Dependencies among Dependencies in Matroids

James Oxley
Mathematics Department
Louisiana State University
Baton Rouge, Louisiana, USA
oxley@math.lsu.edu

Suijie Wang*
Institute of Mathematics
Hunan University
Changsha, Hunan, China
wangsuijie@hnu.edu.cn

Abstract

In 1971, Rota introduced the concept of derived matroids to investigate "dependencies among dependencies" in matroids. In this paper, we study the derived matroid δM of an $\mathbb{F}$-representation of a matroid M. The matroid δM has a naturally associated $\mathbb{F}$-representation, so we can define a sequence δM, $\delta^2 M$, $\dots$. The main result classifies such derived sequences of matroids into three types: finite, cyclic, and divergent. For the first two types, we obtain complete characterizations and thereby resolve some of the questions that Longyear posed in 1980 for binary matroids. For the last type, the divergence is estimated by the coranks of the matroids in the derived sequence.

Keywords: derived matroids, derived sequence, dependencies among dependencies, representable matroids.

1 Introduction

In algebraic topology, homology groups examine the independent holes of topological spaces. It is natural to ask about the dependence relations among these holes. For a 1-dimensional simplicial complex (a graph), this amounts to determining the dependencies among all of the cycles in the graph.

As Judith Q. Longyear wrote in [5] that, at the Bowdoin College Summer 1971 NSF Conference on Combinatorics, Gian-Carlo Rota posed the following question: "The minimal dependent sets of vectors in a space V may be regarded as vectors in the derived space δV over the same field by using the vectors of V as a basis for δV. Can this same sort of process be applied to the dependent sets of a matroid M to investigate the 'dependencies among dependencies'? If so, what properties does δM, the derived matroid, possess?"

Longyear [5] answered the first question when M is a binary matroid. She defined its derived matroid δM to have as its ground set, the set $\mathscr{C}(M)$ of circuits of M where a set X of such circuits is independent in δM exactly when, for each nonempty subset Y of X, the symmetric difference of the circuits in Y is nonempty. Longyear noted that her derived matroid is a binary matroid and she asked the following four questions about this matroid where we have differentiated the parts of these questions that contain more than one part.

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- Question 1. (a) What effect does δ have on the flats of a matroid? (b) On the dual?
- Question 2. How many different (nonisomorphic) binary matroids M are there for which δM has rank r?
- Question 3. (a) When does $\delta M = M$? (b) When is there a matroid N for which $\delta N = M$? (c) If $\delta^{k+1} M = \delta(\delta^k M)$, when can $\delta^k M = \delta^j M$?
- Question 4. If M is $U_{1,3}$, then δM is $U_{2,3}$, $\delta^2 M$ is $U_{1,1}$ and $\delta^3 M$ is $U_{0,0}$. Characterize those M for which $\delta^k M$ can eventually be $U_{0,0}$.

In this paper, we answer Questions 3(a), 3(c), and 4 by proving the following results.

Theorem 1.1. *Let M be a binary matroid. If $\delta^k M \cong M$ for some $k \geq 1$, then $M \cong U_{0,0}$.*

Theorem 1.2. *Let M be a binary matroid such that $\delta^k M \cong U_{0,0}$ for some $k \geq 0$. Then $\delta^3 M \cong U_{0,0}$, and either $M \cong U_{0,0}$, or M is nonempty and each if its components is isomorphic to $U_{1,1}$, a circuit, or the cycle matroid of a theta graph.*

In addition, we extend Longyear's work by defining the derived sequence $M, \delta M, \delta^2 M, \ldots$ of matroids beginning with any $\mathbb{F}$-represented matroid M where $\mathbb{F}$ is an arbitrary field. We show that, when M has no coloops, δM is connected if and only if M is connected. Our main theorems show that derived sequences are either finite, cyclic, or divergent, and they answer Questions 3(a), 3(c), and 4 for arbitrary $\mathbb{F}$-represented matroids. In particular, we prove that $U_{2,4}$ is the unique nonempty connected matroid M for which $\delta M \cong M$. These results appear in Section 4 of the paper. In Sections 2 and 3, we define derived sequences of represented matroids and prove a number of basic properties of derived matroids.

2 The derived sequence of a represented matroid

Our matroid terminology and notation will follow Oxley [6]. For a field $\mathbb{F}$, let M be an $\mathbb{F}$-representable matroid with ground set $E = \{e_1, e_2, \ldots, e_m\}$, and let $\varphi : E \to \mathbb{F}^n$ be a representation of M. The matrix A whose columns are the vectors $\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_m)$ is the matrix corresponding to φ. Moreover, M is $M[A]$, the vector matroid of the matrix A. The matrix A is also referred to as an $\mathbb{F}$-representation of M.

