ON CONNECTIVITY IN MATROIDS AND GRAPHS

BY

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ABSTRACT. In this paper we derive several results for connected matroids and use these to obtain new results for 2-connected graphs. In particular, we generalize work of Murty and Seymour on the number of two-element cocircuits in a minimally connected matroid, and work of Dirac, Plummer and Mader on the number of vertices of degree two in a minimally 2-connected graph. We also solve a problem of Murty by giving a straightforward but useful characterization of minimally connected matroids. The final part of the paper gives a matroid generalization of Dirac and Plummer's result that every minimally 2-connected graph is 3-colourable.

1. Introduction. The structure of minimally 2-connected graphs was determined independently by Dirac [5] and Plummer [16]. Their work led Murty [10] to examine minimally 2-connected matroids and some of the latter's results were generalized by Seymour [17], [18]. In §2 of this paper, we strengthen one such result of Seymour by showing that if C is a circuit in a 2-connected matroid M and for all x in C, the restriction $M \setminus x$ is not 2-connected, then provided $|E(M)| \geqslant 4$, M has at least two disjoint cocircuits of size two contained in C. Several corollaries of this theorem are proved and the theorem is also used to derive the corresponding result for a 2-connected graph G, the conclusion in this case being that the circuit C meets at least two nonadjacent vertices of G of degree two. This result, which extends a result of Dirac [5] and Plummer [16] for minimally 2-connected graphs, is a strengthening in the case $n = 2$ of a result of Mader [8] for n-connected graphs. It has a number of corollaries including a new lower bound on the number of vertices of degree two in a minimally 2-connected graph. Some similar results for minimally n-connected graphs and matroids are also obtained for $n \geqslant 3$.

In [10], Murty asks for a characterization of minimally 2-connected matroids. In §3, we give such a result, showing that unless every element of a minimally 2-connected matroid M is in a cocircuit of size two, M can be obtained from two minimally 2-connected matroids on fewer elements by a join operation which is closely related to series connection. This characterization, which is not difficult to prove, is used to give short proofs of the main results of [10].

Dirac [5] and Plummer [16] have shown that a minimally 2-connected graph is 3-colourable and Dirac's argument [5, p. 215] can be extended to show that a minimally n-connected graph is $(n + 1)$-colourable (see, for example, [1, Corollary 4.7]). In §4, by generalizing this argument, we establish the corresponding result for minimally n-connected matroids when $n = 2$ or 3.

The terminology used here for matroids and graphs will in general follow Welsh [23] and Bondy and Murty [2] respectively. Most of the known graph connectivity results to which we shall refer can be found in Bollobás [1, Chapter 1]. The ground set of a matroid M will be denoted by $E(M)$ and, if $T \subseteq E(M)$, we denote the rank of T by rk T. We shall write rk M for rk($E(M)$) and cork M for the rank of the dual matroid M^* of M. The restriction of M to $E(M) \setminus T$ will sometimes be denoted by $M \setminus T$ or, if $T = \{x_1, x_2, \ldots, x_m\}$, by $M \setminus x_1, x_2, \ldots, x_m$. Likewise, the contraction of M to $E(M) \setminus T$ will sometimes be written as M/T or $M/x_1, x_2, \ldots, x_m$. A cocircuit of M having k elements will be called a k-cocircuit, and a k-element circuit, a k-circuit. If $x, y \in E(M)$ and $\{x, y\}$ is a 2-cocircuit, we say that x and y are *in series*; if instead $\{x, y\}$ is a 2-circuit, then x and y are *in parallel*. A *series class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct elements of A, then a and b are in series. Parallel classes are defined analogously. We call a series or parallel class *nontrivial* if it contains at least two elements.

Familiarity will be assumed with the concept of n-connection for graphs as defined, for example, in [2, p. 42]. We now recall the definition of n-connection for matroids [20]. If k is a positive integer, the matroid M is k-*separated* if there is a subset T of $E(M)$ such that $|T| \geqslant k$, $|E(M) \setminus T| \geqslant k$ and

$$\text{rk } T + \text{rk}(E(M) \setminus T) - \text{rk } M = k - 1.$$

If there is a least positive integer j such that M is j-separated, it is called the *connectivity* $\kappa(M)$ of M. If there is no such integer, we say that $\kappa(M) = \infty$. The matroid M is said to be n-*connected* for any positive integer n such that $n \leqslant \kappa(M)$. It is routine to show [20, (12)] that

$$\kappa(M) = \kappa(M^*). \tag{1.1}$$

The familiar notion of connectivity or nonseparability of matroids is related to n-connectedness as follows.

(1.2) *A matroid is connected if and only if it is 2-connected.*

A matroid or graph H is *minimally n-connected* if H is n-connected and, for all elements e of $E(H)$, $H \setminus e$ is not n-connected.

