of D such that $d_{i''',j} = 0$ if and only if $j \neq 1$. Now construct the matrix B from D by deleting all the rows except those in $K' \cup \{i_1, i'''\}$. It is clear that, after interchanging the first two rows of B if necessary, $B \in \mathscr{B}'_n$. ∎

3. A Result for 3-Connected Matroids

Recall Theorem 1.3. It implies that every large 3-connected matroid M contains a large circuit or a large cocircuit. In the proof of (1.5) in Section 5, we shall use duality to allow us to assume that M has a large circuit. In this section, we prove a result implying that M has a large 3-connected minor M_1 that is hamiltonian. We begin with some terminology.

For a matroid M, the simple matroid associated with M will be denoted by $\tilde{M}$, and the cosimple matroid associated with M will be denoted by $\underset{\sim}{M}$. If $\{e, f\}$ is a cocircuit of M, then e and f are said to be *in series* in M. A *series class* of M is a maximal subset A of $E(M)$ such that A contains no coloops of M and if x and y are in A, then x and y are in series. Let N be a matroid such that $M/T = N$, where every element of T is in series with an element of M not in T. Then M is a *series extension* of N, and N is a *series contraction* of M. The matroid M' is a *series minor* of M if M' can be obtained from M by a sequence of deletions and series contractions.

The following is the main result of this section.

(3.1) Theorem. *Let M be a 3-connected matroid and N be a coloop-free series minor of M. Then M has a 3-connected minor M_1 that has a restriction N_1 such that $r(N_1) = r(M_1)$ and N_1 is isomorphic to a series extension of N.*

The proof of this will use the following result of Lemos [4].

(3.2) LEMMA. *Let M be a 3-connected matroid with at least four elements and let C^* be a cocircuit of M such that, for all e in C^*, the matroid M/e is not 3-connected. Then C^* meets at least two distinct triangles of M.*

Proof of (3.1). As N is a series minor of M, there are subsets X and Y of $E(M)$ such that $N = M\backslash X/Y$, where every element of Y is in series with an element of $M\backslash X$ not in Y (see, for example, [6, Proposition 5.4.2]). Thus $M\backslash X$ is a series extension of N. Since N is coloop-free, so too is $M\backslash X$. Thus it suffices to prove the theorem in the case when N is a coloop-free restriction of M. In that case, since M is certainly simple, so too is N.

Assume that the theorem fails. Take a counterexample in which $|E(M) - E(N)|$ is as small as possible. We may certainly assume that $E(N)$ does not span M. Take a cocircuit C^* of M that avoids $E(N)$. Choose $e \in C^*$ and suppose that $\widetilde{M/e}$ is 3-connected. Then, as N is a simple restriction of M/e, we may assume that N is a restriction of $\widetilde{M/e}$. Then the choice of M and N implies that $\widetilde{M/e}$ has a 3-connected minor M_1 having a restriction N_1 such that $r(N_1) = r(M_1)$ and N_1 is isomorphic to a series extension of N. As M_1 is a minor of M, we obtain the contradiction that the theorem holds for M. Hence, for all e in C^*, the simplification $\widetilde{M/e}$ is not 3-connected. Thus, by [1], for all such e, the cosimplification $\underset{\sim}{M\backslash e}$ is 3-connected. If such a matroid $\underset{\sim}{M\backslash e}$ has N as a restriction, then the choice of M and N leads to a contradiction. Thus, for all e in C^*, the matroid $\underset{\sim}{M\backslash e}$ does not have N as a restriction. Now, either (i) for some e in C^*, every nontrivial series class of $M\backslash e$ avoids $E(N)$, or (ii) for all e in C^*, some element of N is in a 2-cocircuit of $M\backslash e$.

In case (i), let $S_1, S_2, ..., S_k$ be the nontrivial series classes of $M\backslash e$ and let x_i be an element of S_i for all i. Then N is a restriction of $M\backslash(e \cup S_1 \cup S_2 \cup \cdots \cup S_k)$ and we may assume that

$$\underset{\sim}{M\backslash e} = (M\backslash e) \Big/ \bigcup_{i=1}^{k} (S_i - x_i).$$

But $M\backslash e\backslash x_i$ has $S_i - x_i$ as a set of coloops. Thus $M\backslash e\backslash x_i/(S_i - x_i) = M\backslash e\backslash x_i\backslash(S_i - x_i)$. Hence $(\underset{\sim}{M\backslash e})\backslash\{x_1, x_2, ..., x_k\} = M\backslash e\backslash(S_1 \cup S_2 \cup \cdots \cup S_k)$ and so $\underset{\sim}{M\backslash e}$ has N as a restriction; a contradiction. We conclude that (i) does not occur.

