JOURNAL OF COMBINATORIAL THEORY, Series B 52, 301–320 (1991)

Some Extremal Connectivity Results for Matroids*

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Communicated by the Editors

Received June 6, 1989

Let n be an integer exceeding one and M be a matroid having at least $n+2$ elements. In this paper, we prove that every n-element subset X of $E(M)$ is in an $(n+1)$-element circuit if and only if (i) for every such subset, M/X is disconnected, and (ii) for every subset Y with at most n elements, M/Y is connected. Various extensions and consequences of this result are also derived including characterizations in terms of connectivity of the 4-point line and of Murty's Sylvester matroids. The former is a result of Seymour. © 1991 Academic Press, Inc.

1. INTRODUCTION

Minimally connected matroids are those connected matroids for which every single-element deletion is disconnected. Such matroids have been investigated by several authors including Murty [20], Seymour [28], White [34], and Oxley [21, 23, 24]. In particular, Seymour's work on these matroids arose in connection with his proof of the excluded-minor characterization of the class of ternary matroids. A basic tool in that proof is the following characterization of the four-point line [28, Lemma 2.3].

(1.1) THEOREM. *$U_{2,4}$ is the only connected matroid with more than one element in which every 2-element deletion and every 2-element contraction is*

* Both authors were partially supported by a grant from the Louisiana Education Quality Support Fund through the Board of Regents.

0095-8956/91 $3.00

disconnected, but every 1*-element deletion and every* 1*-element contraction is connected.*

This paper investigates a matroid property that generalizes minimal connectedness and is motivated in part by Seymour's theorem. Among the results that are obtained are a generalization of Seymour's theorem and two connectivity characterizations of Murty's *Sylvester matroids* [18, 19]. The latter are those matroids in which every two distinct elements are contained in a 3-element circuit.

The matroid and graph terminology used here will follow Welsh [33] and Bondy and Murty [2], respectively, with the following exceptions. If M is a matroid, then its ground set, rank, and corank will be denoted by $E(M)$, $r(M)$, and $r^*(M)$, respectively. If $T \subseteq E(M)$, then $\bar{T}$ will denote the closure of T, and $r_M(T)$ or $r(T)$ will denote the rank of T. The deletion and contraction of T from M will be denoted by $M \backslash T$ and M/T. The m-element circuits and m-element cocircuits of M will be referred to as simply *m-circuits* and *m-cocircuits*.

If k is a positive integer, a partition $\{X, Y\}$ of $E(M)$ is a k-separation of the matroid M if $|X| \geqslant k$, $|Y| \geqslant k$, and

$$r(X) + r(Y) - r(M) \leqslant k - 1.$$

If equality holds in the last line, then the k-separation $\{X, Y\}$ is said to be *exact*. If n is a positive integer such that M has no k-separation for any $k < n$, then we say that M is *n-connected* [31]. Thus a matroid is 2-connected if and only if it is connected. A *minimally n-connected* matroid is an n-connected matroid for which no single-element deletion is n-connected. We shall use a number of properties of n-connected matroids including the fact that a matroid is n-connected if and only if its dual is n-connected. A survey of these properties can be found in [26, Chap. 8].

Among the matroids that play an important role in this paper are those that are derived from Steiner systems. Recall that a *Steiner system* $S(t, k, v)$ is a pair $(S, \mathscr{D})$, where S is a v-element set and $\mathscr{D}$ is a collection of k-element subsets of S called *blocks* such that every t-element subset of S is contained in exactly one block. To exclude trivial cases, we shall assume throughout this paper that $2 \leqslant t < k < v$. The matroid associated with the Steiner system $(S, \mathscr{D})$ has S as its ground set and $\mathscr{D}$ as its set of hyperplanes. Its rank is $t+1$ and every subset of S with fewer than t elements is an independent flat (see [33, Chap. 12]).

In general, a Steiner system is not uniquely determined by its parameters t, k, and v. However, two of the systems that will appear here, $S(3, 4, 8)$ and $S(5, 6, 12)$, are unique (see, for example, [35, p. 36]). Throughout this paper, the notation $S(t, k, v)$ will be used to refer to both a Steiner system with those parameters and the corresponding matroid. The matroid

$S(3, 4, 8)$ is the rank-4 binary affine space, $AG(3, 2)$. On the other hand, $S(5, 6, 12)$ is a special ternary matroid having numerous attractive properties [25].

The primary focus of this paper will be a new matroid connectivity property that is defined as follows. Let n be a positive integer. A matroid M is *n-element minimally connected* or, for brevity, *n-minimally connected* if it satisfies the following conditions:

(i) $|E(M)| \geqslant n$;

(ii) $M \backslash X$ is disconnected for every n-element subset X of $E(M)$; and

(iii) $M \backslash Y$ is connected for every subset Y of $E(M)$ having fewer than n elements.

Condition (i) is added to ensure that condition (ii) actually takes effect. By taking Y to be empty in (iii), we deduce that every n-minimally connected matroid is connected. Moreover, since all three matroids on fewer than two elements are connected, an n-minimally connected matroid must have at least $n+2$ elements, that is, in the presence of (ii) and (iii), condition (i) implies that $|E(M)| \geqslant n+2$.

The n-minimally connected matroids for $n=1$ are just the minimally connected matroids which were discussed at the outset. In Section 2, we consider the class of 2-minimally connected matroids showing that this essentially coincides with the class of duals of Sylvester matroids. In addition, we characterize the former class in terms of 3-connectedness.

In Section 3, we consider the class of n-minimally connected matroids for $n \geqslant 3$ and show that one of our characterizations of 2-minimally connected matroids generalizes to this class while the other does not. We also discuss the existence of n-minimally connected matroids. The purpose of Section 4 is to characterize those matroids M for which both M and M^* are n-minimally connected. This result generalizes Theorem 1.1. The paper concludes in Section 5 with a discussion of the properties of those n-minimally connected matroids that are not $(n+1)$-connected. In particular, such matroids are characterized when n is 3.

2. The Class of 2-Minimally Connected Matroids

The main result of this section is the following characterization of 2-minimally connected matroids.

