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On Weakly Symmetric Graphs of Order Twice a Prime

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A graph is weakly symmetric if its automorphism group is both vertex-transitive and edge-transitive. In 1971, Chao characterized all weakly symmetric graphs of prime order and showed that such graphs are also transitive on directed edges. In this paper we determine all weakly symmetric graphs of order twice a prime and show that these graphs too are directed-edge transitive. © 1987 Academic Press, Inc.

1. INTRODUCTION

A graph G is *weakly symmetric* if its automorphism group $\mathrm{Aut}(G)$ is transitive on both the vertex-set $V(G)$ and the edge-set $E(G)$ of G. Turner [15] determined all vertex-transitive graphs on a prime number p of vertices. In addition, he made a conjecture as to which of these is weakly symmetric. Turner's conjecture was verified by Chao [7] and later Berggren [2] simplified the proof of this result. A characterization of vertex-transitive graphs of order $2p$ was independently conjectured by Alspach and Sutcliffe [1] and Toida [14] and this characterization was proved by Marušič [11] provided p satisfies certain weak restrictions. The aim of this paper is to characterize weakly symmetric graphs of order $2p$. This characterization will rely on several important group-theoretic results including the recently completed classification of finite simple groups. We shall also need the following result of Liebeck and Saxl [10] that solves a problem of Wielandt [16, p. 94].

(1.1) THEOREM. *Let p be a prime number. A primitive permutation group of degree $2p$ is doubly transitive provided $p \neq 5$.*

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If $p = 5$, the only primitive groups of degree 10 are S_5 and A_5 acting on the set of 2-element subsets of a 5-element set.

The following technique for constructing weakly symmetric graphs is well known (see, for example, [4, p. 86]). Let A be a transitive permutation group acting on a finite set X and consider the action of A on the set $\tilde{X}$ of 2-element subsets of X. If O is an orbit of A under this action, then the graph with vertex-set A and edge-set O is certainly weakly symmetric. Moreover, every weakly symmetric graph G arises in this way by taking $A = \mathrm{Aut}(G)$ and $X = V(G)$. A consequence of the classification of all finite simple groups is that all transitive permutation groups A that act on a set X of size $2p$ are known. From the point of view of permutation groups, this paper describes all possible orbits of such groups on $\tilde{X}$.

Since it is routine to determine the non-simple weakly symmetric graphs on n vertices from a list of the simple weakly symmetric graphs on n vertices, throughout the rest of this paper the term "graph" will mean "simple graph." If G_1 and G_2 are disjoint graphs, we shall denote their union by $G_1 + G_2$. The disjoint union of k copies of G_1 will sometimes be written as kG_1. If v is a vertex of a graph G, then $N_G(v)$ will denote the *neighbor set* of v in G, that is, the set of vertices of G that are adjacent to v. For all other graph-theoretic terminology which is otherwise unexplained we shall follow Bondy and Murty [5].

Before proceeding further we note that Chao [7] uses the term "symmetric graph" for what we have called a "weakly symmetric graph." Our preference for the latter term is based on the widespread use of the term *symmetric graph* to describe a vertex-transitive graph G with the property that for every pair, uv and xy, of edges of G there is an automorphism mapping u to x and v to y (see, for example, [3, p. 104]). Evidently every symmetric graph is weakly symmetric but the converse of this is not true in general (see, for example, [9]). It is true for graphs of prime order by Chao's work and follows for graphs of order twice a prime from the results of this paper. The next result establishes the converse for graphs of odd degree. It is related to a group-theoretic result in [16, Theorem 16.5]. We give a combinatorial proof of it.

(1.2) PROPOSITION. *Let G be a weakly symmetric graph of degree r where r is odd. Then G is symmetric.*

Proof. Evidently if G has an automorphism fixing some edge and interchanging its endpoints, then, as G is edge-transitive, it is symmetric. We assume that G is not symmetric and fix an edge e of G. Now assign a direction to e. Then, for each edge f distinct from e, there is an automorphism mapping e to f and hence inducing a direction on f. Furthermore, if there are two such automorphisms σ_1 and σ_2 inducing different directions on f, then $\sigma_1^{-1}\sigma_2$ fixes the edge f and interchanges its endpoints—a contradiction. It follows that we obtain a directed graph **G** such that all the automorphisms of G are also automorphisms of **G**. Since G is vertex-transitive, the indegrees of all the vertices of **G** are the same. Likewise all the outdegrees are the same. But, since the sum of the indegrees equals the sum of the outdegrees, each vertex has its indegree and outdegree equal. Thus G has even degree—a contradiction. ∎

The following lemma will be used frequently to establish the edge-transitivity of various graphs. The routine proof is omitted.

(1.3) LEMMA. *Let G be a vertex-transitive graph having a vertex x such that if y and y' are in $N_G(x)$, there is an automorphism of G that fixes x and maps y to y'. Then G is symmetric and hence is edge-transitive.*

Two other results that we shall use heavily in our characterization are Chao's classification of all weakly symmetric graphs of prime order [7] and the following theorem of Burnside [13, p. 53]. If a is an element of the group $\mathbb{Z}_p^*$ of non-zero elements of $\mathbb{Z}_p$ and $b \in \mathbb{Z}_p$, we shall abbreviate to $ax+b$ the permutation of $\mathbb{Z}_p$ which maps each element x of $\mathbb{Z}_p$ to the element $ax+b$.

(1.4) THEOREM. *Let A be a transitive permutation group of prime degree p. Then either A is doubly transitive or we can identify A with $\mathbb{Z}_p$ in such a way that*

$$\{x+b\colon b \in \mathbb{Z}_p\} \subseteq A \subsetneqq \{ax+b\colon a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}.$$

Let r be an even positive integer dividing $p-1$ where p is prime and let $H(p,r)$ denote the unique order-r subgroup of $\mathbb{Z}_p^*$. We define $G(p,r)$ to be the graph with vertex-set $\mathbb{Z}_p$ and edge-set $\{xy\colon x, y \in \mathbb{Z}_p$ and $y-x \in H(p,r)\}$.