In case (ii), every element of C^* is in a triad of M which meets $E(N)$. But every element of N is in a circuit of N, so, by orthogonality, every such triad must contain two elements of N. Now, for all e in C^*, the matroid $\widetilde{M/e}$ is not 3-connected, and hence, for all such elements e, the matroid M/e is not 3-connected. But $|C^*| \geqslant 2$ and $|E(N)| \geqslant 2$, so $|E(M)| \geqslant 4$. Thus, by Lemma 3.2, M has a triangle T that meets C^*. Clearly, $|T \cap C^*| \geqslant 2$. Let $T \cap C^* \supseteq \{u, v\}$. As u is in a triad T^* of M that has just one element of C^* in it and $|T^* \cap T| \neq 1$, the triangle T must contain an element, say w, of $E(N)$. Consider $N' = M \mid (E(N) \cup \{u, v\})$. It has $\{u, v\}$ as a cocircuit and $\{u, v, w\}$ as a circuit. Thus N' is the parallel connection of N and $M \mid T$ with respect to the basepoint w. Moreover, $N' \backslash w$ is a restriction of M that is isomorphic to a series extension of N. Hence, by the choice of M and N, there is a 3-connected minor M_1 of M having a restriction N_1 such that $r(N_1) = r(M_1)$ and N_1 is isomorphic to a series extension of $N' \backslash w$. Since $N' \backslash w$ is isomorphic to a series extension of N, we get a contradiction. ∎

Since Theorem 3.1 is interesting in itself, we present its analog for graphs. A graph H is a *topological minor* of G if H can be obtained from G by a sequence of operations each of which is one of the following: deletion of an edge, contraction of an edge incident with a vertex of degree two, and deletion of an isolated vertex. A topological minor is the graph analog of a series minor for matroids, but this analogy is not exact: if H is a topological minor of G, then $M(H)$ is a series minor of $M(G)$, but the reverse implication does not hold in general. Thus the following theorem for graphs, which appears to be new, is not a direct consequence of (3.1), but it can be shown using similar ideas. The proof of this graph result is not given explicitly here as it is not needed to derive other results of this paper.

(3.3) THEOREM. *Let G be a simple 3-connected graph and H be a topological minor of G without isolated vertices and without isthmuses. Then G has a simple 3-connected minor G_1 that has a subgraph H_1 that is isomorphic to a subdivision of H and has the same vertex set as G_1.*

4. THE CROSSING GRAPH OF A MATRIX

For this entire section we assume that F is a finite field. Let $A = (a_{i,j})$ be an F-matrix. We shall define a graph G^A as follows. The set of vertices of G^A is the set of columns of A. Two vertices of G^A are joined by an edge if and only if they cross as columns of A. The main result of this section states that if A arises as a chordal matrix in a normal F-representation of a 3-connected hamiltonian matroid, then G^A is connected.

We begin by presenting an auxiliary lemma.

(4.1) LEMMA. *Suppose that G_0 is a connected component of G^A and that k_0 is an element of $V(G^A)-V(G_0)$ that dominates at least one element of $V(G_0)$. Then k_0 dominates every element of $V(G_0)$.*

Proof. Recall that if k is a column of an F-matrix $B=(b_{i,j})$, then $s_B(k)$, or $s(k)$, denotes the set of rows h of B for which $b_{h,k}\neq 0$. If column k' of B dominates column k'', we shall write $k''\prec k'$. Let

$$X=\{k\in V(G_0)\colon k\prec k_0\},$$

$$Y=\{k\in V(G_0)\colon k_0\prec k\},$$

$$Z=\{k\in V(G_0):s(k)\cap s(k_0)=\varnothing\}.$$

Since k_0 is not in $V(G_0)$, it follows that $V(G_0)=X\cup Y\cup Z$, and, by assumption, X is nonempty. Since the relation $\prec$ is transitive, we have $k'\prec k''$ for all $k'\in X$ and $k''\in Y$. Thus no pair of vertices k' and k'' with $k'\in X$ and $k''\in Y$ is joined by an edge of G_0. It is clear from the definitions of X and Z that $s(k')\cap s(k'')=\varnothing$ for every $k'\in X$ and $k''\in Z$ and, hence, no edge of G_0 joins a vertex of X to a vertex of Z. Therefore, every edge of G_0 having one endpoint in X has its other endpoint in X. Thus, as G_0 is connected, $V(G_0)=X$. The conclusion follows. ▮

The following is the main result of this section. The proof of Theorem 1.5, which appears in the next section, will be based on an examination of the graph G^A.