(2.1) Theorem. *The following statements are equivalent for a matroid M having at least four elements.*

(i) *M is 2-minimally connected.*

(ii) *M is 3-connected and $M\backslash e, f$ is disconnected for every pair $\{e, f\}$ of distinct elements.*

(iii) *Every pair of distinct elements of M is in a 3-cocircuit.*

This result is important for several reasons. First, it provides the basis of an induction argument characterizing n-minimally connected matroids which will appear in the next section. Second, it links 2-minimally connected matroids and minimally 3-connected matroids. If M is a 3-connected matroid having at least five elements and e and f are distinct elements of M for which $M\backslash e, f$ is disconnected, then one easily checks that although $M\backslash e$ is connected, it cannot be 3-connected. Therefore, if M satisfies (2.1) (ii) and has at least five elements, then M is a minimally 3-connected matroid for which every single-element deletion is minimally connected. It has been shown [21] that all minimally 3-connected matroids contain a number of 3-cocircuits. It is not surprising then that the special minimally 3-connected matroids considered here should have such an abundance of 3-cocircuits.

The third reason for the importance of Theorem 2.1 is that it yields the following two characterizations of Sylvester matroids in terms of connectivity conditions. It should be noted that Sylvester matroids have also been investigated by Bryant, Dawson, and Perfect [3, 4, 12] under the name of *hereditary circuit spaces.*

(2.2) COROLLARY. *The following statements are equivalent for a matroid M having at least four elements.*

(i) *M is a Sylvester matroid.*

(ii) *M is 3-connected and $M/e, f$ is disconnected for all pairs $\{e, f\}$ of distinct elements.*

(iii) *M^* is 2-minimally connected.*

The proof of Theorem 2.1 will use the next two lemmas, the first of which is taken from [21, Theorem 2.4]. A *non-trivial series class* in a matroid M is a maximal subset X of $E(M)$ such that $|X| \geqslant 2$ and every 2-element subset of X is a cocircuit.

(2.3) LEMMA. *Let C be a circuit of a connected matroid M such that $M\backslash e$ is disconnected for all e in C. Then either M is the circuit C, or C contains at least two distinct non-trivial series classes of M.*

(2.4) LEMMA. *Let M be a 3-connected matroid having at least four elements such that $M\backslash e, f$ is disconnected for every pair $\{e, f\}$ of distinct elements. If C is a circuit of M and g is an element of M not in C, then M has a 3-cocircuit that contains g and is contained in $C \cup g$.*

Proof. Since M is 3-connected and $|E(M)| \geqslant 4$, $M \backslash g$ is connected. Now C is a circuit of $M \backslash g$ such that $(M \backslash g) \backslash e$ is disconnected for all e in C. Therefore, by Lemma 2.3, C contains a 2-cocircuit $\{a, b\}$ of $M \backslash g$. As M is 3-connected having at least four elements, it has no 2-cocircuits. Thus $\{a, b, g\}$ is a 3-cocircuit of M contained in $C \cup g$. ▮

Proof of Theorem 2.1. It is straightforward to check that (iii) implies (i). We shall complete the proof of the theorem by showing that (i) implies (ii) and that (ii) implies (iii). Suppose that (i) holds but that M is not 3-connected. Then, by [29, (2.6)], for some minors M_1 and M_2 of M each having at least three elements, $M = P((M_1; p), (M_2; p)) \backslash p$, where $P((M_1; p), (M_2; p))$ denotes the parallel connection [5] of M_1 and M_2 with respect to the basepoint p. Now take e in $E(M_1) - p$ and f in $E(M_2) - p$. As $M \backslash e$ is connected and $M \backslash e = P((M_1 \backslash e; p), (M_2; p)) \backslash p$, it follows by [22, (1.13)] that $M_1 \backslash e$ is connected. Similarly, $M_2 \backslash f$ is connected and so, by [22, (1.13)] again, $P((M_1 \backslash e; p), (M_2 \backslash f; p)) \backslash p$ is connected. As the last matroid is equal to $M \backslash e, f$, we have a contradiction. We conclude that (i) implies (ii).

Now suppose that (ii) holds but that M has a subset $\{e, f\}$ that is not contained in a 3-cocircuit. Then $M \backslash e, f$ has no coloops. Since $M \backslash e, f$ certainly has no loops, every component of it must contain a circuit having at least two elements. As $M \backslash e, f$ is disconnected, we can find two distinct components X_1 and X_2 of it. Let g be an element of X_1 and C be a circuit of $M | X_2$. Then, by Lemma 2.4, M has a 3-cocircuit $\{g, a, b\}$ for some subset $\{a, b\}$ of C. Therefore $M \backslash e, f$ has a cocircuit C^* that contains g and is contained in $\{g, a, b\}$. Since $M \backslash e, f$ has no coloops, $|C^*| \geqslant 2$. Thus C^* meets the distinct components X_1 and X_2 of $M \backslash e, f$. This contradiction finishes the proof that (ii) implies (iii), thereby completing the proof of Theorem 2.1. ▮

To conclude this section, we remark that a result of Akkari that strengthens the equivalence of (ii) and (iii) of (2.1) will appear elsewhere [1]. The proof of that result is considerably longer and more difficult than the proof just given. Moreover, we do not need the stronger result for the applications to follow.

3. The Case when n Exceeds Two

The following proposition, which is easily proved using elementary properties of n-connectedness, identifies an important class of n-minimally connected matroids.

(3.1) Proposition. *Suppose that n is an integer exceeding one and M is a matroid having at least $2n$ elements. If M is $(n+1)$-connected and $M \backslash X$*

is disconnected for every n-element subset X of $E(M)$, then M is n-minimally connected.

Theorem 2.1 established that the converse of this lemma holds when n is 2. However, the converse fails for $n=3$. For example, it is not difficult to show that $AG(3,2)$ is 3-minimally connected and yet is not 4-connected. Thus the equivalence of (i) and (ii) of Theorem 2.1 does not generalize to all $n \geqslant 2$. The next result shows that, in contrast, the equivalence of (ii) and (iii) in (2.1) does generalize. In Section 5, we shall look more closely at those n-minimally connected matroids that are not $(n+1)$-connected. In particular, we characterize such matroids when n is 3.

(3.2) THEOREM. *Let n be an integer exceeding one. The following statements are equivalent for a matroid M having at least $n+2$ elements.*

(i) *M is n-minimally connected.*

(ii) *Every n-element subset of $E(M)$ is contained in an $(n+1)$-cocircuit.*

Proof. To prove that (i) implies (ii), we shall argue by induction on n. Thus suppose that (i) holds. If $n=2$, then Theorem 2.1 implies that (ii) holds. Now assume that (i) implies (ii) if $n<k$ and suppose $n=k \geqslant 3$. Let X be an n-element subset of $E(M)$ and e be an element of X. Evidently $M \backslash e$ is $(n-1)$-minimally connected. Therefore, by the induction assumption, $X-e$ is in an n-cocircuit C^* of $M \backslash e$. Hence either $C^* \cup e$ or C^* is a cocircuit of M. Suppose the latter holds and let f be an element of C^*. Then $M \backslash (C^*-f)$ has at least two elements including a coloop f. Thus $M \backslash (C^*-f)$ is disconnected. Since $|C^*-f|<n$, this is a contradiction. We conclude that $C^* \cup e$ is a cocircuit of M. Since $C^* \cup e \supseteq X$, we deduce that every n-element subset of $E(M)$ is in an $(n+1)$-cocircuit. Hence, by induction, (i) implies (ii).

