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On Infinite Antichains of Matroids*

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Robertson and Seymour have shown that there is no infinite set of graphs in which no member is a minor of another. By contrast, it is well known that the class of all matroids does contains such infinite antichains. However, for many classes of matroids, even the class of binary matroids, it is not known whether or not the class contains an infinite antichain. In this paper, we examine a class of matroids of relatively simple structure: $\mathcal{M}_{a,b,c}$ consists of those matroids for which the deletion of some set of at most a elements and the contraction of some set of at most b elements results in a matroid in which every component has at most c elements. We determine precisely when $\mathcal{M}_{a,b,c}$ contains an infinite antichain. We also show that, among the matroids representable over a finite fixed field, there is no infinite antichain in a fixed $\mathcal{M}_{a,b,c}$; nor is there an infinite antichain when the circuit size is bounded.

1. INTRODUCTION

As the culmination of a long sequence of papers, Robertson and Seymour [8] proved that, in any infinite set of graphs, there is one that is a minor of another. However, it is well known that no matroid in the set $\{PG(2, p): p \text{ prime}\}$ is a minor of another matroid in this set. Thus the analogue of Robertson and Seymour's result fails for the class of all matroids. But the answer to the following question, which has recently become popular, is not known even for $q = 2$.

(1.1) *Question.* For a prime power q, is there an infinite set of $GF(q)$-representable matroids none of which is isomorphic to a minor of another?

Brylawski [3] noted that the answer to this question is negative if one replaces $GF(q)$ by any fixed infinite field F: for $n \geqslant 3$, let M_n be the rank-3 matroid on $2n$ elements, $e_1, f_1, e_2, f_2, ..., e_n, f_n$, for which the only non-spanning circuits are $\{e_1, f_1, e_2\}, \{e_2, f_2, e_3\}, ..., \{e_n, f_n, e_1\}$; every member of $\{M_n: n \geqslant 3\}$ is representable over F, but none is a minor of

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another. Since M_n^* is transversal for all n, the analogue of Robertson and Seymour's result also fails for the class of transversal matroids.

Bixby [1] and Seymour [9] proved that the excluded minors for the class of ternary matroids are the 5-point line, $U_{2,5}$; its dual, $U_{3,5}$; the Fano matroid, F_7; and its dual, F_7^*. This motivated Brylawski [3] to ask whether the class of matroids having no minor isomorphic to $U_{2,5}$ or $U_{3,5}$ contains an infinite antichain, an infinite set of matroids none of which is a minor of another. Kahn (in [6, p. 471]) answered this question affirmatively.

Apart from the work noted above, little appears to have been done to identify which classes of matroids contain infinite antichains. In this paper, we shall consider matroids with a relatively simple structure; roughly speaking, each is only a few elements away from being a direct sum of small matroids. Our main result will determine precisely when a class of such matroids contains an infinite antichain.

The terminology used here for graphs and matroids will follow [6]. A binary relation $\leqslant$ on a set Q is a *quasi-order* if it is reflexive and transitive. A sequence $q_1, q_2, \ldots$ of members of Q is *bad* (with respect to $\leqslant$) if there are no indices i and j such that $i<j$ and $q_i \leqslant q_j$. We call $(Q, \leqslant)$ a *well-quasi-order* (or a *wqo* for brevity) if no infinite sequence of members of Q is bad (with respect to $\leqslant$).

The proof that finite graphs are well-quasi-ordered by the minor relation is extremely long and complicated, but the proof for a subclass of graphs that exhibit a certain simple structure (see tree-width in [7]) is somewhat easier. Thus it seems reasonable to seek classes of matroids that are well-quasi-ordered among the matroids that also have relatively simple structure. For non-negative integers a, b, and c, let $\mathcal{M}_{a,b,c}$ consist of those matroids M for which there are disjoint subsets X and Y of $E(M)$ such that $|X| \leqslant a$, $|Y| \leqslant b$, and every connected component of $M\backslash X/Y$ has at most c elements. The main result of the first part of the paper is the following:

(1.2) THEOREM. *The class $\mathcal{M}_{a,b,c}$ is well-quasi-ordered with respect to the minor relation if and only if at least one of the following holds*:

(i) $\min\{a, b\} = 0$ *and* $\max\{a, b\} \leqslant 1$;

(ii) $\min\{a, b\} = 0$, $\max\{a, b\} = 2$, *and* $c = 2$; *and*

(iii) $\max\{a, b\} \leqslant 2$ *and* $c \leqslant 1$.