The notions of n-connectedness of a graph G and n-connectedness of its cycle matroid $M(G)$ do not, in general, coincide (see [6], [21], [23]). However, [21, pp. 1–2]

(1.3) *if G has no loops and at least three vertices, then G is 2-connected if and only if $M(G)$ is 2-connected; and*

(1.4) *if G is simple and has at least four vertices, then G is 3-connected if and only if $M(G)$ is 3-connected.*

If G is a 2-connected graph without loops, then clearly the set of all edges incident with a vertex of G is a cocircuit of $M(G)$. Such a cocircuit will be called a *vertex cocircuit* of $M(G)$.

2. Connectivity results for matroids and graphs.
The following basic lemmas will be used frequently throughout this paper.

(2.1) LEMMA [20, (6.5)]. *If M is a 2-connected matroid and $e \in E(M)$, then either $M \setminus e$ or M/e is 2-connected.*

(2.2) LEMMA [**15**, LEMMA 2.1]. *Let M be a matroid and n be an integer exceeding one. Suppose that e is an element of M for which M/e is n-connected but M is not. Then either e is a loop of M or M has a cocircuit containing e and having fewer than n elements.*

In this section we shall give two extensions of a result of Seymour [**18**, (2.3)]. The next lemma is the first of these.

(2.3) LEMMA. *Let M be a 2-connected matroid having at least two elements and $\{x_1, x_2, \ldots, x_m\}$ be a circuit of M such that $M \setminus x_i$ is not 2-connected for all i in $\{1, 2, \ldots, m - 1\}$. Then $\{x_1, x_2, \ldots, x_{m-1}\}$ contains a 2-cocircuit of M.*

PROOF. We argue by induction on m. If $m = 1$ or 2, the result is vacuously true, so suppose $m > 2$. As $M \setminus x_1$ is not 2-connected, it follows by Lemma 2.1 that M/x_1 is 2-connected. Moreover, $\{x_2, x_3, \ldots, x_m\}$ is a circuit of M/x_1. Thus, if for all i in $\{2, 3, \ldots, m - 1\}$, $M/x_1 \setminus x_i$ is not 2-connected, then, by the induction assumption, $\{x_2, x_3, \ldots, x_{m-1}\}$ contains a 2-cocircuit of M/x_1, and hence, as required, $\{x_1, x_2, x_3, \ldots, x_{m-1}\}$ contains a 2-cocircuit of M. We may therefore suppose that for some i in $\{2, 3, \ldots, m - 1\}$, $M/x_1 \setminus x_i$ is 2-connected. Thus, as $M \setminus x_i$ is not 2-connected, by Lemma 2.2, x_1 is a loop or a coloop of $M \setminus x_i$ and so $\{x_1, x_i\}$ is a 2-cocircuit of M and the required result holds. $\square$

In the notation of Lemma 2.3, Seymour [**18**, (2.3)] proved that $\{x_1, x_2, \ldots, x_m\}$ contains a 2-cocircuit provided M satisfies the additional condition that $M \setminus x_m$ is not 2-connected. Earlier he [**17**, (2.2)], Murty [**10**, Lemma 3.1] and White [**24**, Lemma 4.6] had independently shown that a minimally 2-connected matroid with at least two elements has a 2-cocircuit.

The next theorem, an alternative strengthening of Seymour's result, underlies most of the remaining results in this section.

(2.4) THEOREM. *Let C be a circuit of a 2-connected matroid M such that for all x in C, $M \setminus x$ is not 2-connected. Then either M is the circuit C or C contains at least two distinct nontrivial series classes of M.*

PROOF. We argue by induction on $|C|$. If $|C| = 1$, then the result is trivially true. Now suppose that $|C| \geqslant 2$. Then, by Lemma 2.3, M has a 2-cocircuit $\{x, y\}$ meeting C. If every element of C is in series with x, then it follows, since M is 2-connected, that $C = E(M)$; for, if there is an element e in $E(M) \setminus C$, then M has a circuit C' containing e and meeting C. But, as a circuit and a cocircuit of M cannot have exactly one common element, we obtain the contradiction that $C' \supsetneq C$. It follows that we may suppose that C contains an element z which is not in series with x. Consider M/z. By Lemma 2.1, this matroid is 2-connected. Moreover, $C \setminus z$ is a circuit of M/z. Now, if a is an element of $C \setminus z$, then consider $M/z \setminus a$. If this is 2-connected, then, as $M \setminus a$ is not 2-connected, it follows from Lemma 2.2 that $\{a, z\}$ is a 2-cocircuit of M. Evidently $\{a, z\} \subseteq C$, and as z is not in series with x, the required result holds. Thus we may assume that $M/z \setminus a$ is not 2-connected for all a in $C \setminus z$. The required result now follows on applying the induction assumption. $\square$

The preceding result is best-possible for if we take the cycle matroid of the plane graph G shown in Figure 1, the cycle C bounding the exterior face of G contains only two distinct nontrivial series classes. It is not difficult to extend this to give examples of matroids in which $|C|$ is arbitrarily large and yet C still contains only two nontrivial series classes.
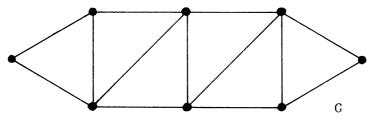

FIGURE 1

The next result, which generalizes a graph-theoretic result of Dirac [**5**, Proposition 3] and Plummer [**16**, Corollary 1b], is an immediate consequence of the preceding theorem.