The pair (M, φ), or equivalently the pair (M, A), denotes an $\mathbb{F}$*-represented matroid*. For such a pair, associated with each circuit C of M, there is a vector $\mathbf{c}_C = (c_1, c_2, \ldots, c_m)$ in $\mathbb{F}^m$ such that $\sum_{i=1}^m c_i \varphi(e_i) = 0$ where $c_i \neq 0$ if and only if $i \in C$. Moreover, as one easily checks, $\mathbf{c}_C$ is unique to within a non-zero constant scalar multiple. It follows that, associated with the $\mathbb{F}$-represented matroid (M, φ), there is an $\mathbb{F}$-represented matroid $(\delta M, \delta\varphi)$ with ground set $\mathscr{C}(M)$, the set of circuits of M, such that $(\delta\varphi)(C) = \mathbf{c}_C$ for all C in $\mathscr{C}(M)$. We call the $\mathbb{F}$-represented matroid $(\delta M, \delta\varphi)$ the *derived matroid* of (M, φ); the vector $\mathbf{c}_C$ is the *circuit vector* of C. We shall frequently write δM for $(\delta M, \delta\varphi)$.

Let $(\delta^0 M, \delta^0\varphi) = (M, \varphi)$. Inductively, for any positive integer k, the *kth derived matroid* $(\delta^k M, \delta^k\varphi)$ of M is the derived matroid of $(\delta^{k-1} M, \delta^{k-1}\varphi)$. The *derived sequence* of (M, φ) is the sequence $(\delta^0 M, \delta^0\varphi), (\delta^1 M, \delta^1\varphi), (\delta^2 M, \delta^2\varphi), \ldots$. Since over $GF(2)$, taking linear combinations of vectors coincides with taking symmetric differences of their supports, this definition is easily seen to extend Longyear's definition of derived matroids of binary matroids. For such matroids, Recski [7] denotes δM by $\theta(M)$ and refers to the operation $M \to \theta(M)$ as the *theta-operation* on M.

For a field $\mathbb{F}$, let (M, φ) be an $\mathbb{F}$-represented matroid and A be the matrix corresponding to φ. Elementary row operations on A clearly do not alter the circuit vectors of (M, φ). This means that, when $r(M) > 0$, for any ordered basis $(e_1, e_2, \dots, e_r)$ of M, we can assume, after potentially permuting columns, that A is a standard representative matrix $[I_r|D]$ for M, where I_r is the $r \times r$ identity matrix and its columns are labelled, in order, $e_1, e_2, \dots, e_r$.

For a basis B in a matroid M and an element e of $E(M) - B$, the unique circuit $C(e, B)$ contained in $B \cup e$ is called the *fundamental circuit of e with respect to B*.

Lemma 2.1. *For a field $\mathbb{F}$, let (M, φ) be an $\mathbb{F}$-represented matroid. Then δM is a simple matroid of rank $r^*(M)$. In particular, if B is a basis of M, then $\{C(e, B) : e \in E(M) - B\}$ is a basis of δM.*

Proof. It is clear that δM is simple since no circuit vector is the zero vector while no two distinct circuits have circuit vectors that are scalar multiples of each other. If $r(M) = 0$, then $M \cong U_{0,m}$, so $\delta M \cong U_{m,m}$ and the result holds. Now suppose $r(M) > 0$. As noted above, we can transform the matrix A corresponding to the representation φ into the form $[I_r|D]$. Let the columns of this matrix be labelled, in order, by $e_1, e_2, \dots, e_m$ where $\{e_1, e_2, \dots, e_r\}$ is a basis B of M. For each i in $\{1, 2, \dots, m - r\}$, consider $C(e_{r+i}, B)$. Clearly the $m - r$ circuit vectors of $C(e_{r+1}, B), C(e_{r+2}, B), \dots, C(e_m, B)$ are linearly independent. As all of the circuit vectors of M are in the solution space of the equation $[I_r|D]X = 0$, we deduce that δM has $\{C(e_i, B) : r + 1 \le i \le m\}$ as a basis. Thus $r(\delta M) = m - r = |E(M)| - r(M) = r^*(M)$. $\square$

For a basis B of an $\mathbb{F}$-represented matroid (M, φ), we shall call $\{C(e, B) : e \in E(M) - B\}$ the *circuit basis* of δM associated with B. Because the fundamental circuits do not depend on the representation φ, this circuit basis also does not depend on the representation.

Corollary 2.2. *For a field $\mathbb{F}$, let (M, φ) be an $\mathbb{F}$-represented matroid. Then*

$$|\mathscr{C}(M)| = r^*(M) + r^*(\delta M).$$

The following matrix A represents $M(K_4)$ over both $GF(2)$ and $GF(3)$ where, of course, $-1 = 1$ in the former:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

We will write A_2 and A_3 for the interpretations of A over $GF(2)$ and $GF(3)$, respectively. Hence we can view $M[A_2]$ and $M[A_3]$ as $GF(2)$- and $GF(3)$-represented matroids.