The proof of (1.2) is given in Section 2. In Section 3, we shall show that, among the matroids representable over a fixed finite field, there is no infinite antichain in a fixed $\mathcal{M}_{a,b,c}$; nor is there an infinite antichain when the circuit size is bounded. The remainder of this section will present some auxiliary notation and results about well-quasi-orders and matroids that will be used in Sections 2 and 3.

Let $Q^{<\omega}$ be the set of all finite sequences of members of Q and let $\leqslant^{<\omega}$ be the binary relation on $Q^{<\omega}$ defined as follows. For any two members $p=(p_1, p_2, ..., p_m)$ and $q=(q_1, q_2, ..., q_n)$ of $Q^{<}$, write $p \leqslant^{<\omega} q$ if there are indices $i_1, i_2, ..., i_m$ with $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$, such that $p_j \leqslant q_{i_j}$ for all $j \in \{1, 2, ..., m\}$. The following fundamental result is due to Higman [5].

(1.3) THEOREM. *If $(Q, \leqslant)$ is a wqo, then so is $(Q^{<\omega}, \leqslant^{<\omega})$.*

Let $(Q_1, \leqslant_1), (Q_2, \leqslant_2), ..., (Q_k, \leqslant_k)$ be quasi-orders and let $Q = Q_1 \times Q_2 \times \cdots \times Q_k$. Then a binary relation $\leqslant$ can be defined naturally on Q as follows: For any two members $p=(p_1, p_2, ..., p_k)$ and $q=(q_1, q_2, ..., q_k)$ of Q, let $p \leqslant q$ if $p_j \leqslant_j q_j$ for all $j \in \{1, 2, ..., k\}$. We shall refer to $\leqslant$ as $\leqslant_1 \times \leqslant_2 \times \cdots \times \leqslant_k$. The next result is a well-known and straightforward consequence of (1.3).

(1.4) COROLLARY. *If $(Q_i, \leqslant_i)$ is a wqo for all $i \in \{1, 2, ..., k\}$, then $(Q_1 \times Q_2 \times \cdots \times Q_k, \leqslant_1 \times \leqslant_2 \times \cdots \times \leqslant_k)$ is also a wqo.*

The following is another useful fact about well-quasi-orders.

(1.5) LEMMA. *Let $(Q_1 \cup Q_2 \cup \cdots \cup Q_k, \leqslant)$ be a quasi-ordering and let each $(Q_i, \leqslant)$ be a wqo. Then $(Q_1 \cup Q_2 \cup \cdots \cup Q_k, \leqslant)$ is a wqo.*

In the proof of (1.2.), we will use the concepts of a labeled matroid and of a labeled minor. Let M be a matroid and let Q be a non-empty set. A *Q-labeling* of M is a mapping f from $E(M)$ to Q. We call the pair (M, f) a *Q-labeled matroid.* If $\leqslant$ is a quasi-order on Q, then, for any two Q-labeled matroids (M_1, f_1) and (M_2, f_2), we define $(M_1, f_1) \leqslant_{\mathrm{m}} (M_2, f_2)$ if there are disjoint subsets X and Y of $E(M_2)$ and an isomoprhism σ from M_1 to $M_2 \backslash X/Y$ such that $f_1(e) \leqslant f_2(\sigma(e))$ for all the elements e of $E(M_1)$. If $\mathcal{M}$ is a class of matroids, we shall denote by $\mathcal{M}_0$ the class of connected members of $\mathcal{M}$. We also denote by $\mathcal{M}(Q)$ the class of all Q-labeled matroids (M, f) such that M is in $\mathcal{M}$. Evidently if $\leqslant$ is a quasi-order on Q, then $\leqslant_{\mathrm{m}}$ is a quasi-order on $\mathcal{M}(Q)$. The following is a straightforward consequence of (1.3).

