BOUNDING THE NUMBER OF BASES OF A MATROID

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Abstract. Thomassen proved in 2010 that the number of spanning trees of a graph with vertex degrees $d_1, d_2, \dots, d_n$ is at most $d_1 d_2 \dots d_{n-1}$. This note generalizes this result to show that if A is a matrix representing a rank-r matroid M over a field and $S_1, S_2, \dots, S_r$ are the supports of the rows of A, then the number of bases of M is at most $|S_1||S_2| \dots |S_r|$. More generally, it is shown that if $C_1^*, C_2^*, \dots, C_r^*$ are cocircuits of a rank-r matroid N such that the deletion of any k of these cocircuits from N drops the rank by at least k, then the number of bases of N is at most $|C_1^*||C_2^*| \dots |C_r^*|$.

1. Introduction

A number of authors including Alon [1] have given bounds on the number $\tau(G)$ of spanning trees in a graph G in terms of the degree sequence of the graph. In particular, Kostochka [4] proved that, when G is simple having n vertices of degrees $d_1, d_2, \dots, d_n$, we have

$$\tau(G) \leq \tfrac{1}{n-1} d_1 d_2 \dots d_n.$$

Thomassen [7] proved a similar result for an arbitrary graph G showing that

$$\tau(G) \leq d_1 d_2 \dots d_{n-1}.$$

Recently, Klee, Naranyan, and Sauermann [3] have proved that, when G is simple,

$$\tau(G) \leq \tfrac{1}{n^2}(d_1 + 1)(d_2 + 1) \dots (d_n + 1).$$

The purpose of this note is to prove a bound on the number $b(M)$ of bases of a matroid M that generalizes Thomassen's bound. The terminology used here for graphs and matroids will follow [6]. When G is a loopless 2-connected graph, the set of edges meeting a fixed vertex of G forms a cocircuit of its cycle matroid $M(G)$. Since $r(M(G)) = |V(G)| - 1$, the following consequence of a result of Bondy and Welsh [2, Lemma 3.2] is a matroid analogue of Thomassen's result.

Theorem 1.1. *Let M be a rank-r matroid. Let $C_1^*, C_2^*, \dots, C_r^*$ be a collection of cocircuits of M such that no C_j^* is contained in $\cup_{i \neq j} C_i^*$. Then*

$$b(M) \leq |C_1^*||C_2^*| \dots |C_r^*|.$$

Bondy and Welsh [2, Lemma 3.1] proved that the condition on cocircuits in the last theorem is equivalent to the assertion that $\{C_1^*, C_2^*, \dots, C_r^*\}$ is the set of fundamental cocircuits with respect to some cobasis of M. Thus, such a collection of cocircuits has the property that the deletion of any k of them drops the rank by at least k. With this condition alone, we can get the same conclusion as in the last theorem. This is our main result.

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Theorem 1.2. *Let M be a rank-r matroid. Let $C_1^*, C_2^*, \dots, C_r^*$ be a collection of cocircuits of M such that, for all k in $\{1, 2, \dots, r\}$, the deletion of the union of k of these cocircuits from M has rank at most $r - k$. Then $b(M) \leq |C_1^*||C_2^*| \dots |C_r^*|$. Moreover, equality holds if and only if M is the direct sum of r rank-one uniform matroids with ground sets $C_1^*, C_2^*, \dots, C_r^*$ and a rank-zero matroid.*

2. The proof and some consequences

Let $(T_1, T_2, \dots, T_r)$ be a collection of subsets of the ground set of a rank-r matroid M such that $r(M \backslash (T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k})) \leq r - k$ for all subsets $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, r\}$. Then each T_i must contain a cocircuit of M. We call $(T_1, T_2, \dots, T_r)$ a *full codependent family* of M. For instance, if $(S_1, S_2, \dots, S_r)$ is a presentation of a rank-r transversal matroid N, then $(S_1, S_2, \dots, S_r)$ is a full codependent family of N. Theorem 1.2 follows immediately from the next result, which also implies the subsequent corollary.

Proposition 2.1. *Let M be a rank-r matroid. Let $(T_1, T_2, \dots, T_r)$ be a full codependent family of M. Then $b(M) \leq |T_1||T_2| \dots |T_r|$. Moreover, equality holds if and only if M is the direct sum of r rank-one uniform matroids with ground sets $T_1, T_2, \dots, T_r$ and a rank-zero matroid.*

Proof. Let N be the transversal matroid having $(T_1, T_2, \dots, T_r)$ as a presentation and having ground set $E(M)$. Clearly, $b(N) \leq |T_1||T_2| \dots |T_r|$. Let B be a basis of M. We shall show that B is a basis of N. Consider the family $(B \cap T_1, B \cap T_2, \dots, B \cap T_r)$. Suppose that $|(T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k}) \cap B| < k$ for some collection of k of the sets T_i. By assumption, $r(M \backslash (T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k})) \leq r - k$, so B is not a basis of M, a contradiction. Thus $|(T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_j}) \cap B| \geq k$. Hence, by Hall's Marriage Theorem, the family $(B \cap T_1, B \cap T_2, \dots, B \cap T_r)$ has a transversal, so B is a basis of N, as desired. Therefore $\mathscr{B}(M) \subseteq \mathscr{B}(N)$, so $b(M) \leq b(N) \leq |T_1||T_2| \dots |T_r|$.