Now $M(K_4)$ has exactly seven circuits. These label the columns of the matrix in Figure 1, where, for example, 124 is an abbreviation for $\{1, 2, 4\}$. Row i of this matrix is labelled by the column vector $\varphi(i)$ corresponding to column i in A. The columns of this matrix are the circuit vectors of the corresponding circuits. Thus $\delta M[A_2]$ and $\delta M[A_3]$ are represented by this matrix interpreted over $GF(2)$ and $GF(3)$, respectively. By Lemma 2.1, $\delta M[A_2]$ and $\delta M[A_3]$ both have rank three. Clearly each is simple having seven elements. Thus $\delta M[A_2] \cong F_7$, the Fano matroid. Since $\delta M[A_3]$ is ternary, we deduce that $\delta M[A_3] \not\cong \delta M[A_2]$. It is not difficult to check that $\delta M[A_3] \cong F_7^-$, the non-Fano matroid.

	124	135	236	456	1346	1256	2345
$\varphi(1)$	1	1	0	0	1	1	0
$\varphi(2)$	1	0	1	0	-1	0	1
$\varphi(3)$	0	1	-1	0	0	-1	-1
$\varphi(4)$	-1	0	0	1	0	-1	-1
$\varphi(5)$	0	-1	0	-1	-1	0	1
$\varphi(6)$	0	0	-1	-1	1	1	0

Figure 1: A matrix representation of $\delta M(K_4)$.

In contrast to the above, where we considered representations of a binary matroid over two different fields, if we fix the field $\mathbb{F}$, then the derived matroid of a binary matroid does not depend on the representation.

Lemma 2.3. *Let M be a binary matroid and let φ and ψ be $\mathbb{F}$-representations of M for some field $\mathbb{F}$. Then $(\delta M, \delta\varphi) = (\delta M, \delta\psi)$.*

Proof. By a theorem of Brylawski and Lucas [1] (see [6, Proposition 6.6.5]), as M is binary, all $\mathbb{F}$-representations of M are projectively equivalent. Thus if A_1 and A_2 are the matrices corresponding to $\delta\varphi$ and $\delta\psi$, then there are non-singular matrices X and Y, where Y is diagonal, such that $A_2 = XA_1Y$. It follows, by using determinants to compare the sets of bases, that $M[A_2] = M[A_1]$; that is, $(\delta M, \delta\varphi) = (\delta M, \delta\psi)$. □

The derived matroid of the smallest non-binary matroid, $U_{2,4}$, depends neither on the representation nor the field. More generally, we have the following result.

Lemma 2.4. *For a field $\mathbb{F}$ and $n \geq 3$, let φ be an $\mathbb{F}$-representation of $U_{n-2,n}$. Then $\delta U_{n-2,n} \cong U_{2,n}$. In particular, $\delta U_{2,4} \cong U_{2,4}$.*

Proof. Clearly $\delta U_{n-2,n}$ has n elements. By Lemma 2.1, this matroid is simple of rank two and so it is isomorphic to $U_{2,n}$. □

The derived matroids of rank-one matroids are the cycle matroids of complete graphs.

Lemma 2.5. *For a field $\mathbb{F}$ and $n \geq 1$, let φ be an $\mathbb{F}$-representation of $U_{1,n}$. Then $\delta U_{1,n} \cong M(K_n)$.*

Proof. As $U_{1,n}$ is binary, by Lemma 2.3, we may assume that $\mathbb{F} = GF(2)$. Then $\delta U_{1,n}$ is represented over that field by the $n \times \binom{n}{2}$ matrix whose columns are all distinct vectors of length n having exactly two non-zero entries. This matrix also represents $M(K_n)$. □

For a fixed field $\mathbb{F}$, we know that δM does not depend on the $\mathbb{F}$-representation of M when M is binary. We now show that this does not hold in general.

Theorem 2.6. *Let $\mathbb{F}$ be a field. Then, for all $\mathbb{F}$-represented matroids (M, φ) the derived matroid δM does not depend on the $\mathbb{F}$-representation φ if and only if $\mathbb{F}$ is $GF(2)$ or $GF(3)$.*

Proof. If $\mathbb{F} = GF(2)$, then, as noted above, δM does not depend on the $GF(2)$-representation of M. Now let $\mathbb{F} = GF(3)$. Then, by a theorem of Brylawski and Lucas [1] (see [6, Corollary 14.6.1]), all $GF(3)$-representations of a ternary matroid M are projectively equivalent. Hence δM does not depend on the representation. For the converse, we use two examples.

