GENERALIZED LAMINAR MATROIDS

TARA FIFE AND JAMES OXLEY

Abstract. Nested matroids were introduced by Crapo in 1965 and have appeared frequently in the literature since then. A flat of a matroid M is Hamiltonian if it has a spanning circuit. A matroid M is nested if and only if its Hamiltonian flats form a chain under inclusion; M is laminar if and only if, for every 1-element independent set X, the Hamiltonian flats of M containing X form a chain under inclusion. We generalize these notions to define the classes of k-closure-laminar and k-laminar matroids. This paper focuses on structural properties of these classes noting that, while the second class is always minor-closed, the first is if and only if $k \leq 3$. The main results are excluded-minor characterizations of the classes of 2-laminar and 2-closure-laminar matroids.

1. Introduction

Our matroid terminology follows Oxley [18]. A transversal matroid is *nested* if it has a nested presentation, that is, a transversal presentation $(B_1, B_2, \dots B_n)$ such that $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n$. These matroids were introduced by Crapo [7] and have appeared under a variety of names including freedom matroids [8], generalized Catalan matroids [4], shifted matroids [1], and Schubert matroids [21]

A family $\mathcal{A}$ of subsets of a set E is *laminar* if, for every two intersecting sets A and B in $\mathcal{A}$, either $A \subseteq B$ or $B \subseteq A$. Let $\mathcal{A}$ be a laminar family of subsets of a finite set E and c be a function from $\mathcal{A}$ into the set of non-negative integers. Define $\mathcal{I}$ to be the set of subsets I of E such that $|I \cap A| \leq c(A)$ for all A in $\mathcal{A}$. It is well known (see, for example, [11, 12, 14, 17]) and easily checked that $\mathcal{I}$ is the set of independent sets of a matroid on E. We write this matroid as $M(E, \mathcal{A}, c)$. A matroid M is *laminar* if it is isomorphic to $M(E, \mathcal{A}, c)$ for some set E, laminar family $\mathcal{A}$, and function c.

Laminar matroids have appeared often during the last fifteen years particularly in relation to their behavior for the matroid secretary problem and other optimization problems [2, 5, 9, 14, 16, 22]. Huynh [13] reviewed this work, while Finkelstein [11] investigated some of the structural properties of laminar matroids. In [10], we characterized laminar matroids both constructively and via excluded minors. We also showed that all nested matroids are laminar and noted a number of similarities between the classes of nested and laminar matroids. Here we exploit some of these similarities to define two natural infinite families of classes of matroids, each having the classes of nested and laminar matroids as their smallest members. Every matroid belongs to a member of each of these families.

We say that a flat in a matroid is *Hamiltonian* if it has a spanning circuit. In [19], it was shown that a matroid is nested if and only if its Hamiltonian flats form a chain under inclusion. This immediately yields the following result.

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Proposition 1.1. *A matroid is nested if and only if, for all circuits C_1 and C_2, either $C_1 \subseteq \mathrm{cl}(C_2)$, or $C_2 \subseteq \mathrm{cl}(C_1)$.*

This parallels the following characterization of laminar matroids found in [10].

Theorem 1.2. *A matroid is laminar if and only if, for all circuits C_1 and C_2 with $|C_1 \cap C_2| \geq 1$, either $C_1 \subseteq \mathrm{cl}(C_2)$, or $C_2 \subseteq \mathrm{cl}(C_1)$.*

Using circuit elimination, it can quickly be shown that we get a similar description in terms of Hamiltonian flats.

Corollary 1.3. *A matroid is laminar if and only if, for every 1-element independent set X, the Hamiltonian flats containing X form a chain under inclusion.*

In light of these results, for any non-negative integer k, we define a matroid M to be *k-closure-laminar* if, for any k-element independent subset X of $E(M)$, the Hamiltonian flats of M containing X form a chain under inclusion. We say that M is *k-laminar* if, for any two circuits C_1 and C_2 of M with $|C_1 \cap C_2| \geq k$, either $C_1 \subseteq \mathrm{cl}(C_2)$ or $C_1 \subseteq \mathrm{cl}(C_2)$. The following observation is straightforward.

Lemma 1.4. *A matroid M is k-closure-laminar if and only if, whenever C_1 and C_2 are circuits of M with $r\big(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)\big) \geq k$, either $C_1 \subseteq \mathrm{cl}(C_2)$, or $C_2 \subseteq \mathrm{cl}(C_1)$.*

Observe that the class of nested matroids coincides with the classes of 0-laminar matroids and 0-closure-laminar matroids, while the class of laminar matroids coincides with the classes of 1-laminar matroids and 1-closure-laminar matroids. It is easy to see that k-closure-laminar matroids are also k-laminar. For $k \geq 2$, consider the matroid that is obtained from a $(k+1)$-element circuit C by attaching, via parallel connection, a single triangle at each of two different elements of C. This matroid is k-laminar, but not k-closure-laminar. Thus, for all $k \geq 2$, the class of k-laminar matroids strictly contains the class of k-closure-laminar matroids. Our hope is that, for small values of k, the classes of k-laminar and k-closure-laminar matroids will enjoy some of the computational advantages of laminar matroids.

It is not hard to show that the class of k-laminar matroids is minor-closed. This implies the previously known fact that the class of k-closure-laminar matroids is minor-closed for $k \in \{0, 1\}$. We show that the latter class is also minor-closed for $k \in \{2, 3\}$. Somewhat surprisingly, for all $k \geq 4$, the class of k-closure-laminar matroids is not minor-closed. This is shown in Section 2. In Section 3, we prove our main results, excluded-minor characterizations of the classes of 2-laminar matroids and 2-closure-laminar matroids. In Section 4, we consider the intersection of the classes of k-laminar and k-closure-laminar matroids with other well-known classes of matroids. In particular, we show that these intersections with the class of paving matroids coincide. Moreover, although all nested and laminar matroids are representable, we note that, for all $k \geq 2$, the classes of k-laminar and k-closure-laminar matroids both contain members that are not representable.

2. Preliminaries

In this section, we establish some basic properties of k-laminar and k-closure-laminar matroids. The first result summarizes some of these properties. Its straightforward proof is omitted.

Proposition 2.1. *Let M be a matroid and k be a non-negative integer.*

(i) *If M is k-closure-laminar, then M is k-laminar.*
(ii) *If M is k-closure-laminar, then M is $(k+1)$-closure-laminar.*
(iii) *If M is k-laminar, then M is $(k+1)$-laminar.*
(iv) *M is k-closure-laminar if and only if, whenever C_1 and C_2 are non-spanning circuits of M with $r\big(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)\big) \geq k$, either $C_1 \subseteq \mathrm{cl}(C_2)$, or $C_2 \subseteq \mathrm{cl}(C_1)$.*
(v) *M is k-laminar if and only if, whenever C_1 and C_2 are non-spanning circuits of M with $|C_1 \cap C_2| \geq k$, either $C_1 \subseteq \mathrm{cl}(C_2)$ or $C_2 \subseteq \mathrm{cl}(C_1)$.*
(vi) *If M has at most one non-spanning circuit, then M is k-laminar and k-closure-laminar.*

Clearly, for all k, the classes of k-laminar and k-closure-laminar matroids are closed under deletion. Next, we investigate contractions of members of these classes. We omit the routine proof of the following.

Lemma 2.2. *The class of k-laminar matroids is minor-closed.*

As we will see, the class of k-closure-laminar matroids is not closed under contraction when $k \geq 4$. The next lemma will be useful in proving that the classes of 2-closure-laminar and 3-closure-laminar matroids are minor-closed.

