

# UNBREAKABLE MATROIDS

JAMES OXLEY AND SIMON PFEIL

ABSTRACT. A matroid  $M$  is unbreakable if  $M$  is connected and, for each flat  $F$ , the matroid  $M/F$  is connected. Equivalently,  $M$  is unbreakable if its dual has no two skew circuits. This paper characterizes unbreakable matroids in terms of excluded parallel minors and determines all regular unbreakable matroids.

## 1. INTRODUCTION

This paper explores connected matroids that remain connected upon contracting any flat. These matroids were first investigated [8] as potential analogues to graphs with no two vertex-disjoint cycles; however, and perhaps unsurprisingly given that the latter definition is entirely dependent on vertex information, such graphs do not generalize easily to matroids. Nevertheless, these matroids have proved interesting in their own right, as evidenced by the attention given to them in [3] and [4]. We follow Oxley [7] for notation and terminology. A matroid  $M$  *unbreakable* if  $M$  is connected and, for every flat  $F$  of  $M$ , the matroid  $M/F$  is also connected. Thus the matroid  $U_{0,1}$  is unbreakable. Indeed, because it is the unique unbreakable matroid having a loop, we restrict attention in our main results to loopless matroids.

Our first theorem gives several characterizations of unbreakable matroids including one in terms of excluded parallel minors.

**Theorem 1.1.** *The following statements are equivalent for a loopless matroid  $M$  of rank  $r$ .*

- (i)  $M$  is unbreakable.
- (ii)  $M^*$  has no two skew circuits.
- (iii) Every rank- $(r-2)$  flat of  $M$  is contained in at least three hyperplanes.
- (iv)  $M$  does not have  $U_{2,2}$  as a parallel minor.
- (v)  $\text{si}(M) \not\cong U_{2,2}$  and  $M/F$  is unbreakable for all rank-1 flats  $F$  of  $M$ .
- (vi) For every partition  $(X, Y)$  of  $E(M)$ , if  $X'$  is a flat that is properly contained in  $X$  and  $Y' \subseteq Y$ , then  $\square(X', Y') < \square(X, Y)$ .

---

*Date:* November 3, 2020.

*1991 Mathematics Subject Classification.* 05B35.

*Key words and phrases.* unbreakable matroid, round matroid, skew circuits.

The following theorem, the main result of the paper, is proved in Section 3.

**Theorem 1.2.** *A loopless regular matroid is unbreakable if and only if its simplification is isomorphic to  $U_{1,1}$ ;  $M^*(K_{3,3})$ ;  $R_{10}$ ; or  $M(C_n)$  or  $M(K_n)$  for  $n \geq 3$ .*

A matroid is *round* if its dual has no two disjoint circuits. In view of Theorem 1.1, we see that if a loopless matroid is round, then it is unbreakable. By Theorem 1.2,  $R_{10}$  is unbreakable. But, because it has a pair of disjoint cocircuits, it is not round. The projective geometry  $PG(r-1, q)$  is round. Moreover, as noted in [7, p.326], so is every matroid that is obtained from  $PG(r-1, q)$  by deleting at most  $q^{r-1} - q^{r-2} - 1$  elements. Hence all such matroids are unbreakable. Similarly, so are all matroids that are obtained from  $AG(r-1, q)$  by deleting at most  $q^{r-2} - q^{r-3} - 1$  elements. Thus, even in the case of unbreakable binary matroids, we do not have the same kind of characterization as in Theorem 1.2.

## 2. EQUIVALENT CHARACTERIZATIONS OF UNBREAKABLE MATROIDS

To prove Theorem 1.1, we shall show that (i) implies (iv), that (iv) implies (iii), that (iii) implies (ii), and that (ii) implies (i). Then we show the equivalence of (i) and (v), and finally the equivalence of (i) and (vi).

*Proof of Theorem 1.1.* Let  $M$  be a loopless matroid with ground set  $E$ . To show that (i) implies (iv), let  $M$  be unbreakable. Then, for  $X \subset E(M)$ , we see that  $\text{si}(M/X)$  is not isomorphic to the disconnected matroid  $U_{2,2}$  as  $\text{si}(M/X) \cong \text{si}(M/\text{cl}(X))$ . Thus  $M$  does not have  $U_{2,2}$  as a parallel minor, so (i) implies (iv). Next observe that it follows by the Scum Theorem (see, for example, [7, Theorem 3.3.1]) that (iv) implies (iii).