We view the field $GF(4)$ as $GF(2)(\omega)$ where $\omega^2+\omega+1=0$. Kahn [3] noted that $U_{2,4}\oplus_2 U_{2,4}$ is represented over $GF(4)$ by the matrix A_x for each x in $\{\omega, \omega+1\}$ where

$$A_x = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & x \end{bmatrix} \end{array}.$$

Then the matrix representing $\delta M[A_x]$ has the following submatrix:

$$\begin{array}{c} \begin{matrix} 1246 & 1345 & 2356 \end{matrix} \\ \begin{bmatrix} 1 & \omega & 0 \\ 1 & 0 & (\omega+1)(x+1) \\ 0 & 1 & \omega(x+1) \\ x & 1 & 0 \\ 0 & 1 & x \\ 1 & 0 & 1 \end{bmatrix} \end{array}.$$

These three columns are linearly dependent when $x = \omega + 1$ but are linearly independent when $x = \omega$. Thus δM may depend on the $\mathbb{F}$-representation of M when $\mathbb{F} = GF(4)$.

For a field $\mathbb{F}$ with at least four elements, the matroid $U_{3,6}$ is represented by the matrix

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & x_3 & x_4 \end{bmatrix} \end{array}$$

where x_1, x_2, x_3, and x_4 are elements of $\mathbb{F} - \{0, 1\}$ such that

$$\{x_1, x_4\} \cap \{x_2, x_3\} \neq \emptyset, \quad x_1x_4 \neq x_2x_3, \quad \text{and} \quad (x_1-1)(x_4-1) \neq (x_2-1)(x_3-1).$$

We know that $\delta M[A]$ has fifteen elements and rank three and is spanned by the fundamental circuits of the elements 4, 5, and 6 with respect to the basis $\{1, 2, 3\}$ of $M[A]$. If we consider the matrix δA whose columns are labelled by the circuits of $\delta M[A]$ and whose rows are labelled by the columns of A, we see that the submatrix of this matrix whose columns are labelled by these fundamental circuits, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, and $\{1, 2, 3, 6\}$, and whose rows are labelled by $4, 5$, and 6 is I_3. It follows that $\delta M[A]$ is represented by the submatrix of δA obtained by removing its first three rows. This submatrix has the following submatrix:

$$\begin{array}{c} \begin{matrix} 1246 & 1356 & 2345 \end{matrix} \\ \begin{bmatrix} -x_4 & 0 & -1 \\ 0 & -x_2 & 1 \\ 1 & x_1 & 0 \end{bmatrix} \end{array}.$$

As the determinant of this matrix is $x_1x_4 - x_2$, it follows that $\delta M[A]$ does depend on the representation provided the field is large enough to allow us to choose two different 4-tuples (x_1, x_2, x_3, x_4) such that $x_1x_4 - x_2 = 0$ for one of these 4-tuples but $x_1x_4 - x_2 \neq 0$ for the other. As the reader can easily check, this is possible provided $\mathbb{F}$ has at least five elements. Thus, for all such fields, δM may depend on the $\mathbb{F}$-representation of M. □

The last result is not surprising since it is well known that all $\mathbb{F}$-representations of an $\mathbb{F}$-representable matroid are projectively equivalent if and only if $\mathbb{F}$ is $GF(2)$ or $GF(3)$.

Next we observe that adding an element in series to an existing element of M does not alter δM.

Lemma 2.7. *For a field $\mathbb{F}$, let (M, φ) be an $\mathbb{F}$-represented matroid. If $\{e, f\}$ is a cocircuit of M, then $\delta(M/e) = \delta(M)$ where the representation of M/e is that induced by φ.*

Proof. As M has $\{e, f\}$ as a cocircuit, it has a basis B that contains e but not f. We can transform the matrix A corresponding to φ into a matrix of the form $[I_r|D]$ where the columns of the identity matrix are labelled by the elements of B, and the first columns of I_r and D are labelled by e and f, respectively. Then M/e is represented by the matrix that is obtained from $[I_r|D]$ by deleting the first row and the first column. As $e \cup (E(M) - B)$ contains a unique cocircuit of M and $\{e, f\}$ is a cocircuit contained in this set, the only non-zero entries in the first row of $[I_r|D]$ are in the columns labelled by e and f. Hence a circuit of M contains e if and only if it contains f. Thus, in the representation of δM in which the columns are the circuit vectors and the rows are labelled by the elements of M, the row labelled by f is a non-zero scalar multiple of the first row, the row labelled by e. Thus deleting the row labelled by f gives a representation for the same matroid. As $(c_1, c_2, \dots, c_n)$ is a circuit vector of M if and only if $(c_2, c_3, \dots, c_n)$ is a circuit vector of M/e, we deduce that $\delta(M/e) = \delta M$. $\square$

The next lemma specifies what effect the presence of a loop or a coloop has on δM.

Lemma 2.8. *Let (M, φ) be an $\mathbb{F}$-represented matroid.*

(i) If e is a coloop of M, then $\delta M = \delta(M \backslash e)$.