(1.5) THEOREM [7]. *The graph $G(p,r)$ is symmetric and, provided $r<p-1$, $\mathrm{Aut}(G(p,r)) = \{ax+b\colon a \in H(p,r), b \in \mathbb{Z}_p\}$. Moreover, every non-null weakly symmetric graph of order p is isomorphic to $G(p,r)$ for some even integer r dividing $p-1$.*

In the next section we describe constructions for several classes of weakly symmetric graphs on $2p$ vertices and state the main theorem of the paper, that all weakly symmetric graphs on $2p$ vertices are contained in one of these classes. The proof of this theorem occupies the remainder of the paper. It falls naturally into two cases and these are treated separately in Sections 3 and 4. The last section lists the automorphism groups of all the weakly symmetric graphs of order $2p$.

2. CONSTRUCTIONS

The purpose of this section is to describe several classes of symmetric graphs of order $2p$. Our main theorem will prove that every weakly symmetric graph of order $2p$ is in one of these classes. Four obvious examples of symmetric graphs on $2p$ vertices are the null graph, the complete graph K_{2p}, the complete bipartite graph $K_{p,p}$, and the complement of $K_{p,p}$, $2K_p$. When $p=2$, it is not difficult to check that these 4 graphs are the only weakly symmetric graphs of order $2p$. From now on, we shall assume that $p>2$.

When $p=5$, we get two special graphs that are well known to be symmetric (see, for example, [4, p. 87]), but which do not belong to any of the general classes of symmetric graphs of order $2p$ that we shall describe in this section. These graphs are the Petersen graph O_3 and its complement O_3^c.

The next three classes of graphs that we shall construct are based on the symmetric graphs $G(p, r)$ of order p. Just as $2K_p$ is symmetric, it is clear that in general $2G(p, r)$, the disjoint union of two copies of $G(p, r)$, is also symmetric.

Next let A and A' be two disjoint copies of $\mathbb{Z}_p$. For each element i of $\mathbb{Z}_p$, we shall denote the corresponding elements of A and A' by i and i', respectively. Two natural permutations on $A \cup A'$ which we shall use frequently in this paper are defined as follows: for all i in $\mathbb{Z}_p$, $\tau(i)=i+1$ and $\tau(i')=(i+1)'$, and $\rho(i)=(-i)'$ and $\rho(i')=-i$.

Now let r be a positive integer dividing $p-1$ and recall that $H(p, r)$ is the unique subgroup of $\mathbb{Z}_p^*$ of order r. We define the graph $G(2p, r)$ to have vertex-set $A \cup A'$ and edge-set $\{xy' : x, y \in \mathbb{Z}_p \text{ and } y-x \in H(p, r)\}$. It is easy to check that both τ and ρ are automorphisms of $G(2p, r)$ and using these automorphisms and Lemma 1.3 it is not difficult to prove that

(2.1) LEMMA. *The graph $G(2p, r)$ is symmetric.*

Note that $G(2p, 1)$ is isomorphic to pK_2, a complete matching, and $G(2p, p-1)$ is the bipartite complement of a complete matching.

Next assume that r is an *even* positive integer dividing $p-1$. We define $G(2, p, r)$ to be the graph with vertex-set $A \cup A'$ and edge-set $\{xy, x'y, xy', x'y' : x, y \in \mathbb{Z}_p \text{ and } y-x \in H(p, r)\}$. Again, both τ and ρ are automorphisms of $G(2, p, r)$ and one easily checks that

(2.2) LEMMA. *The graph $G(2, p, r)$ is symmetric.*

When $r=p-1$, $G(2, p, r)$ is the complementary graph of a complete matching.

The remaining symmetric graphs which we shall consider are obtained from certain symmetric (v, k, λ)-designs. Let D be such a design, A be its set of points, and A' be its set of blocks. The incidence graph $B(D)$ of this design has vertex-set $A \cup A'$ and edge-set $\{xy: x \in A, y \in A' \text{ and } x \in y\}$. We shall denote by $B'(D)$ the incidence graph of the complementary design of D. Thus $B'(D)$ has vertex-set $A \cup A'$ and edge-set $\{xy: x \in A, y \in A' \text{ and } x \notin y\}$.

Let n be an integer greater than two and D be the symmetric design $PG(n-1, q)$ that has as its points and blocks the points and hyperplanes respectively of the $(n-1)$-dimensional projective space over $GF(q)$ with the incidence relation being determined by inclusion. It is a routine exercise in linear algebra to verify that both $B(PG(n-1, q))$ and $B'(PG(n-1, q))$ are symmetric graphs. Each has $2(q^n-1)/(q-1)$ vertices and so, when $(q^n-1)/(q-1)$ is a prime p, we have two symmetric graphs of order $2p$. We note here that in the special case that $n=3$ and $q=2$, the graph $B(PG(n-1, q))$ is isomorphic to $G(2 \cdot 7, 3)$.

The last example of a symmetric graph of order twice a prime arises from the unique symmetric $(11, 5, 2)$-design $H(11)$. The points of this design are the elements of $\mathbb{Z}_{11}$ and the blocks are the 11 sets $R+i=\{x+i: x \in R\}$ where $i \in \mathbb{Z}_{11}$ and R is the set of non-zero quadratic residues modulo 11, namely $\{1, 3, 4, 5, 9\}$. We note that $B(H(11)) \cong G(2 \cdot 11, 5)$. Hence $B(H(11))$ is included amongst the symmetric graphs noted earlier. We also have that