Lemma 2.3. *Let C be a circuit of a k-laminar matroid M such that $|C| \geq 2k-1$. If $e \in E(M) - \mathrm{cl}(C)$ and $r(\mathrm{cl}(C \cup e) - \mathrm{cl}(C)) \geq 2$, then $\mathrm{cl}(C \cup e)$ is a Hamiltonian flat of M.*

Proof. Take an element f of $\mathrm{cl}(C \cup e) - (\mathrm{cl}(C) \cup \mathrm{cl}(\{e\}))$. Then M has a circuit D such that $\{e, f\} \subseteq D \subseteq C \cup \{e, f\}$. As $f \notin \mathrm{cl}(\{e\})$, we may choose an element d in $D - \{e, f\}$. By circuit elimination, M has a circuit D' such that $f \in D' \subseteq (C \cup D) - d$. Then $e \in D'$ as $f \notin \mathrm{cl}(C)$. Applying circuit elimination again gives a circuit C' contained in $(D \cup D') - e$. As $f \notin \mathrm{cl}(C)$, it follows that $C' = C$. Hence $D' \supseteq C - D$. As $|C| \geq 2k-1$, either $|D \cap C|$ or $|D' \cap C|$ is at least k. Since neither D nor D' is contained in $\mathrm{cl}(C)$, it follows that C is contained in $\mathrm{cl}(D)$ or $\mathrm{cl}(D')$. Thus D or D' is a spanning circuit of $\mathrm{cl}(C \cup e)$, so this flat is Hamiltonian. □

Theorem 2.4. *The classes of 2-closure-laminar and 3-closure-laminar matroids are minor-closed.*

Proof. For some k in $\{2, 3\}$, let e be an element of a k-closure-laminar matroid M, and let C_1 and C_2 be distinct circuits in M/e with $r_{M/e}(\mathrm{cl}_{M/e}(C_1) \cap \mathrm{cl}_{M/e}(C_2)) \geq k$. We aim to show that $\mathrm{cl}_{M/e}(C_1) \subseteq \mathrm{cl}_{M/e}(C_2)$ or $\mathrm{cl}_{M/e}(C_2) \subseteq \mathrm{cl}_{M/e}(C_1)$. This is certainly true if $r_M(C_1) = k$ or $r_M(C_2) = k$, so assume each of $|C_1|$ and $|C_2|$ is at least $k+2$. As $k \in \{2, 3\}$, it follows that $|C_i| \geq 2k-1$ for each i.

2.4.1. *For each i in $\{1, 2\}$, there is a circuit D_i of M such that $\mathrm{cl}_M(C_i \cup e) - \mathrm{cl}(\{e\}) = \mathrm{cl}_M(D_i) - \mathrm{cl}(\{e\})$.*

To see this, first note that C_i or $C_i \cup e$ is a circuit of M. In the latter case, we take $D_i = C_i \cup e$. In the former case, by Lemma 2.3, the result is immediate unless $\mathrm{cl}(C_i \cup e) = \mathrm{cl}(C_i) \cup \mathrm{cl}(\{e\})$, in which case we can take $D_i = C_i$. Thus 2.4.1 holds.

Now $r((\mathrm{cl}_M(C_1 \cup e) \cap \mathrm{cl}_M(C_2 \cup e))) \geq k+1$ as $r(\mathrm{cl}_{M/e}(C_1) \cap \mathrm{cl}_{M/e}(C_2)) \geq k$. Hence, by 2.4.1, $r(\mathrm{cl}_M(D_1) \cap \mathrm{cl}_M(D_2)) \geq k$. Thus $\mathrm{cl}_M(D_i) \subseteq \mathrm{cl}_M(D_j)$ for some $\{i, j\} = \{1, 2\}$. Hence $\mathrm{cl}_M(C_i \cup e) - \mathrm{cl}_M(\{e\}) \subseteq \mathrm{cl}_M(C_j \cup e) - \mathrm{cl}_M(\{e\})$, so $\mathrm{cl}_{M/e}(C_i) - \mathrm{cl}_M(\{e\}) \subseteq \mathrm{cl}_{M/e}(C_j) - \mathrm{cl}_M(\{e\})$. As each element of $\mathrm{cl}_M(\{e\}) - e$ is a loop in M/e, we deduce that $\mathrm{cl}_{M/e}(C_i) \subseteq \mathrm{cl}_{M/e}(C_j)$. Thus the theorem holds. □

Theorem 2.5. *For all $k \geq 4$, the class of k-closure-laminar matroids is not minor-closed.*

The proof of this theorem will use Bonin and De Mier's characterization of matroids in terms of their collections of cyclic flats [3, Theorem 3.2].

Theorem 2.6. *Let $\mathcal{Z}$ be a collection of subsets of a set E and let r be an integer-valued function on $\mathcal{Z}$. There is a matroid for which $\mathcal{Z}$ is the collection of cyclic flats and r is the rank function restricted to the sets in $\mathcal{Z}$ if and only if*

(Z0) *$\mathcal{Z}$ is a lattice under inclusion;*
(Z1) $r(0_{\mathcal{Z}}) = 0$;
(Z2) $0 < r(Y) - r(X) < |Y - X|$ *for all sets X, Y in $\mathcal{Z}$ with $X \subsetneq Y$; and*
(Z3) *for all sets X, Y in $\mathcal{Z}$,*

$$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.$$

Proof of Theorem 2.5. Let A, B, and C be disjoint sets with $A = \{a_1, a_2, \ldots, a_{k-1}\}$, $B = \{b_1, b_2, \ldots, b_{k-1}\}$, and $C = \{c_1, c_2, \ldots, c_{k-1}\}$. Let $D = \{e, a_1, b_1, c_1\}$ where $e \notin A \cup B \cup C$. Let $E = A \cup B \cup C \cup D$ and let $\mathcal{Z}$ be the following collection of subsets of E having the specified ranks and cardinalities.

Rank t	Cardinality	Members of $\mathcal{Z}$ of rank t
0	0	$\emptyset$
k	$k+1$	$A\triangle D$, $B\triangle D$, $C\triangle D$
$2k-3$	$2k-2$	$A \cup B$, $A \cup C$, $B \cup C$
$2k-2$	$2k$	$A \cup B \cup D$, $A \cup C \cup D$, $B \cup C \cup D$
$2k-1$	$3k+1$	E

We will show that $\mathcal{Z}$ is the collection of cyclic flats of a matroid M on E. We then show that M is k-closure-laminar but that M/e is not. Observe that $A\triangle D$, $B\triangle D$, $C\triangle D$, $A \cup B$, $A \cup C$, and $B \cup C$ form an antichain, and that, for example, $A \cup B \cup D$ contains exactly three members of this antichain, $A \cup B$, $A\triangle D$, and $B\triangle D$. It is straightforward to see that $\mathcal{Z}$ is a lattice obeying (Z1) (see Figure 1). We can quickly check that $\mathcal{Z}$ obeys (Z2). To check that $\mathcal{Z}$ obeys (Z3), we see by symmetry that we need only check (Z3) when (X, Y) is one of $(A \cup C, A\triangle D)$, $(A \cup C, B\triangle D)$, $(A \cup C, B \cup C \cup D)$, $(A \cup C, B \cup C)$, $(A\triangle D, B\triangle D)$, $(A\triangle D, B \cup C \cup D)$, and $(A \cup B \cup D, A \cup C \cup D)$. Calculating $r(X) + r(Y) - r(X \vee Y) - r(X \wedge Y) - |(X \cap Y) - (X \wedge Y)|$ for each of these pairs, we find that the first and fifth give 0, and the other five give $k - 4$. Hence $\mathcal{Z}$ obeys (Z3) for all $k \geq 4$, so M is a matroid. As noted in [3], its circuits are the minimal subsets S of E such that $\mathcal{Z}$ contains an element Z containing S with $|S| = r(Z) + 1$.