(ii) If e is a loop of M, then $\delta M = U_{1,1} \oplus \delta(M \backslash e)$.

Proof. The first part is immediate from the fact that a coloop is in no circuits. Now suppose e is a loop of M. The only circuit vector with a non-zero entry in the coordinate corresponding to e is the circuit vector of the circuit $\{e\}$. Hence $\{e\}$ is a coloop of δM. $\square$

Recall that, for a matroid M, its *cosimplification* $\text{co}(M)$ is the matroid that is obtained from M by deleting all coloops and then contracting all but one element from each series class. The following result is an immediate consequence of the last two lemmas.

Corollary 2.9. *For a field $\mathbb{F}$, let (M, φ) be an $\mathbb{F}$-represented matroid. Then $\delta M = \delta(\text{co}(M))$.*

By combining the last corollary with Lemma 2.4, we obtain the following result.

Corollary 2.10. *For a field $\mathbb{F}$, let (M, φ) be a connected $\mathbb{F}$-represented matroid with $r^*(M) = 2$. Then $\text{co}(M) \cong U_{n-2,n}$ for some $n \geq 3$ and $\delta M \cong U_{2,n}$.*

We observe that the connected matroids M for which $\text{co}(M) \cong U_{1,3}$ coincide with the cycle matroids of theta graphs.

3 Connected matroids

In this section, we prove some results that will be used in the proofs of the main theorems.

Lemma 3.1. *Let e be an element of a connected matroid M. Then M has at least $r^*(M)$ circuits containing e.*

Proof. The result is true if M is a circuit or a coloop and so holds if $|E(M)| \in \{1, 2\}$. Assume the result is true for $|E(M)| < k$ and let $|E(M)| = k \geq 3$. Suppose $f \in E(M) - e$. Assume M/f is connected. Then, by the induction assumption, as $r^*(M/f) = r^*(M)$, we see that M/f, and hence M, has at least $r^*(M)$ circuits containing e. We may now assume that M/f is disconnected. Then $M\backslash f$ is connected and so has at least $r^*(M\backslash f)$ circuits containing e. As the connected matroid M certainly has a circuit containing $\{e, f\}$, we deduce that M has at least $r^*(M\backslash f) + 1$, that is, $r^*(M)$, circuits containing e. The result follows by induction. □

Lemma 3.2. *Let M be a nonempty connected matroid. Then*

$$|\mathscr{C}(M)| \geq \binom{r^*(M)+1}{2} \qquad \text{and} \qquad r^*(\delta M) \geq \binom{r^*(M)}{2}.$$

Proof. We prove the first inequality by induction on $|E(M)|$. The result clearly holds for $r^*(M) = 1$ and so holds for $|E(M)| \in \{1, 2\}$. Assume it holds for $|E(M)| < k$ and let $|E(M)| = k \geq 3$. Take e in $E(M)$. Suppose $M\backslash e$ is connected. Then, by the induction assumption,

$$|\mathscr{C}(M\backslash e)| \geq \binom{r^*(M\backslash e)+1}{2} = \binom{r^*(M)}{2}.$$

By Lemma 3.1, M has at least $r^*(M)$ circuits containing e. Thus

$$|\mathscr{C}(M)| \geq \binom{r^*(M)}{2} + \binom{r^*(M)}{1} = \binom{r^*(M)+1}{2}.$$

We may now assume that $M\backslash e$ is disconnected. Then M/e is connected, so

$$|\mathscr{C}(M)| \geq |\mathscr{C}(M/e)| \geq \binom{r^*(M/e)+1}{2} = \binom{r^*(M)+1}{2}.$$

The first inequality follows by induction. The second inequality is a straightforward consequence of the first since, by Corollary 2.2, $|\mathscr{C}(M)| = r^*(\delta M) + r^*(M)$. □

We omit the proof of the following straightforward consequence of the last lemma.

Corollary 3.3. *Let M be a connected matroid. Then*

$$|\mathscr{C}(M)| \geq 3(r^*(M) - 1).$$

For a matroid M and a basis B of M, the graph $G_B(M)$ is the simple bipartite graph having B and $E(M) - B$ as its vertex classes where a vertex x of $E(M) - B$ is adjacent to a vertex y of B if and only if $y \in C(x, B)$. Cunningham [2] and Krogdahl [4] proved the following result (see also [6, Proposition 4.3.2]).