Now suppose that M satisfies (ii) and let X be an n-element subset of $E(M)$. Since X is in an $(n+1)$-cocircuit, $M \backslash X$ has a coloop. As $|E(M)| \geqslant n+2$, it follows that $M \backslash X$ is disconnected. To complete the proof that M is n-minimally connected, we need to show that $M \backslash Y$ is connected for all subsets Y of $E(M)$ with at most $n-1$ elements. We show this first when $|Y|=j \leqslant n-2$. In that case, by (ii), every $(n-j)$-element subset of $M \backslash Y$ is contained in a cocircuit. Since $|E(M \backslash Y)| \geqslant n-j+2$ and $n-j \geqslant 2$, it follows that $M \backslash Y$ is connected.

Next suppose that $|Y|=n-1$ but that $M \backslash Y$ is disconnected. Let e and f be elements of distinct components of $M \backslash Y$ and let y be an element of Y. Then $(Y-y) \cup \{e, f\}$ has n elements and so is contained in an $(n+1)$-cocircuit, say $(Y-y) \cup \{e, f, g\}$. In $M \backslash (Y-y)$, the set $\{e, f, g\}$ is a cocircuit. But $M \backslash Y$ does not have a cocircuit containing $\{e, f\}$. Therefore,

$y \neq g$ and $M\backslash(Y-y)$ has a cocircuit $D \cup y$, where D is a proper subset of $\{e, f, g\}$ and $D \neq \{e, f\}$. Since every n-element subset of $E(M)$ is in an $(n+1)$-cocircuit, $M\backslash Y$ has no coloops. Thus $|D| \neq 1$ and so $|D| = 2$. Without loss of generality, we may assume that $D = \{e, g\}$. By cocircuit elimination using $\{e, f, g\}$ and $D \cup y$, we deduce that $M\backslash(Y-y)$ has a cocircuit $D' \cup y$, where $g \notin D'$ and D' is a non-empty subset of $\{e, f, g\}$. Since $M\backslash Y$ has no coloops, $|D'| \neq 1$, hence $D' = \{e, f\}$; a contradiction. We conclude that $M\backslash Y$ is connected and this completes the proof that (ii) implies (i). ∎

The restriction that n exceeds one is needed in the last theorem since, as can be seen from the cycle matroid of the graph in Fig. 1, a 1-minimally connected matroid need not have every element in a 2-cocircuit. Such a matroid M still has a large number of 2-cocircuits. Indeed, it can be shown using Lemma 2.3 that M has at least $r^*(M)+1$ pairwise disjoint 2-cocircuits [21, Corollary 2.7].

If n is a positive integer, we shall call a matroid M *n-cyclic* if it has at least n elements and every n-element subset of $E(M)$ is contained in an $(n+1)$-circuit. If M^* is n-cyclic, we shall call M *n-cocyclic*. Now suppose that M is n-cyclic. Then clearly $|E(M)| \neq n$, while if $|E(M)| = n+1$, then $M \cong U_{n,n+1}$. Moreover, the following result is an immediate consequence of Theorem 3.2 and duality.

(3.3) COROLLARY. *Let n be an integer exceeding one and M be a matroid having at least $n+2$ elements. Then M is n-cyclic if and only if M^* is n-minimally connected.*

This corollary and its dual will be used frequently in what follows as will the following consequence of the dual.

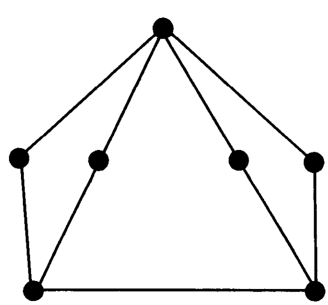

FIGURE 1

(3.4) COROLLARY. *Let n be an integer exceeding one. If M is an n-minimally connected matroid, then every n-element subset of $E(M)$ is coindependent.*

Evidently an n-cyclic matroid has rank at least n. If such a matroid M has rank equal to n, then it is isomorphic to $U_{n,n+m}$ for some $m \geqslant 1$. Thus all n-cyclic matroids of rank n are known. For this reason, when n-cyclic matroids are considered in what follows, we shall often assume that their ranks exceed n.

To conclude this section, we make some observations concerning the existence of n-minimally connected matroids. A constructive characterization of all 1-minimally connected matroids is given in [21, Theorem 3.1]. For $n=2$, we know by Corollary 2.2 that if M is n-minimally connected, then M^* is a Sylvester matroid. Numerous examples of Sylvester matroids are given by Bryant, Dawson, and Perfect [4]. These include Steiner systems $S(t, k, v)$ for $t=2$, projective geometries of rank at least two, and affine geometries $AG(r-1, q)$, where $r \geqslant 2$ and $q \geqslant 3$. Murty [19] established the following lower bound on the number of elements in a Sylvester matroid.

(3.5) THEOREM. *Let M be a Sylvester matroid of rank r where $r \geqslant 3$. Then $|E(M)| \geqslant 2^r - 1$. Moreover, the only matroid for which equality is attained here is $PG(r-1, 2)$.*

To find n-minimally connected matroids for $n \geqslant 3$, we look for the duals of n-cyclic matroids with at least $n+2$ elements. No matroid of the latter type is regular:

(3.6) PROPOSITION. *Let n be an integer exceeding one and M be an n-cyclic matroid having at least $n+2$ elements. Then M is not regular.*

Proof. If T is a j-element subset of $E(M)$ where $n-j \geqslant 1$, then M/T is $(n-j)$-cyclic having at least $(n-j)+2$ elements. Therefore, it suffices to prove the proposition in the case that $n=2$. Assume that we are in this case and suppose also that M is regular. Let $r(M)=r$. If $r \leqslant 2$, then, as M is simple, it has a 4-point line as a minor; a contradiction. Hence $r \geqslant 3$. Therefore, by [14], as M is simple and regular, $|E(M)| \leqslant \binom{r+1}{2}$. But, by Theorem 3.5, $|E(M)| \geqslant 2^r - 1$. Thus $2^r - 1 \leqslant \binom{r+1}{2}$. Since $r \geqslant 3$, this is a contradiction. ∎

The next result is the analogue of Theorem 3.5 for 3-cyclic matroids.