To show that M is k-closure-laminar, we first note that $A \cup C \cup D$ is non-Hamiltonian for $2 = |A \cup C \cup D| - r(A \cup C \cup D)$, yet there is no element of $A \cup C \cup D$ that is in all three non-spanning circuits of $M|(A \cup C \cup D)$. By symmetry, $A \cup B \cup D$ and $B \cup C \cup D$ are non-Hamiltonian. All of the other cyclic flats of M are Hamiltonian. By symmetry, if (X, Y) is a pair of incomparable Hamiltonian flats of M, then we may assume that (X, Y) is $(A \cup C, A\triangle D)$, $(A \cup C, B\triangle D)$, $(A \cup C, B \cup C)$, or $(A\triangle D, B\triangle D)$. For each such pair, we need to check that $r(X \cap Y) \leq k - 1$. For the first and third pairs, $|X \cap Y| = k - 1$; for the second and fourth pairs, $|X \cap Y| = 2$. Hence M is indeed k-closure-laminar. To see that M/e is not k-closure-laminar, note that $A \cup C$ and $B \cup C$ are circuits of this matroid.

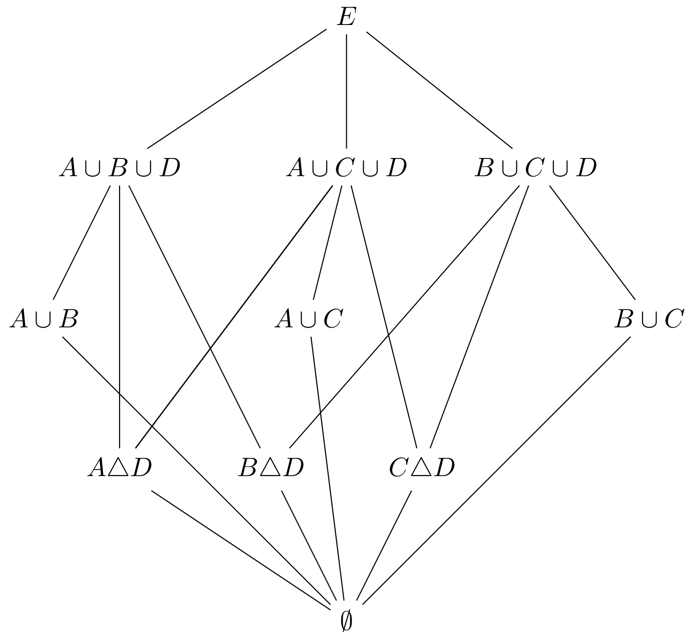

FIGURE 1. The lattice $\mathcal{Z}$ of cyclic flats of the matroid in the proof of Theorem 2.5.

Moreover, $\mathrm{cl}_{M/e}(A \cup C) = A \cup C \cup b_1$ and $\mathrm{cl}_{M/e}(B \cup C) = B \cup C \cup a_1$. Then $\mathrm{cl}_{M/e}(A \cup C) \cap \mathrm{cl}_{M/e}(B \cup C) = C \cup \{a_1, b_1\}$. The last set has rank k in M/e as $(C \triangle D) - e$ is the only circuit of M/e contained in it. Thus M/e is not k-closure-laminar as neither $\mathrm{cl}_{M/e}(A \cup C)$ nor $\mathrm{cl}_{M/e}(B \cup C)$ is contained in the other. $\square$

3. Excluded Minors

We now note some excluded minors for the classes of k-laminar and k-closure-laminar matroids. For $n \geq k+2$, let $M_n(k)$ be the truncation to rank n of the cycle matroid of the graph consisting of two vertices that are joined by three internally disjoint paths P, X_1, and X_2 of lengths k, $n-k$, and $n-k$, respectively. In particular, $M_4(2) \cong M(K_{2,3})$. Observe that, when $k = 0$, the path P has length 0 so its endpoints are equal. Thus $M_n(0)$ is the truncation to rank n of the direct sum of two n-circuits. Let $M^-(K_{2,3})$ be the unique matroid that is obtained by relaxing a circuit-hyperplane of $M(K_{2,3})$. For $n \geq k+3 \geq 5$, let $N_n(k)$ be the truncation to rank n of the graphic matroid that is obtained by attaching two $(n-k)$-circuits to distinct elements of a $(k+2)$-circuit via parallel connection. For $n \geq k+2 \geq 4$, let $P_n(k)$ be the truncation to rank n of the graphic matroid that is obtained by attaching two $(n-k+1)$-circuits to distinct elements of a $(k+1)$-circuit via parallel connection. Thus $P_n(k)$ is a single-element contraction of $N_{n+1}(k)$. Moreover, $P_4(2)$ is isomorphic to the matroid that is obtained by deleting a rim element from a rank-4 wheel.

Lemma 3.1. *For all $n \geq k+2$, the matroid $M_n(k)$ is an excluded minor for the classes of k-laminar matroids and k-closure-laminar matroids.*

Proof. We may assume that $k \geq 2$, as the lemma holds for $k = 0$ and for $k = 1$ by results in [19] and [10]. Clearly $M_n(k)$ is not k-laminar so is not k-closure-laminar. If we delete an element of $M_n(k)$, then we get a matroid with at most one non-spanning circuit. By Proposition 2.1(vi), such a matroid is k-closure-laminar and hence is k-laminar. If we contract an element of P from $M_n(k)$, we get a matroid that is k-closure-laminar since in it the closures of the only two non-spanning circuits meet in $k-1$ elements. Instead, if we contract an element of X_1 or X_2, we again get a matroid with exactly one non-spanning circuit. Thus the lemma holds. $\square$

Similar arguments give the following result.

Lemma 3.2.

(i) The matroid $M^-(K_{2,3})$ is an excluded minor for the classes of 2-laminar and 2-closure-laminar matroids.

(ii) For all $n \geq k+3 \geq 5$, the matroid $N_n(k)$ is an excluded minor for the class of k-laminar matroids.

(iii) For all $n \geq k+2 \geq 4$, the matroid $P_n(k)$ is an excluded minor for the class of k-closure-laminar matroids.

The main results of this paper show that we have now identified all of the excluded minors for the classes of 2-laminar and 2-closure-laminar matroids. We will use the following basic results. We omit the elementary proof of the second one.

Lemma 3.3. *Let C be a circuit of a matroid M. If there is a partition $\{C_1, C_2\}$ of C and distinct elements x and y for which $C_1 \cup x$, $C_2 \cup x$, $C_1 \cup y$, and $C_2 \cup y$ are all circuits, then x and y are parallel.*

Proof. By submodularity and the fact that C is a circuit,

$$r\big(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)\big) \leq |C_1| + |C_2| - (|C| - 1) = 1.$$

From this, the lemma is immediate. $\square$

Lemma 3.4. *Let C and D be distinct circuits of a matroid M.*

(i) If $D \not\subseteq \mathrm{cl}(C)$, then $|D - \mathrm{cl}(C)| \geq 2$.

(ii) If $|D - C| = 1$ and D' is a circuit contained in $C \cup D$ other than C or D, then $C - D \subseteq D'$.

Lemma 3.5. *Let M be an excluded minor for $\mathcal{M}$ where $\mathcal{M}$ is the class of 2-laminar or 2-closure-laminar matroids. Let C_1 and C_2 be circuits of M neither of which is contained in the closure of the other such that $|C_1 \cap C_2| \geq 2$ when $\mathcal{M}$ is the class of 2-laminar matroids while $r(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)) \geq 2$ otherwise. Then*

(i) $E(M) = C_1 \cup C_2$ if $\mathcal{M}$ is the class of 2-laminar matroids;

(ii) $E(M) = C_1 \cup C_2 \cup (\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2))$ if $\mathcal{M}$ is the class of 2-closure-laminar matroids;

(iii) M has $\mathrm{cl}(C_1)$ and $\mathrm{cl}(C_2)$ as hyperplanes, so $|C_1| = |C_2|$; and

(iv) if C is a circuit of M that meets both $C_1 - \mathrm{cl}(C_2)$ and $C_2 - \mathrm{cl}(C_1)$, then either C is spanning, or C contains $C_1 \triangle C_2$.