Lemma 3.4. *The vertex sets of the components of the graph $G_B(M)$ coincide with the components of the matroid M.*

Lemma 3.5. *Let (M, φ) be an $\mathbb{F}$-represented matroid such that M has no coloops. If $M = M_1 \oplus M_2$, then $\delta M = \delta M_1 \oplus \delta M_2$.*

Proof. For a basis B of M, we know that δM is spanned by the set $\{C(f, B) : f \in E - B\}$ of fundamental circuits of B. By Lemma 3.4, the components of $G_B(M)$ coincide with the components of M. In particular, if $B_i = E(M_i) \cap B$ for each i, then $\{C(f, B_i) : f \in E(M_i) - B_i\}$ spans δM_i. Since M has no circuit that meets both $E(M_1)$ and $E(M_2)$, it follows that no circuit of δM meets both $E(\delta M_1)$ and $E(\delta M_2)$. We deduce that $\delta M = \delta M_1 \oplus \delta M_2$. □

Theorem 3.6. *Let (M, φ) be an $\mathbb{F}$-represented matroid with no coloops. If $\delta M = N_1 \oplus N_2$, then there are matroids M_1 and M_2 such that $M = M_1 \oplus M_2$ and $N_i = \delta M_i$ for each i.*

Proof. Let $E(M) = \{e_1, e_2, \dots, e_m\}$ where $\{e_1, e_2, \dots, e_r\}$ is a basis B of M. Then δM is spanned by $\{C(e_{r+1}, B), C(e_{r+2}, B), \dots, C(e_m, B)\}$. We may assume that N_1 and N_2 are spanned by $\{C(e_i, B) : r + 1 \le i \le s\}$ and $\{C(e_i, B) : s + 1 \le i \le m\}$, respectively. Let $E_1 = \cup_{i=r+1}^{s} C(e_i, B)$ and $E_2 = \cup_{i=s+1}^{m} C(e_i, B)$.

Suppose $E_1 \cap E_2 \ne \emptyset$. Then we may assume that $C(e_{r+1}, B) \cap C(e_m, B) \ne \emptyset$. Then M has a circuit C containing $\{e_{r+1}, e_m\}$. Writing the circuit vector $\mathbf{c}_C$ as a linear combination of the circuit vectors $\mathbf{c}_{r+1}, \mathbf{c}_{r+2}, \dots, \mathbf{c}_m$ where $\mathbf{c}_j$ is the circuit vector of $C(e_j, B)$, we see that the coefficients of $\mathbf{c}_{r+1}$ and $\mathbf{c}_m$ must both be non-zero. We deduce that, in δM, there is a circuit containing the elements $C(e_{r+1}, B)$ and $C(e_m, B)$. This is a contradiction as these elements are in different components of δM. We conclude that $E_1 \cap E_2 = \emptyset$.

Because M has no coloops, every element of M is in E_1 or E_2. Letting $M_i = M|E_i$, we see that $\delta M_i = N_i$. □

Corollary 3.7. *For a connected representable matroid M and all $k \ge 0$, the matroid $\delta^k M$ is connected and*

$$r^*(\delta^k M) \ge 2^k(r^*(M) - 3) + 3.$$

Proof. We argue by induction on k. The result is immediate if $k = 0$. Now assume the result holds for $k - 1$, which is non-negative. By Corollaries 2.2 and 3.3,

$$r^*(\delta^k M) + r^*(\delta^{k-1} M) = |\mathscr{C}(\delta^{k-1} M)| \ge 3(r^*(\delta^{k-1} M) - 1).$$

Thus $r^*(\delta^k M) \ge 2r^*(\delta^{k-1} M) - 3$. Hence, by the induction assumption, as $\delta^{k-1} M$ is connected, it is either a coloop or has no coloops. Using Theorem 3.6, we see that, in each case, $\delta^k M$ is connected. Moreover, by the induction assumption again,

$$r^*(\delta^k M) \ge 2(2^{k-1}(r^*(M) - 3) + 3) - 3 = 2^k(r^*(M) - 3) + 3.$$

Thus the result holds by induction. □

4 The classification of derived sequences

In this section, we classify derived sequences into finite, cyclic, and divergent types, and characterize each of them. By Theorem 3.6, we may focus on connected matroids.

Lemma 4.1. *Let (M, φ) be a connected $\mathbb{F}$-represented matroid. Then*

(i) $\delta M \cong U_{0,0}$ if and only if $M \cong U_{1,1}$;

(ii) $\delta M \cong U_{1,1}$ *if and only if* $M \cong U_{n,n+1}$ *for some* $n \geq 0$*; and*

(iii) $\delta M \cong U_{n,n+1}$ *for some* $n \geq 0$ *if and only if* $n = 2$ *and* $M \cong M(G)$ *where* G *is a theta graph.*