(4.2) THEOREM. *Let M be a 3-connected hamiltonian matroid that is representable over F and has rank and corank at least two, and let $A=(a_{i,j})$ be a chordal matrix of some normal F-representation of M. Then G^A is connected.*

Proof. Let H be a normal F-representation of M that has A as its chordal matrix, let I denote the set of elements of M that correspond to the identity submatrix of H, and let $\mathbf{1}$ denote the column of H consisting of all ones. If J is a nonempty set of columns of A, then let $s(J)=\bigcup_{j\in J}s(j)$, and let $t(J)$ be the number of elements α of F for which $a_{i,j}=\alpha$ for some row i of A and some $j\in J$. Suppose that G^A is disconnected. From among all components of G^A, pick G_1 according to the following rules:

(1) If G is a component of G^A, then $s(V(G))$ is not a proper subset of $s(V(G_1))$.

(2) If G is a component of G^A with $s(V(G))=s(V(G_1))$, then $|V(G)|\leqslant|V(G_1)|$.

(3) If G is a component of G^A with $s(V(G))=s(V(G_1))$ and $|V(G)|=|V(G_1)|$, then $t(V(G))\leqslant t(V(G_1))$.

Observe that, since M is 3-connected,

(4) *the columns of A are distinct.*

We now prove that

(5) *if j_0 and j_1 are elements of $V(G^A) - V(G_1)$ and $V(G_1)$, respectively, such that $s(j_0) \cap s(j_1) \neq \emptyset$, then j_0 dominates j_1.*

Since j_0 is not a vertex of G_1, it does not cross j_1. This fact, together with the assumption that $s(j_0) \cap s(j_1) \neq \emptyset$ implies that either $j_0 \prec j_1$ or $j_1 \prec j_0$. To prove (5), we need only show that $j_0 \prec j_1$ does not hold. Suppose that j_1 dominates j_0. Then, by (4.1), j_1 dominates all elements of $V(G_0)$, where G_0 is the connected component of G^A containing j_0. Thus $s(V(G_0)) \subseteq s(j_1) \subseteq s(V(G_1))$, which by (1) implies that $s(V(G_0)) = s(j_1) = s(V(G_1))$. Therefore, as j_1 dominates all elements of $V(G_0)$ and (4) holds, the entries of j_1 take exactly one nonzero value. But then j_1 dominates all elements of $V(G_1)$, and, as G_1 is connected, $V(G_1) = \{j_1\}$. From (2) it follows that $V(G_0) = \{j_0\}$. Thus $s(j_0) = s(j_1)$, and, by (3), all nonzero entries of j_0 are equal. Thus $\{j_0, j_1\}$ is a two-element circuit of M. This contradiction to the assumption that M is 3-connected shows that j_1 fails to dominate j_0 and, hence, (5) holds.

Next we show that

(6) *if $j \in V(G^A) - V(G_1)$, then there is an element α in F such that $a_{i,j} = \alpha$ for all $i \in s(V(G_1))$.*

Suppose j_0 is a counterexample to (6). Then clearly $s(j_0) \cap s(V(G_1)) \neq \emptyset$ and hence $s(j_0) \cap s(j_1) \neq \emptyset$ for some $j_1 \in V(G_1)$. It follows from (5) that $j_1 \prec j_0$. Thus, by (4.1), j_0 dominates all elements of $V(G_1)$. This fact together with (4) contradicts the choice of j_0 and establishes (6).

Now observe that $s(V(G_1))$ is a subset of the set of rows of A, and hence it may be viewed as a subset of I. We define the subset X of $E(M)$ as the union of $V(G_1)$ with $s(V(G_1))$. Note that as neither $V(G_1)$ nor $s(V(G_1))$ is empty, $|X| \geqslant 2$. Also, $|E(M) - X| \geqslant 2$ as $E(M) - X$ contains an element of $V(G^A) - V(G_1)$ as well as the element $\mathbf{1}$. Now $r(X) = |s(V(G_1))|$, and, by (6), every element of $V(G^A) - V(G_1)$ is spanned by $(I - s(V(G_1))) \cup \{\mathbf{1}\}$. Thus,

$$r(E(M) - X) \leqslant r((I - s(V(G_1))) \cup \{\mathbf{1}\}) \leqslant |I| - |s(V(G_1))| + 1$$

$$= r(M) - r(X) + 1.$$

Hence $(X, E(M) - X)$ is a 2-separation of M, contrary to the assumption that M is 3-connected. ▌