Proof. For (ii), if $f \in E(M) - (C_1 \cup C_2 \cup (\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)))$, then $C_i \subseteq \mathrm{cl}_{M\backslash f}(C_j)$ for some $\{i, j\} = \{1, 2\}$. Thus $C_i \subseteq \mathrm{cl}_M(C_j)$, a contradiction. Hence (ii) holds. Part (i) follows similarly. Certainly $C_2 - \mathrm{cl}(C_1)$ contains an element e. As $e \notin \mathrm{cl}(C_1)$, if $\{x, y\}$ is an independent subset of $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$, then $\{x, y\}$ is independent in M/e.

It follows, since $M/e \in \mathcal{M}$, that either $\mathrm{cl}_{M/e}(C_2 - e) \supseteq C_1$ or $\mathrm{cl}_{M/e}(C_1) \supseteq C_2 - e$. The former yields a contradiction. Hence $\mathrm{cl}_M(C_1 \cup e) \supseteq C_2$, so $\mathrm{cl}_M(C_1 \cup e) = E(M)$. Thus $\mathrm{cl}(C_1)$ is a hyperplane of M. By symmetry, so is $\mathrm{cl}(C_2)$. Hence $|C_1| = |C_2|$, so (iii) holds.

Now let C be a circuit of M that meets both $C_1 - \mathrm{cl}(C_2)$ and $C_2 - \mathrm{cl}(C_1)$. As $C - \mathrm{cl}(C_2)$ is non-empty, $|C - \mathrm{cl}(C_2)| \geq 2$, so $|C \cap C_1| \geq 2$. Suppose C is non-spanning. As $\mathrm{cl}(C_1)$ is a hyperplane and C meets $C_2 - \mathrm{cl}(C_1)$, it follows that $\mathrm{cl}(C_1) \not\supseteq \mathrm{cl}(C)$ and $\mathrm{cl}(C) \not\supseteq \mathrm{cl}(C_1)$. Since $|C \cap C_1| \geq 2$, if $E(M) - (C \cup C_1)$ contains an element e, then, as $M \backslash e \in \mathcal{M}$, we get a contradiction. Therefore $E(M) = C \cup C_1$. By symmetry, $E(M) = C \cup C_2$. Thus C contains $C_1 \triangle C_2$, so (iv) holds. $\square$

Theorem 3.6. *The excluded minors for the class of 2-laminar matroids are $M^-(K_{2,3})$, $M_n(2)$ for all $n \geq 4$, and $N_n(2)$ for all $n \geq 5$.*

Proof. Suppose that M is an excluded minor for the class of 2-laminar matroids. Then M has circuits C_1 and C_2 with $|C_1 \cap C_2| \geq 2$ such that neither C_1 nor C_2 is contained in the closure of the other. Thus each of $C_1 - \mathrm{cl}(C_2)$ and $C_2 - \mathrm{cl}(C_1)$ contains at least two elements. By Lemma 3.5(i), $E(M) = C_1 \cup C_2$. Moreover, $|C_1 \cap C_2| = 2$, otherwise we could contract an element of $C_1 \cap C_2$ and still get a matroid that is not 2-laminar. Let $\{a, b\} = C_1 \cap C_2$.

Suppose $g \in \mathrm{cl}(C_1) - C_1$. Leading up to 3.6.5, we shall prove four preliminary results.

3.6.1. *Suppose D is a circuit contained in $C_1 \cup g$ and containing $\{g, a, b\}$. Then $D \subseteq \mathrm{cl}(C_2)$.*

To see this, note that, as $\{a, b, g\} \subseteq D \cap C_2$, we deduce, since M/g is 2-laminar having $D - g$ and $C_2 - g$ as circuits, that $D - g \subseteq \mathrm{cl}_{M/g}(C_2 - g)$ or $C_2 - g \subseteq \mathrm{cl}_{M/g}(D - g)$. The latter implies that $C_2 \subseteq \mathrm{cl}(D)$. But $\mathrm{cl}(D) \subseteq \mathrm{cl}(C_1)$, so this yields a contradiction. Hence 3.6.1 holds.

3.6.2. *If D_1 and D_2 are circuits contained in $C_1 \cup g$ and containing $\{g, a, b\}$, then $D_1 = D_2$.*

Suppose $D_1 \neq D_2$. By Lemma 3.4(ii), $D_1 \cup D_2 = C_1 \cup g$. By 3.6.1, $D_i \subseteq \mathrm{cl}(C_2)$ for each i in $\{1, 2\}$. Thus $C_1 \subseteq \mathrm{cl}(C_2)$. This contradiction implies that 3.6.2 holds.

3.6.3. *If D_1 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g, then D_1 or D_2 contains $\{a, b\}$.*

Assume that this fails. By Lemma 3.4(ii), $D_1 \cup D_2 = C_1 \cup g$. We may suppose that $D_1 \cap \{a, b\} = \{a\}$ and $D_2 \cap \{a, b\} = \{b\}$. For each i in $\{1, 2\}$, assume that D_i avoids some element d_i of $C_1 - C_2$. Now $\mathrm{cl}(D_i) \subseteq \mathrm{cl}(C_1)$, so $C_2 \not\subseteq \mathrm{cl}(D_i)$. As $M \backslash d_i$ is 2-laminar, it follows that $D_i \subseteq \mathrm{cl}(C_2)$ for each i. Thus $C_1 \subseteq \mathrm{cl}(C_2)$, a contradiction. It follows that we may assume that $C_1 - C_2 \subseteq D_1$, so $D_1 = (C_1 - b) \cup g$.

Now $\{b, g\} \subseteq D_2$. As $\{a, b, g\} \subsetneq C_2$, we see that D_2 contains an element w of $C_1 - \{a, b\}$. Then $w \in D_1 \cap D_2$ and $b \in D_2 - D_1$. Hence there is a circuit D_3 contained in $(D_1 \cup D_2) - w$ and containing b. As g must be in D_3, it follows by Lemma 3.4(ii) that $D_3 \cup D_2 = C_1 \cup g$. Thus $a \in D_3$, so $\{a, g\} \subseteq D_3$. Hence, by 3.6.1, $D_3 \subseteq \mathrm{cl}(C_2)$. As $\{g, b\} \subseteq D_2 \cap D_3$ and $M \backslash e$ is 2-laminar for e in $C_2 - \mathrm{cl}(C_1)$, we deduce that $D_3 \subseteq \mathrm{cl}(D_2)$ otherwise $D_2 \subseteq \mathrm{cl}(D_3) \subseteq \mathrm{cl}(C_2)$, so $C_1 \subseteq \mathrm{cl}(C_2)$, a contradiction. We conclude that D_2 spans C_1. As $a \notin D_2$, we see that $D_2 = (C_1 - a) \cup g$.

Next take an element v of $D_3 - \{g, a, b\}$. Then $v \in D_1$ and $b \in D_3 - D_1$. Thus M has a circuit D_4 that contains b and is contained in $(D_1 \cup D_3) - v$. Clearly $g \in D_4$ and $D_4 \neq D_3$. Thus, by 3.6.2, $\{a, b\} \not\subseteq D_4$. Hence $D_4 \cap \{a, b\} = \{b\}$. But D_2 and D_4 are distinct circuits contained in $C_1 \cup g$ and each contains g. Yet $D_2 \cup D_4$ avoids a, which contradicts Lemma 3.4(ii). Thus 3.6.3 holds.