(3.7) PROPOSITION. *Let M be a 3-cyclic matroid of rank r, where $r \geqslant 4$. Then $|E(M)| \geqslant 2^{r-1}$. Moreover, the only matroid for which equality if attained here is $AG(r-1, 2)$.*

Proof. If $e \in E(M)$, then M/e is a Sylvester matroid of rank $r-1$. Since $r-1 \geqslant 3$, it follows by Theorem 3.5 that $|E(M/e)| \geqslant 2^{r-1}-1$ with equality holding here if and only if $M/e \cong PG(r-2, 2)$. Therefore, $|E(M)| \geqslant 2^{r-1}$.

Now suppose that $|E(M)| = 2^{r-1}$. Then, from the above, every single-element contraction of M is binary. But $r(M) \geqslant 4$. Therefore, by Crapo and Rota's Scum Theorem [11] and Tutte's excluded-minor characterization of binary matroids [30], M is binary. Let $[I_r | A]$ be a representation for M over $GF(2)$, where the columns of I_r are labelled by the elements of a basis B of M. Since M is 3-cyclic, for every 3-element subset X of B, there is an element e_X of $E(M)-B$ such that $X \cup e_X$ is a 4-circuit of M. Thus every member of $V(r, 2)$ with exactly three ones is a column of A. A straightforward induction argument can now be used to show that every member of $V(r, 2)$ with an odd number of ones is a column of $[I_r | A]$. For instance, if $B = \{1, 2, ..., r\}$ and $f = e_X$, where $X = \{1, 2, 3\}$, then $\{f, 4, 5\}$ is contained in a 4-circuit C of M. The element of $V(r, 2)$ corresponding to the fourth member of C has ones as its first five coordinates and zeros elsewhere.

The number of members of $V(r, 2)$ with an odd number of ones is exactly 2^{r-1}. Hence the columns of $[I_r | A]$ are precisely the members of $V(r, 2)$ with an odd number of ones. Thus $M \cong AG(r-1, 2)$. ∎

Apart from the binary affine geometries $AG(r-1, 2)$ for $r \geqslant 3$, examples of 3-cyclic matroids include truncations of these binary affine geometries to some rank exceeding three, and Steiner systems $S(t, k, v)$ for $t=3$. Other less symmetric 3-cyclic matroids can be obtained in several ways. For example, one can delete at most $k-4$ elements from an $S(3, k, v)$ and still have a 3-cyclic matroid. Alternatively, let $\{H_1, H_2, ..., H_m\}$ be a subset of the set of blocks of an $S(3, k, v)$ and suppose that an $S(3, k', k)$ exists. If we replace each H_i by the blocks of an $S(3, k', k)$ on H_i, then the blocks of the original $S(3, k, v)$ along with the blocks of each $S(3, k', k)$ are easily seen to be the hyperplanes of a 3-cyclic matroid.

By combining and iterating these two operations of deleting elements and replacing blocks, one can obtain further 3-cyclic matroids. These examples will all have rank four. Indeed, for all $n \geqslant 3$, we know of no examples of n-cyclic matroids of rank exceeding $n+1$ except for the binary affine geometries and their truncations that were noted above. For $n \geqslant 4$, the n-cyclic matroids of rank equal to $n+1$ include the Steiner systems $S(n, k, v)$ and those matroids obtained by deleting at most $k-n-1$ elements from such a system. However, relatively few Steiner systems $S(t, k, v)$ with $t \geqslant 4$ are known and none at all is known for $t \geqslant 6$ (see Cameron [8] and Denniston [13]). Thus for $n \geqslant 4$, the set of known n-cyclic matroids is small, and it would be of interest to find more examples of such matroids particularly if these examples have rank exceeding $n+1$. In Section 5, we shall show that all examples of the latter type must be $(n+1)$-connected.

It is clear that Proposition 3.7 can be extended to give a lower bound on the number of elements in an n-cyclic matroid of rank exceeding n for all $n \geqslant 4$. In particular, a 4-cyclic matroid of rank r exceeding 4 must have at least $2^{r-2}+1$ elements. By using Proposition 3.7 again, it is not difficult to show that equality is never attained in this bound. To close this section, we remark that, for $n \geqslant 4$, Proposition 3.6 can be strengthened to give that no n-cyclic matroid with more than $n+1$ elements is binary. The proof of this is an easy consequence of the fact that, in a binary matroid, the symmetric difference of two distinct circuits contains a circuit.

4. The n-Cyclic, n-Cocyclic Matroids

The main goal of this section is to generalize Theorem 1.1 by determining, for all $n \geqslant 3$, those matroids M such that both M and M^* are n-minimally connected or, equivalently, such that M is both n-cyclic and n-cocyclic. The Steiner systems $S(n, n+1, 2n+2)$ with $n \geqslant 2$ will feature prominently here and we now note some properties of such matroids. First, Mendelsohn [16] has shown that in every such matroid, the complement of every block is a block. Hence the matroid associated with an $S(n, n+1, 2n+2)$ is identically self-dual. Second, the only Steiner systems $S(n, n+1, 2n+2)$ that are known to exist are $S(3, 4, 8)$ and $S(5, 6, 12)$, and, as noted in the introduction, there is a unique system of each of these types. Finally, by the well-known divisibility conditions for Steiner systems (see, for example, [8, p. 47]), a necessary condition for an $S(n, n+1, 2n+2)$ to exist is that $n+2$ is prime. This condition is, however, not a sufficient condition for existence; for example, Mendelsohn and Hung [17] have shown that there is no $S(9, 10, 20)$.

Evidently every $S(n, n+1, 2n+2)$ is n-cyclic. Since such a matroid is identically self-dual, it is also n-cocyclic. Another matroid that is both n-cyclic and n-cocyclic is $U_{n,2n}$. The main result of this section is that these are the only examples of matroids that are both n-cyclic and n-cocyclic.

(4.1) Theorem. *Let M be a connected matroid having at least n elements, where n is an integer exceeding one. Suppose that, for all n-element subsets X of $E(M)$, both $M \backslash X$ and M/X are disconnected, but, for all subsets Y of $E(M)$ with fewer than n elements, both $M \backslash Y$ and M/Y are connected. Then M is isomorphic to $U_{n,2n}$ or $S(n, n+1, 2n+2)$, where the latter can only occur if $n+2$ is a prime.*

Since we know about the existence and uniqueness of the matroids $S(n, n+1, 2n+2)$ for all n with $2 \leqslant n \leqslant 10$, we can specify, for all such n, precisely which matroids have the property that both they and their duals are n-minimally connected. For $n=2$, this is the content of Theorem 1.1. For $n=3$, the result is as follows.