5. Proof of the Main Result

The goal of this section is to prove the main theorem of the paper. Before embarking on this proof, we present three auxiliary results. The first of these is a result of Tucker [10], and the other two are results for graphs. For a positive integer n, let each of C'_n, C''_n, and C'''_n be a $\{0, 1\}$-matrix with, respectively, $n+2$, $n+3$, and $n+2$ columns of the form depicted in Fig. 2. The same figure shows two other matrices, $C^{(4)}$ and $C^{(5)}$.

The following is a result of Tucker [10].

(5.1) Theorem. *Let A be a $\{0, 1\}$-matrix. Then the following are equivalent*:

(i) *the rows of A may be permuted so that the ones in each column appear consecutively*;

$$\begin{pmatrix}
1 & & & & & & & 1\\
1 & 1 & & & & 0 & & 0\\
 & 1 & 1 & & & & & 0\\
 & & 1 & 1 & & & & 0\\
 & & & \ddots & \ddots & & & \vdots\\
 & & & & \ddots & 1 & & 0\\
 & 0 & & & & 1 & 1 & 0\\
 & & & & & & 1 & 1
\end{pmatrix}$$

C'_n

$$\begin{pmatrix}
1 & & & & & & & 1 & 0\\
1 & 1 & & & & 0 & & 1 & 1\\
 & 1 & 1 & & & & & 1 & 1\\
 & & 1 & 1 & & & & 1 & 1\\
 & & & \ddots & \ddots & & & \vdots & \vdots\\
 & & & & \ddots & 1 & & 1 & 1\\
 & 0 & & & & 1 & 1 & 1 & 1\\
 & & & & & & 1 & 0 & 1\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1
\end{pmatrix}$$

C''_n

$$\begin{pmatrix}
1 & & & & & & & 0\\
1 & 1 & & & & 0 & & 1\\
 & 1 & 1 & & & & & 1\\
 & & 1 & 1 & & & & 1\\
 & & & \ddots & \ddots & & & \vdots\\
 & & & & \ddots & 1 & & 1\\
 & 0 & & & & 1 & 1 & 1\\
 & & & & & & 1 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{pmatrix}$$

C'''_n

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
1 & 0 & 0 & 1\\
0 & 1 & 0 & 0\\
0 & 1 & 0 & 1\\
0 & 0 & 1 & 0\\
0 & 0 & 1 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 1\\
1 & 1 & 0 & 0\\
0 & 1 & 1 & 0\\
0 & 1 & 1 & 1\\
0 & 0 & 0 & 1
\end{pmatrix}$$

$C^{(4)}$ $\qquad$ $C^{(5)}$

Figure 2.

(ii) *no matrix obtained from a submatrix of A by permuting rows and permuting columns is in* $\bigcup_{p \geqslant 1} \{C'_p, C'''_p, C''''_p\} \cup \{C^{(4)}, C^{(5)}\}$.

Recall from Section 4 that, for a matrix C, the graph G^C has vertex set equal to the set of columns of C with two of its vertices being joined by an edge if and only if the columns cross in A. Observe the following.

(5.2) COROLLARY. *If a matrix C is in $\{C'_p, C''''_p\}$ for some positive integer p, then G^C is isomorphic to a cycle on $p+2$ vertices; if $C = C''_p$, then G^C is isomorphic to a cycle on $p+3$ vertices; and if C is in $\{C^{(4)}, C^{(5)}\}$, then G^C is isomorphic to $K_{1,3}$.*

The next result describes unavoidable induced subgraphs of large simple connected graphs. Recall the Ramsey function R from Theorem 1.1, and, for a positive integer n, let P_n denote a path on n vertices.

(5.3) THEOREM. *Let n be a positive integer, and let G be a simple connected graph on $(R(n, 2))^n$ vertices. Then G has an induced subgraph isomorphic to K_n, $K_{1,n}$, or P_n.*

Proof. Suppose first that G has a vertex v and a set K of $R(n, 2)$ vertices each of which is adjacent to v. Upon applying Ramsey's Theorem to K, we conclude that there is an n-element subset K' of K such that either every pair of vertices in K' is adjacent in G, or no pair of vertices of K' is adjacent in G. In the first case, K' induces a subgraph isomorphic to K_n; in the second case, $K' \cup \{v\}$ induces a subgraph isomorphic to $K_{1,n}$.