Now suppose that (iii) holds. To show that (ii) holds, let  $D_1$  and  $D_2$  be cocircuits of  $M$ , and let  $H_i = E - D_i$  for each  $i$  in  $\{1, 2\}$ . Then

$$\begin{aligned} \square_{M^*}(D_1, D_2) &= r_{M^*}(D_1) + r_{M^*}(D_2) - r_{M^*}(D_1 \cup D_2) \\ &= |D_1| + |D_2| - 2 - [r_M(H_1 \cap H_2) + |D_1 \cup D_2| - r(M)] \\ &= |D_1 \cap D_2| - 2 - r_M(H_1 \cap H_2) + r(M) \\ &\geq |D_1 \cap D_2| - 2 - (r(M) - 2) + r(M) \\ &= |D_1 \cap D_2|. \end{aligned}$$

Since equality holds only when  $r_M(H_1 \cap H_2) = r(M) - 2$ , we need only argue that, in this case,  $|D_1 \cap D_2| \neq 0$ . Let  $F = H_1 \cap H_2$ . As  $F$  is in at least three distinct hyperplanes by assumption,  $\text{cl}(\{e\} \cup F) \notin \{H_1, H_2\}$  for some  $e$  in  $E - F$ . Thus  $|D_1 \cap D_2| = |E - (H_1 \cup H_2)| \geq 1$ . Hence (iii) implies (ii).

Next, suppose that (ii) holds, but (i) does not. Then  $M$  has a flat  $F$  such that  $M/F$  is disconnected. Now,  $M/F = [M^*|(E - F)]^*$ , so  $M^*|(E - F)$  is disconnected. By, for example, [7, Exercise 2.1.13(a)],  $E - F$  is a union of circuits of  $M^*$ . Thus  $M^*|(E - F) = M_1 \oplus M_2$  where each of  $E(M_1)$  and  $E(M_2)$  is a union of circuits of  $M^*$ . Hence  $M^*$  has two skew circuits, a contradiction. Thus (ii) implies (i).

To show that (i) implies (v), assume  $M$  is unbreakable. Then  $M$  is connected, so  $\text{si}(M) \not\cong U_{2,2}$ . Moreover,  $M/F$  is unbreakable for all rank-1 flats  $F$  of  $M$ . Thus (i) implies (v). Now assume (v) holds. Then it is straightforward to see that  $M$  is unbreakable provided  $M$  is connected. As  $M/F$  is connected for every rank-1 flat  $F$  of  $M$  but  $\text{si}(M) \not\cong U_{2,2}$ , it follows that  $M$  is connected. Hence (v) implies (i).

Next, we show that (i) implies (vi). Assume  $M$  is unbreakable and suppose that  $(X, Y)$  partitions  $E$ , that  $X'$  is a flat of  $M$  properly contained in  $X$ , and that  $Y' \subseteq Y$ . Further, suppose  $\square(X', Y') = \square(X, Y)$ . Then  $\square(X', Y) = \square(X, Y)$ . Thus

$$r(X') = r(X) - r(X \cup Y) + r(X' \cup Y) = r(X) - r(M) + r(X' \cup Y). \quad (1)$$

Now we consider  $M' = M/X'$ . Then

$$\begin{aligned} \square_{M'}(X - X', Y) &= r_{M'}(X - X') + r_{M'}(Y) - r_{M'}((X - X') \cup Y) \\ &= r_M(X) - r_M(X') + r_{M'}(Y) - (r(M) - r_M(X')) \\ &= r_M(X) + r_M(Y \cup X') - r_M(X') - r(M) \\ &= 0, \text{ by (1)}. \end{aligned}$$

Hence  $M/X'$  is disconnected, a contradiction. Thus (i) implies (vi).

