UNBREAKABLE MATROIDS

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Abstract. A matroid M is unbreakable if M is connected and, for each flat F, the matroid M/F is connected. Equivalently, M is unbreakable if its dual has no two skew circuits. This paper characterizes unbreakable matroids in terms of excluded parallel minors and determines all regular unbreakable matroids.

1. Introduction

This paper explores connected matroids that remain connected upon contracting any flat. These matroids were first investigated [8] as potential analogues to graphs with no two vertex-disjoint cycles; however, and perhaps unsurprisingly given that the latter definition is entirely dependent on vertex information, such graphs do not generalize easily to matroids. Nevertheless, these matroids have proved interesting in their own right, as evidenced by the attention given to them in [3] and [4]. We follow Oxley [7] for notation and terminology. A matroid M *unbreakable* if M is connected and, for every flat F of M, the matroid M/F is also connected. Thus the matroid $U_{0,1}$ is unbreakable. Indeed, because it is the unique unbreakable matroid having a loop, we restrict attention in our main results to loopless matroids.

For an element e of a matroid M, we call $M\backslash e$ a *parallel deletion* of M if e is in a 2-circuit of M. A matroid N is a *parallel minor* of M if N can be obtained from M by a sequence of contractions and parallel deletions. Our first theorem gives several characterizations of unbreakable matroids including one in terms of excluded parallel minors.

Theorem 1.1. *The following statements are equivalent for a loopless matroid M of rank r.*

(i) *M is unbreakable.*
(ii) *M^* has no two skew circuits.*
(iii) *Every rank-$(r-2)$ flat of M is contained in at least three hyperplanes.*

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(iv) *M does not have $U_{2,2}$ as a parallel minor.*
(v) $\mathrm{si}(M) \ncong U_{2,2}$ *and M/F is unbreakable for all rank-1 flats F of M.*
(vi) *For every partition (X, Y) of $E(M)$, if X' is a flat that is properly contained in X, then $\sqcap(X', Y) < \sqcap(X, Y)$.*

Unbreakable matroids are linked to the well-studied class of round matroids where a matroid is *round* if its dual has no two disjoint circuits. In view of Theorem 1.1, we see that if a loopless matroid is round, then it is unbreakable; however, an unbreakable matroid need not be round. For example, R_{10} has two disjoint cocircuits so it is not round. But, as the next theorem shows, it is unbreakable. This theorem, which is the main result of the paper, is proved in Section 3.

Theorem 1.2. *A loopless regular matroid is unbreakable if and only if its simplification is isomorphic to $U_{1,1}$, to $M^*(K_{3,3})$, to R_{10}, or, for $n \geq 3$, to $M(C_n)$ or $M(K_n)$.*

One might hope for a theorem similar to Theorem 1.2 for unbreakable representable matroids. Observe that the projective geometry $PG(r-1, q)$ is round. Moreover, as noted in [7, p.326], so is every matroid that is obtained from $PG(r-1, q)$ by deleting at most $q^{r-1} - q^{r-2} - 1$ elements. Hence all such matroids are unbreakable. Thus, even in the case of unbreakable binary matroids, we do not have the same kind of characterization as in Theorem 1.2.

2. Equivalent Characterizations of Unbreakable Matroids

To prove Theorem 1.1, we shall show that (i) implies (iv), that (iv) implies (iii), that (iii) implies (ii), and that (ii) implies (i). Then we show the equivalence of (i) and (v), and finally the equivalence of (i) and (vi).

Proof of Theorem 1.1. Let M be a loopless matroid with ground set E. To show that (i) implies (iv), let M be unbreakable. Then, for $X \subseteq E(M)$, we see that $\mathrm{si}(M/X)$ is not isomorphic to the disconnected matroid $U_{2,2}$ as $\mathrm{si}(M/X) \cong \mathrm{si}(M/\mathrm{cl}(X))$. Thus M does not have $U_{2,2}$ as a parallel minor, so (i) implies (iv).

We show that (iv) implies (iii) by proving the contrapositive. Suppose F is a rank-$(r-2)$ flat of M contained in fewer than three hyperplanes of M. This implies that F is contained in exactly two hyperplanes. Call them H_1 and H_2. Then $F = H_1 \cap H_2$ and $r(M/F) = 2$, so M/F must consist of two disjoint rank-1 flats, that is, $\mathrm{si}(M/F) \cong U_{2,2}$. Thus (iv) implies (iii).