Let Δ denote the maximum vertex degree in G. From the argument in the previous paragraph, we may assume that $\Delta < R(n, 2)$. Let v be a vertex of G, and, for all nonnegative integers i, let V_i be the set of vertices x of G such that the shortest path between v and x has exactly $i+1$ vertices. Let p be the largest integer such that $V_p \neq \emptyset$, and let w be an element of V_p. Then a shortest path in G that joins v to w is an induced subgraph of G that is isomorphic to a path on $p+1$ vertices.

It remains to show that $p+1 \geqslant n$. It is clear that we need only consider the case $n \geqslant 2$. Then the graph G has more than two vertices, and thus $\Delta \geqslant 2$. Notice that $V(G)$ is the disjoint union of $V_0, V_1, ..., V_p$, and, clearly, $|V_0| = 1$, and $|V_i| \leqslant \Delta(\Delta-1)^{i-1}$ for $i \in [p]$. Hence,

$$\begin{aligned}\Delta^n < |V(G)| &\leqslant 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{p-1} \\ &\leqslant 1 + \Delta + \Delta^2 + \cdots + \Delta^p \\ &\leqslant \Delta^{p+1},\end{aligned}$$

and $n \leqslant p+1$, as desired. ∎

Let G be a simple graph and let C be a hamiltonian cycle of G. The edges of G not in C are *chords* of C. A chord e' *crosses* another chord e'' if e' meets both components obtained upon deleting the endvertices of e'' from C. Clearly, the binary relation of crossing on the set of chords of C is symmetric. The *chord graph of G with respect to C*, denoted by $\Omega(G, C)$, has the set of chords of C as its vertex set with two of its vertices joined by an edge if and only if the chords cross.

(5.4) THEOREM. *Let n be an integer greater than two, let G be a simple graph and let C be a hamiltonian cycle of G. If $\Omega(G, C)$ is a path on at least $6n$ vertices, then G has a minor isomorphic to W_n.*

Proof. Evidently, we may assume that

(1) *G and C are such that, for every nonempty subset X of $E(C)$, the graph $\Omega(G/X, C/X)$ fails to be a path on at least $6n$ vertices.*

The proof of (5.4) begins by introducing some new terminology, then proceeds with presenting properties of G and $\Omega(G, C)$ numbered (2)–(6), and finishes by applying these properties to derive the conclusion.

A chord e is *confined* to a subgraph D of C if both endvertices of e are in D. It is clear that, for every chord e, the graph $C \cup \{e\}$ has exactly two cycles distinct from C. Denote these two cycles by $Z'(e)$ and $Z''(e)$. Denote the path $\Omega(G, C)$ by P.

(2) *If e, e', and e'' are distinct chords of C such that e' is confined to $Z'(e)$, and e'' is confined to $Z''(e)$, then e' and e'' belong to distinct connected components of $P \backslash e$.*

To see (2), let D denote the set of chords that cross e. Then neither e' nor e'' is in D. Moreover, as no edge confined to $Z'(e)$ can cross an edge confined to $Z''(e)$, it follows that e' and e'' are in distinct connected components of $P \backslash D$. Now (2) follows easily since P is a path.

(3) *No vertex of G is incident with more than two chords.*

Suppose that a vertex v is incident with three distinct chords e_x, e_y, and e_z whose endvertices distinct from v are, respectively, x, y, and z. Without loss of generality, we may assume that v, x, y, and z appear on C in the (cyclic) order listed. By (2), the chords e_x and e_z are in distinct connected components of $P \backslash e_y$. Consequently, e_y is not an endvertex of P. Let e'_x and e'_z be the chords crossing e_y that belong to the connected components of $P \backslash e_y$ containing, respectively, e_x and e_z. Let $Z(e'_x)$ denote the element of $\{Z'(e'_x), Z''(e'_x)\}$ whose vertex set contains v. Similarly, let $Z(e'_z)$ denote the element of $\{Z'(e'_z), Z''(e'_z)\}$ whose vertex set contains v. Since e'_x and e'_z do not cross, either $v \in V(Z(e'_x)) \subseteq V(Z(e'_z))$ or $v \in V(Z(e'_z)) \subseteq V(Z(e'_x))$. By symmetry, we may assume that the former holds. Then e'_z is confined to the element of $\{Z'(e'_x), Z''(e'_x)\}$ other than $Z(e'_x)$, whereas, since e_z and e'_x do not cross and e_z is incident with v, the chord e_z is confined to $Z(e'_x)$. But by letting e, e', and e'' be e'_x, e_z, and e'_z, respectively, we obtain a contradiction to (2). Hence, (3) follows.