Now assume that (vi) holds, but (i) does not. Let  $F$  be a flat of  $M$  having  $(X_F, Y_F)$  as a 1-separation. Consider  $(X_F \cup F, Y_F)$ , a partition of  $E(M)$ . We will show that  $\square_M(X_F \cup F, Y_F) = \square_M(F, Y_F)$ . Observe that

$$\begin{aligned} 0 &= \square_{M/F}(X_F, Y_F) \\ &= r_M(X_F \cup F) - r_M(F) + r_M(Y_F \cup F) - r(M). \end{aligned}$$

Thus  $r_M(X_F \cup F) - r(M) = r_M(F) - r_M(Y_F \cup F)$ , so

$$\begin{aligned} \square_M(X_F \cup F, Y_F) &= r_M(X_F \cup F) - r(M) + r_M(Y_F) \\ &= r_M(F) - r_M(Y_F \cup F) + r_M(Y_F) \\ &= \square_M(F, Y_F). \end{aligned}$$

This contradicts (vi), so (vi) implies (i). Hence the theorem holds.  $\square$

The following is an immediate consequence of Theorem 1.1.

**Corollary 2.1.** *A loopless parallel minor of an unbreakable matroid is unbreakable.*

## 3. UNBREAKABLE REGULAR MATROIDS

To determine the unbreakable regular matroids, we will find the unbreakable graphic and cographic matroids and then apply Seymour's Decomposition Theorem. We will use the following elementary result.

**Lemma 3.1.** *If  $M$  is an unbreakable matroid and  $N$  is a loopless matroid such that  $\text{si}(N) \cong M$ , then  $N$  is unbreakable.*

**Proposition 3.2.** *A non-empty loopless graphic matroid  $M$  is unbreakable if and only if  $\text{si}(M) \cong M(C_n)$  or  $\text{si}(M) \cong M(K_n)$  for some  $n \geq 2$ .*

*Proof.* One easily checks that if  $\text{si}(M) \cong M(C_n)$  or  $\text{si}(M) \cong M(K_n)$  for some  $n \geq 2$ , then  $M$  is unbreakable. Now, suppose that  $M$  is unbreakable and simple and let  $G$  be a connected graph such that  $M(G) \cong M$ . If  $|V(G)| < 3$ , then  $G \cong K_2$  so we may assume that  $|V(G)| \geq 3$ .

Suppose that  $G$  is 3-connected and that  $G \not\cong K_n$ . For two non-adjacent vertices,  $v_1$  and  $v_2$ , in  $G$ , we see that  $G - \{v_1, v_2\}$  is connected, and  $\text{si}(M(G/E(G - \{v_1, v_2\}))) \cong U_{2,2}$ . Thus  $M$  is not unbreakable. We may now suppose that  $G$  is not 3-connected. Assume that  $G \not\cong C_n$ . Then  $G$  can be written as the 2-sum of two 2-connected graphs  $G_1$  and  $G_2$  across a common edge  $p$  having endpoints  $u$  and  $v$ . Then  $G_1$ , say, has a vertex  $w$  and a path joining  $u$  and  $v$  that avoids  $w$ . Let  $w'$  be a vertex not in  $V(G_1)$ . Let  $S$  be the set of all edges not incident with  $w$  or  $w'$ . Then  $\text{si}(M(G/S)) \cong U_{2,2}$ . Thus  $M$  is not unbreakable.  $\square$

Lovász [6] (see also [1, Theorem III.2.2]) proved the following result.

**Theorem 3.3.** *Let  $G$  be a loopless graph with no two vertex-disjoint cycles. Suppose every vertex of  $G$  has degree at least three and no vertex meets every cycle. Then*

- (a)  $G$  has three vertices and multiple edges join every pair of vertices;  
or
- (b)  $G$  is a  $K_4$  in which one triangle may have multiple edges; or
- (c)  $G \cong K_5$ ; or
- (d)  $G \cong K_5 \setminus e$  where some of the edges not adjacent to the missing edge may be multiple edges; or
- (e)  $G$  is a wheel whose spokes may be multiple edges; or
- (f)  $G$  is obtained from  $K_{3,p}$  for some  $p \geq 2$  by adding edges joining vertices in the 3-vertex class.