Proof. Parts (i) and (ii) are straightforward to check. Now assume that $\delta M \cong U_{n,n+1}$ for some $n \geq 0$. Then, by Corollary 2.2, $n+1 = r^*(M) + r^*(\delta M) = r^*(M) + 1$, so $r^*(M) = n$. As δM is simple, $n \geq 2$. By Corollary 3.3, $|\mathscr{C}(M)| \geq 3(r^*(M) - 1)$, so $r^*(M) + 1 \geq 3(r^*(M) - 1)$. Hence $r^*(M) \leq 2$. Thus $r^*(M) = 2$, so, by Corollary 2.10, for some $t \geq 3$, we have that $\mathrm{co}(M) \cong U_{t-2,t}$ and $\delta M \cong U_{2,t}$. We deduce that $n = 2$ and $t = 3$. Thus $M \cong M(G)$ where G is a theta graph. The converse is established in Corollary 2.10. □

Lemma 4.2. *Let* (M, φ) *be a connected* $\mathbb{F}$*-represented matroid. Then* δM *is not the cycle matroid of a theta graph.*

Proof. Suppose δM is the cycle matroid of a theta graph. Then $r^*(\delta M) = 2$. Thus, by Corollary 2.2, $|\mathscr{C}(M)| = r^*(M) + 2$. But, by Corollary 3.3, $|\mathscr{C}(M)| \geq 3(r^*(M) - 1)$. Hence $r^*(M) \leq 2$. By Lemma 4.1, $r^*(M) \notin \{0, 1\}$. Thus $r^*(M) = 2$, so, by Corollary 2.10, $\delta M \cong U_{2,n}$ for some $n \geq 3$, a contradiction. □

Lemma 4.3. *Let* M *be a* 6*-element rank-*3 *simple matroid. Then* $|\mathscr{C}(M)| \geq 7$ *with equality if and only if* $M \cong M(K_4)$*.*

Proof. If M is not 3-connected, then it is either the 2-sum of $U_{2,3}$ and $U_{2,5}$ or the parallel connection of $U_{2,3}$ and $U_{2,4}$. One easily checks that, in these cases, $|\mathscr{C}(M)| = 10$ and $|\mathscr{C}(M)| = 8$, respectively. If M is 3-connected, then M is isomorphic to $M(K_4)$, $\mathcal{W}^3$, Q_6, P_6, or $U_{3,6}$ (see, for example, [6, Corollary 12.2.19]) where each of the last four matroids is obtained from its predecessor by relaxing a circuit-hyperplane. Since each such relaxation eliminates one circuit but adds r^* new circuits, we deduce that $|\mathscr{C}(M)| \geq |\mathscr{C}(M(K_4)|$ with equality if and only if $M \cong M(K_4)$. As $M(K_4)$ has exactly seven circuits, the lemma follows □

The next theorem answers Longyear's Questions 3(a) and 3(c) for represented matroids over arbitrary fields. In particular, Theorem 1.1 is an immediate consequence of this result.

Theorem 4.4. *Let* (M, φ) *be a nonempty* $\mathbb{F}$*-represented matroid. If* $\delta^k M \cong M$ *for some* $k \geq 1$*, then* M *is a direct sum of matroids each of which is isomorphic to* $U_{2,4}$*.*

Proof. It suffices to show that if $\delta^k M \cong M$ for some $k \geq 1$ and M is connected, then $M \cong U_{2,4}$. Thus we assume that M is connected. We have $r^*(M) = r^*(\delta^k M)$, so, by Corollary 3.7,

$$r^*(M) = r^*(\delta^k M) \geq 2^k(r^*(M) - 3) + 3.$$

Hence $0 \geq (2^k - 1)(r^*(M) - 3)$, so $r^*(M) \leq 3$. By Lemma 4.1, $r^*(M) \notin \{0, 1\}$.

Suppose $r^*(M) = 3$. Then $r^*(\delta^k M) = 3$. As $r^*(\delta^i M) \geq 2r^*(\delta^{i-1} M) - 3$ for all i in $[k]$, it follows by induction that $r^*(\delta^i M) = 3$ for all such i. Since $r(\delta^k M) = r^*(\delta^{k-1} M)$, we deduce that $r(M) = 3$. Hence M is a 6-element rank-3 matroid having exactly six circuits. As $\delta^k M$ is simple, so is M and we have a contradiction to Lemma 4.3. Thus $r^*(M) \neq 3$, so $r^*(M) = 2$.

Now $r(\delta M) = r^*(M) = 2$. Since δM is simple, it follows that $\delta M \cong U_{2,n}$ for some $n \geq 3$. If $n = 3$, then δM, $\delta^2 M$, and $\delta^3 M$ are $U_{2,3}$, $U_{1,1}$, and $U_{0,0}$, so, for all $i \geq 1$, no $\delta^i M$ is isomorphic to M. If $n \geq 5$, then $r^*(\delta M) \geq 3 > r^*(M)$, so $k \geq 2$. Then

$$r^*(\delta^k M) \geq 2^{k-1}(r^*(\delta M) - 3) + 3 \geq 3 > r^*(M),$$

a contradiction. We conclude that $n = 4$, that is, $\delta M \cong U_{2,4}$. By Lemma 2.4, $\delta^k M \cong U_{2,4}$ for all $k \geq 1$, so $r(\delta^k M) = r^*(\delta^{k-1} M) = 2$. Hence $r(M) = 2$, so M is a simple 4-element matroid of rank two, that is, $M \cong U_{2,4}$. □