Let $e_1, e_2, ..., e_m$ be the chords of C listed here in the order they appear on P. Since e_1 and e_m do not cross, we may assume that e_1 is confined to $Z''(e_m)$, and e_m is confined to $Z''(e_1)$. Since e_2 is the only chord crossing e_1, and e_{m-1} is the only chord crossing e_m, we conclude from (1) and (2) that $Z'(e_1)$ and $Z'(e_m)$ are triangles. Let s denote the vertex of $Z'(e_1)$ not incident with e_1, and let t denote the vertex of $Z'(e_m)$ not incident with e_m.

(4) *Every vertex in $V(G)-\{s, t\}$ has degree four in G.*

By (1), every vertex of G has degree at least three, and, by (3), every vertex has degree at most four. Suppose v is a vertex of degree three. Let u and w be the vertices that are adjacent to v in C, and let v' be the remaining neighbor of v. By (1), if G' and C' are obtained by contracting the edge $\{v, u\}$ in, respectively, G and C, then $\Omega(G', C') \neq P$. Hence the chord $\{v, v'\}$ crosses a chord incident with u, say $\{u, u'\}$. Similarly, $\{v, v'\}$ crosses some chord $\{w, w'\}$. Since the chords $\{v, v'\}$, $\{u, u'\}$, and $\{w, w'\}$ do not pairwise cross, as P is a path, at least one of $\{u, u'\}$ and $\{w, w'\}$ is $\{u, w\}$. It is clear that the chord $\{u, w\}$ is an endvertex of P, as it crosses only $\{v, v'\}$. Thus $v \in \{s, t\}$, and (4) follows.

For each $i \in [m]$, let V_i be the set of vertices incident with at least one of the edges $e_1, e_2, ..., e_i$.

(5) *For every $i \in \{3, 4, ..., m-2\}$, the chord e_i is incident with exactly one vertex in V_{i-1}.*

Let $i \in \{3, 4, ..., m-2\}$ and let u and v be the endvertices of e_i. Since each of e_{i-1} and e_{i+1} crosses e_i, neither is incident with v. Let $Z(e_{i-1})$ denote the element of $\{Z'(e_{i-1}), Z''(e_{i-1})\}$ whose vertex set contains v. Similarly, let $Z(e_{i+1})$ denote the element of $\{Z'(e_{i+1}), Z''(e_{i+1})\}$ whose vertex set contains v. As e_{i-1} and e_{i+1} do not cross, either $Z(e_{i-1}) \subseteq Z(e_{i+1})$ or $Z(e_{i+1}) \subseteq Z(e_{i-1})$. By symmetry, we may assume that the latter holds. Then $v \notin V_{i-1}$, as otherwise e_j would be incident with v for some $j < i-1$, thereby contradicting (2) when e, e', and e'' are e_{i+1}, e_{i-1}, and e_j. To see that $u \in V_{i-1}$, note that, from the choice of i, it follows that $u \notin \{s, t\}$, and thus, by (4), u is incident with some chord e_j. Upon applying (2) with e, e', and e'' equal to e_j, e_{i-1}, and e_{i+1}, respectively, we conclude that $j < i-1$, as required. Hence (5) has been proved.

(6) *Let i, j, and k be distinct elements of $\{3, 4, ..., m-2\}$ such that e_j is adjacent in G to both e_i and e_k. Then* $\min\{i, k\} < j < \max\{i, k\}$.

Observe first that, by (4), e_i and e_k are incident with distinct endvertices of e_j. Clearly, $j < \max\{i, k\}$ as otherwise e_j is incident with two vertices in V_{j-1} contradicting (5). Similarly, $j > \min\{i, k\}$ as otherwise e_j is incident with no vertices of V_{j-1}, also contradicting (5). Thus (6) holds.