(4.2) COROLLARY. *$U_{3,6}$ and $AG(3,2)$ are the only matroids M with at least three elements such that $M\backslash X$ and M/X are disconnected for all 3-element subsets X of $E(M)$, but $M\backslash Y$ and M/Y are connected for all subsets Y of $E(M)$ with at most two elements.*

As $AG(3,2)$ is not 4-connected, an immediate consequence of this is the following:

(4.3) COROLLARY. *$U_{3,6}$ is the only 4-connected matroid M with at least three elements such that $M\backslash X$ and M/X are disconnected for all 3-element subsets X of $E(M)$.*

The proof of Theorem 4.1 will require a number of preliminary lemmas, the first of which is a straightforward consequence of circuit elimination.

(4.4) LEMMA. *Let I be an independent set in a matroid M. If I_1 and I_2 are distinct subsets of I, then $E(M)-I$ does not contain an element e such that both $I_1 \cup e$ and $I_2 \cup e$ are circuits of M.*

Evidently every circuit in an n-cyclic matroid M has at least $n+1$ elements. We show next that the same is true for every cocircuit provided $r(M) \geqslant n+1$.

(4.5) LEMMA. *Let M be an n-cyclic matroid having rank at least $n+1$. Then every cocircuit C^* of M has at least $n+1$ elements.*

Proof. Let e be an element of C^* and B be a basis of the hyperplane $E(M)-C^*$ of M. Then $|B| = r(M)-1 \geqslant n$. If X is an $(n-1)$-element subset of B, then there is an element e_X of $E(M)$ such that $X \cup e \cup e_X$ is a circuit. As $E(M)-C^*$ is a flat, $e_X \in C^*-e$. By Lemma 4.4, if X and Y are distinct $(n-1)$-element subsets of B, then $e_X \neq e_Y$. Thus $|C^*-e| \geqslant \binom{n}{n-1}$ and so $|C^*| \geqslant n+1$. ∎

The next lemma puts tight bounds on the rank of an n-cyclic, n-cocyclic matroid.

(4.6) LEMMA. *Suppose $n \geqslant 2$ and let M be a matroid that is both n-cyclic and n-cocyclic. Then $r(M)=r^*(M)$. Moreover, $r(M)$ is n or $n+1$, so $|E(M)|$ is $2n$ or $2n+2$.*

Proof. Let $r(M)=r$ and $r^*(M)=r^*$, and assume, without loss of generality, that $r \geqslant r^*$. As every n-element subset of $E(M)$ is coindependent,

$r^* \geqslant n$. Now let B be a basis of M. As M is n-cyclic, for every n-element subset X of B, there is an element e_X of $E(M)-B$ such that $X \cup e_X$ is a circuit of M. By Lemma 4.4, if X and Y are distinct n-element subsets of B, then $e_X \neq e_Y$. Thus $r^* = |E(M)-B| \geqslant \binom{r}{n}$. But $r \geqslant r^*$. Hence $r \geqslant \binom{r}{n}$. As $r \geqslant n \geqslant 2$, it follows that r is n or $n+1$. In the first case, since $r \geqslant r^* \geqslant n$, we deduce that $r = r^*$. In the second case, $n+1 = r \geqslant r^* \geqslant \binom{r}{n} = r$ and, again, $r = r^*$. ▮

We shall need just one more lemma before proving the main result of this section.

(4.7) LEMMA. *Suppose $n \geqslant 2$ and let M be a rank-$(n+1)$ matroid that is both n-cyclic and n-cocyclic. Then, for every circuit C of M, $|C| = n+1$, and $E(M)-C$ is a cocircuit of M.*

Proof. As M is n-cyclic, $|C| \geqslant n+1$. Moreover, as $r(M) = n+1$, we must have that $|C| \leqslant n+2$. Therefore, since Lemma 4.6 implies that $|E(M)| = 2n+2$, we deduce that $E(M)-C$ contains an n-element subset X. As M is n-cocyclic, there is an element e of $E(M)-X$ such that $X \cup e$ is a cocircuit. Since $|(X \cup e) \cap C|$ cannot be one, $e \notin C$. Thus $X \cup e = E(M)-C$ and $|C| = n+1$. ▮

Proof of Theorem 4.1. Let M be a matroid that is both n-cyclic and n-cocyclic. Then, by Lemma 4.6, $r(M) = r^*(M)$ and $r(M)$ is n or $n+1$. In the first case, since every n-element subset of $E(M)$ is in an $(n+1)$-circuit, $M \cong U_{n,2n}$. Now suppose that $r(M) = n+1$. Then $|E(M)| = 2n+2$. Moreover, as M is n-cyclic, if X is an n-element subset of $E(M)$, then X is contained in an $(n+1)$-circuit C. By Lemma 4.7, C is a hyperplane of M. Thus $C = \bar{X}$ and so C is the unique hyperplane of M containing X. We conclude that M is an $S(n, n+1, 2n+2)$.

5. The Connectivity of n-Cyclic Matroids

Proposition 3.1 noted that if M is an $(n+1)$-connected matroid with $|E(M)| \geqslant 2n$, and $M \backslash X$ is disconnected for every n-element subset X of $E(M)$, then M is n-minimally connected. We also observed earlier that the converse of this proposition fails for $n=3$. In this section, we shall show that if $r^*(M) > n+1$, then the converse holds. We also consider what happens if $r^*(M) = n+1$ and, in particular, characterize those matroids for which the converse fails in the case when $n=3$. The arguments here will be given for n-cyclic matroids rather than for n-cocyclic matroids since we prefer to argue in terms of bases and rank rather than in terms of cobases and corank. Moreover, since the set of n-cyclic matroids of rank n is

$\{U_{n,n+m}: m \geqslant 1\}$, we shall concentrate in this section on n-cyclic matroids of rank at least $n+1$. The next lemma, a preliminary to the main result of the section, shows that such a matroid M is n-connected. The main result will show that if $r(M)$ exceeds $n+1$, then M is $(n+1)$-connected.