(2.3) LEMMA. *$B'(H(11))$ is a symmetric graph.*

Proof. The 11 blocks of the complementary design $H'(11)$ of $H(11)$ are the sets $R'+i$ where $i \in \mathbb{Z}_{11}$ and $R'=\{0, 2, 6, 7, 8, 10\}=\{2x^2: x \in \mathbb{Z}_{11}\}$. Since $B(H(11))$ is vertex-transitive, $B'(H(11))$ is also vertex-transitive. By Lemma 1.3, it will follow that $B'(H(11))$ is edge-transitive if we can find an automorphism of the graph which fixes R' and maps $2x^2$ to $2y^2$ for any x and y in $\mathbb{Z}_{11}$. We shall first show that such an automorphism exists when neither x nor y is zero. In that case, let $z=x^{-2}y^2$ and define the permutation σ by $\sigma(u)=uz$ for all u in $\mathbb{Z}_{11}$. Then $\sigma(2x^2)=2x^2z=2x^2x^{-2}y^2=2y^2$. Moreover, $\sigma(R'+i)=\sigma\{2t^2+i: t \in \mathbb{Z}_{11}\}=\{(2t^2+i)z: t \in \mathbb{Z}_{11}\}=\{2(tx^{-1}y)^2+iz: t \in \mathbb{Z}_{11}\}=R'+iz$. Hence, in particular, $\sigma(R')=R'$. Thus σ induces the required automorphism of $B'(H(11))$. To complete the proof of the lemma we need only show that $B'(H(11))$ has an automorphism that fixes R' and maps 0 to 2. It is not difficult to check that such an automorphism is induced by the permutation $(0\ 2)(1\ 9)(3\ 4)(7\ 10)(5)(6)(8)$. ∎

We shall conclude this section by stating the main theorem of the paper and beginning its proof. Before doing this, however, we note that, as a consequence of the classification of all finite simple groups, one can characterize all doubly transitive groups of prime degree [6, 8] and hence determine all doubly transitive symmetric designs on a prime number of points. The latter consist of the trivial $(p, 1, 0)$-design and its complementary design, together with the examples noted above: $H(11)$ and its complementary design and $PG(n-1, q)$ and its complementary design where, for the last two, $p = (q^n - 1)/(q-1)$.

(2.4) THEOREM. *Let G be a weakly symmetric graph of order $2p$ where p is a prime. Then either*

(i) *G is isomorphic to the null graph on $2p$ vertices, the complete graph K_{2p}, or the complete bipartite graph $K_{p,p}$;*

(ii) *G is isomorphic to $2G(p, r)$ or $G(2, p, r)$ for some even integer r dividing $p-1$;*

(iii) *G is isomorphic to $G(2p, r)$ for some integer r dividing $p-1$;*

(iv) *G is isomorphic to $B(PG(n-1, q))$ or $B'(PG(n-1, q))$, and $p = (q^n - 1)/(q-1)$;*

(v) *G is isomorphic to $B'(H(11))$ and $p = 11$; or*

(vi) *G is isomorphic to the Petersen graph O_3 or its complement O_3^c, and $p = 5$.*

Before commencing the proof of this theorem, we recall that $G(2 \cdot 7, 3) \cong B(PG(2, 2))$. In Section 5, we shall determine the automorphism groups of all the graphs in (i)–(vi) above. It will follow from this that, except for the one coincidence just noted, all the graphs listed above are non-isomorphic.

Proof of Theorem 2.4. Assume that G is a weakly symmetric graph of order $2p$ where p is a prime. The theorem was verified earlier for $p = 2$ and is straightforward to check when $p = 3$ (alternatively, see [12]). We therefore suppose that $p \geqslant 5$.

By Theorem 1.1, either $\mathrm{Aut}(G)$ is doubly transitive, $\mathrm{Aut}(G)$ is imprimitive, or $\mathrm{Aut}(G)$ is primitive and $p = 5$. In the first case, G is either the null graph or the complete graph. In the last case we recall from the introduction that the only primitive groups of degree 10 are S_5 and A_5 acting on the set of 2-element subsets of a 5-element set. Thus $\mathrm{Aut}(G)$ is one of these two groups and it is not difficult to check that G is either the Petersen graph or its complement.

For the rest of the proof we shall assume that $\mathrm{Aut}(G)$ is imprimitive. Thus $\mathrm{Aut}(G)$ has a block containing p vertices or a block containing 2 vertices. For the major part of the proof we shall treat these cases separately, the first in Section 3 and the second in Section 4. However, the next two lemmas will be used in both cases. The first of these is an easy consequence of edge-transitivity and we omit the proof.

(2.5) LEMMA. *The graph G has the property that either no edge joins two vertices in different blocks of* $\mathrm{Aut}(G)$, *or no edge joins two vertices in the same block of* $\mathrm{Aut}(G)$.

(2.6) LEMMA. *Suppose that the graph G is bipartite having as its vertex classes two distinct copies, $\{0, 1,..., p-1\}$ and $\{0', 1',..., (p-1)'\}$, of $\mathbb{Z}_p$. If G has an automorphism τ which, for all i in $\mathbb{Z}_p$, maps i to $i+1$ and i' to $(i+1)'$, then G is a symmetric graph.*

Proof. Consider the permutation ρ of $V(G)$ which, for all i in $\mathbb{Z}_p$, maps i to $(-i)'$ and i' to $-i$. If $ij' \in E(G)$, then so is $\rho(ij')$ since $\rho(ij') = (-j)(-i)' = \tau^{-i-j}(ij')$. Thus ρ is an automorphism of G. Since G is vertex-transitive, the lemma will follow if we can show that for any elements i' and j' of $N_G(0)$, there is an automorphism fixing 0 and mapping i' to j'. Since G is edge-transitive, it certainly has an automorphism μ that maps $\{0, i'\}$ to $\{0, j'\}$. If μ fixes 0, μ is the required automorphism. We therefore suppose that μ maps 0 to j' and i' to 0. But then the automorphism $\rho\tau^{-j}\mu$ has the desired effect. ∎

3. BLOCKS OF SIZE p

In this section, we assume that G is a weakly symmetric graph of order $2p$ and that $\mathrm{Aut}(G)$ has a block A of size p. Evidently $V(G) - A$ is also a block of $\mathrm{Aut}(G)$ and we denote it by A'.

The following result follows easily from Lemma 2.5 and Theorem 1.5. Its proof is omitted.