Clearly $b(M) = |T_1||T_2| \dots |T_r|$ when M is the direct sum of r rank-one uniform matroids with ground sets $T_1, T_2, \dots, T_r$ and a rank-zero matroid. Now suppose that $b(M) = |T_1||T_2| \dots |T_r|$. Then $\mathscr{B}(M) = \mathscr{B}(N)$, and $b(N) = |T_1||T_2| \dots |T_r|$, so the sets $T_1, T_2, \dots, T_r$ in the presentation of the transversal matroid N are pairwise disjoint. Thus N is the direct sum of r rank-one uniform matroids with ground sets $T_1, T_2, \dots, T_r$ and a rank-zero matroid with ground set $E(M) - (T_1 \cup T_2 \cup \cdots \cup T_r)$. As $\mathscr{B}(M) = \mathscr{B}(N)$, we deduce that $M = N$, so M is the specified direct sum. $\square$

Corollary 2.2. *Let $S_1, S_2, \dots, S_r$ be the supports of the rows of a matrix A that represents a rank-r matroid M. Then $b(M) \leq |S_1||S_2| \dots |S_r|$.*

We remark that Theorem 1.2 has Thomassen's result as a consequence while Theorem 1.1 does not. For example, the set consisting of all but the largest vertex bond in a 2-connected bipartite graph G does not have the property that no cocircuit in the collection is contained in the union of the others.

For matroids of rank two, we can obtain tight bounds on $b(M)$ that can be leveraged to improve on the bound on $b(M)$ for representable matroids.

Proposition 2.3. *Let M be a rank-2 matroid and let C_1^* and C_2^* be distinct cocircuits of M. Then*

$$b(M) \leq |C_1^*||C_2^*| - \binom{|C_1^* \cap C_2^*| + 1}{2}$$

and equality holds if and only if no element of $C_1^* \cap C_2^*$ *is in a* 2*-circuit. Furthermore, if* M *is binary, then* $b(M) = |C_1^*||C_2^*| - |C_1^* \cap C_2^*|^2$.

Proof. Let $(A_1, A_2, A_3) = (C_1^* - C_2^*, C_2^* - C_1^*, C_1^* \cap C_2^*)$. Then any basis of M is a member of exactly one of the following sets:

$$\mathscr{B}_1 = \{\{x_1, x_2\} : x_1 \in A_1, x_2 \in A_2\};\ \mathscr{B}_2 = \{\{x_1, x_2\} : x_1 \in A_1, x_2 \in A_3\},$$
$$\mathscr{B}_3 = \{\{x_1, x_2\} : x_1 \in A_3, x_2 \in A_2\};\ \mathscr{B}_4 = \{\{x_1, x_2\} : x_1, x_2 \in A_3, x_1 \neq x_2\}.$$

Therefore, $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3 \subseteq \mathscr{B}(M) \subseteq \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3 \cup \mathscr{B}_4$. It follows that $|\mathscr{B}(M)| \leq |A_1||A_2| + |A_1||A_3| + |A_2||A_3| + \binom{A_3}{2} = |C_1^*||C_2^*| - \binom{|C_1^* \cap C_2^*|+1}{2}$. Moreover, equality holds in the last bound if and only if no two elements of A_3 are in parallel.

When M is binary, the set $\mathscr{B}_4$ contains no bases of M, so $|\mathscr{B}(M)| = |A_1||A_2| + |A_1||A_3| + |A_2||A_3| = |C_1^*||C_2^*| - |C_1^* \cap C_2^*|^2$. □

Lemma 2.4. *Let* A *be a matrix over a field such that* A *has* r *rows and rank* r*. For some* k *with* $1 \leq k \leq r-1$*, let* C *and* D *be the submatrices of* A *consisting of the first* k *and the last* $r-k$ *rows, respectively. Then* $b(M[A]) \leq b(M[C])b(M[D])$.

Proof. Let B be a basis of $M[A]$. Then the submatrix A' of A whose columns are labelled by the elements of B has non-zero determinant. Using a Laplace expansion of $\det A'$ (see, for example, [5, p. 180]), we see that this determinant is a sum of terms each of which is plus or minus the product of the determinants of a $k \times k$ submatrix C' of C and an $(r-k) \times (r-k)$ submatrix D' of D such that every column label of A' is a column label of exactly one of C' and D'. Because $\det A'$ is non-zero, there must be such a pair (C', D') for which both $\det C'$ and $\det D'$ are non-zero. Hence B can be written as the disjoint union of a basis of $M[C]$ and a basis of $M[D]$. Thus $b(M[A]) \leq b(M[C])b(M[D])$. □

As an example of how we can combine the last two results, we have the following result, whose straightforward proof we omit.

Proposition 2.5. *Let* A *be an* $r \times n$ *binary matrix representing a rank-*r *binary matroid* M *where* r *is even. Let* $S_1, S_2, \dots, S_r$ *be the supports of the rows of* A*. Then* $b(M)$ *is at most*

$$(|S_1||S_2| - |S_1 \cap S_2|^2)(|S_3||S_4| - |S_3 \cap S_4|^2) \dots (|S_{(r/2)-1}||S_{r/2}| - |S_{(r/2)-1} \cap S_{r/2}|^2).$$

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