To prove (5.4), consider the graph G' obtained from G by deleting the edges of C and deleting the edges e_1 and e_m. From (6), it follows that G' has no cycles. Observe from (4) that the graph obtained from G by deleting the edges of C and deleting e_1 has exactly four vertices of degree one. Thus G' has either six vertices of degree one, or four vertices of degree one and one vertex of degree zero. It follows that G' consists of three disjoint paths P_1, P_2, and P_3, at most one of which has just a single vertex, and each of which has its endvertices in $V(Z'(e)) \cup V(Z'(e_m))$. Let E' be the set of edges of C that are not in $Z'(e_1) \cup Z'(e_m)$. By (1), each element e of E' must be adjacent to two chords that cross each other. But, by (6), no two chords from the same P_i, for $i \in \{1, 2, 3\}$, cross each other, and so e joins a vertex of P_i to a vertex of P_j for some $i \neq j$. By (4), we have $|E'| = |E(C)| - 4 = |V(P)| - 3 \geqslant 6n - 3$. From the choice of E', it contains a matching E'' with at least $3n-1$ edges. Thus, using the pigeon-hole principle, i and j can be chosen so that E'' has an n-element subset E''' all of whose elements join a vertex of P_i to a vertex of P_j. Now it is straightforward to verify that the edges in E''' together with $Z'(e_1)$, $Z'(e_m)$, P_1, P_2, and P_3 form a graph having a W_n minor. ∎

In the next five easy lemmas, we examine hamiltonian matroids whose normal binary representations have chordal matrices of the form described in the conclusions of some results from Section 2. For all these lemmas, assume that m is an integer greater than two, and that M is a hamiltonian matroid whose normal binary representation H has a chordal matrix A with m columns. Let $\mathbf{1}$ denote the all-ones column of H.

(5.5) Lemma. *If A has the form depicted in* (1) *or* (2) *of Fig.* 3, *then M is isomorphic to $M(W_{m+1})$.*

Proof. If A has form (1), then consider the following sequence of columns of H: the first column of I_{m+1}, followed by all but the first column of A, followed by the last column of I_{m+1}. It is easy to verify that this sequence defines the circular order of the spokes of a W_{m+1}. If A has form (2), then, upon pivoting in H on the last entry of $\mathbf{1}$, we obtain a matrix H' whose chordal matrix has form (1). ∎

(5.6) Lemma. *If A has the form depicted in* (3) *of Fig.* 3, *then M has a minor isomorphic to $M^*(K_{3,m-1})$.*

Proof. Let A' be the matrix obtained from A by adjoining $\mathbf{1}$ at the end and then transposing the resulting matrix. Let K be the graph obtained

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

(1) (2) (3)

FIGURE 3

from $K_{3,m-1}$ by adding two more edges so that the vertex class of $K_{3,m-1}$ with three vertices induces a path P in K. Let u be an endvertex of P. Define the spanning tree T of K to have as its edges all the edges of P and all the edges of $K_{3,m-1}$ incident with u. It is easy to check that $M(K)$ can be represented by the matrix $(I_{m+1}|A')$ by taking the edges of T to correspond to the first $m+1$ columns. The lemma follows. ∎

(5.7) LEMMA. *Suppose A has $2m$ rows, the first m of which form a* $(0, 1, 1)$*-diagonal matrix, and the last m of which form a* $(1, 1, 0)$*-diagonal matrix. Then M has a minor isomorphic to $M(W_{m+1})$.*

Proof. This follows from (5.5) since, upon deleting rows numbered 2, 3, ..., m from A, and then placing the first row at the bottom of the resulting matrix, we obtain a matrix of the form (1) in Fig. 3. ∎

(5.8) LEMMA. *Suppose A has $m+1$ rows, the first m of which form an identity matrix and the last of which has all ones. Then M has a minor isomorphic to S_m.*

Proof. Let j denote the element of M corresponding to the all-ones column of H. Pivot in H on the last entry of $\mathbf{1}$. From the resulting matrix, delete columns j and $m+1$ and the last row to obtain the matrix $(I_m \mid J_m - I_m)$. ∎

(5.9) LEMMA. *If A is* $(1, 1)$*-complete, then M has a minor isomorphic to $M^*(K_{3,m-2})$.*

Proof. By the definition preceding Theorem 2.7, A is the vertex-edge incidence matrix of a complete graph on m vertices. Hence, if e denotes the element of M that corresponds to $\mathbf{1}$, then $M\backslash e$ is isomorphic to $M^*(K_{m+1})$. As K_{m+1} certainly has $K_{3,m-2}$ as a minor, the lemma follows. ∎

We are now ready to prove the main theorem.