(3.1) LEMMA. *Suppose that G has an edge that joins two vertices in A or joins two vertices in A'. Then $G \cong 2G(p, r)$ for some even integer r dividing $p-1$.*

In view of this lemma, we may assume for the remainder of this section that G is a bipartite graph having A and A' as its vertex classes. One possibility here is that G is the complete bipartite graph on A and A' but we shall also assume from now on that this is not the case. As G is vertex-transitive, G has an automorphism π of order p. As p is odd, $\pi(A) = A$ and $\pi(A') = A'$. Thus π is a member of the subgroup $\mathrm{Aut}(G)^0$ of $\mathrm{Aut}(G)$ consisting of those automorphisms which fix both of the sets A and A'. Every such automorphism θ can be represented by a pair (θ_1, θ_2), where θ_1 is the restriction $\theta|_A$ of θ to A and θ_2 is $\theta|_{A'}$. A key step in the proof of this case of the theorem will be to establish that the groups $\mathrm{Aut}(G)^0$, $\mathrm{Aut}(G)^0|_A$, and $\mathrm{Aut}(G)^0|_{A'}$ are all isomorphic. The next two lemmas are steps in the proof of this fact.

(3.2) LEMMA. *Both π_1 and π_2 have order p.*

Proof. Suppose, without loss of generality, that π_2 is trivial. Then π_1 is a p-cycle. From this it follows that every vertex in A has the same neighbor set and hence that $G \cong K_{p,p}$—a contradiction. ∎

(3.3) LEMMA. *If a and b are distinct vertices of G, then $N_G(a) \neq N_G(b)$.*

Proof. As G is a non-null bipartite graph, if $a \in A$ and $b \in A'$, $N_G(a) \neq N_G(b)$. Now assume that $N_G(a) = N_G(b)$ for some pair a and b of distinct elements of A. Using the automorphism π it is easy to show that every element of A has the same neighbor set, namely $N_G(a)$. Then, as in the preceding proof, we get the contradiction that $G \cong K_{p,p}$. ∎

We shall now verify the isomorphism of $\mathrm{Aut}(G)^0$, $\mathrm{Aut}(G)^0|_A$, and $\mathrm{Aut}(G)^0|_{A'}$.

(3.4) LEMMA. *The restriction maps from* $\mathrm{Aut}(G)^0$ *onto* $\mathrm{Aut}(G)^0|_A$ *and onto* $\mathrm{Aut}(G)^0|_{A'}$ *are both isomorphisms.*

Proof. It suffices to show that the restriction map of $\mathrm{Aut}(G)^0$ onto $\mathrm{Aut}(G)^0|_A$ is an isomorphism. This map is clearly a surjective homomorphism. To prove it is injective, let θ be an element in the kernel. Then θ_1 acts trivially on A. Suppose that θ_2 does not act trivially on A'. Then there are distinct elements a' and b' of A' such that $\theta_2(a') = b'$. Thus $\theta(a') = b'$. Now, since both $N_G(a')$ and $N_G(b')$ are subsets of A and θ fixes every element of A, we have

$$N_G(b') = N_G(\theta(a')) = \theta(N_G(a')) = N_G(a').$$

But this contradicts the preceding lemma. Thus θ_2 is the identity and so θ is the identity of $\mathrm{Aut}(G)^0$. We conclude that the restriction map from $\mathrm{Aut}(G)^0$ onto $\mathrm{Aut}(G)^0|_A$ is indeed an isomorphism. ∎

We now show that G must be symmetric. Evidently A and A' may be identified with distinct copies, $\{0, 1, \ldots, p-1\}$ and $\{0', 1', \ldots, (p-1)'\}$ of $\mathbb{Z}_p$. Moreover, by Lemma 3.2, these identifications can be made in such a way that the permutation τ of $V(G)$ which, for all i in $\mathbb{Z}_p$, maps i to $i+1$ and i' to $(i+1)'$ is an automorphism of G. It now follows immediately from Lemma 2.6 that G is symmetric.

Now $\mathrm{Aut}(G)^0|_A$ is a transitive permutation group of degree p. The rest of this section will distinguish the cases when $\mathrm{Aut}(G)^0|_A$ is doubly transitive and when it is not. In the first of these cases we shall show that G is the incidence graph of a doubly transitive symmetric (v, k, λ)-design that has the elements of A as its points and the elements of A' as its blocks. Certainly this incidence structure D has the same number of points as blocks.

Moreover, as G is vertex-transitive, G is regular of degree d, say. Thus all the blocks of D are incident with d points and all the points of D are incident with d blocks. Next suppose that x and y are any two distinct points of D. The number of blocks that are incident with both x and y is $|N_G(x) \cap N_G(y)|$. Since $\mathrm{Aut}(G)^0|_A$ is doubly transitive, this number is independent of x and y. This proves that D is indeed a doubly transitive symmetric (v, k, λ)-design having G as its incidence graph $B(D)$. It follows from Section 2 that G is isomorphic to one of pK_2, $G(2p, p-1)$, $B(H(11))$, $B'(H(11))$, $B(PG(n-1, q))$ or $B'(PG(n-1, q))$ where, for that last two, $p = (q^n - 1)/(q-1)$.

We now suppose that $\mathrm{Aut}(G)^0|_A$ is not doubly transitive. We shall continue to use our earlier identification of A and A' with distinct copies $\{0, 1, ..., p-1\}$ and $\{0', 1', ..., (p-1)'\}$ of $\mathbb{Z}_p$. By Theorem 1.4, there are permutations of the labels on the elements of A and on the elements of A' so that

$$\{x+b: b \in \mathbb{Z}_p\} \subseteq \mathrm{Aut}(G)^0|_A \subsetneqq \{ax+b: a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

and

$$\{(x+b)': b \in \mathbb{Z}_p\} \subseteq \mathrm{Aut}(G)^0|_{A'} \subsetneqq \{(ax+b)': a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}.$$

Let $\pi = (\pi_1, \pi_2)$ be an order-p element of $\mathrm{Aut}(G)^0$ that, for all i in $\mathbb{Z}_p$, maps i to $i+1$. Then, by Lemma 3.4, π is the unique such element and, moreover, $\mathbb{Z}_p^*$ has an element m such that, for all i in $\mathbb{Z}_p$, π maps i' to $(i+m)'$. Using the automorphism π we may now determine the edge-set of G in terms of $N_G(0)$.