3.6.4. *Suppose D_1 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g. If $\{a, b\} \subseteq D_1$, then $D_1 \cap D_2 = \{g\}$, so $D_2 = C_1 \triangle D_1$.*

By 3.6.1, $D_1 \subseteq \mathrm{cl}(C_2)$. Now suppose that $\{a, b\} \cap D_2 \neq \emptyset$. Then, by 3.6.2, we may assume that $\{a, b\} \cap D_2 = \{b\}$. As $D_1 \cup D_2 = C_1 \cup g$, we see that $D_2 \not\subseteq \mathrm{cl}(C_2)$, otherwise $C_1 \subseteq \mathrm{cl}(C_2)$. If D_2 does not contain $C_1 - C_2$, then, for f in $(C_1 - C_2) - D_2$, the matroid $M\backslash f$ is 2-laminar and so $C_2 \subseteq \mathrm{cl}(D_2) \subseteq \mathrm{cl}(C_1)$, a contradiction. Thus $C_1 - C_2 \subseteq D_2$, so $D_2 = (C_1 - a) \cup g$. Take x in $(D_1 \cap D_2) - \{g, b\}$. Then M has a circuit D_3 contained in $(D_1 \cup D_2) - x$ and containing a. Clearly $g \in D_3$, so D_3 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g. It follows by 3.6.3 that D_3 or D_2 contains $\{a, b\}$. Thus, by 3.6.2, D_3 or D_2 is D_1. But $D_3 \neq D_1$ as $x \in D_1 - D_3$; and $D_2 \neq D_1$ by assumption. We conclude that $\{a, b\} \cap D_2 = \emptyset$.

Finally, suppose that $h \in (D_1 \cap D_2) - g$. Then, as $M\backslash e$ is 2-laminar for e in $C_2 - \mathrm{cl}(C_1)$, we deduce that $D_1 \subseteq \mathrm{cl}(D_2)$ or $D_2 \subseteq \mathrm{cl}(D_1)$. As $D_1 \subseteq \mathrm{cl}(C_2)$ and $C_1 \not\subseteq \mathrm{cl}(C_2)$, it follows that $D_1 \subseteq \mathrm{cl}(D_2)$, so D_2 spans C_1. But D_2 avoids $\{a, b\}$, so $r(D_2) \leq |C_1| - 2$, a contradiction. Thus 3.6.4 holds.

On combining 3.6.1–3.6.4, we obtain the following.

3.6.5. *If $g \in \mathrm{cl}(C_1) - C_1$, then there are circuits G and G' that meet in $\{g\}$ such that $G \cup G' = C_1 \cup g$ and $\{a, b\} \subseteq G \subseteq \mathrm{cl}(C_2)$. Furthermore, G, G', and C_1 are the only circuits contained in $C_1 \cup g$, so $\mathrm{cl}(G) - C_2 = G - C_2$.*

As there are at least two elements in each of $C_2 - \mathrm{cl}(C_1)$, $C_1 - \mathrm{cl}(C_2)$, and $C_1 \cap C_2$, it follows that $r(M) \geq 4$. Next we show

3.6.6. $|\mathrm{cl}(C_1) - C_1| = |\mathrm{cl}(C_2) - C_2| \leq 1$.

Suppose that $\mathrm{cl}(C_1) - C_1 = \{g_1, g_2, \ldots, g_t\}$ where $t \geq 2$. For each i in $\{1, 2, \ldots, t\}$, let G_i and G'_i be the associated circuits given by 3.6.5 whose union is $C_1 \cup g_i$, where $\{a, b, g_i\} \subseteq G_i$ and $G'_i = C_1 \triangle G_i$. By 3.6.1, $G_i \subseteq \mathrm{cl}(C_2)$. For distinct i and j in $\{1, 2, \ldots, t\}$, as G_i and G_j are distinct circuits contained in the 2-laminar matroid $M|\mathrm{cl}(C_1)$, and $|G_i \cap G_j| \geq 2$, the closures of $G_1, G_2, \ldots, G_t$ form a chain under inclusion. Say $\mathrm{cl}(G_1) \supseteq \mathrm{cl}(G_2) \supseteq \cdots \supseteq \mathrm{cl}(G_t)$. Since $\mathrm{cl}(G_i) - C_2 = G_i - C_2$, it follows that $G_1 - C_2 \supseteq G_2 - C_2 \supseteq \cdots \supseteq G_t - C_2$. Now let $\{f_1, f_2, \ldots, f_s\} = \mathrm{cl}(C_2) - C_2$. For each f_i, there are circuits F_i and F'_i whose union is $C_2 \cup f_i$ such that $\{a, b\} \subseteq F_i$ and $F'_i = C_2 \triangle F_i$. Moreover, we may assume that $\mathrm{cl}(F_1) \supseteq \mathrm{cl}(F_i)$ for all i.

By 3.6.5, for all i,

$$F_i - \{a, b, f_i\} \subseteq \mathrm{cl}(C_1) - C_1 = \{g_1, g_2, \ldots, g_t\} \subseteq \mathrm{cl}(G_1).$$

Thus $F_i - f_i \subseteq \mathrm{cl}(G_1)$ so $f_i \in \mathrm{cl}(G_1)$. Hence, by 3.6.5, $f_i \in G_1$. Moreover, as $F_i \subseteq \{f_i, a, b, g_1, g_2, \ldots, g_t\}$ and $\{a, b, g_1, g_2, \ldots, g_t\} \subseteq \mathrm{cl}(G_1)$, we see that $\mathrm{cl}(F_i) \subseteq \mathrm{cl}(G_1)$. Since $\{f_1, f_2, \ldots, f_s\} \subseteq G_1$, we deduce, since $G_1 \subseteq \mathrm{cl}(C_2)$, that $G_1 = \{g_1, a, b, f_1, f_2, \ldots, f_s\}$. As $\mathrm{cl}(F_1) \subseteq \mathrm{cl}(G_1)$, it follows by symmetry that $\mathrm{cl}(F_1) = \mathrm{cl}(G_1)$. Moreover, symmetry also gives that $F_1 = \{f_1, a, b, g_1, g_2, \ldots, g_t\}$. Since G_1

and F_1 are both circuits spanning the same set, they have the same cardinality, so $t = s$; that is,

$$|\mathrm{cl}(C_1) - C_1| = |\mathrm{cl}(C_2) - C_2|.$$

By Lemma 3.3, since $\{g_1, g_2, \dots, g_t\}$ is independent, we get that $G_i - g_i \neq G_j - g_j$ for distinct i and j. Thus

$$t + 3 = |G_1| > |G_2| > \cdots > |G_t| \geq 4$$

where the last inequality follows because G_t is not a proper subset of C_2.

Now suppose that $|G_2| = |G_1| - 1$ where $(G_1 - g_1) - (G_2 - g_2) = \{f_i\}$. Choose $e \in C_1 - \mathrm{cl}(C_2)$. As $f_i \in G_2' - G_1'$, strong circuit elimination on G_1' and G_2', both of which contain e, yields a circuit D containing f_i and avoiding e. Since D avoids $\{a, b\}$, it follows that $\{g_1, g_2\} \subseteq D$. As $e \notin C_2 \cup D$, we deduce that $D \subseteq \mathrm{cl}(C_2)$, otherwise we obtain the contradiction that $C_2 \subseteq \mathrm{cl}(D) \subseteq \mathrm{cl}(C_1)$. But $(G_2' - g_2) - (G_1' - g_1) = \{f_i\}$, so $D \subseteq G_2' \cup g_1$, and $(G_2' \cup g_1) \cap \mathrm{cl}(C_2) = \{f_i, g_1, g_2\}$. As $D \subseteq \mathrm{cl}(C_2)$, it follows that $D \subseteq \{f_i, g_1, g_2\}$. This is a contradiction to 3.6.5 because $D \notin \{F_i, F_i'\}$. We deduce that $|G_2| \leq |G_1| - 2$. Thus $|G_2| \leq t + 3 - 2 = t + 1$. Hence $|G_t| \leq 3$, a contradiction. We conclude that 3.6.6 holds.