(5.1) LEMMA. *Let M be an n-cyclic matroid having rank at least $n+1$. If $\{X, Y\}$ is a k-separation of M with $|X| \geqslant |Y|$ and $k \leqslant n$, then $\{\bar{X}, E(M)-\bar{X}\}$ is an exact k-separation of M, $k=n$, and $\bar{X}$ is a hyperplane of M. Therefore M is n-connected.*

Proof. Since $\{X, Y\}$ is a k-separation of M with $|X| \geqslant |Y|$, we have

$$|X| \geqslant |Y| \geqslant k \tag{1}$$

and

$$r(X)+r(Y)-r(M) \leqslant k-1. \tag{2}$$

Now, as every n-element subset of $E(M)$ is independent and $k \leqslant n$, we deduce that $r(Y) \geqslant k$ and so, by (2),

$$r(X) \leqslant r(M)-1. \tag{3}$$

Next we consider $\{\bar{X}, E(M)-\bar{X}\}$. By (1),

$$|\bar{X}| \geqslant |X| \geqslant |Y| \geqslant |E(M)-\bar{X}|. \tag{4}$$

Now, by Lemma 4.5, every cocircuit of M contains at least $n+1$ elements. Hence, by (3),

$$|E(M)-\bar{X}| \geqslant n+1 \geqslant k+1. \tag{5}$$

Therefore, since (2) implies that

$$r(\bar{X})+r(E(M)-\bar{X})-r(M) \leqslant k-1, \tag{6}$$

we conclude that $\{\bar{X}, E(M)-\bar{X}\}$ is a k-separation of M.

Assume next that $\bar{X}$ is not a hyperplane of M. Then $E(M)-\bar{X}$ contains a cocircuit C^* and an element e that is not in C^*. By (4) and (5), $|\bar{X}| \geqslant n+1$. Therefore $\bar{X}$ contains an $(n-2)$-element subset Z and an element f that is not in Z. As M is n-cyclic, it has an element g_f such that $Z \cup \{e, f, g_f\}$ is a circuit. Since $\bar{X}$ is a flat of M and $(Z \cup \{e, f\})-\bar{X}=\{e\}$, the element g_f is not in $\bar{X}$. Moreover, since the circuit $Z \cup \{e, f, g_f\}$ cannot have exactly one element in common with the cocircuit C^*, this cocircuit does not contain g_f. Hence $g_f \in E(M)-(\bar{X} \cup C^* \cup e)$. If f and h are distinct elements of $\bar{X}-Z$, then, as $Z \cup \{f, h\}$ has exactly n elements, it is independent. Therefore, as $e \notin \bar{X}$, the set $Z \cup \{f, h, e\}$ is also independent. Now, applying Lemma 4.4 to the subsets $Z \cup \{f, e\}$ and $Z \cup \{h, e\}$ of $Z \cup \{f, h, e\}$, we deduce that $g_f \neq g_h$. Thus $|E(M) - (\bar{X} \cup C^* \cup e)| \geqslant |\bar{X} - Z| = |\bar{X}| - (n-2)$. Since Lemma 4.5 implies that $|C^*| \geqslant n+1$, it follows that $|E(M) - \bar{X}| \geqslant |\bar{X}| + 4$, a contradiction to (4). We conclude that $\bar{X}$ is a hyperplane of M.

Finally, we note that, since $r(\bar{X}) = r(M) - 1$, a consequence of (6) is that $r(E(M) - \bar{X}) \leqslant k$. But $|E(M) - \bar{X}| \geqslant n+1$, so, as M is n-cyclic, $r(E(M) - \bar{X}) \geqslant n$. Since $k \leqslant n$, it follows that $n \geqslant k \geqslant r(E(M) - \bar{X}) \geqslant n$. Thus $k = n$ and equality holds in (6). Hence $\{\bar{X}, E(M) - \bar{X}\}$ is an exact n-separation of M and M is n-connected. ▌

The next theorem, the main result of this section, shows that if M has corank exceeding $n+1$, then M is n-minimally connected if and only if $M \backslash X$ is minimally $(n+1-j)$-connected for all j-element subsets X of $E(M)$ and all j in $\{0, 1, ..., n-1\}$.

(5.2) THEOREM. *Let n be an integer exceeding one. The following statements are equivalent for a matroid M having rank exceeding $n+1$.*

(i) *M is n-cyclic.*

(ii) *M is $(n+1)$-connected and M/X is disconnected for all n-element subsets X of $E(M)$.*

Proof. Assume that (ii) holds. By Corollary 3.3 and the dual of Proposition 3.1, to show that (i) holds, it suffices to show that $|E(M)| \geqslant 2n$. As M is $(n+1)$-connected, it has no k-separation for any $k < n+1$. If M has a k-separation for some $k \geqslant n+1$, then $|E(M)| \geqslant 2k$ and hence $|E(M)| \geqslant 2n$. If M has no such k-separation, then, by [27, 15], M is isomorphic to $U_{m,2m-1}$, $U_{m,2m}$, or $U_{m,2m+1}$ for some m. But $r(M) > n+1$, so again $|E(M)| \geqslant 2n$. We conclude that (ii) implies (i).

Now assume that (i) does not imply (ii). Let n be the least integer exceeding one for which (i) holds but (ii) does not, and, for this n, let M be a matroid that satisfies (i) but not (ii). Then M is not $(n+1)$-connected. By Theorem 2.1 and Corollary 3.3, $n > 2$. Moreover, by Lemma 5.1, M has an exact n-separation $\{X, Y\}$ in which X is a hyperplane. Clearly,

$$r(Y) = n. \tag{7}$$

We show next that Y is a flat of M. Suppose that $e \in X$. Then M/e is $(n-1)$-cyclic and has rank exceeding $(n-1)+1$. Thus, by the choice of n, M/e is n-connected and so

$$r_{M/e}(X-e) + r_{M/e}(Y) \geqslant r(M/e) + n - 1,$$

that is,

$$r(X)+r(Y\cup e)\geqslant r(M)+n.$$

But $r(X)=r(M)-1$, so $r(Y\cup e)\geqslant n+1$. We conclude, by (7), that Y is indeed a flat of M.

Next we give a lower bound on $|Y|$.

(5.3) LEMMA. $|Y|\geqslant\binom{n+1}{n-1}+1\geqslant 2n-5$.