(1.6) COROLLARY. *Let $(Q, \leqslant)$ be a wqo and let $\mathcal{M}$ be a class of matroids with the property that if M is in $\mathcal{M}$, then all the connected components of M are also in $\mathcal{M}$. Then $(\mathcal{M}(Q), \leqslant_{\mathrm{m}})$ is a wqo if and only if $(\mathcal{M}_0(Q), \leqslant_{\mathrm{m}})$.*

Let $\mathcal{M}^*$ be the class of duals of members of $\mathcal{M}$. Then the following is clear.

(1.7) LEMMA. *Let* $(Q, \leqslant)$ *be a wqo. Then* $(\mathcal{M}(Q), \leqslant_m)$ *is a wqo if and only if* $(\mathcal{M}^*(Q), \leqslant_m)$ *is.*

Let $\tilde{M}$ denote the simple matroid associated with a matroid M; let $\tilde{\mathcal{M}}$ denote the class of simple matroids associated with members of $\mathcal{M}$; and let $\bar{\mathcal{M}}$ be the class of matroids M such that $\tilde{M}$ is in $\tilde{\mathcal{M}}$.

(1.8) LEMMA. *Let* $(Q, \leqslant)$ *be a wqo and let* $\preccurlyeq$ *be* $\leqslant^{<\omega}$. *If* $(\tilde{\mathcal{M}}(Q^{<\omega}), \preccurlyeq_m)$ *is a wqo, then so is* $(\bar{\mathcal{M}}(Q), \leqslant_m)$.

Proof. Let $(M_1, f_1), (M_2, f_2), ...$ be an infinite sequence of members of $\bar{\mathcal{M}}(Q)$. We shall prove that there are indices i and j such that $i<j$ and $(M_i, f_i) \leqslant_m (M_j, f_j)$. For each M_i, if $e_1, e_2, ..., e_m$ are the loops of M_i, then we define p_i to be sequence $f_i(e_1), f_i(e_2), ..., f_i(e_m)$. Next we define a $Q^{<\omega}$-labeling $\tilde{f}_i$ of $\tilde{M}_i$, where we view $\tilde{M}_i$ here as a matroid on the set of parallel classes of M_i. If X is an element of $\tilde{M}_i$, let $x_1, x_2, ..., x_n$ be an arbitrary ordering of the members of the parallel class X and define $\tilde{f}_i(X)$ to be the sequence $f_i(x_1), f_i(x_2), ..., f_i(x_n)$. Clearly, p_i is in $Q^{<\omega}$ and $(\tilde{M}_i, \tilde{f}_i)$ is in $\tilde{\mathcal{M}}(Q^{<\omega})$. Let q_i denote $(p_i, (\tilde{M}_i, \tilde{f}_i))$. Then q_i is in $Q^{<\omega} \times \tilde{\mathcal{M}}(Q^{<\omega})$. We first observe from (1.3) that $(Q^{<\omega}, \preccurlyeq)$ is a wqo and then we conclude from (1.4) that $(Q^{<\omega} \times \tilde{\mathcal{M}}(Q^{<\omega}), \preccurlyeq \times \preccurlyeq_m)$ is a wqo. Thus, there are indices i and j with $i<j$ such that $p_i \preccurlyeq p_j$ and $(\tilde{M}_i, \tilde{f}_i) \preccurlyeq_m (\tilde{M}_j, \tilde{f}_j)$. From the definitions of $p_i, p_j, \tilde{f}_i$, and $\tilde{f}_j$, it is easy to verify that $(M_i, f_i) \leqslant_m (M_j, f_j)$, as required. ∎

Let k be an integer exceeding one and let $M_1, M_2, ..., M_k$ be connected matroids such that each of them has at least two elements and $E(M_i) \cap E(M_j) = \{e\}$ for all $1 \leqslant i < j \leqslant k$. Then we define the *parallel sum* $PS(M_1, M_2, ..., M_k)$ of $M_1, M_2, ..., M_k$ with respect to the *basepoint* e as the matroid obtained from the parallel connection [2] of $M_1, M_2, ..., M_k$ by deleting e. Observe that if each of M_1 and M_2 has at least three elements, then $PS(M_1, M_2)$ is exactly the 2-sum of M_1 and M_2 [10]. It is not difficult to show [2, 3] that parallel sum has the following properties.