By Lemma 3.5(iii), $\mathrm{cl}(C_1)$ and $\mathrm{cl}(C_2)$ are hyperplanes of M, and $|C_1| = |C_2|$. Suppose that $\mathrm{cl}(C_1) = C_1$. Then, by 3.6.6, $\mathrm{cl}(C_2) = C_2$. As $E(M) = C_1 \cup C_2$, every circuit of M other than C_1 or C_2 must meet both $C_1 - C_2$ and $C_2 - C_1$. Assume M has such a circuit C that is non-spanning. Then, by Lemma 3.5(iii), $C_1 \triangle C_2 \subseteq C$. As $|C_1 \cap C_2| = 2$ but C is non-spanning, it follows that $C = C_1 \triangle C_2$. Thus $r(C) = r(M) - 1$, so $r(M) - 1 = |C_1| + |C_2| - 5$. But $r(M) - 1 = r(C_1) = |C_1| - 1$. Hence $|C_2| = 4$, so $|C_1| = 4$. It follows easily that $M \cong M(K_{2,3}) \cong M_4(2)$. Now suppose that every circuit other than C_1 or C_2 is spanning. Then, letting $|C_1| = n$, we see that $|C_2| = n$ and $r(M) = r(C_1) + 1 = n$. It follows that $M \cong M(K_{2,3}^-)$ when $n = 4$, while $M \cong M_n(2)$ when $n \geq 5$.

By 3.6.6, we may now suppose that $\mathrm{cl}(C_1) - C_1 = \{g\}$. Then $\mathrm{cl}(C_2) - C_2 = \{f\}$, say. By 3.6.5, $\{a, b, g, f\}$ is a circuit of M as are both $G' = (C_1 - \{a, b, f\}) \cup \{g\}$ and $F' = (C_2 - \{a, b, g\}) \cup \{f\}$. All circuits of M other than $C_1, C_2, \{a, b, g, f\}, G'$, and F' must meet both $C_1 - \mathrm{cl}(C_2)$ and $C_2 - \mathrm{cl}(C_1)$. Hence, by Lemma 3.5(iv), every such circuit is spanning as $C_1 \triangle C_2$ properly contains G'. Again letting $|C_1| = n$, we see that $|C_2| = n$ and $r(M) = n$. Thus $M \cong N_n(2)$ for some $n \geq 5$. □

Theorem 3.7. *The excluded minors for the class of 2-closure-laminar matroids are $M^-(K_{2,3})$, $M_n(2)$ for all $n \geq 4$, and $P_n(2)$ for all $n \geq 4$.*

Proof. Let M be an excluded minor for the class of 2-closure-laminar matroids. Clearly M is simple. Now M has two circuits C_1 and C_2 with $r(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)) \geq 2$ such that neither is a subset of the closure of the other. By Lemma 3.5(ii),

3.7.1. $E(M) = C_1 \cup C_2 \cup (\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2))$.

Clearly $|C_1 \cap C_2| \leq 2$, otherwise we could contract an element of $C_1 \cap C_2$ and still have a matroid that is not 2-closure-laminar. We break the rest of the proof into three cases based on the size of $C_1 \cap C_2$.

3.7.2. $C_1 \cap C_2 \neq \emptyset$.

Assume the contrary. Let $\{x, y\}$ be a subset of $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. To show 3.7.2, we first establish that

3.7.3. $\{x, y\} \not\subseteq C_2$.

Suppose $\{x, y\} \subseteq C_2$. As $M|(C_1 \cup \{x, y\})$ is connected, there is a circuit D_1 with $\{x, y\} \subseteq D_1 \subseteq C_1 \cup \{x, y\}$. Then, for c in $C_1 - D_1$, the matroid $M\backslash c$ is 2-closure-laminar. Now $C_2 \not\subseteq \mathrm{cl}(D_1)$ since $\mathrm{cl}(D_1) \subseteq \mathrm{cl}(C_1)$. Thus $D_1 \subseteq \mathrm{cl}(C_2)$, so $C_1 \cap \mathrm{cl}(C_2)$ is non-empty. Choose an element z in $C_1 \cap \mathrm{cl}(C_2)$. Now M has circuits C_x and C'_x, with $x \in C_x \cap C'_x$, and $C_x \cup C'_x = C_1 \cup x$. It also has circuits C_y, and C'_y with $y \in C_y \cap C'_y$ and $C_y \cup C'_y = C_1 \cup y$. We may assume that $z \in C_x \cap C_y$. Then $\{x, z\} \subseteq \mathrm{cl}(C_x) \cap \mathrm{cl}(C_2)$. As $C_1 - (C_x \cup C_2)$ is non-empty, this implies that $C_x \subseteq \mathrm{cl}(C_2)$ since $C_2 \not\subseteq \mathrm{cl}(C_x)$ because $\mathrm{cl}(C_x) \subseteq \mathrm{cl}(C_1)$. Similarly, $C_y \subseteq \mathrm{cl}(C_2)$.

Suppose $(C_x - x) \cap (C'_x - x)$ is non-empty and choose e in this set. Then, as $\{e, x\} \subseteq C_x \cap C'_x$ and $y \notin C_x \cup C'_x$, either $C_x \subseteq \mathrm{cl}(C'_x)$ or $C'_x \subseteq \mathrm{cl}(C_x)$. In the latter case, $C'_x \subseteq \mathrm{cl}(C_x) \subseteq \mathrm{cl}(C_2)$, so $C_1 \subseteq \mathrm{cl}(C_2)$, a contradiction. Thus $C_x \subseteq \mathrm{cl}(C'_x)$. But then C_1 and C'_x have the same rank, and hence the same size. Then $C'_x = C_1 \triangle \{x, c\}$ for some $c \in C_1$. Now consider the 2-closure-laminar matroid $M\backslash c$. In it, C'_x and C_2 are circuits as $c \notin C_2$. Then $r_{M\backslash c}(\mathrm{cl}_{M\backslash c}(C'_x) \cap \mathrm{cl}_{M\backslash c}(C_2)) \geq 2$ so $C'_x \subseteq \mathrm{cl}_{M\backslash c}(C_2)$ or $C_2 \subseteq \mathrm{cl}_{M\backslash c}(C'_x)$. As $C_x \subseteq \mathrm{cl}_M(C_2)$ and $\mathrm{cl}_{M\backslash c}(C'_x) \subseteq \mathrm{cl}_M(C_1)$, we obtain the contradiction that $C_1 \subseteq \mathrm{cl}(C_2)$ or $C_2 \subseteq \mathrm{cl}(C_1)$. We conclude that $C_x \cap C'_x = \{x\}$. Likewise $C_y \cap C'_y = \{y\}$.

If there is some element f in $C_x \cap C'_y$, then, as $f \in C_x$, we have $f \in \mathrm{cl}(C_2)$. But then $\{f, y\} \subseteq \mathrm{cl}(C'_y) \cap \mathrm{cl}(C_2)$ Thus either $C'_y \subseteq \mathrm{cl}(C_2)$ or $C_2 \subseteq \mathrm{cl}(C'_y)$. The former cannot occur as $C_y \subseteq \mathrm{cl}(C_2)$; nor can the latter as $\mathrm{cl}(C'_y) \subseteq \mathrm{cl}(C_1)$. Hence $C_x \cap C'_y = \emptyset$. Likewise, $C_y \cap C'_x = \emptyset$. But then $C_x \triangle \{x, y\} = C_y$ and $C'_x \triangle \{x, y\} = C'_y$. Hence, by Lemma 3.3, x and y are parallel, a contradiction. Thus 3.7.3 holds.

Next we suppose that $x \in C_2$ and $y \notin C_2$. Choose a circuit D with $\{x, y\} \subseteq D \subseteq C_2 \cup y$. Then $D \subseteq \mathrm{cl}(C_1)$, since $C_2 - (D \cup C_1) \neq \emptyset$ and $\mathrm{cl}(D) \cap \mathrm{cl}(C_1)$ has rank at least two, while $\mathrm{cl}(D) \subseteq \mathrm{cl}(C_2)$. Now $(D \cap C_2) - x$ certainly contains some element d. Then $\{x, d\} \subseteq \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Applying 3.7.3 gives a contradiction.