Proof of (1.5). Let g_5 be the function from (2.7), let g_6 be the function from (2.8), and let R be the Ramsey function. Let $m = \max\{g_5(n+2, 2), g_6(n+1, 2), 6n\}$, let $k = (R(m, 2))^m$, let $N = 4^{2^k}$, and let M be a 3-connected binary matroid with more than N elements. By (1.3), M has a circuit or cocircuit with at least $2^k + 1$ elements. We may assume that the former holds; otherwise we apply the argument that follows to the dual of M, rather than to M itself, noting that $\{M(K_{3,n}), M^*(K_{3,n}), M(W_n), S_n\}$ is closed under the taking of duals. Let C be a circuit of M of maximal cardinality. Now, it follows by (3.1) that M has a 3-connected minor M_1 that has a circuit of size $|C|$. Without loss of generality, we may assume that C spans M_1. Choose a normal $\{0, 1\}$-representation of M_1, denote the chordal matrix of this representation by A, and denote the all-ones column by $\mathbf{1}$. Since $|C| \geqslant 2^k + 1$, it follows that A has at least 2^k rows, and, as M_1 has no two-element cocircuits, no two rows of A are identical. Thus A has at least k columns.

Recall that G^A is the graph whose vertex set is the set of columns of A with two of its vertices being joined by an edge if and only if the columns cross in A. Then G^A has at least $(R(m, 2))^m$ vertices, and, by (4.2), it is connected. By (5.3), G^A has an induced subgraph isomorphic to one of K_m, $K_{1,m}$, and P_m. We shall consider these three cases separately.

Suppose first that G^A has an induced subgraph K isomorphic to K_m. Upon deleting all columns of A, except those corresponding to the vertices of K, we obtain a submatrix A' of A with m columns that pairwise cross. Upon applying (2.7) to A' we conclude that A' has a row-permuted submatrix B that has $n+2$ columns and satisfies one of the following conditions:

(i) B is $(1, 0, 1)$-diagonal;

(ii) B has $2n+4$ rows, the first $n+2$ of which form a $(0, 1, 1)$-diagonal matrix and the last $n+2$ of which form a $(1, 1, 0)$-diagonal matrix;

(iii) B has $n+3$ rows, the first $n+2$ of which form the identity matrix and the last of which has all entries equal to 1;

(iv) B is $(1, 1)$-complete.

It is immediate that if B satisfies (i), then $M_1 \backslash e$ is isomorphic to S_{n+2} where e is the all-ones column; if B satisfies (ii), then, by (5.7), M_1 has a minor isomorphic to $M(W_{n+2})$; if B satisfies (iii), then, by (5.8), M_1 has a minor isomorphic to S_{n+2}; and if B satisfies (iv), then, by (5.9), M_1 has a minor isomorphic to $M^*(K_{3,n})$.

Suppose now that G^A has an induced subgraph K that is isomorphic to $K_{1,m}$. Construct a matrix A' from A by deleting all columns except those corresponding to the vertices of K, and permuting the columns, if necessary, so that the first column crosses all the other columns. Then, upon applying (2.8) to A', we conclude that A' has a row-permuted submatrix B that has $n+1$ columns and, up to a permutation of columns, is of one of the forms illustrated in Fig. 3.

From (5.5), we conclude that if B is of the form (1) or (2), then M_1 contains a minor isomorphic to $M(W_{n+2})$; and, by (5.6), if B is of the form (3), then M_1 contains a minor isomorphic to $M^*(K_{3,n+1})$.

Suppose finally that G^A has an induced subgraph K that is a path on m vertices. Let A' be the matrix obtained from A by deleting all columns except those corresponding to the vertices of K. Observe that, for every submatrix D of A', the graph G^D is a subgraph of K. Thus, by (5.2), no submatrix of A' is in $\bigcup_{p\geqslant 1}\{C'_p, C''_p, C'''_p\}\cup\{C^{(4)}, C^{(5)}\}$, and hence, by (5.1), we may assume that the ones in each column of A' appear consecutively. Thus, for every column i of A', there are integers x_i and y_i such that $s(i)=[y_i]-[x_i-1]$, where $1\leqslant x_i\leqslant y_i\leqslant r$, with r being the number of rows of A'.

Define a graph G whose vertex set is $[r+1]$ by arranging all of its vertices on a cycle H in the natural order, and then, for each column i of A', joining the vertices x_i and y_i+1. It is clear that A' is a chordal matrix in a normal $\{0, 1\}$-representation of $M(G)$, where the basis is formed by all but one of the edges of H. Moreover, two edges in $E(G)-E(H)$ cross if and only if the corresponding columns of A' cross. Hence $\Omega(G, H)$ is isomorphic to K, and, by (5.4), G contains a minor isomorphic to W_n. ∎

Acknowledgment

This research was partially supported by a grant from the Louisiana Education Quality Support Fund through the Board of Regents.

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