(2.5) COROLLARY. *If a 2-connected matroid M has a 3-circuit T such that for all x in T, the matroid $M \setminus x$ is not 2-connected, then M is the 3-circuit T.*

(2.6) COROLLARY. *Let M be a 2-connected matroid other than a single circuit. Suppose that $A \subseteq E(M)$ such that for all a in A, $M \setminus a$ is not 2-connected. Then either A is independent or A contains at least $|A| -$ rk $A + 1$ nontrivial series classes of M.*

PROOF. Suppose that A is dependent and let A' be the union of all circuits of M contained in A. We shall prove that A' contains at least $|A'| -$ rk $A' + 1$ nontrivial series classes of M. The required result will then follow since $|A| -$ rk $A = |A'| -$ rk A'. As a circuit and a cocircuit of a matroid cannot have exactly one common element, if C^* is a 2-cocircuit of M meeting A', then $C^* \subseteq A'$, and moreover, C^* is a 2-cocircuit of $M|A'$. Now let X be the set of elements of A' which are contained in a 2-cocircuit of M. Then, by Theorem 2.4, X meets every circuit of $M|A'$. Since the cobases of a matroid are the minimal sets meeting every circuit, it follows that X contains a cobase B^* of $M|A'$. Choose an element b of B^* and let C_b be the fundamental circuit of b in $M|A'$ with respect to $A' \setminus B^*$. By the choice of B^*, M has a 2-cocircuit, say $\{b, c\}$, containing b. Moreover, by Theorem 2.4, C_b contains another 2-cocircuit $\{d, e\}$ which is contained in a different series class from $\{b, c\}$. Now let $B^* = \{x_1, x_2, \ldots, x_m\}$ where $x_1 = b$, and let $\{x_1, y_1\}$, $\{x_2, y_2\}, \ldots, \{x_m, y_m\}$ be 2-cocircuits of M where $y_1 = c$. We complete the proof by showing that $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}, \{d, e\}$ are contained in different series classes of M. Since B^* is a cobase of $M|A'$, it contains at most one element of each series class of M and therefore $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}$ are contained in distinct nontrivial series classes of M. It remains to show that none of these classes contains $\{d, e\}$. But, if for some i, $\{d, e, x_i, y_i\}$ is contained in a series class of M, then this series class is contained in C_b. It follows that $i = 1$, hence

$\{x_i, y_i\} = \{b, c\}$ and we have a contradiction to the fact that $\{b, c\}$ and $\{d, e\}$ are contained in different series classes of M. □

The next result is an immediate consequence of the preceding corollary, slightly strengthens [**14**, Proposition 4.13].

(2.7) COROLLARY. *Let M be a minimally 2-connected matroid which is not a circuit. Then M has at least* cork $M + 1$ *nontrivial series classes and therefore has at least* cork $M + 1$ *pairwise disjoint 2-cocircuits.*

We now come to apply Theorem 2.4 to graphs. First, however, we note the following result of Mader.

(2.8) THEOREM [**8**, SATZ 1]. *Let C be a cycle of a simple n-connected graph G such that for all edges x of C, $G \setminus x$ is not n-connected. Then C meets a vertex which has degree n in G.*

In the case $n = 2$, we can strengthen this result as follows.

(2.9) THEOREM. *Let G be a 2-connected graph without loops and C be a cycle of G such that for all edges x of C, the graph $G \setminus x$ is not 2-connected. Then either G is the cycle C or C meets two vertices of G of degree two which are separated on C by vertices of degree greater than two.*

PROOF. As G is 2-connected, $|V(G)| \geqslant 3$ and therefore, by (1.3), $M(G)$ is 2-connected. The required result will follow if we can show that either G is a cycle or C contains at least two nontrivial series classes of $M(G)$ each of which contains a vertex 2-cocircuit of G. We shall show this by induction on $|C|$ noting that the result holds for $|C| \leqslant 3$ (see Corollary 2.5). Now assume that G is not a cycle. Then, by Theorem 2.4, C contains at least two nontrivial series classes, A_1 and A_2 say. We may assume that A_1 does not contain a vertex 2-cocircuit of $M(G)$. Let $\{a, b\}$ be a 2-cocircuit contained in A_1. Then $G \setminus \{a, b\}$ has precisely two components, H_1 and H_2 say. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then clearly a_1, a_2, b_1 and b_2 are distinct. For $i = 1, 2$, assume that $a_i, b_i \in H_i$, and form G_i as follows. Let $V(G_i) = V(H_i) \cup \{i\}$ and $E(G_i) = E(H_i) \cup \{(a_i, i), (b_i, i)\}$. Then clearly G_1 and G_2 are both 2-connected. Moreover, neither G_1 nor G_2 is a cycle as otherwise A_1 contains a vertex 2-cocircuit of G. Now for $i = 1, 2$, let

$$C_i = (C \cap E(H_i)) \cup \{(a_i, i), (b_i, i)\}.$$

Then evidently C_i is a cycle of G_i. In addition, it is not difficult to check that for all x in C_i, the graph $G_i \setminus x$ is not 2-connected. Then, as $|C_i| < |C|$, it follows by the induction assumption that C_i contains at least two nontrivial series classes $A_{i,1}$ and $A_{i,2}$ of $M(G_i)$ each of which contains a vertex 2-cocircuit of G_i. Assume without loss of generality that $\{(a_i, i), (b_i, i)\} \subseteq A_{i,2}$. Then it is straightforward to show that $A_{1,1}$ and $A_{2,1}$ are distinct nontrivial series classes of $M(G)$ each containing a vertex 2-cocircuit of $M(G)$. Thus by induction the required result is established. □