We may now assume that $\{x, y\} \cap C_2 = \emptyset$. Let D be a circuit with $\{x, y\} \subseteq D \subseteq C_2 \cup \{x, y\}$. Then $D \subseteq \mathrm{cl}(C_1)$ as $C_2 - (D \cup C_1) \neq \emptyset$. By replacing y by an element of $D \cap C_2$, we revert to the case eliminated in the last paragraph. Hence 3.7.2 holds.

Now, we consider the case when $|C_1 \cap C_2| = 1$. Let $C_1 \cap C_2 = \{x\}$ and choose y in $(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)) - (C_1 \cap C_2)$. Suppose $y \notin C_1 \cup C_2$. Then M has circuits D_1 and D_2 containing $\{x, y\}$ and contained in $C_1 \cup y$ and $C_2 \cup y$, respectively. Without loss of generality, as $E(M) - (D_1 \cup D_2)$ is non-empty, we may assume that $D_1 \subseteq \mathrm{cl}(D_2)$. Since $|D_1| \geq 3$, there is an element z of $D_1 - \{x, y\}$. Then z is in $\mathrm{cl}(D_2)$ and so is in $\mathrm{cl}(C_2)$. Thus $\{x, z\} \subseteq C_1 \cap \mathrm{cl}_{M\backslash y}(C_2)$ and we obtain a contradiction. It follows, by 3.7.1, that $C_1 \cup C_2 = E(M)$.

We may now assume that $y \in C_1 \cap \mathrm{cl}(C_2)$. Then M has a circuit D such that $\{x, y\} \subseteq D \subseteq C_2 \cup y$. Clearly $D \subseteq \mathrm{cl}(C_2)$. To see that $D \subseteq \mathrm{cl}(C_1)$, we note that $C_2 - (D \cup C_1) \neq \emptyset$, and $C_1 \not\subseteq \mathrm{cl}(D)$ as $D \subseteq \mathrm{cl}(C_2)$. We now have $D - x \subseteq \mathrm{cl}_{M/x}(C_1 - x) \cap \mathrm{cl}_{M/x}(C_2 - x)$. Thus $r_{M/x}(D - x) \leq 1$, otherwise, for some $\{i, j\} = \{1, 2\}$, we have $\mathrm{cl}_{M/x}(C_i - x) \subseteq \mathrm{cl}_{M/x}(C_j - x)$, a contradiction. As $y \in D - x$, we see that $r_M(D) = 2$, so $D = \{x, y, y'\}$ for some y'. We deduce that

3.7.4. $\{x, y, y'\}$ *is the only circuit of* $M|(C_2 \cup y)$ *containing* $\{x, y\}$.

We show next that

3.7.5. $(C_2 - \{x, y'\}) \cup y$ *is the only circuit of* $M|(C_2 \cup y)$ *containing* y *but not* x.

By Lemma 3.4(ii), every circuit D' of M that contains y, avoids x, and is contained in $C_2 \cup y$ must contain $(C_2 - \{x, y'\}) \cup y$. If $y' \in D'$, then $D' = (C_2 \cup y) - x$. Using D' and D, we find a circuit D'' containing x and contained in $(C_2 \cup y) - y'$. As D'' must also contain y, we see that $\{x, y\} \subseteq D''$ and we showed in 3.7.4 that M has no such circuit. We conclude that 3.7.5 holds.

By 3.7.5 and symmetry, M has $(C_1 - \{x, y\}) \cup y'$ as a circuit, say C_1'. Let C_2' be the circuit $(C_2 - \{x, y'\}) \cup y$. Next we note that

3.7.6. $\mathrm{cl}(C_2) - C_2 = \{y\}$ *and* $\mathrm{cl}(C_1) - C_1 = \{y'\}$.

Assume there is an element y_1 in $(\mathrm{cl}(C_2) - C_2) - y$. Then $\{y, y_1\}$ is a subset of $\mathrm{cl}_{M/x}(C_1 - x) \cap \mathrm{cl}_{M/x}(C_2 - x)$ that is independent in M/x. Thus $C_i - x \subseteq \mathrm{cl}_{M/x}(C_j - x)$ for some $\{i, j\} = \{1, 2\}$, so $C_i \subseteq \mathrm{cl}_M(C_j)$, a contradiction. It follows that $\mathrm{cl}(C_2) - C_2 = \{y\}$. By symmetry, $\mathrm{cl}(C_1) - C_1 = \{y'\}$.

By Lemma 3.5(iii), M has $\mathrm{cl}(C_1)$ and $\mathrm{cl}(C_2)$ as hyperplanes, and $|C_1| = |C_2|$. Let C be a circuit of M that is not C_1, C_2, C_1', C_2', or D. If $y \in C \subseteq C_2 \cup y$, then, by 3.7.4 and 3.7.5, C is D or C_2'. We deduce that C meets both $C_1 - \mathrm{cl}(C_2)$ and $C_2 - \mathrm{cl}(C_1)$. Then, by Lemma 3.5(iii), either C is spanning, or C contains $C_1 \triangle C_2$. But $|C_1 \cap C_2| = 1$ so C is spanning. We conclude that C_1, C_2, C_1', C_2', and D are the only non-spanning circuits of M. Hence $M \cong P_n(2)$ for some $n \geq 4$.

Finally, suppose $|C_1 \cap C_2| = 2$. Then M is not 2-laminar so it has as a minor one of the matroids identified in Theorem 3.6. But M cannot have a $N_n(2)$-minor for any $n \geq 5$ as this matroid has $P_{n-1}(2)$, an excluded minor for the class of 2-closure-laminar matroids, as a proper minor. Thus M has as a minor $M^-(K_{2,3})$ or $M_n(2)$ for some $n \geq 4$. The result follows by Lemmas 3.1 and 3.2. □

Our methods for finding the excluded minors for the classes of k-laminar and k-closure-laminar matroids for $k = 2$ do not seem to extend to larger values of k.

4. Intersections with other classes of matroids

We now discuss how the classes of k-closure-laminar and k-laminar matroids relate to some other well-known classes of matroids. Finkelstein [11] showed that all laminar matroids are gammoids, so they are representable over all sufficiently large fields [20, 15]. An immediate consequence of the following easy observation is that, for all $k \geq 2$, if M is a k-closure-laminar matroid or a k-laminar matroid, then M need not be representable and hence M need not be a gammoid.

Proposition 4.1. *If $r(M) \leq k + 1$, then M is k-laminar and k-closure-laminar.*

We use the next lemma to describe the intersection of the classes of 2-laminar and 2-closure-laminar matroids with the classes of binary and ternary matroids. Recall that $M_4(2) \cong M(K_{2,3})$ and that the definitions of $P_n(2)$ and $N_n(2)$ require that $n \geq 4$ and $n \geq 5$, respectively.

Lemma 4.2. *The matroid $M^-(K_{2,3})$ is ternary and non-binary; $P_n(2)$ has a $U_{n,2n-3}$-minor; $N_n(2)$ has a $U_{n,2n-4}$-minor; and $M_n(2)$ has a $U_{n,2n-3}$-minor when $n \geq 5$.*

Proof. The first part follows because $M^-(K_{2,3})$ can be obtained from $U_{2,4}$ by adding elements in series to two elements of the latter. Next we note that we get $U_{n,2n-3}$ from $P_n(2)$ by deleting the basepoints of the parallel connections involved in its construction. Deleting the basepoints of the parallel connections involved in producing $N_n(2)$ gives $U_{n,2n-4}$. Finally, when $n \geq 5$, we get $U_{n,2n-3}$ from $M_n(2)$ by deleting an element of the path P. $\square$

The next two results follow without difficulty by combining the last lemma with Theorems 3.6 and 3.7 as the set of excluded minors for $\mathcal{M} \cap \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are minor-closed classes of matroids consists of the minor-minimal matroids that are excluded minors for $\mathcal{M}$ or $\mathcal{N}$ (see, for example, [18, Lemma 14.5.1]). Recall that $N_5(2)$ and $P_4(2)$ are the matroids obtained by adjoining, via parallel connection, two triangles across distinct elements of a 4-circuit and a triangle, respectively.