**Proposition 3.4.** *Let  $M$  be a loopless matroid that is cographic but not graphic. Then  $M$  is unbreakable if and only if  $\text{si}(M) \cong M^*(K_{3,3})$ .*

*Proof.* Let  $M = M^*(G)$  for some graph  $G$ . By Theorem 1.1(ii),  $M$  is unbreakable if and only if all cycles of  $G$  share at least two vertices. From Theorem 3.3, we deduce, since  $M$  is graphic, that  $G$  can be obtained from  $K_{3,3}$  by possibly replacing some edges by paths.  $\square$

The proof of Theorem 1.2 will use two more lemmas. An element in a matroid is *free* if it is not a coloop and the only circuits containing it are spanning. Thus a loopless matroid with a free element is connected.

**Lemma 3.5.** *If a loopless matroid  $M$  has a free element, then  $M$  is unbreakable.*

*Proof.* Suppose  $p$  is a free element of  $M$  but  $M$  is not unbreakable. Then  $M$  has a flat  $F$  such that  $M/F$  is disconnected. Let  $B_F$  be a basis of  $F$  where  $p \in B_F$  if  $p \in F$ . If  $p \notin F$ , then  $B_F \cup p$  is independent. Thus  $B$  has a basis that contains both  $B_F$  and  $p$ . As  $M$  is connected, by a result of Cunningham [2] and Krogdahl [5] (see also [7, Proposition 4.3.2]), for some  $e$  in  $E(M) - B$ , the fundamental circuit  $C(e, B)$  contains  $p$ . As  $p$  is free,  $C(e, B) = B \cup e$ . Thus  $C(e, B) - B_F$  is a spanning circuit of  $M/F$ , so this matroid is connected, a contradiction.  $\square$

**Lemma 3.6.** *Let  $M_1$  and  $M_2$  be loopless matroids with  $E(M_1) \cap E(M_2) = \{p\}$  and  $r(M_i) \geq 2$  for each  $i$ . The 2-sum of  $M_1$  and  $M_2$  with respect to the basepoint  $p$  is unbreakable if and only if  $p$  is free in both  $M_1$  and  $M_2$ .*

*Proof.* Suppose  $M_1 \oplus_2 M_2$  is unbreakable, but  $p$  is not free in  $M_1$ . Let  $C$  be a non-spanning circuit of  $M_1$  containing  $p$ , and let  $F = \text{cl}_M(C - p)$ . Then  $M/F$  is disconnected, a contradiction. To prove the converse, suppose that  $p$  is free in both  $M_1$  and  $M_2$ . By Lemma 3.5, both  $M_1$  and  $M_2$  are unbreakable. Suppose  $M$  has a flat  $F$  such that  $M/F$  is disconnected. Let  $M'$  be the parallel connection  $P(M_1, M_2)$ . Then the closure of  $F$  in  $M'$  is  $F$  or  $F \cup p$ . In the latter case,  $F$  spans  $E(M_j) - p$  for  $j = 1$ , say. Then  $M/F$  can be obtained by contracting a flat of  $M_2/p$  and so it is connected, a contradiction. Thus  $F$  does not span  $p$  in  $M'$  so  $F = F_1 \cup F_2$  where  $F_i$  is a flat of  $M_i$  avoiding  $p$ . As  $M_i/F_i$  is connected containing  $p$ , it follows that  $(M_1/F_1) \oplus_2 (M_2/F_2)$ , that is  $M/F$ , is connected, a contradiction.  $\square$

We can now prove the main result of the paper.

*Proof of Theorem 1.2.* By Propositions 3.2 and 3.4, we need only show that  $R_{10}$  is unbreakable to ensure that each of the listed matroids is unbreakable. As  $R_{10}$  has rank five and its smallest circuit has four elements,  $R_{10}$  has no skew circuits. Since  $R_{10}$  is self-dual, it is unbreakable, by Theorem 1.1(ii).

Now let  $M$  be an unbreakable regular matroid that is not listed in the theorem so that  $|E(M)|$  is a minimum among such matroids. Then  $M$  is simple. By Seymour's Regular Matroids Decomposition Theorem [9], as  $M$  is connected,  $M$  can be obtained using 2- and 3-sums from graphic matroids, cographic matroids, and copies of  $R_{10}$ .

Suppose  $M$  is the 2-sum with basepoint  $p$  of  $M_1$  and  $M_2$  where each  $M_i$  has at least three elements and has rank at least two. By Lemmas 3.6 and 3.5,  $p$  is free in each  $M_i$ , so each  $M_i$  is unbreakable and hence is a member of the specified list of unbreakable matroids. But the only such matroids having a free element are those whose simplification is a circuit, and a 2-sum of circuits is a circuit. This contradiction implies that  $M$  is 3-connected.