Proof. Since X is a hyperplane of M and $r(M)>n+1$, X has an $(n+1)$-element independent subset, say I. Now choose an element e from Y. Then, for every $(n-1)$-element subset J of I, there is an element y_J such that $J\cup e\cup y_J$ is a circuit of M. As this circuit cannot have exactly one element in common with the cocircuit Y, it follows that $y_J\in Y-e$. By Lemma 4.4, if J and K are distinct $(n-1)$-element subsets of I, then $y_J\neq y_K$. Thus

$$|Y-e|\geqslant\binom{n+1}{n-1}\qquad\text{and so}\qquad |Y|\geqslant\binom{n+1}{n-1}+1.$$

It is now straightforward to complete the proof of the lemma by showing that $\binom{n+1}{n-1}+1\geqslant 2n-5$. ∎

Since Y is a flat of M of rank n and $n<r(M)-1$, the set X contains a cocircuit C^* and an element e that is not in C^*. Let I be an $(n-2)$-element subset of Y. Then, for every element y of $Y-I$, there is an element f_y of $E(M)$ such that $I\cup\{e, y, f_y\}$ is a circuit of M. This circuit cannot have exactly one element outside of the flat Y; nor can it have exactly one element in the cocircuit C^*. Thus $f_y\in X-e$ and $f_y\notin C^*$. If y and z are distinct elements of $Y-I$ and $f_y=f_z$, then, by circuit elimination, $[(I\cup\{e, y, f_y\})\cup(I\cup\{e, z, f_y\})]-e$ contains a circuit of M. As Y is a flat and $I\cup\{y, z\}\subseteq Y$, this circuit does not contain f_y. Hence it has at most n elements; a contradiction. Therefore $f_y\neq f_z$ and so $|Y-I|\leqslant|X-(C^*\cup e)|$, that is, $|Y|-(n-2)\leqslant|X-C^*|-1$. By Lemma 5.3, $|Y|-(n-2)\geqslant n-3$, and hence

$$|X-C^*|\geqslant n-2.$$

By the last inequality, $X-C^*$ has an $(n-2)$-element subset A. Now choose f in Y and c in C^*. Then M has an element g such that $A\cup\{c, f, g\}$ is a circuit. This circuit cannot have exactly one element in common with Y or with C^*. Therefore, $g\in Y\cap C^*$. But $Y\cap C^*$ is empty and this contradiction completes the proof that (i) implies (ii) thereby finishing the proof of the theorem. ∎

On combining Lemma 5.1 and Theorem 5.2, we immediately obtain the following:

(5.4) COROLLARY. *Assume that M is an n-cyclic matroid having rank at least $n+1$ and suppose that M is not $(n+1)$-connected. Then $r(M)=n+1$ and M has an exact n-separation $\{X, Y\}$ in which $|X| \geqslant |Y|$ and X is a hyperplane.*

The remaining results in this section concentrate on the case when $n=3$. In particular, we explicitly describe the structure of those 3-cyclic matroids of rank at least 4 that are not 4-connected.

(5.5) LEMMA. *Let $\{X, Y\}$ be a 3-separation of a 3-cyclic matroid M and suppose that X is a flat of M. Then $|Y| \geqslant |X|$.*

Proof. Choose elements x and y in X and Y, respectively. Then, as M is 3-cyclic, for all e in $X-x$, there is an element g_e in $E(M)$ such that $\{x, y, e, g_e\}$ is a 4-circuit. As X is a flat, $g_e \in Y-y$. Now suppose that $g_e = g_f$ for some pair $\{e, f\}$ of distinct elements of $X-x$. Then, by circuit elimination, $(\{x, y, e, g_e\} \cup \{x, y, f, g_f\}) - g_e$ contains a circuit. Since this circuit has at least four elements, it must be $\{x, y, e, f\}$. Thus $y \in \bar{X}$. Since X is a flat, this is a contradiction. We conclude that $g_e \neq g_f$ and hence that $|Y| \geqslant |X|$. ∎

(5.6) LEMMA. *Let M be a 3-cyclic matroid that is not 4-connected. Then $r(M)=4$. Moreover, M has a 3-separation $\{X, Y\}$ in which $|X| = |Y|$ and both X and Y are hyperplanes.*

Proof. By Corollary 5.4, $r(M)=4$ and M has a 3-separation $\{X, Y\}$ in which $|X| \geqslant |Y|$ and X is a hyperplane. As X is a flat, Lemma 5.5 implies that $|Y| \geqslant |X|$. Thus, by Lemma 5.1, $\{\bar{Y}, E(M)-\bar{Y}\}$ is a 3-separation of M and $\bar{Y}$ is a hyperplane. By Lemma 5.5 again, $|E(M)-\bar{Y}| \geqslant |\bar{Y}|$. On combining the above inequalities and using the fact that $E(M)-\bar{Y} \subseteq X$, we obtain that

$$|Y| \geqslant |X| \geqslant |E(M)-\bar{Y}| \geqslant |\bar{Y}| \geqslant |Y|.$$

Thus $Y=\bar{Y}$, so Y is a hyperplane. ∎

We can, in fact, give a more precise description of the 3-cyclic matroids that have rank at least four and are not 4-connected. To do this, we begin by looking at the one concrete example we know of such a matroid, namely $AG(3, 2)$. It is well known that this matroid can be constructed by sticking together two copies of the Fano matroid F_7 along a line and then deleting the points of that line (see Fig. 2). More precisely, $AG(3,2)$ is the *modular sum* [6, p. 186] of two copies of F_7 across a 3-point line. A generalization of this construction produces all 3-cyclic matroids of rank 4 that are not 4-connected.