If one strengthens the hypothesis of the preceding result by insisting that the graph G be minimally 2-connected, then one obtains a result of Dirac [**5**, Theorem 5] and Plummer [**16**, Corollary 2a]. The next three results follow from Theorem 2.9

in the same way that Corollaries 2.6 and 2.7 follow from Theorem 2.4. For a graph G, the number of connected components of G will be denoted by $k(G)$.

(2.10) COROLLARY. *Let G be a 2-connected loopless graph other than a cycle. Suppose that $A \subseteq E(G)$ such that for all a in A, $G \setminus a$ is not 2-connected. Then either A is the edge-set of a forest, or $V(A)$ contains at least $|A| - |V(A)| + k(A) + 1$ pairwise nonadjacent vertices having degree two in G.*

(2.11) COROLLARY. *Let G be a minimally 2-connected graph other than a cycle. Then there is a set U containing at least $|E(G)| - |V(G)| + 2$ vertices such that each member of U has degree two and every path in G joining two members of U contains a vertex of degree greater than two.*

(2.12) COROLLARY. *A minimally 2-connected graph G having at least four edges has at least $|E(G)| - |V(G)| + 2$ pairwise nonadjacent vertices of degree two.*

The last result can also be deduced from the results of Dirac [5, Theorem 5] and Plummer [16, Corollary 2a] which Theorem 2.9 extends. We now compare the bound in Corollary 2.12 with the following bound which comes from a result of Dirac [5, (6), (5)], noting that the last result can also be used to give an alternative derivation of Corollary 2.12.

(2.13) THEOREM [5, (6), (5)]. *A minimally 2-connected graph G has at least $(|V(G)| + 4)/3$ vertices of degree two.*

The number of vertices of degree k in a graph G will be denoted by ν_k.

(2.14) PROPOSITION. *Let G be a minimally 2-connected graph having at least four edges. Then*

$$
\nu_2 \geqslant
\begin{cases}
\dfrac{|V(G)| + 5}{3} & \text{for } |V(G)| \leqslant |E(G)| < \dfrac{4|V(G)| - 2}{3}, \\[2ex]
|E(G)| - |V(G)| + 2 & \text{for } \dfrac{4|V(G)| - 2}{3} \leqslant |E(G)| \leqslant 2|V(G)| - 4.
\end{cases}
$$

PROOF. Dirac [5, (7)] has shown that $E(G)$ contains at least $|V(G)|$ and at most $2|V(G)| - 4$ edges. By Corollary 2.12 and Theorem 2.13,

$$\nu_2 \geqslant \max\{(|V(G)| + 4)/3, |E(G)| - |V(G)| + 2\},$$

and it is straightforward to check that $|E(G)| - |V(G)| + 2 \geqslant (|V(G)| + 4)/3$ if and only if $|E(G)| \geqslant (4|V(G)| - 2)/3$. Thus

$$
\nu_2 \geqslant
\begin{cases}
\dfrac{|V(G)| + 4}{3} & \text{for } |V(G)| \leqslant |E(G)| < \dfrac{4|V(G)| - 2}{3}, \\[2ex]
|E(G)| - |V(G)| + 2 & \text{for } \dfrac{4|V(G)| - 2}{3} \leqslant |E(G)| \leqslant 2|V(G)| - 4.
\end{cases}
$$

But if $|E(G)| < (4|V(G)| - 2)/3$, then

$$2|E(G)| < (8|V(G)| - 4)/3. \tag{2.15}$$

Now, by summing the vertex degrees of G we get

$$2|E(G)| \geqslant 2\nu_2 + 3(|V(G)| - \nu_2). \tag{2.16}$$

On combining (2.15) and (2.16), we get $(8|V(G)| - 4)/3 > 3|V(G)| - \nu_2$. Hence $\nu_2 > (|V(G)| + 4)/3$, and therefore, $\nu_2 \geqslant (|V(G)| + 5)/3$. $\square$

The graph $K_{2,3}$ is one of infinitely many graphs which show that in Theorem 2.13 we cannot replace $(|V(G)| + 4)/3$ by $(|V(G)| + 5)/3$ (see [1, pp. 15–16 and p. 48, Exercise 20]), although clearly this replacement can be made unless $|V(G)| \equiv 2$ (mod 3). The structure of those minimally 2-connected graphs G having exactly $\{(|V(G)| + 4)/3\}$ vertices of degree two will be determined elsewhere.