Now suppose that (iii) holds. To show that (ii) holds, let D_1 and D_2 be cocircuits of M, and let $H_i = E - D_i$ for each i in $\{1,2\}$. Then

$$\begin{aligned}
\sqcap_{M^*}(D_1, D_2) &= r_{M^*}(D_1) + r_{M^*}(D_2) - r_{M^*}(D_1 \cup D_2)\\
&= |D_1| + |D_2| - 2 - [r_M(H_1 \cap H_2) + |D_1 \cup D_2| - r(M)]\\
&= |D_1 \cap D_2| - 2 - r_M(H_1 \cap H_2) + r(M)\\
&\geq |D_1 \cap D_2| - 2 - (r(M) - 2) + r(M)\\
&= |D_1 \cap D_2|.
\end{aligned}$$

Since equality holds only when $r_M(H_1 \cap H_2) = r(M) - 2$, we need only argue that, in this case, $|D_1 \cap D_2| \neq 0$. Let $F = H_1 \cap H_2$. As F is in at least three distinct hyperplanes by assumption, $\mathrm{cl}(\{e\} \cup F) \notin \{H_1, H_2\}$ for some e in $E - F$. Thus $|D_1 \cap D_2| = |E - (H_1 \cup H_2)| \geq 1$. Hence (iii) implies (ii).

Next, suppose that (ii) holds, but (i) does not. Then M has a flat F such that M/F is disconnected. Now, $M/F = [M^*|(E-F)]^*$, so $M^*|(E-F)$ is disconnected. By, for example, [7, Exercise 2.1.13(a)], $E - F$ is a union of circuits of M^*. Thus $M^*|(E-F) = M_1 \oplus M_2$ where each of $E(M_1)$ and $E(M_2)$ is a union of circuits of M^*. Hence M^* has two skew circuits, a contradiction. Thus (ii) implies (i).

To show that (i) implies (v), assume M is unbreakable. Then M/F is connected for all flats F of M. In particular, M is connected, so $\mathrm{si}(M) \not\cong U_{2,2}$. Moreover, M/F is unbreakable for all rank-one flats F of M. Thus (i) implies (v). Now assume (v) holds. Then it is straightforward to see that M is unbreakable provided M is connected. As M/F is connected for every rank-1 flat F of M but $\mathrm{si}(M) \not\cong U_{2,2}$, it follows that M is connected. Hence (v) implies (i).

Next, we show that (i) implies (vi). Let M be unbreakable and suppose that (X, Y) partitions E and that X' is a flat of M properly contained in X. Seeking a contradiction, assume that $\sqcap(X', Y) = \sqcap(X, Y)$. Thus

$$r(X') = r(X) - r(X \cup Y) + r(X' \cup Y) = r(X) - r(M) + r(X' \cup Y). \quad (1)$$

Now we consider $M' = M/X'$. Then

$$\begin{aligned}
\sqcap_{M'}(X - X', Y) &= r_{M'}(X - X') + r_{M'}(Y) - r_{M'}((X - X') \cup Y)\\
&= r_M(X) - r_M(X') + r_{M'}(Y) - (r(M) - r_M(X'))\\
&= r_M(X) + r_M(Y \cup X') - r_M(X') - r(M)\\
&= 0, \quad \text{by (1)}.
\end{aligned}$$

Hence M/X' is disconnected, a contradiction. Thus (i) implies (vi).

Now assume that (vi) holds, but (i) does not. Then there is a flat F of M such that M/F is not connected. Let (X_F, Y_F) be a 1-separation of M/F. Consider $(X_F \cup F, Y_F)$, a partition of $E(M)$. We will show that

$\sqcap_M(X_F \cup F, Y_F) = \sqcap_M(F, Y_F)$. Observe that

$$\begin{aligned} 0 &= \sqcap_{M/F}(X_F, Y_F) \\ &= r_M(X_F \cup F) - r_M(F) + r_M(Y_F \cup F) - r(M). \end{aligned}$$

Thus $r_M(X_F \cup F) - r(M) = r_M(F) - r_M(Y_F \cup F)$, so

$$\begin{aligned} \sqcap_M(X_F \cup F, Y_F) &= r_M(X_F \cup F) - r(M) + r_M(Y_F) \\ &= r_M(F) - r_M(Y_F \cup F) + r_M(Y_F) \\ &= \sqcap_M(F, Y_F). \end{aligned}$$

This contradicts (vi), so (vi) implies (i). Hence the theorem holds. □

The following is an immediate consequence of Theorem 1.1.