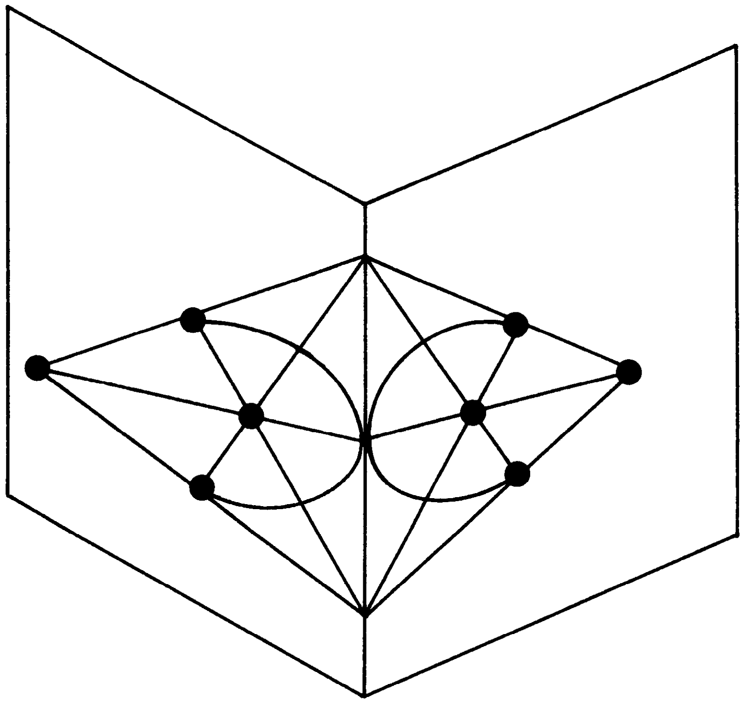

FIGURE 2

Let m be an integer exceeding one. Suppose that G is a complete graph having vertex set $\{x_1, x_2, ..., x_{2m}\}$, and let $\{p_1, p_2, ..., p_{2m-1}\}$ be a partition π of $E(G)$ into perfect matchings, that is, π is a 1-factorization of G. Then it is not difficult to see that one obtains a rank-3 paving matroid $N_m(\pi)$ on $\{x_1, x_2, ..., x_{2m}, p_1, p_2, ..., p_{2m-1}\}$ by taking the dependent lines to be $\{p_1, p_2, ..., p_{2m-1}\}$ together with all sets of the form $\{x_i, x_j, p_k\}$ such that, in the 1-factorization π of G, the edge $x_i x_j$ is in the perfect matching p_k. In this matroid, the line $\{p_1, p_2, ..., p_{2m-1}\}$ will be called the *distinguished line*. It is modular because it meets every other line and $N_m(\pi)$ has rank 3. This means that we can form the generalized parallel connection [6, p. 185] of two matroids of the form $N_m(\pi)$ by identifying the points on the distinguished line of one with the points on the distinguished line of the other. To form the corresponding modular sum, one deletes these $2m-1$ composite points. This modular sum is certainly 3-cyclic of rank 4 and, because its ground set is the union of two rank-3 flats, it is not 4-connected. We show that every 3-cyclic matroid of rank 4 that is not 4-connected can be constructed in this way.

(5.7) THEOREM. *Let M be a 3-cyclic matroid that has rank at least four* *and is not 4-connected. Then $r(M)=4$ and $|E(M)|=4m$ for some integer m exceeding one. Moreover, M is the modular sum across their distinguished $(2m-1)$-point lines of two matroids of the form $N_m(\pi)$.*

Before proving this theorem, we remark that it is well known (see, for example, [10, p. 236]) that K_{2m} has a 1-factorization. Thus, for all $m \geqslant 2$, there is at least one matroid of the form $N_m(\pi)$. For m in $\{2, 3\}$, there is, up to isomorphism, only a single 1-factorization of K_{2m} (see, for example, [9, p. 56]), but for larger values of m, there is more than one [32]. Indeed, the number $f(m)$ of non-isomorphic 1-factorizations of K_{2m} satisfies the equation $\lim_{m \to \infty} \ln(f(m))/m^2 \ln(m) = \frac{1}{2}$ [7, p. 66].

Proof of Theorem 5.7. By Lemma 5.6, M has rank 4 and there is a partition $\{X, Y\}$ of $E(M)$ into hyperplanes such that $|X| = |Y|$. Since X and Y are both flats, every 4-circuit of M meets both X and Y in an even number of elements.

(5.8) Lemma. *If $\{x_1, x_2\} \subseteq X$, then there is a partition $\{Y_1, Y_2, ..., Y_m\}$ of Y into 2-element subsets such that $Y_i \cup \{x_1, x_2\}$ is a circuit of M for all i in $\{1, 2, ..., m\}$. Moreover, M has no other circuits that contain $\{x_1, x_2\}$ and meet Y.*

Proof. Suppose $y \in Y$. Then $\{x_1, x_2, y\}$ is in a 4-circuit of M, the fourth element of which must be in Y. If $\{x_1, x_2, y, z\}$ and $\{x_1, x_2, y, w\}$ are circuits of M, where z and w are distinct elements of $Y - y$, then, by circuit elimination, $\{x_2, y, z, w\}$ contains a circuit C of M. Since C must have size at least four, $C = \{x_2, y, z, w\}$. But this is a contradiction since C meets Y in an odd number of elements. The lemma follows immediately. ∎

An easy consequence of this lemma is that $|Y| = 2m$. Thus, as $|X| = |Y|$, we conclude that $|E(M)| = 4m$. Moreover, by interchanging X and Y in the last lemma, we deduce that every 2-element subset $\{y_1, y_2\}$ of Y induces a partition $\{X_1, X_2, ..., X_m\}$ of X into 2-element subsets such that $\{X_i \cup \{y_1, y_2\}: 1 \leqslant i \leqslant m\}$ is the set of 4-circuits that contain $\{y_1, y_2\}$ and meet X. It is not difficult to see that the following lemma completes the proof of Theorem 5.7.

(5.9) Lemma. *Suppose that $\{y_1, y_2, y_3, y_4\}$ is a subset of Y such that both $\{y_1, y_2, x_1, x_2\}$ and $\{y_3, y_4, x_1, x_2\}$ are circuits of M for some subset $\{x_1, x_2\}$ of X. If $\{y_1, y_2, x_3, x_4\}$ is a circuit for some $\{x_3, x_4\} \subseteq X$, then so is $\{y_3, y_4, x_3, x_4\}$.*

Proof. Assume that $\{y_3, y_4, x_3, x_4\}$ is not a 4-circuit of M. Then, as M is 3-cyclic, for some x_5 in $X - x_4$, the set $\{y_3, y_4, x_3, x_5\}$ is a circuit. By the strong circuit elimination axiom, $(\{y_1, y_2, x_3, x_4\} \cup$

$\{y_3, y_4, x_3, x_5\}) - x_3$ contains a circuit C containing x_5. As Y is a flat, x_5 cannot be the only element in $C - Y$. Therefore, $x_4 \in C$. Moreover, by Lemma 5.8, $|C| = 4$. By Lemma 5.8 again, M does not have two distinct circuits meeting both X and Y that have exactly three common elements. Therefore, C does not contain $\{y_1, y_2\}$ or $\{y_3, y_4\}$. Assume, without loss of generality, that $C = \{y_1, y_3, x_4, x_5\}$. Then, by the strong elimination axiom using this circuit and $\{y_3, y_4, x_3, x_5\}$, we deduce that $\{y_1, y_3, y_4, x_3, x_4\}$ contains a circuit C' containing x_4. Arguing as for C, we deduce that $x_3 \in C'$ and $|C'| = 4$. Since C' cannot have exactly three elements in common with $\{y_1, y_2, x_3, x_4\}$, we deduce that $C' = \{y_3, y_4, x_3, x_4\}$. This contradiction completes the proof of Lemma 5.9 and thereby that of Theorem 5.7. ∎

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