Suppose  $M \cong M_1 \oplus_3 M_2$ . By Seymour's Theorem, each of  $M_1$  and  $M_2$  is isomorphic to a loopless minor of  $M$ . It follows that each  $M_i$  is a loopless parallel minor of  $M$  and so, by Corollary 2.1, each is unbreakable. Thus each  $M_i$  must be one of the previously identified unbreakable matroids and must contain a triangle. As  $M$  is not graphic, we may assume that  $\text{si}(M_1) \cong M^*(K_{3,3})$  and that either  $\text{si}(M_2) \cong M^*(K_{3,3})$  or  $\text{si}(M_2) \cong M(K_n)$  for some  $n \geq 4$ . Let  $\{a_1, a_2, a_3\}$  be the common triangle  $T$  of  $M_1$  and  $M_2$ . Suppose that  $\text{si}(M_2) \cong M^*(K_{3,3})$ . For each  $i$ , take a triangle  $T_i$  of  $M_i$  that meets  $T$  in  $\{a_i\}$  where  $\cap(T_i, T) = 1$ . Then one easily checks that  $M/\text{cl}((T_1 - a_1) \cup (T_2 - a_2))$  is disconnected, a contradiction.

We may now assume that  $\text{si}(M_2) \cong M(K_n)$  for some  $n \geq 4$ . Now  $M$  is obtained from the generalized parallel connection of  $M_1$  and  $M_2$  across the triangle  $T$ . If  $M_1$  or  $M_2$  has an element  $x$  in parallel to some element of  $T$ , then there are elements  $y$  and  $z$  of  $E(M_1) - \text{cl}_{M_1}(T)$  such that  $\{x, z\}$  is independent in  $M/y$  and  $\{x, z\} \subseteq \text{cl}_P(T)$ . Then  $M/\text{cl}(\{x, y, z\})$  is disconnected, a contradiction. Thus we may assume that no such element  $x$  exists, so  $P$  is simple. Then, taking  $\{y, z\}$  in  $E(M_1) - \text{cl}_{M_1}(T)$  such that  $\{y, z\}$  is a flat of  $M_1$  that is skew to  $T$ , we deduce that  $\{y, z\}$  is a flat of  $M$  and, in  $M/\{y, z\}$ , the elements of  $E(M_1) - (T \cup \{y, z\})$  form two skew 2-circuits. Then  $M/\{y, z\} = M(G)$  where  $G$  is obtained from  $K_n \setminus e$  by adding an edge in parallel to each of  $f$  and  $g$  for some triangle  $\{e, f, g\}$  of  $K_n$ . By Proposition 3.2,  $M(G)$  is not unbreakable, a contradiction.  $\square$

## REFERENCES

- [1] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978. Reprinted by Dover, Mineola, New York, 2004.
- [2] W.H. Cunningham, *A Combinatorial Decomposition Theory*, Ph.D. thesis, University of Waterloo, 1973.
- [3] G. Drummond, T. Fife, K. Grace, and J. Oxley, Circuit-difference matroids, *Electron. J. Combin.* 27 (2020) #P3.11.

- [4] T. Fife, D. Mayhew, J. Oxley, C. Semple, The unbreakable frame matroids, *SIAM J. Discrete Math.* 34 (2020) 1522–1537.
- [5] S. Krogdahl, The dependence graph for basis in matroids, *Discrete Math.* 19 (1977) 47–59.
- [6] L. Lovász, On graphs not containing independent circuits (Hungarian), *Mat. Lapok* 16 (1965), 289–299.
- [7] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.
- [8] S. Pfeil, On Properties of Matroid Connectivity, Ph.D. thesis, Louisiana State University, 2016.
- [9] P.D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* 28 (1980) 305–359.

MATHEMATICS DEPARTMENT, LOUISIANA STATE UNIVERSITY, BATON ROUGE,  
LOUISIANA, USA

*E-mail address:* `oxley@math.lsu.edu`

MATHEMATICS DEPARTMENT, ANGELO STATE UNIVERSITY, SAN ANGELO, TEXAS,  
USA

*E-mail address:* `spfeil@angelo.edu`