It is natural to ask whether Proposition 2.14 can be extended to minimally n-connected graphs for $n \geqslant 3$. The proof of Corollary 2.12 relied heavily on the fact that in a minimally 2-connected graph with at least four edges, each cycle meets at least two nonadjacent vertices of degree two. The corresponding, but slightly weaker, result for minimally n-connected graphs is an immediate consequence of Theorem 2.8. Mader [8, Satz 2] used this result to show that in a minimally n-connected graph G,

$$\nu_n \geqslant \frac{(n - 1)|V(G)| + 2}{2n - 1}. \tag{2.17}$$

Recently, Mader [9, Theorem 17] has strengthened (2.17) to get that

$$\nu_n \geqslant \frac{(n - 1)|V(G)| + 2n}{2n - 1}. \tag{2.18}$$

The following application of Theorem 2.8, although straightforward, seems new, yet the bound it gives frequently sharpens (2.18). The proof is similar to the proof of Corollary 2.6.

(2.19) PROPOSITION. *Let G be a minimally n-connected graph where $n \geqslant 2$. Then*

$$\nu_n \geqslant \frac{|E(G)| - |V(G)| + 1}{n - 1}.$$

PROOF. Let X be the set of edges of G which are incident with some vertex of degree n. Then, by Theorem 2.8, X meets every cycle of G. That is, X meets every circuit of $M(G)$, and thus X contains a cobase B^* of $M(G)$. Now the set of edges incident with a vertex v of G is a cocircuit of $M(G)$, so B^* contains at most $n - 1$ of the edges of a vertex n-cocircuit of G. Hence $\nu_n \geqslant |B^*|/(n - 1)$. As $|B^*| = |E(G)| - |V(G)| + 1$, the required result follows. $\square$

In the case $n = 3$, (2.18) was first proved by Halin [7, Satz 6]. His proof was based on the fact that a cycle in a minimally 3-connected graph meets at least two vertices of degree three [7, Satz 5]. Using this, the proof method of Propositions 2.14 and 2.19 can now be extended to give the following analogue of Proposition 2.14.

(2.20) PROPOSITION. *Let G be a minimally 3-connected graph. Then*

$$\nu_3 \geqslant \begin{cases} \dfrac{2|V(G)| + 7}{5} & \textit{for } \dfrac{3|V(G)|}{2} \leqslant |E(G)| < \dfrac{9|V(G)| - 3}{5}, \\[2ex] \dfrac{|E(G)| - |V(G)| + 3}{2} & \textit{for } \dfrac{9|V(G)| - 3}{5} \leqslant |E(G)| \leqslant 3|V(G)| - 6. \end{cases}$$

PROOF. As every vertex of G has degree at least three, $|E(G)| \geq 3|V(G)|/2$. Moreover, $|E(G)| \leq 3|V(G)| - 6$ (see, for example, [14, Theorem 4.7]). We shall now show that $\nu_3 \geq \frac{1}{2}(|E(G)| - |V(G)| + 3)$. The rest of the proof then follows by generalizing the proof of Proposition 2.14. Let X and B^* be as in the proof of Proposition 2.19. Then we need to show that $\nu_3 \geq \frac{1}{2}(|B^*| + 2)$. Choose an element a of B^*. Then a is in a vertex 3-cocircuit C_1^*. Now let C_a be the fundamental circuit of a with respect to $E(G) \setminus B^*$. Then C_a meets a vertex 3-cocircuit C_2^* which is different from C_1^*. Since $|C_i^* \cap C_a| \geq 2$ for $i = 1, 2$, $|B^* \cap (C_1^* \cup C_2^*)| \leq 3$. Therefore, as B^* contains at most two elements of any vertex 3-cocircuit, $\nu_3 \geq \frac{1}{2}(|B^* \setminus (C_1^* \cup C_2^*)|) + 2$. That is, $\nu_3 \geq \frac{1}{2}(|B^*| + 1)$ with equality holding only if $|B^* \cap C_1^*| = 2$ and $|B^* \cap (C_1^* \cup C_2^*)| = 3$. Thus we may assume that there is an element b in $(B^* \cap C_2^*) \setminus C_1^*$, for otherwise the required result holds. Now let C_b be the fundamental circuit of b with respect to $E(G) \setminus B^*$. Then C_b meets a vertex 3-cocircuit C_3^* different from C_2^*. Moreover, since $b \notin C_1^*$, it follows that $C_3^* \neq C_1^*$. Now $|B^* \setminus (C_1^* \cup C_2^* \cup C_3^*)| \geq |B^*| - 4$, so $\nu_3 \geq \frac{1}{2}(|B^*| - 4) + 3$; that is, $\nu_3 \geq \frac{1}{2}(|B^*| + 2)$, as required. $\square$

A natural question now is what can be said when $n \geq 3$ about the occurrence of n-cocircuits in minimally n-connected matroids. Some progress has been made in this direction for $n = 3$ and $n = 4$ [14], [25].