Corollary 4.3. *A matroid M is binary and 2-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, or $N_5(2)$.*

Corollary 4.4. *A matroid M is binary and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, or $P_4(2)$.*

Similarly, we find the excluded minors for the classes of ternary 2-laminar matroids and ternary 2-closure-laminar matroids by noting that deleting an element from F_7^* produces $M(K_{2,3})$, so F_7^* is not 2-laminar.

Corollary 4.5. *A matroid M is ternary and 2-laminar if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7, $M^-(K_{2,3})$, $M(K_{2,3})$, or $N_5(2)$.*

Corollary 4.6. *A matroid M is ternary and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7, $M^-(K_{2,3})$, $M(K_{2,3})$, or $P_4(2)$.*

Next we describe the intersection of the class of graphic matroids with the classes of 2-laminar and 2-closure-laminar matroids both constructively and via excluded minors.

Corollary 4.7. *A matroid M is graphic and 2-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, F_7, $M^*(K_{3,3})$, or $N_5(2)$.*

Corollary 4.8. *A matroid M is graphic and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, F_7, or $P_4(2)$.*

Lemma 4.9. *Let M be a simple, connected, graphic matroid. Then M is 2-laminar if and only if M is a coloop, M is isomorphic to $M(K_4)$, or M is the cycle matroid of a graph consisting of a cycle with at most two chords such that, when there are two chords, they are of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2.*

Proof. Clearly each of the specified matroids is 2-laminar. Now let G be a simple, 2-connected graph. Suppose first that G is not outerplanar. By a theorem of Chartrand and Harary [6], either G is K_4 or G has $K_{2,3}$ as a minor. In the latter case, $M(G)$ is not 2-laminar. Now suppose that G is outerplanar. If G has two chords that are not of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2, then $M(G)$ has $N_4(2)$ as a minor, and so is not 2-laminar. $\square$

Proposition 4.10. *Let M be a simple, connected, graphic matroid. Then M is 2-closure-laminar if and only if M is a coloop, M is isomorphic to $M(K_4)$, or M is the cycle matroid of a cycle with at most one chord.*

Proof. This follows from Lemma 4.9 by noting that the cycle matroid of a cycle with two chords of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2 has $P_4(2)$ as a minor. $\square$

We now show that the intersections of the classes of k-laminar and k-closure-laminar matroids with the class of paving matroids coincide.

Theorem 4.11. *Let M be a paving matroid, and k be a non-negative integer. Then M is k-laminar if and only if M is k-closure-laminar.*

Proof. By Proposition 2.1(i), it suffices to prove that if M is not k-closure-laminar, then M is not k-laminar. We use the elementary observation that, since M is paving, for every flat F, either $F = E(M)$, or $M|F$ is uniform. Suppose that C_1 and C_2 are circuits of M for which $r(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)) \geq k$ but neither $\mathrm{cl}(C_1)$ nor $\mathrm{cl}(C_2)$ is contained in the other. Then neither $\mathrm{cl}(C_1)$ nor $\mathrm{cl}(C_2)$ is spanning. Hence both $M|\mathrm{cl}(C_1)$ and $M|\mathrm{cl}(C_2)$ are uniform. Let X be a basis of $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Then M has circuits C_1' and C_2' containing X such that $\mathrm{cl}(C_i') = \mathrm{cl}(C_i)$ for each i. Thus M is not k-laminar. □

It is well known that the unique excluded minor for the class of paving matroids is $U_{0,1} \oplus U_{2,2}$. Using this, in conjunction with Theorems 3.6 and 4.11, it is not difficult to obtain the following.

Corollary 4.12. *The following are equivalent for a matroid M.*

(i) *M is 2-laminar and paving;*
(ii) *M is 2-closure-laminar and paving;*
(iii) *M has no minor in $\{U_{0,1} \oplus U_{2,2}, M^-(K_{2,3})\} \cup \{M_n(2) : n \geq 4\}$.*

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References

[1] Ardila, F., The Catalan matroid, *J. Combin. Theory Ser. A* **104** (2003), 49–62.

[2] Babaioff, M., Immorlica, N., and Kleinberg, R., Matroids, secretary problems, and online mechanisms, Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 434–443, ACM, New York, 2007.

[3] Bonin, J. and De Mier, A., The lattice of cyclic flats of a matroid, *Ann. Combin.* **12** (2008), 155–170.

[4] Bonin, J., De Mier, A., and Noy, M., Lattice path matroids: enumerative aspects and Tutte polynomials, *J. Combin. Theory Ser. A* **104** (2003), 63–94.

[5] Chakraborty, S. and Lachish, O., Improved competitive ratio for the matroid secretary problem, *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, 1702–1712, ACM, New York, 2012.

[6] Chartrand, G. and Harary, F., Planar permutation graphs, *Ann. Inst. Henri Poincaré* **3** (1967), 433–438.

[7] Crapo, H., Single-element extensions of matroids. *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 55–65.

[8] Crapo, H. and Schmitt, W., A free subalgebra of the algebra on matroids, *European J. Combin.* **26** (2005), 1066–1085.

[9] Feldman, M., Svensson, O., and Zenklusen, R., A simple $O(loglog(rank))$-competitive algorithm for the matroid secretary problem, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1189–1201, SIAM, Philadelphia, PA, 2015.

[10] Fife, T. and Oxley, J., Laminar matroids, *European J. Combin.* **62** (2017), 206–216.

[11] Finkelstein, L., Two algorithms for the matroid secretary problem, Master's thesis, Technion, Israel Institute of Technology, Haifa, 2011.

[12] Gabow, H. and Kohno, T., A network-flow-based scheduler: design, performance history and experimental analysis, Second Workshop on Algorithmic Engineering and Experiments (ALENEX 00) (San Francisco, CA), *ACM J. Exp. Algorithmics* **6** (2001), 1–14.

[13] Huynh, T., The matroid secretary problem, The Matroid Union, Posted on September 7, 2015, http://matroidunion.org/?p=1487

[14] Im, S. and Wang, Y., Secretary problems: laminar matroids and interval scheduling, *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, 1265–1274, SIAM, Philadelphia, PA, 2011.

[15] Ingleton A. W. and Piff, M. J., Gammoids and transversal matroids, *J. Combin. Theory Ser. B* **15** (1973), 51–68.

[16] Jaillet, P., Soto, J., and Zenklusen, R., Advances on matroid secretary problems: free order model and laminar case, *Integer programming and combinatorial optimization*, 254–265, Lecture Notes in Comput. Sci., 7801, Springer, Heidelberg, 2013.

[17] Kaparis, K. and Letchford, A., On laminar matroids and b-matchings, submitted.

[18] Oxley, J., *Matroid Theory,* Second edition, Oxford University Press, New York, 2011.

[19] Oxley, J., Prendergast, K., and Row, D., Matroids whose ground sets are domains of functions, *J. Austral. Math. Soc. Ser. A* **32** (1982), 380–387.

[20] Piff, M. J. and Welsh, D. J. A., On the vector represntation of matroids, *J. London Math. Soc. (2)* **2** (1970), 284–288.

[21] Sohoni, M., Rapid mixing of some linear matroids and other combinatorial objects, *Graphs Combin.* **15** (1999), 93–107.

[22] Soto, J., Matroid secretary problem in the random assignment model, *SIAM J. Comput.* **42** (2013), 178–211.

E-mail address: tfife2@lsu.com

E-mail address: oxley@math.lsu.edu

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana, USA