(1.9) LEMMA. (i) *If* $1 \leqslant i \leqslant k$, *and* X *and* Y *are disjoint subsets of some* $E(M_i) - \{e\}$ *such that* $M_i \backslash X/Y$ *is connected and has at least two elements, then*

$$PS(M_1, M_2, ..., M_{i-1}, M_i \backslash X/Y, M_{i+1}, ..., M_k) = PS(M_1, M_2, ..., M_k) \backslash X/Y.$$

(ii) *If* $k \geqslant 3$ *and* $1 \leqslant i \leqslant k$, *then*

$$PS(M_1, M_2, ..., M_{i-1}, M_{i+1}, ..., M_k) = PS(M_1, M_2, ..., M_k) \backslash (E(M_i) - \{e\}).$$

If $\mathcal{M}$ is a class of matroids, we shall denote by $PS(\mathcal{M})$ the class of matroids M such that either M is in $\mathcal{M}$, or M equals $PS(M_1, M_2, ..., M_k)$ for some connected matroids $M_1, M_2, ..., M_k$ in $\mathcal{M}$.

(1.10) LEMMA. *If $(\mathcal{M}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$, then $(PS(\mathcal{M})(Q), \leqslant_{\mathrm{m}})$ is also a wqo for every wqo $(Q, \leqslant)$.*

Proof. Let $\mathcal{N} = PS(\mathcal{M}) - \mathcal{M}$ and let $(Q, \leqslant)$ be a wqo. It follows from (1.5) that we need only show that $(\mathcal{N}(Q), \leqslant_{\mathrm{m}})$ is a wqo.

Let S be the set that is obtained from Q by adjoining a new element q'. Clearly, the relation $\leqslant$ on Q can be extended to a relation on S by insisting that q' is incomparable with all the elements of Q. Evidently, $(S, \leqslant)$ is a wqo, where we have used the same symbol for the extended relation as for the original one. Thus we conclude from (1.3) and the hypothesis of (1.10) that $((\mathcal{M}(S))^{<\omega}, \leqslant^{<\omega})$ is a wqo. Now suppose that (M, f) is a member of $\mathcal{N}(Q)$. It follows that M equals $PS(M_1, M_2, ..., M_k)$ for some matroids $M_1, M_2, ..., M_k$ in $\mathcal{M}$. Let e be the basepoint of the parallel sum. Then, for every $i \in \{1, 2, ..., k\}$, we define an S-labeling f_i of M_i by letting $f_i(e) = q'$ and letting $f_i(x) = f(x)$ for all elements x of $E(M_i) - \{e\}$. Let $h(M, f)$ be the sequence of S-labeled matroids (M_1, f_1), (M_2, f_2), ..., (M_k, f_k). Then $h(M, f)$ belongs to $(\mathcal{M}(S))^{<\omega}$. Moreover, for every member (N, g) of $\mathcal{N}(Q)$, it is not difficult to verify by (1.9) that if $h(M, f) \leqslant^{<\omega} h(N, g)$, then $(M, f) \leqslant_{\mathrm{m}} (N, g)$. Therefore the result follows from the observation that $((\mathcal{M}(S))^{<\omega}, \leqslant^{<\omega})$ is a wqo. ∎

2. MATROIDS THAT BREAK INTO SMALL PIECES

This section will be devoted to proving (1.2). We begin with the following simple observations. Suppose that $a' \leqslant a$, that $b' \leqslant b$, that $c' \leqslant c$, and that $\mathcal{M}_{a, b, c}$ is well-quasi-ordered by the minor relation. Then $\mathcal{M}_{a', b', c'}$ is also well-quasi-ordered by the minor relation. Moreover, (1.7) implies that $\mathcal{M}_{b, a, c}$ is well-quasi-ordered since $\mathcal{M}_{b, a, c} = (\mathcal{M}_{a, b, c})^*$. Hence, to verify (1.2), it suffices to prove the following statements, and these proofs will occupy the remainder of this section.