(3.5) LEMMA. *$E(G) = \{ij': i, j \in \mathbb{Z}_p$ and $(j - mi)' \in N_G(0)\}$.*

Proof. Since G is a bipartite graph on A and A', every edge has the form ij' for some i and j in $\mathbb{Z}_p$. Moreover, $N_G(i) = N_G(\pi^i(0)) = \pi^i(N_G(0))$. Thus $j' \in N_G(i)$ if and only if $j' \in \pi^i(N_G(0))$. Since the latter occurs precisely when $(j - mi)' \in N_G(0)$, the lemma is proved. ∎

Our ultimate goal in this case is to show that $G \cong G(2p, r)$ for some integer r dividing $p-1$. To do this, we shall first determine $N_G(0)$ in terms of the stabilizer T_0 of the vertex 0. Evidently, $T_0 \subseteq \mathrm{Aut}(G)^0$ and $T_0|_A \subseteq \{ax: a \in \mathbb{Z}_p^*\}$. As the last group is cyclic, $T_0|_A$ is also cyclic. Therefore, by Lemma 3.4, T_0 is cyclic. Now let v be a generator for T_0. Then $v = (v_1, v_2) = (ax, (cx+d)')$ for some elements a and c of $\mathbb{Z}_p^*$ and some element d of $\mathbb{Z}_p$. If $c = 1$, then it follows by Lemma 3.4 that v_1 has order p—a contradiction. Hence $c \neq 1$.

The next two technical lemmas concern v. The proof of the second of these is a routine induction argument.

(3.6) LEMMA. *If* $k' \in N_G(0)$, *then* $(ck+d)' \neq k'$.

Proof. If $(ck+d)' = k'$, then T_0 fixes k'. But, as G is symmetric, T_0 acts transitively on $N_G(0)$, hence $N_G(0) = \{k'\}$ and G is a complete matching pK_2. This contradicts the fact that G has a block of size p. ∎

(3.7) LEMMA. *If* $k' \in A'$ *and* $i \in \mathbb{Z}^+$, *then*

$$v^i(k') = \left(c^i k + \frac{c^i - 1}{c-1} d \right)'.$$

We now determine $N_G(t)$ for all t in A. Let the order of c in $\mathbb{Z}_p^*$ be r and fix an element k' of $N_G(0)$. Then, by Lemmas 3.6 and 3.7, $N_G(0) = \{(c^i k + ((c^i - 1)/(c-1))\, d)' : 0 \leqslant i < r\}$. Thus, for all t in A,

$$N_G(t) = \left\{ \left(mt + c^i k + \frac{c^i - 1}{c-1} d \right)' : 0 \leqslant i < r \right\}. \tag{3.8}$$

The following lemma completes the proof of the theorem in the case when $\mathrm{Aut}(G)$ has a block of size p.

(3.9) LEMMA. $G \cong G(2p, r)$.

Proof. We begin by recalling the structure of $G(2p, r)$. Let B and B' be two distinct copies of $\mathbb{Z}_p$. For each element i of $\mathbb{Z}_p$, we denote the corresponding elements in B and B' by i and i', respectively. Since the element c of $\mathbb{Z}_p^*$ has order r, this element generates the unique subgroup $H(p, r)$ of $\mathbb{Z}_p^*$ of order r. The graph $G(2p, r)$ has vertex-set $B \cup B'$ and edge-set $\{xy' : x, y \in \mathbb{Z}_p \text{ and } y - x = c^i \text{ for some } 0 \leqslant i < r\}$.

Define $\eta : B \cup B' \to A \cup A'$ by

$$\eta(x) = \frac{(c-1)\,k + d}{(c-1)\,m}\, x \qquad \text{for all } x \text{ in } B,$$

$$\eta(x') = \left(\frac{(c-1)\,k + d}{c-1}\, x - \frac{d}{c-1} \right)' \qquad \text{for all } x' \text{ in } B'.$$

We shall show that η is an isomorphism between $G(2p, r)$ and G. Since, by Lemma 3.6, $(c-1)\,k + d \neq 0$, η is a bijection. Now, by (3.8) and the definition of $G(2p, r)$, we have that, for all t in $\mathbb{Z}_p$,

$$N_{G(2p,r)}(t) = \{(t + c^i)' : 0 \leqslant i < r\},$$

$$N_G(t) = \left\{ \left(mt + c^i k + \frac{c^i - 1}{c-1} d \right)' : 0 \leqslant i < r \right\}.$$

It is straightforward to show that

$$\eta(N_{G(2p,r)}(t)) = N_G(\eta(t)).$$

Therefore, under η, every edge of $G(2p, r)$ is mapped to an edge of G. Since $|E(G(2p, r))| = |E(G)|$, it follows easily that every edge of G is the image of an edge of $G(2p, r)$ under η. We conclude that η is an isomorphism. ∎

4. Blocks of Size 2

In this section, we assume that G is a weakly symmetric graph of order $2p$ and that $\mathrm{Aut}(G)$ has a non-trivial block A_0 of size 2 and has no blocks of size p. As $\mathrm{Aut}(G)$ is transitive on $V(G)$, there is an automorphism τ of G of order p. Moreover, $A_0, \tau(A_0), \tau^2(A_0), ..., \tau^{p-1}(A_0)$ are distinct blocks of $\mathrm{Aut}(G)$. For convenience, we shall denote these blocks by $A_0, A_1, ..., A_{p-1}$, where, for all i in $\mathbb{Z}_p$, $A_i = \{i, i'\}$ and τ maps i to $i+1$ and i' to $(i+1)'$. As before, we denote the sets $\{0, 1, ..., p-1\}$ and $\{0', 1', ..., (p-1)'\}$ by A and A', respectively.