Corollary 2.1. *A loopless parallel minor of an unbreakable matroid is unbreakable.*

3. Unbreakable Regular Matroids

To determine the unbreakable regular matroids, we will find the unbreakable graphic and cographic matroids and then apply Seymour's Decomposition Theorem. We will use the following elementary result.

Lemma 3.1. *If M is an unbreakable matroid and N is a loopless matroid such that $si(N) \cong M$, then N is unbreakable.*

Proposition 3.2. *A non-empty loopless graphic matroid M is unbreakable if and only if $\mathrm{si}(M) \cong M(C_n)$ or $\mathrm{si}(M) \cong M(K_n)$ for some $n \geq 2$.*

Proof. One easily checks that if $\mathrm{si}(M) \cong M(C_n)$ or $\mathrm{si}(M) \cong M(K_n)$ for some $n \geq 2$, then M is unbreakable. Now, suppose that M is unbreakable and simple and let G be a connected graph such that $M(G) \cong M$. If $|V(G)| < 3$, then $G \cong K_2$ so we may assume that $|V(G)| \geq 3$.

Suppose that G is 3-connected and that $G \ncong K_n$. For two non-adjacent vertices, v_1 and v_2, in G, we see that $G - \{v_1, v_2\}$ is connected, and $\mathrm{si}(M(G/E(G - \{v_1, v_2\}))) \cong U_{2,2}$. Thus M is not unbreakable. We may now suppose that G is not 3-connected. Assume that $G \ncong C_n$. Then G can be written as the 2-sum of two 2-connected graphs G_1 and G_2 across a common edge p having endpoints u and v. Then G_1, say, has a vertex w and a path joining u and v that avoids w. Let w' be a vertex not in $V(G_1)$. Let S be the set of all edges not incident with w or w'. Then $\mathrm{si}(M(G/S)) \cong U_{2,2}$. Thus M is not unbreakable. □

Lovász [6] (see also [1, Theorem III.2.2]) proved the following result.

Theorem 3.3. *Let G be a loopless graph with no two vertex-disjoint cycles. Suppose every vertex of G has degree at least three and no vertex meets every cycle. Then*

(a) *G has three vertices and multiple edges join every pair of vertices; or*
(b) *G is a K_4 in which one triangle may have multiple edges; or*
(c) *$G \cong K_5$; or*
(d) *$G \cong K_5\backslash e$ where some of the edges not adjacent to the missing edge may be multiple edges; or*
(e) *G is a wheel whose spokes may be multiple edges; or*
(f) *G is obtained from $K_{3,p}$ for some $p \geq 3$ by possibly adding some of the edges joining two vertices in the first vertex class.*

Proposition 3.4. *Let M be a loopless matroid that is cographic but not graphic. Then M is unbreakable if and only if $\mathrm{si}(M) \cong M^*(K_{3,3})$.*

Proof. Let $M = M^*(G)$ for some graph G. By Theorem 1.1(ii), M is unbreakable if and only if all cycles of G share at least two vertices. From Theorem 3.3, we deduce, since M is not graphic, that G can be obtained from $K_{3,3}$ by possibly replacing some edges by paths. □

The proof of Theorem 1.2 will use two more lemmas. An element in a matroid is *free* if it is not a coloop and the only circuits containing it are spanning. Thus a loopless matroid with a free element is connected.

Lemma 3.5. *If a loopless matroid M has a free element, then M is unbreakable.*

Proof. Suppose p is a free element of M but M is not unbreakable. Then M has a flat F such that M/F is disconnected. Let B_F be a basis of F where $p \in B_F$ if $p \in F$. If $p \notin F$, then $B_F \cup p$ is independent. Thus B has a basis that contains both B_F and p. As M is connected, by a result of Cunningham [2] and Krogdahl [5] (see also [7, Proposition 4.3.2]), for some e in $E(M) - B$, the fundamental circuit $C(e, B)$ contains p. As p is free, $C(e, B) = B \cup e$. Thus $C(e, B) - B_F$ is a spanning circuit of M/F, so this matroid is connected, a contradiction. □

Lemma 3.6. *Let M_1 and M_2 be loopless matroids with $E(M_1) \cap E(M_2) = \{p\}$ and $r(M_i) \geq 2$ for each i. The 2-sum of M_1 and M_2 with respect to the basepoint p is unbreakable if and only if p is free in both M_1 and M_2.*

