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A Note on the Non-spanning Circuits of a Matroid

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In this note we determine when a collection of subsets of a set E is the set of non-spanning circuits of a matroid on E. This result is then used to characterize when a matroid on E is uniquely determined by its set of non-spanning circuits.

Certain sets in a matroid are forced to be dependent simply because their size exceeds the matroid's rank. On the other hand, the small dependent sets, those whose sizes do not exceed the rank, convey the crucial information about the matroid's structure. More specifically, if $\mathscr{C}'$ is the set of non-spanning circuits of a rank-k matroid on a set E, then clearly the set $\mathscr{C}$ of circuits of M is given by

$$\mathscr{C} = \mathscr{C}' \cup \{X \subseteq E: |X| = k+1 \text{ and } X \text{ contains no member of } \mathscr{C}'\}. \tag{1}$$

Thus a matroid on E is uniquely determined by its set of non-spanning circuits provided that its rank has been specified. If the rank has not been fixed, then the set of non-spanning circuits may not uniquely determine the matroid. For example, all uniform matroids on E have $\varnothing$ as their set of non-spanning circuits. In this note, we study the non-spanning circuits of a matroid. We shall begin by specifying the relationship between two matroids on E that have the same set of non-spanning circuits. Then we determine when a collection of subsets of E is the set of non-spanning circuits of some matroid on E. Finally, we combine these two results to specify precisely when a matroid on E is uniquely determined by its set of non-spanning circuits.

If M is a matroid, then we shall denote its rank and ground set by $r(M)$ and $E(M)$, respectively. We shall follow Welsh [1] for any unexplained matroid terminology. If $r(M) \geqslant 1$, then $T(M)$, the *truncation* of M, is the matroid on $E(M)$ the bases of which are the independent sets of M of cardinality $r(M)-1$. This operation can be iterated so that if $r(M) \geqslant j$, then $T^j(M) = T(T^{j-1}(M))$. The operation of truncation is the key to the next result.

(2) PROPOSITION. *Let $\mathscr{C}'$ be a collection of subsets of a set E. Suppose that there is a non-empty set $\mathscr{T}$ of matroids on E for which $\mathscr{C}'$ is the set of non-spanning circuits. Then, for some non-negative integer n and some matroid M,*

$$\mathscr{T} = \{M, T(M), T^2(M), \ldots, T^n(M)\}.$$

PROOF. Let $r_1 = \max\{j: \mathscr{C}' \text{ is the set of non-spanning circuits of a rank-}j \text{ matroid on } E\}$. If $\mathscr{C}' \neq \varnothing$, let $r_2 = \max\{|C|: C \in \mathscr{C}'\}$; otherwise let $r_2 = 0$. Evidently if $N \in \mathscr{T}$, then $r_2 \leqslant r(N) \leqslant r_1$. Moreover, for all j in $\{0, 1, 2, \ldots, r_1 - r_2\}$, $T^j(M)$ is a rank-$(r_1 - j)$ matroid on E having $\mathscr{C}'$ as its set of non-spanning circuits. As $T^j(M)$ must be the unique such matroid, the proposition follows. □

The following theorem, which characterizes when a collection of sets is the set of non-spanning circuits of some matroid, should be compared with the well-known characterization of a matroid in terms of its full set of circuits. Since the complement of

a non-spanning circuit is a dependent hyperplane of the dual, this theorem can be used to characterize when a collection of sets is the set of dependent hyperplanes of some matroid.

(3) THEOREM. *Let $\mathscr{C}'$ be a collection of subsets of a set E and k be a non-negative integer. Then $\mathscr{C}'$ is the set of non-spanning circuits of a rank-k matroid on E iff $\mathscr{C}'$ has the following properties*:
(3.1) *No member of $\mathscr{C}'$ properly contains another.*
(3.2) *If C_1 and C_2 are distinct members of $\mathscr{C}'$, $e \in C_1 \cap C_2$, and $|(C_1 \cup C_2) - e| \leq k$, then $(C_1 \cup C_2) - e$ contains a member of $\mathscr{C}'$.*
(3.3) *All members of $\mathscr{C}'$ have at most k elements.*
(3.4) *E has a k-element subset that contains no member of $\mathscr{C}'$.*

PROOF. Evidently, if $\mathscr{C}'$ is the collection of non-spanning circuits of a rank-k matroid on E, then (3.1)–(3.4) hold. To prove the converse, suppose that $\mathscr{C}'$ satisfies (3.1)–(3.4) and define $\mathscr{C}$ as in (1). Clearly, $\varnothing \notin \mathscr{C}$ and no member of $\mathscr{C}$ properly contains another. To show that $\mathscr{C}$ is the set of circuits of a matroid on E, it remains to check that $\mathscr{C}$ satisfies the circuit elimination axiom. Suppose that C_1 and C_2 are distinct members of $\mathscr{C}$ and $e \in C_1 \cap C_2$. The elimination axiom follows by (3.2) unless $|(C_1 \cup C_2) - e| \geq k+1$. In the exceptional case, the definition of $\mathscr{C}$ guarantees that if $(C_1 \cap C_2) - e$ does not contain a member of $\mathscr{C}'$, then it must contain a member of $\mathscr{C} - \mathscr{C}'$. We conclude that $\mathscr{C}$ is indeed the set of circuits of a matroid M on E. Moreover, the definition of $\mathscr{C}$ implies that $r(M) \leq k$ and, by (3.4), $r(M) \geq k$. Hence M has rank k and $\mathscr{C}'$ is its set of non-spanning circuits. □

On combining the last two results, we obtain the following characterization of when a matroid on a fixed non-empty set is uniquely determined by its set of non-spanning circuits.