Theorem 4.5. *Let M be a connected represented matroid that is not isomorphic to $U_{0,0}$, $U_{1,1}$, a circuit, the cycle matroid of a theta graph, or a matroid whose cosimplification is $U_{2,4}$. Then, for all $k \geq 1$,*

$$r^*(\delta^{k+1} M) > r^*(\delta^k M).$$

Moreover, $r^*(\delta^k M) \geq 2^{k-1} + 3$ unless $M \cong U_{1,4}$. In the exceptional case, $r^*(M) = 3 = r^*(\delta M)$ and $r^*(\delta^2 M) = 4$, so $r^*(\delta^k M) \geq 2^{k-2} + 3$ for all $k \geq 2$.

Proof. We may assume that $r^*(M) \geq 2$ otherwise M is $U_{0,0}$, $U_{1,1}$, or a circuit. By Corollary 3.7,

$$r^*(\delta^{k+1} M) \geq 2r^*(\delta^k M) - 3. \tag{1}$$

Then, for $k \geq 1$,

$$r^*(\delta^k M) \geq 2^{k-1}(r^*(\delta M) - 3) + 3 \geq 2^{k-1}(2r^*(M) - 6) + 3.$$

Thus $r^*(\delta^k M) \geq 2^{k-1} + 3 \geq 4$ provided $r^*(\delta M) \geq 4$ or $r^*(M) \geq 4$. In each of these cases, the lemma holds since, by (1), $r^*(\delta^{k+1} M) > r^*(\delta^k M) + r^*(\delta^k M) - 4 \geq r^*(\delta^k M)$.

We may now assume that $r^*(\delta M) < 4$ and $r^*(M) < 4$. Suppose $r^*(M) = 3$. Then $r(\delta M) = 3$. As $r^*(\delta M) \leq 3$, we see that $|E(\delta M)| \leq 6$, that is, $|\mathscr{C}(M)| \leq 6$. Hence, by Lemma 4.3, $|E(M)| \leq 5$. Suppose $|E(M)| = 4$. Then $M \cong U_{1,4}$ and, by Lemma 2.5, $\delta M \cong M(K_4)$. Thus $r^*(\delta M) = 3$ and $r^*(\delta^2 M) = 4$. Therefore the first inequality in the lemma holds when $k = 1$. For $k \geq 2$, we have $r^*(\delta^k M) \geq 2^{k-2}(r^*(\delta^2 M) - 3) + 3$ and, using (1), we see that the lemma follows when $M \cong U_{1,4}$ since $r^*(\delta^2 M) = 4$.

Now suppose that $r^*(M) = 3$ and $|E(M)| = 5$. Then M is isomorphic to $U_{2,5}$ or the matroid that is obtained from $U_{2,4}$ by adding an element in parallel to an existing element. In each of these cases, $|\mathscr{C}(M)| \geq 8$, so $r^*(\delta M) \geq 5$, a contradiction.

We may now assume that $r^*(M) = 2$ and $r^*(\delta M) \leq 3$. Then $r(\delta M) = 2$ and, by Corollary 2.10, $\delta M \cong U_{2,4}$. Since M^* is a rank-2 connected matroid with exactly four rank-one flats, we deduce that $\mathrm{co}(M) \cong U_{2,4}$, a contradiction. □

We now answer Longyear's Question 4 for arbitrary represented matroids and thereby prove Theorem 1.2.

Theorem 4.6. *Let M be a represented matroid such that $\delta^k M \cong U_{0,0}$ for some $k \geq 0$. Then $\delta^3 M \cong U_{0,0}$ and each component of M is isomorphic to $U_{1,1}$, a circuit, or the cycle matroid of a theta graph.*

Proof. By Lemma 4.1, if each component of M is isomorphic to $U_{1,1}$, a circuit, or the cycle matroid of a theta graph, then $\delta^3 M \cong U_{0,0}$. Now let N be a component of M. By Lemma 2.4, if $\mathrm{co}(N) \cong U_{2,4}$, then $\delta^k M$ has $U_{2,4}$ as a component for all $k \geq 1$, so $\delta^k M \not\cong U_{0,0}$. We may now assume that N is not isomorphic to $U_{1,1}$, a circuit, the cycle matroid of a theta graph, or a matroid whose cosimplification is $U_{2,4}$ Then, for all $k \geq 1$, by Theorem 4.5, $r^*(\delta^k M) \geq 3$, so $\delta^k M \not\cong U_{0,0}$. □

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