We first note that, by Lemma 2.5, if G has an edge ii' for some i in $\mathbb{Z}_p$, then $G \cong pK_2$. For the remainder of this section, we shall assume that, for all i in $\mathbb{Z}_p$, ii' is not an edge of G. Let $\bar{G}$ be the graph induced on the blocks of $\mathrm{Aut}(G)$ by G. Thus $\bar{G}$ has vertex-set $\{\bar{i}: i \in \mathbb{Z}_p\}$ and edge-set $\{\bar{i}\bar{j}: G$ has an edge between the blocks A_i and $A_j\}$. If $\theta \in \mathrm{Aut}(G)$, then θ induces an automorphism $\bar{\theta}$ on $\bar{G}$. Thus the map from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\bar{G})$ that sends θ to $\bar{\theta}$ is a group homomorphism. Let $\overline{\mathrm{Aut}(G)}$ be the image of $\mathrm{Aut}(G)$ under this homomorphism. Evidently $\overline{\mathrm{Aut}(G)}$ acts transitively on both $V(\bar{G})$ and $E(\bar{G})$ and so $\bar{G}$ is a weakly symmetric graph of order p.

Next we consider the number $e(A_i, A_j)$ of edges between the blocks A_i and A_j of $\mathrm{Aut}(G)$. Clearly $0 \leqslant e(A_i, A_j) \leqslant 4$. Moreover, it is not difficult to check that, for all pairs of adjacent blocks, A_i and A_j, $e(A_i, A_j)$ takes the same value, say $e(G)$.

(4.1) Lemma. *Suppose that* $e(G) \geqslant 3$. *Then* $e(G) = 4$.

Proof. For any pair A_i and A_j of blocks, the subgraph induced on $A_i \cup A_j$ is bipartite and edge-transitive. No such graph has exactly 3 edges. ∎

The next result is a straightforward consequence of Theorem 1.5.

(4.2) Lemma. *If* $e(G) = 4$, *then* $G \cong G(2, p, r)$ *for some even integer* r *dividing* $p-1$.

As a consequence of the last two lemmas, we may assume, for the rest of this section, that $e(G)$ is 1 or 2. Now $\overline{\mathrm{Aut}(G)}$ acts transitively on $V(\bar{G})$. In the rest of the proof we shall distinguish the cases when $\overline{\mathrm{Aut}(G)}$ acts doubly transitively on $V(\bar{G})$ and when it does not. Suppose that the first of these occurs. Then $\bar{G}$ is complete. Moreover,

(4.3) LEMMA. *If i and j are distinct elements of $\mathbb{Z}_p$, then $ij \in E(G)$ or $ij' \in E(G)$.*

Proof. Let π and $\bar{\pi}$ be the permutation characters of $\mathrm{Aut}(G)$ on $V(G)$ and $\overline{\mathrm{Aut}(G)}$ on $V(\bar{G})$, respectively. As $\overline{\mathrm{Aut}(G)}$ is doubly transitive, $\bar{\pi} = 1 + \chi$, where χ is an irreducible character of degree $p-1$. Since $\mathrm{Aut}(G)$ is transitive on $V(G)$, we have that $\langle \pi, 1 \rangle = 1$ and $\langle \pi, \bar{\pi} \rangle \leqslant 3$. Suppose that $\langle \pi, \bar{\pi} \rangle = 3$. Then $\pi = 1 + 2\chi + \varepsilon$, where ε is a linear character. The kernel $\ker \varepsilon$ of ε is non-trivial. Hence it is intransitive. Now the orbits of $\ker \varepsilon$ form blocks of $\mathrm{Aut}(G)$ [16, Proposition 7.1]. Since we have assumed in this section that $\mathrm{Aut}(G)$ has no blocks of size p, $\ker \varepsilon$ must have p orbits. This contradicts the fact that $\pi|_{\ker \varepsilon} = 2 + 2\chi|_{\ker \varepsilon}$. Hence $\langle \pi, \bar{\pi} \rangle = 2$ and so, by the Frobenius reciprocity theorem, the stabilizer of a block has exactly 2 orbits, one of which is the block itself. If neither ij nor ij' is in $E(G)$, then the stabilizer of the block A_j has at least three orbits on $V(G)$, namely A_j, an orbit contained in $N_G(j) \cup N_G(j')$, and an orbit contained in the set of remaining vertices. This contradiction completes the proof. ∎

By this lemma, $e = 2$ and, moreover, if A_i, A_j and A_k are distinct blocks, then the subgraph induced on $A_i \cup A_j \cup A_k$ is either $2K_3$ or C_6. The next lemma shows that the subgraphs induced on any two sets of three distinct blocks are isomorphic.

(4.4) LEMMA. *If $G[A_i \cup A_j \cup A_k] \cong 2K_3$ for some set $\{A_i, A_j, A_k\}$ of distinct blocks, then the subgraph induced on every set of 3 distinct blocks is isomorphic to $2K_3$.*

Proof. Suppose that $G[A_r \cup A_s \cup A_t] \not\cong 2K_3$ for some set $\{A_r, A_s, A_t\}$ of distinct blocks. Then $G[A_r \cup A_s \cup A_t] \cong C_6$. Now consider the graph G' that is obtained from the complement of G by deleting the p edges $00', 11', \ldots, (p-1)(p-1)'$. It is not difficult to check that G' is weakly symmetric and has the same automorphism group as G. Moreover, $G'[A_r \cup A_s \cup A_t] \cong 2K_3$.

In the argument that follows, the graph H is one of G and G'. Let $\{x, y, z\}$ be the vertex-set of a triangle of H and n be the number of K_4-subgraphs of H that contain this triangle. The number m of triangles of H containing a fixed edge does not depend on the edge. Now H has $m-n-1$ vertices that are joined to x and y but not z, and therefore has $3(m-n-1)$ vertices that are joined to exactly two of x, y, and z. Since every block of H other than those containing x, y, and z contains exactly one vertex that is joined to at least two of x, y, and z, H has exactly $3(m-n-1)+n+3$ blocks. But H has p blocks so $p=3m-2n$. Applying this formula to both G and G' and adding, we get that $2p=3(m(G)+m(G'))-2(n(G)+n(G'))$. But $m(G)+m(G')=p-2$, hence $p=2(n(G)+n(G')+3)$—a contradiction since p is odd. ▮

By this lemma, either every subgraph of G induced on the union of 3 blocks is isomorphic to $2K_3$, or every such subgraph is isomorphic to C_6. It is straightforward to show that, in the first case, $G \cong 2K_p$, while in the second, $G \cong G(2p, p-1)$.