(2.21) THEOREM [14, THEOREM 2.5]. *If C is a circuit of a minimally 3-connected matroid M and $|E(M)| \geq 4$, then M has at least two distinct 3-cocircuits meeting C.*

It is not difficult to find examples showing that this result is best-possible. Using it and the proof technique of Proposition 2.20, one can deduce that a minimally 3-connected matroid M with at least four elements has at least $\frac{1}{2}$cork $M + 1$ 3-cocircuits. This result is also best-possible. Some partial results on the question of whether a minimally 4-connected matroid has a 4-cocircuit have been obtained by Wong [25].

The results earlier in this section generalize known results for minimally 2-connected matroids and graphs by weakening the requirement of minimal 2-connectedness. In this context one is led to ask whether the hypothesis of Theorem 2.21 can be weakened so that one does not require the matroid M to be minimally 3-connected, but only that the deletion of elements of the circuit C should destroy 3-connectedness. The corresponding problem for graphs, which if true would strengthen Theorem 2.8 in the case $n = 3$, also seems to be open.

3. A characterization of minimally 2-connected matroids. The main result of this section solves a problem of Murty [10, p. 53] by providing a rather elementary yet useful characterization of minimally 2-connected matroids. This characterization is then used to give short proofs of the main results of Murty's paper [10]. We shall use the operation of series connection of matroids and a number of properties of this operation proved by Brylawski [3]. Let M_1 and M_2 be matroids having disjoint ground sets S_1 and S_2 respectively. Choose an element p_1 from S_1 and an element p_2 from S_2. These elements are called the *basepoints* of M_1 and M_2 respectively. Let p be a new element. Then the *series connection* $S((M_1; p_1), (M_2; p_2))$ of M_1 and M_2

with respect to the basepoint p_1 and p_2 is the matroid on $(S_1 \setminus p_1) \cup (S_2 \setminus p_2) \cup p$ whose circuits are the circuits of M_1 not containing p_1, the circuits of M_2 not containing p_2, and all sets of the form $(C_1 \setminus p_1) \cup (C_2 \setminus p_2) \cup p$ where $p_1 \in C_1$, $p_2 \in C_2$ and C_1 and C_2 are circuits of M_1 and M_2 respectively. If the basepoints are clear the series connection will sometimes be denoted $S(M_1, M_2)$. Series connection and the corresponding dual operation, parallel connection, were studied in detail by Brylawski [3]. More recently, Seymour [19] has looked at the closely related operation of 2-*sum* of matroids, the 2-sum of M_1 and M_2 being precisely $S(M_1, M_2)/p$ [22].

(3.1) THEOREM. *A matroid M is minimally 2-connected if and only if $|E(M)| \geqslant 3$, and either M is 2-connected and every element of M is in a 2-cocircuit, or $M = S((M_1/q_1; p_1), (M_2/q_2; p_2))$ where both M_1 and M_2 are minimally 2-connected having at least four elements and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are cocircuits of M_1 and M_2 respectively.*

PROOF. Evidently if M is a 2-connected matroid for which $|E(M)| \geqslant 3$ and every element is in a 2-cocircuit, then M is minimally 2-connected. Now if M_1 and M_2 are minimally 2-connected matroids each having at least three elements, then, by Theorem 2.4, each certainly possesses a 2-cocircuit. Consider $M = S((M_1/q_1; p_1), (M_2/q_2; p_2))$ where $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are cocircuits of M_1 and M_2 respectively. Then since, by Lemma 2.1, M_1/q_1 and M_2/q_2 are 2-connected, it follows by [3, Proposition 4.6] that M is 2-connected.

We now show that M is minimally 2-connected. Firstly, $M \setminus p = (M_1/q_1) \setminus p_1 \oplus (M_2/q_2) \setminus p_2$ [3, Proposition 4.9], so $M \setminus p$ is not 2-connected. Now suppose that $e \in E(M_1/q_1) \setminus p_1$ and consider $M \setminus e$. We have $M \setminus e = S(M_1/q_1, M_2/q_2) \setminus e = S(M_1/q_1 \setminus e, M_2/q_2)$ where the last step holds by [3, Proposition 4.7]. If $M_1/q_1 \setminus e$ is not 2-connected, then by [3, Proposition 4.6], $M \setminus e$ is not 2-connected, as required. Thus suppose that $M_1/q_1 \setminus e$ is 2-connected. Then by Lemma 2.2, q_1 is a loop or a coloop of $M_1 \setminus e$. But M_1 has no loops or coloops, so $\{e, q_1\}$ is a cocircuit of M_1. Therefore, as $\{p_1, q_1\}$ is a cocircuit, $\{e, p_1\}$ is also a cocircuit of M_1 and hence is a cocircuit of M_1/q_1. Thus p_1 is a coloop of $M_1/q_1 \setminus e$ and thus, by [3, Proposition 4.5], $M \setminus e$ is not 2-connected. It follows that M is minimally 2-connected, as required.