Proof. Suppose $M_1 \oplus_2 M_2$ is unbreakable, but p is not free in M_1. Let C be a non-spanning circuit of M_1 containing p, and let $F = \mathrm{cl}_M(C - p)$. Then M/F is disconnected, a contradiction. To prove the converse, suppose that p is free in both M_1 and M_2. By Lemma 3.5, both M_1 and M_2 are unbreakable. Suppose M has a flat F such that M/F is disconnected. Let M' be the parallel connection $P(M_1, M_2)$. Then the closure of F in M' is F or $F \cup p$. In the latter case, F spans $E(M_j) - p$ for $j = 1$, say. Then M/F can be obtained by contracting a flat of M_2/p and so it is connected, a contradiction. Thus F does not span p in M' so $F = F_1 \cup F_2$ where F_i is a flat of M_i avoiding p. As M_i/F_i is connected containing p, it follows that $(M_1/F_1) \oplus_2 (M_2/F_2)$, that is M/F, is connected, a contradiction. □

We can now prove the main result of the paper.

Proof of Theorem 1.2. By Propositions 3.2 and 3.4, we need only show that R_{10} is unbreakable to ensure that each of the listed matroids is unbreakable. As R_{10} has rank five and its smallest circuit has four elements, R_{10} has no skew circuits. Since R_{10} is self-dual, it is unbreakable, by Theorem 1.1(ii).

Now let M be an unbreakable regular matroid that is not listed in the theorem so that $|E(M)|$ is a minimum among such matroids. Then M is simple. By Seymour's Regular Matroids Decomposition Theorem [9], as M is connected, M can be obtained using 2- and 3-sums from graphic matroids, cographic matroids, and copies of R_{10}.

Suppose M is the 2-sum with basepoint p of M_1 and M_2 where each M_i has at least three elements and has rank at least two. By Lemmas 3.6 and 3.5, p is free in each M_i, so each M_i is unbreakable and hence is a member of the specified list of unbreakable matroids. But the only such matroids having a free element are those whose simplification is a circuit, and a 2-sum of circuits is a circuit. This contradiction implies that M is 3-connected.

Suppose $M \cong M_1 \oplus_3 M_2$. By Seymour's Theorem, each of M_1 and M_2 is isomorphic to a loopless minor of M. It follows that each M_i is a loopless parallel minor of M and so, by Corollary 2.1, each is unbreakable. Thus each M_i must be one of the previously identified unbreakable matroids and must contain a triangle. As M is not graphic, we may assume that $\mathrm{si}(M_1) \cong M^*(K_{3,3})$ and that either $\mathrm{si}(M_2) \cong M^*(K_{3,3})$ or $\mathrm{si}(M_2) \cong M(K_n)$ for some $n \geq 4$. Let $\{a_1, a_2, a_3\}$ be the common triangle T of M_1 and M_2. Suppose that $\mathrm{si}(M_2) \cong M^*(K_{3,3})$. For each i, take a triangle T_i of M_i that meets T in $\{a_i\}$ where $\sqcap(T_i, T) = 1$. Then one easily checks that $M/\mathrm{cl}((T_1 - a_1) \cup (T_2 - a_2))$ is disconnected, a contradiction.

We may now assume that $\mathrm{si}(M_2) \cong M(K_n)$ for some $n \geq 4$. Let P be the generalized parallel connection of M_1 and M_2 across the triangle T. If M_1 or M_2 has an element x in parallel to some element of T, then there are elements y and z of $E(M_1) - \mathrm{cl}_{M_1}(T)$ such that $\{x, z\}$ is independent in M/y and $\{x, z\} \subseteq \mathrm{cl}_{P/y}(T)$. Then $M/\mathrm{cl}(\{x, y, z\})$ is disconnected, a contradiction.

Thus we may assume that no such element x exists, so P is simple. Then, taking $\{y, z\}$ in $E(M_1) - \mathrm{cl}_{M_1}(T)$ such that $\{y, z\}$ is a flat of M_1 that is skew to T, we deduce that $\{y, z\}$ is a flat of M and, in $M/\{y, z\}$, the elements of $E(M_1) - (T \cup \{y, z\})$ form two skew 2-circuits. Then $M/\{y, z\} = M(G)$ where G is obtained from $K_n \backslash e$ by adding an edge in parallel to each of f and g for some triangle $\{e, f, g\}$ of K_n. By Proposition 3.2, $M(G)$ is not unbreakable, a contradiction. □

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