To complete the proof of Theorem 2.4 for Aut(G) having blocks of size 2, we now consider the case when $\overline{\text{Aut}(G)}$ is not doubly transitive. By Theorems 1.4 and 1.5, we may identify $V(\bar{G})$ with $\mathbb{Z}_p$ so that

$$\{\overline{x+b}: b \in \mathbb{Z}_p\} \subseteq \overline{\text{Aut}(G)} \subsetneqq \{\overline{ax+b}: a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\} \tag{4.5}$$

and

$$N_{\bar{G}}(\bar{0}) \qquad \textit{is a subgroup of } \mathbb{Z}_p^*. \tag{4.6}$$

One important consequence of (4.5) that we shall use frequently is that a member of $\overline{\text{Aut}(G)}$ that fixes two distinct members of $V(\bar{G})$ must be the identity. We note that no generality is lost in assuming that the automorphism τ of G acts as previously defined, that is, for all i in $\mathbb{Z}_p$, $\tau(i)=i+1$ and $\tau(i')=(i+1)'$. We also recall that $e(G)$ is 1 or 2. The next lemma shows that no vertex of G is joined to two vertices in the same block.

(4.7) Lemma. *If i and j are in $\mathbb{Z}_p$, then at least one of ij and ij' is not in $E(G)$ and at least one of $i'j$ and $i'j'$ is not in $E(G)$.*

Proof. It suffices to show that if $a \in \mathbb{Z}_p$, then $0a$ and $0a'$ cannot both be in $E(G)$. Assume the contrary. Then, by applying τ^a to $0a$ and $0a'$, we get that both $a(2a)$ and $a(2a)'$ are edges. Moreover, since $e(G) \leqslant 2$, neither $a'(2a)$ nor $a'(2a)'$ is in $E(G)$. Also, neither $0'a$ nor $0'a'$ is in $E(G)$. Now G has an automorphism θ that maps $\{0, a\}$ to $\{0, a'\}$. Hence either (i) $\theta(0)=0$ and $\theta(a)=a'$, or (ii) $\theta(0)=a'$ and $\theta(a)=0$. In case (i), $\bar{\theta}$ fixes both $\bar{0}$ and $\bar{a}$ and hence is the identity. Thus $\bar{\theta}$ fixes $\overline{2a}$. Since $\theta(a)=a'$, $\theta(a')=a$ and so $\theta(\{a', 2a, (2a)'\})=\{a, 2a, (2a)'\}$. However, $G[\{a', 2a, (2a)'\}]$ is null, while $G[\{a, 2a, (2a)'\}]$ is not—a contradiction. In case (ii), $\theta(0')=a$ and $\theta(a')=0'$. Thus $\theta(\{0, a'\})=\{a', 0'\}$. Since $0a' \in E(G)$, $0'a' \in E(G)$—a contradiction. ▮

On combining the last lemma with the following result we get that no vertex of G is joined to vertices in both A and A' where we recall that $A=\{0, 1, 2,..., p-1\}$ and $A'=\{0', 1', 2',..., (p-1)'\}$.

(4.8) LEMMA. *If i, j, and k are distinct elements of $\mathbb{Z}_p$, then at least one of ki and kj' is not in $E(G)$ and at least one of $k'i$ and $k'j'$ is not in $E(G)$.*

Proof. It suffices to prove the first assertion. Assume that $\mathbb{Z}_p$ does contain distinct elements i, j, and k such that both ki and kj' are in $E(G)$. As the automorphism τ^{-k} maps k to 0, we lose no generality in assuming that $k=0$. Then, as G is edge-transitive, there is an automorphism θ of G such that $\theta(\{0, i\}) = \{0, j'\}$. Hence either (i) $\theta(0)=0$ and $\theta(i)=j'$, or (ii) $\theta(0)=j'$ and $\theta(i)=0$.

Consider case (i). As $\overline{\mathrm{Aut}(G)} \subseteq \{\overline{ax+b}: a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_p\}$, $\bar{\theta}(\bar{x}) = \overline{ax+b}$ for all x in $\mathbb{Z}_p$. Now $\bar{\theta}(\bar{0}) = \bar{0}$ and $\bar{\theta}(\bar{i}) = \bar{j}$, so $\bar{\theta}(\bar{x}) = \overline{i^{-1}jx}$ for all x in $\mathbb{Z}_p$. In particular, $\bar{\theta}(\overline{ti}) = \overline{tj}$ for all t in $\mathbb{Z}$. Thus $\theta(ti) \in \{tj, (tj)'\}$ and we show next that, for all nonnegative integers t,

$$\theta(ti) = tj \textit{ when } t \textit{ is even and } \theta(ti) = (tj)' \textit{ when } t \textit{ is odd.} \tag{4.9}$$

This is certainly true if t is 0 or 1. Suppose now that (4.9) holds for all integers not exceeding t. We also assume initially that t is odd. We want to prove that, in that case, $\theta((t+1)\,i) = (t+1)\,j$. If not, then $\theta((t+1)\,i) = ((t+1)\,j)'$ and so $\tau^{-tj}\theta\tau^{ti}(\{0, i\}) = \tau^{-tj}\,\theta(\{ti, (t+1)\,i\}) = \tau^{-tj}(\{(tj)', ((t+1)\,j)'\}) = \{0', j'\}$. Since $0i \in E(G)$, $0'j' \in E(G)$. But $0j'$ is also in $E(G)$ and we have a contradiction to the previous lemma. If t is even, a similar argument shows that $\theta((t+1)\,i) = ((t+1)\,j)'$. We therefore conclude, by induction, that (4.9) holds. Hence, as p is odd, $\theta(pi) = (pj)'$, that is, $\theta(0) = 0'$—a contradiction.

In case (ii), we let $\sigma = \tau^{-j}\theta$. Then $\sigma(0) = 0'$ and $\sigma(i) = -j$. The rest of the argument is similar to that given in case (i). ∎

We now establish that every edge of G must join a vertex in A to a vertex in A'.