For the converse, suppose that M has an element p which is not in a 2-cocircuit. Now $M \setminus p$ is not 2-connected and since M has no 2-cocircuits containing p, the matroid $M \setminus p$ has no component with fewer than two elements. Let $M \setminus p = N_1 \oplus N_2$. Then, by [3, Proposition 4.10], $M = S(M/E(N_1), M/E(N_2))$ where p is the basepoint of both $M/E(N_1)$ and $M/E(N_2)$. Let $N_3 = M/E(N_1)$ and $N_4 = M/E(N_2)$. Then each of N_3 and N_4 has at least three elements. Now, as $M = S(N_3, N_4)$ and M is 2-connected, by [3, Proposition 4.6] again, each of N_3 and N_4 is also 2-connected. If $e \in E(M)$ and $e \neq p$, then $e \in E(N_3) \setminus p$ or $e \in E(N_4) \setminus p$, so suppose the former. Then as $M \setminus e = S(N_3 \setminus e, N_4)$ and $M \setminus e$ is not 2-connected, $N_3 \setminus e$ is not 2-connected. Thus for all elements e of N_3, except possibly p, the matroid $N_3 \setminus e$ is not 2-connected. Similarly, $N_4 \setminus f$ is not 2-connected for all

elements f of N_4 except possibly p. Now for $i = 1, 2$, add an element q_i in series with p in N_{i+2} to get a new matroid M_i which is clearly 2-connected. In fact, it is not difficult to check that M_i is minimally 2-connected. Then M can be obtained by contracting q_1 and q_2 from M_1 and M_2 respectively and then taking the series connection of M_1/q_1 and M_2/q_2 with respect to the basepoint p. Finally, as each of N_3 and N_4 has at least three elements, each of M_1 and M_2 has at least four elements. $\square$

(3.2) COROLLARY [10, THEOREMS 3.2 AND 3.4]. *If $r \geq 3$, a minimally 2-connected matroid M of rank r has at most $2r - 2$ elements, the upper bound being attained if and only if $M \cong M(K_{2,r-1})$.*

PROOF. We argue by induction on $|E(M)|$. If every element of M is in a 2-cocircuit, then consider M^*. Deleting a single element from every parallel class of M^* leaves a 2-connected matroid N having the same rank as M^*. Evidently N has at least rk $M^* + 1$ elements with equality being attained only if N is a circuit. Thus $|E(M^*)| = |E(M)| \geq 2(\text{rk } M^* + 1) = 2(|E(M)| - r + 1)$, and therefore $|E(M)| \leq 2(r - 1)$ with equality being attained only if $M^* \cong C_{r-1}^2$, an $(r - 1)$-circuit in which each element has been replaced by a pair of parallel elements. But if $M^* \cong C_{r-1}^2$, then $M \cong M(K_{2,r-1})$.

We may now suppose that M has an element p which is not in a 2-cocircuit. Then, by Theorem 3.1, $M = S((M_1/q_1;\ p_1),\ (M_2/q_2;\ p_2))$ where, for $i = 1, 2$, $\{p_i, q_i\}$ is a cocircuit of M_i, and M_i is minimally 2-connected having at least four elements and hence having rank at least three. Now, by [3, Theorem 6.16(i)], rk $M = \text{rk}(M_1/q_1) + \text{rk}(M_2/q_2)$ and thus

$$\text{rk } M = \text{rk } M_1 + \text{rk } M_2 - 2. \tag{3.3}$$

Moreover,

$$|E(M)| = |E(M_1)| + |E(M_2)| - 3, \tag{3.4}$$

and, by the induction assumption, $|E(M_i)| \leq 2 \text{ rk } M_i - 2$ for $i = 1, 2$. Thus $|E(M_1)| + |E(M_2)| \leq 2(\text{rk } M_1 + \text{rk } M_2 - 2)$, and, by (3.3) and (3.4), $|E(M)| \leq 2r - 3$. Hence, by induction, the required result is proved. $\square$

Theorem 3.1 may also be used to give alternative proofs of several results for minimally 2-connected matroids such as Corollary 2.7.

4. Colouring. The chromatic number of a loopless graph is the least positive integer at which the value of its chromatic polynomial is positive. For loopless matroids in general there are some difficulties in defining the chromatic number (see [23, p. 264]). However, such problems do not arise for regular matroids. Thus if M is a loopless regular matroid having chromatic polynomial $P(M; \lambda)$ (see, for example, [23, p. 262]) its *chromatic number* $\chi(M)$ is $\min\{j \in \mathbf{Z}^+: P(M; j) > 0\}$. It can be shown (see, for example, [11, p. 17]) that, as for graphs, $P(M; k) > 0$ for all integers k such that $k \geq \chi(M)$.

For a loopless matroid M representable over a finite field $GF(q)$, Crapo and Rota [4, Chapter 16] introduced an important invariant which, when M is regular, is closely related to its chromatic number. The *critical exponent* $c(M; q)$ of M is

$\min\{j \in \mathbf{Z}^+: P(M; q^j) > 0\}$. It follows from [4, p. 16.4] that $P(M; q^k) \geqslant 0$ for all positive integers k and moreover, $P(M; q^k) > 0$ if $k \geqslant c(M; q)$.

The main result of this section is the following.

(4.1) THEOREM. *Let M be a minimally n-connected matroid where $n = 2$ or 3. If M is representable over $GF(q)$, then $c(M; q) \leqslant 2$ for $q \leqslant n$ and $c(M; q) = 1$ for $q > n$. Moreover, if M is regular, then $\chi(M) \leqslant n + 1$.*

The proof of this will use three lemmas.