(4) THEOREM. *Let M be a matroid on a non-empty set E and $\mathscr{C}'$ be its set of non-spanning circuits. Then M is the unique matroid on E having $\mathscr{C}'$ as its set of non-spanning circuits iff the following conditions hold*:
(4.1) *$\mathscr{C}'$ is non-empty and* $\max\{|C|: C \in \mathscr{C}'\} = r(M)$.
(4.2) *Either (i) every $(r(M)+1)$-element subset of E contains a member of $\mathscr{C}'$; or (ii) there is an $(r(M)+1)$-element subset X of E and an element e of $E - X$ such that X contains no member of $\mathscr{C}'$ but $X \cup e$ contains at least two members of $\mathscr{C}'$.*

PROOF. Suppose that M is the only matroid on E having $\mathscr{C}'$ as its set of non-spanning circuits. Then clearly $\mathscr{C}'$ is non-empty. Moreover, $r(M) = \max\{|C|: C \in \mathscr{C}'\}$; otherwise $T(M)$ is a matroid that is distinct from M but has $\mathscr{C}'$ as its set of non-spanning circuits. Assume next that neither (4.2)(i) nor (4.2)(ii) holds, and let $k = r(M)+1$. If (3.1)–(3.4) hold with this choice of k, then Theorem 3 implies the contradiction that $\mathscr{C}'$ is the set of non-spanning circuits of a rank-$(r(M)+1)$ matroid on E. Thus at least one of (3.1)–(3.4) fails for $\mathscr{C}'$. But $\mathscr{C}'$ clearly satisfies both (3.1) and (3.3). Moreover, as (4.2)(i) fails, (3.4) holds. Therefore (3.2) fails for k equal to $r(M)+1$, although it holds for k equal to $r(M)$ because $\mathscr{C}'$ is the set of non-spanning circuits of M. It follows that there are distinct members C_1 and C_2 of $\mathscr{C}'$ and an element e of $C_1 \cap C_2$ such that $|(C_1 \cup C_2) - e| = r(M)+1$ and $(C_1 \cup C_2) - e$ does not contain a member of $\mathscr{C}'$. Hence, taking X equal to $(C_1 \cup C_2) - e$, we obtain the contradiction that (4.2)(ii) holds. We conclude that if M is the only matroid on E with $\mathscr{C}'$ as its set of non-spanning circuits, then (4.1) and (4.2) hold.

The proof of the converse follows without difficulty from Proposition 2 and Theorem 3. □

Joseph P. S. Kung (private communication) has noted that a simple binary matroid is uniquely determined by its set of non-spanning circuits unless it is a free matroid or a circuit. The proof of this is a straightforward combination of Proposition 2, the Scum Theorem, and the excluded-minor characterization of binary matroids.

To conclude this note, we sketch briefly what happens when one is allowed to vary the ground set. The arguments here, which extend those given above, are not difficult and are omitted. If $\mathscr{C}$ is the set of all circuits of a matroid and E is $\bigcup_{C\in\mathscr{C}} C$, then there is a unique matroid M on E having $\mathscr{C}$ as its set of circuits. Moreover, the other matroids that have $\mathscr{C}$ as their set of circuits consist precisely of those matroids that can be obtained from M by adjoining some set of coloops. Now consider the problem of describing all matroids for which some fixed set $\mathscr{C}'$ is the set of non-spanning circuits, where we note that, this time, the ground set of the matroid is not being specified. Let $E=\bigcup_{C\in\mathscr{C}'} C$ and assume that the set $\mathscr{S}$ of matroids for which $\mathscr{C}'$ is the set of non-spanning circuits is non-empty.

We suppose first that there is a member of $\mathscr{S}$ having E as its ground set. In this case, let N be the unique such matroid of largest rank. Then, by Proposition 2, the members of $\mathscr{S}$ having ground set E are $\{N, T(N), \ldots, T^n(N)\}$, where $r(T^n(N))$ is $\max\{|C|: C\in\mathscr{C}'\}$ if $\mathscr{C}'\neq\varnothing$ and is 0 otherwise. Moreover, the other members of $\mathscr{S}$ consist of all the matroids that can be obtained in either of the following ways: (i) by adjoining i coloops to $T^m(N)$ and then truncating the resulting matroid i times for some non-negative i and m with $m\leqslant n$; and (ii) provided that N has no spanning circuits, by adjoining i coloops to N and then truncating the resulting matroid j times for some i and j with $0\leqslant j<i$.

Finally, suppose that no member of $\mathscr{S}$ has E as its ground set. Then if $e\notin E$, there is a unique matroid P on $E\cup e$ that is in $\mathscr{S}$. The element e is a coloop of P. Moreover, every other member of $\mathscr{S}$ is isomorphic to a matroid that can be obtained by adjoining i coloops to P and then truncating the resulting matroid j times for some i and j with $0\leqslant j\leqslant i$.

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Reference

1. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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