(4.10) LEMMA. *Both $G[A]$ and $G[A']$ are null.*

Proof. Suppose that $ij \in E(G)$ for some pair i and j of distinct elements of $\mathbb{Z}_p$. Then $\tau^{-i}(ij) \in E(G)$, that is, $0(j-i) \in E(G)$. If $0k' \in E(G)$ for some k in $\mathbb{Z}_p$, then we have a contradiction to one of the last two lemmas. Hence $N_G(0) \subseteq A$ and so, for all m in $\mathbb{Z}_p$,

$$N_G(m) = N_G(\tau^m(0)) = \tau^m(N_G(0)) \subseteq A.$$

Thus G has no edge joining A and A'. It follows easily that, for all i, A_i is not a block of $\mathrm{Aut}(G)$. This contradiction establishes that $G[A]$ is null and, by symmetry, $G[A']$ is also null. ∎

By the last lemma, G is a bipartite graph on A and A'. The next two lemmas complete the argument in the cases when $e(G)=2$ and $e(G)=1$ respectively and thereby finish the proof of Theorem 2.4. We omit the routine proof of the first lemma; the second proof uses the fact that G is symmetric which follows from Lemma 2.6, since G is a bipartite graph on A and A'.

(4.11) LEMMA. *If* $e(G)=2$, *then* G *is isomorphic to* $G(2p, r)$ *for some even integer* r *dividing* $p-1$.

(4.12) LEMMA. *If* $e(G)=1$, *then* G *is isomorphic to* $G(2p, r_1)$ *for some odd integer* r_1 *dividing* $p-1$.

Proof. As $\bar{G} \cong G(p, r)$ for some integer r dividing $p-1$, $\bar{0}\,\bar{1} \in E(\bar{G})$. Therefore, since $e(G)=1$, exactly one of $01'$ and $0'1$ is in $E(G)$. By symmetry, we may assume that $01' \in E(G)$. Now, by (4.6), $N_{\bar{G}}(\bar{0})$ is a subgroup of $\mathbb{Z}_p^*$. Let $\bar{D}=\{\bar{i}: i' \in N_G(0)\}$. As $|N_G(0)|=|N_G(0')|$ and $e(G)=1$, we have that $|\bar{D}|=\frac{1}{2}|N_{\bar{G}}(\bar{0})|$. We shall show next that $\bar{D}$ is a subgroup of $N_{\bar{G}}(\bar{0})$. Evidently $\bar{1} \in \bar{D}$. Now suppose that $\bar{i}$ and $\bar{j}$ are in $\bar{D}$. Then i' and j' are in $N_G(0)$. Thus, as G is symmetric, it has an automorphism θ that fixes 0 and maps i' to j'. Since $\bar{\theta} \in \overline{\mathrm{Aut}(G)}$ and the latter is a subgroup of $\{\overline{ax+b}$: $a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$, it follows that $\bar{\theta}(\bar{x})=\overline{i^{-1}jx}$ for all x in $\mathbb{Z}_p$. Thus, in particular, $\theta(\bar{1})=\overline{i^{-1}j}$, and so, as G is bipartite, $\theta(01')=0(i^{-1}j)'$. Therefore $(i^{-1}j)' \in N_G(0)$ and so $\overline{i^{-1}j} \in \bar{D}$. We conclude that $\bar{D}$ is indeed a subgroup of $N_{\bar{G}}(\bar{0})$. Since $|\bar{D}|=\frac{1}{2}|N_{\bar{G}}(\bar{0})|$ and $N_{\bar{G}}(\bar{0})$ is the cyclic group $H(p, r)$, it follows that $\bar{D}=H(p, r/2)$ and hence that $G \cong G(2p, r/2)$. Finally, we note that $r/2$ must be odd otherwise $-1 \in \bar{D}$ and $e(G)=2$. ∎

TABLE 1

The Automorphism Groups of the Symmetric Graphs of Order $2p$.

G	$\mathrm{Aut}(G)$	$\lvert\mathrm{Aut}(G)\rvert$
Null graph; K_{2p}	S_{2p}	$(2p)!$
$K_{p,p}$; $2K_p=2G(p, p-1)$	$S_p^2 \rtimes \mathbb{Z}_2$	$2(p!)^2$
$2G(p, r)$; $r<p-1$	$T(p, r)^2 \rtimes \mathbb{Z}_2$	$2r^2p^2$
$G(2, p, r)$; $r<p-1$	$\mathbb{Z}_2^p \rtimes T(p, r)$	$2^p rp$
$G(2p, r)$; $1<r<p-1$	$T(p, r) \rtimes \mathbb{Z}_2$	$2pr$
$(p, r) \neq (7, 3), (11, 5)$		
$G(2p, 1)=pK_2$	$\mathbb{Z}_2^p \rtimes S_p$	$2^p p!$
$G(2p, p-1)$	$S_p \rtimes \mathbb{Z}_2$	$2p!$
$B(H(11))=G(2\cdot 11, 5)$; $B'(H(11))$	$PSL(2, 11) \rtimes \mathbb{Z}_2$	1320
O_3; O_3^c	S_5	120
$B(PG(n-1, q))$; $B'(PG(n-1, q))$	$P\Gamma L(n, q) \rtimes \mathbb{Z}_2$	$2tq^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1)$
$[B(PG(2, 2))=G(2\cdot 7, 3)]$		where $q=s^t$
		for s prime.

5. The Automorphism Groups of the Graphs

In this section we list the automorphism groups of the symmetric graphs in our main theorem. The determination of these automorphism groups is not difficult and we omit the details of the argument. Recall from Theorem 1.5 that, for $r<p-1$, $\mathrm{Aut}(G(p,r))$ is the group $\{ax+b: a\in H(p,r), b\in \mathbb{Z}_p\}$ of permutations of $\mathbb{Z}_p$. In the table above we have denoted $\mathrm{Aut}(G(p,r))$ by $T(p,r)$. In addition, we have written $X \rtimes Y$ for the semidirect product of the group X by the group Y, and X^n for the direct product of n copies of X. The structure of the graphs listed in Table 1 is given in Section 2.

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