(4.2) LEMMA. *Let M be a minimally n-connected matroid where $n = 2$ or 3 and suppose $T \subseteq E(M)$. Then $M|T$ has a cocircuit having at most n elements.*

PROOF. If T does not contain a circuit of M, then $M|T$ is free, so $M|T$ has a coloop. If T does contain a circuit C of M, then, by Theorems 2.4 and 2.21, C meets an n-cocircuit C^* of M. Now C^* certainly contains a cocircuit of $M|T$ and the required result follows. □

(4.3) LEMMA [12, LEMMA 5]. *If M is a matroid representable over $GF(q)$ and M is minimal having critical exponent $k + 1$, then every cocircuit of M has at least q^k elements.*

The analogue of the preceding result for regular matroids is as follows.

(4.4) LEMMA [13, THEOREM 3]. *If M is a regular matroid which is minimal having chromatic number $k + 1$, then every cocircuit of M has at least k elements.*

PROOF OF THEOREM 4.1. We shall prove the result for M representable over $GF(q)$ by using Lemmas 4.2 and 4.3. The result for M regular follows similarly by using Lemma 4.4 in place of Lemma 4.3.

Suppose $c(M; q) = k + 1$. Then by deleting elements from M, we obtain a restriction N which is minimal having critical exponent $k + 1$. By Lemma 4.3, every cocircuit of N has at least q^k elements. But, by Lemma 4.2, N has a cocircuit having at most n elements. Thus $n \geqslant q^k$ and the required result follows. □

REFERENCES

1. B. Bollobás, *Extremal graph theory*, London Math. Soc. Mono., no. 11, Academic Press, London, New York and San Francisco, 1978.

2. J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan, London; American Elsevier, New York, 1976.

3. T. H. Brylawski, *A combinatorial model for series-parallel networks*, Trans. Amer. Math. Soc. **154** (1971), 1–22.

4. H. H. Crapo and G.-C. Rota, *On the foundations of combinatorial theory: combinatorial geometries*, Preliminary ed., MIT Press, Cambridge, Mass. and London, 1970.

5. G. A. Dirac, *Minimally 2-connected graphs*, J. Reine Angew. Math. **228** (1967), 204–216.

6. J. E. Graver and M. E. Watkins, *Combinatorics with emphasis on the theory of graphs*, Springer-Verlag, New York, Heidelberg and Berlin, 1977.

7. R. Halin, *Untersuchungen über minimale n-fach zusammenhängende Graphen*, Math. Ann. **182** (1969), 175–188.

8. W. Mader, *Ecken vom Grad n in minimalen n-fach zusammenhängenden Graphen*, Arch. Math. (Basel) **23** (1972), 219–224.

9. _____, *Connectivity and edge-connectivity in finite graphs*, Surveys in Combinatorics (B. Bollobás, ed.), Cambridge Univ. Press, New York, 1979, pp. 66–95.

10. U. S. R. Murty, *Extremal critically connected matroids*, Discrete Math. **8** (1974), 49–58.

11. J. G. Oxley, *Colouring, packing and the critical problem*, Quart. J. Math. Oxford Ser. (2) **29** (1978), 11–22.

12. _____, *On a covering problem of Mullin and Stanton for binary matroids*, Aequationes Math. **20** (1980), 104–112.

13. _____, *A generalization of a covering problem of Mullin and Stanton for matroids*, Proc. Sixth Austral. Combinatorics Conf., Lecture Notes in Math., vol. 748, Springer-Verlag, Berlin, Heidelberg and New York, 1979, pp. 92–97.

14. _____, *On matroid connectivity*, Quart. J. Math. Oxford Ser. (2) (to appear).

15. _____, *On 3-connected matroids*, Canad. J. Math. (to appear).

16. M. D. Plummer, *On minimal blocks*, Trans. Amer. Math. Soc. **134** (1968), 85–94.

17. P. D. Seymour, *Matroid representation over GF*(3), J. Combin. Theory. Ser. B **26** (1979), 159–173.

18. _____, *Packing and covering with matroid circuits*, J. Combin. Theory Ser. B **28** (1980), 237–242.

19. _____, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), 305–359.

20. W. T. Tutte, *Connectivity in matroids*, Canad. J. Math. **18** (1966), 1301–1324.

21. _____, *Wheels and whirls*, Lecture Notes in Math., vol. 211, Springer-Verlag, Berlin, Heidelberg and New York, 1971, pp. 1–4.

22. P. N. Walton and D. J. A. Welsh, *On the chromatic number of binary matroids*, Mathematika **27** (1980), 1–9.

23. D. J. A. Welsh, *Matroid theory*, London Math. Soc. Mono., no. 8, Academic Press, London, New York and San Francisco, 1976.

24. N. L. White, *The bracket ring of a combinatorial geometry*. II: *Unimodular geometries*, Trans. Amer. Math. Soc. **214** (1975), 233–248.

25. P.-K. Wong, *On certain n-connected matroids*, J. Reine Angew. Math. **299/300** (1978), 1–6.

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