(2.1) *$\mathcal{M}_{0, 3, 1}$ is not well-quasi-ordered by the minor relation.*

(2.2) *$\mathcal{M}_{1, 1, 2}$ is not well-quasi-ordered by the minor relation.*

(2.3) *$\mathcal{M}_{0, 2, 3}$ is not well-quasi-ordered by the minor relation.*

(2.4) *$(\mathcal{M}_{0, 1, c}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$ and every non-negative integer c.*

(2.5). *$(\mathcal{M}_{2, 0, 2}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$.*

(2.6). *$(\mathcal{M}_{2, 2, 1}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$.*

Observe that, to prove (1.2), we only need unlabeled versions of (2.4)–(2.6). These clearly follows from the statements above. We remark here that, as steps towards proving (2.5) and (2.6), we shall also show that each of $(\mathcal{M}_{2,0,1}(Q), \leqslant_m)$ and $(\mathcal{M}_{2,1,1}(Q), \leqslant_m)$ is a wqo.

Proof of (2.1). As noted in the Introduction, it is well known that the class of projective planes $PG(2, p)$, where p is a prime, is not well-quasi-ordered by the minor relation. Since every projective plane has rank 3 and so is in $\mathcal{M}_{0,3,1}$, statement (2.1) follows immediately. ∎

Proof of (2.2). We shall construct a bad infinite sequence of matroids in $\mathcal{M}_{1,1,2}$ by modifying an example of Kahn (in [6, p. 471]). Let n be a positive integer and let $e_i, e_2, ..., e_{2n+1}$ be the unit vectors in $V(2n+1, 2)$, the vector space over $GF(2)$ of dimension $2n+1$. Let $f_{2n+1} = \sum_{i=1}^{2n+1} e_i$ and, for $i = 1, 2, ..., 2n$, let $f_i = f_{2n+1} - e_i$. Let $E_n = \{e_1, f_1, e_2, f_2, ..., e_{2n+1}, f_{2n+1}\}$ and let M_n be the vector matroid on E_n. Let $C_1 = \{e_1, e_{2n+1}, f_2, f_3, ..., f_{2n}\}$ and $C_2 = \{f_1, e_2, e_3, ..., e_{2n+1}\}$. Then it is clear that both C_1 and C_2 are circuit-hyperplanes of M_n. Let N_n be the matroid obtained from M_n by relaxing these two circuit-hyperplanes. We shall prove that $N_1, N_2, ...$ is a bad sequence (with respect to the minor relation). Note that all the connected components of $N_n \backslash e_{2n+1}/f_{2n+1}$ are of size two. Thus each of the matroids N_n is a member of $\mathcal{M}_{1,1,2}$.

Observe that if e is an element in $E_n - \{e_{2n+1}\}$ and if C is the element of $\{C_1, C_2\}$ that does not contain e, then $N_n \backslash e$ is the matroid that is obtained from $M_n \backslash e$ by relaxing C. It follows from [6, p. 344, Ex. 2] that $N_n \backslash e$ is non-binary. However, since $N_n \backslash e_{2n+1}$ equals $M_n \backslash e_{2n+1}$, the matroid $N_n \backslash e_{2n+1}$ is binary. Now suppose that, for some indices m and n with $0 < m < n$, there are disjoint subsets X and Y of $E(N_n)$ with $N_m = N_n \backslash X/Y$. It is clear that we may assume that X and Y are independent in N_n^* and N_n, respectively. Thus

$$|Y| = r(N_n) - r(N_m) = 2(n-m) = r(N_n^*) - r(N_m^*) = |X|.$$

Since N_m is non-binary, X does not meet both C_1 and C_2, for otherwise $N_n \backslash X = M_n \backslash X$, and the latter is certainly binary. Since $|E(N_n) - (C_1 \cup C_2)| = 1$ but $|X| = 2(n-m) \geqslant 2$, X meets $C_1 \cup C_2$. Thus X meets C_i for some $i \in \{1, 2\}$. Let $\{j\} = \{1, 2\} - \{i\}$. Then $X \cup \{e\}$ meets both C_1 and C_2 for every $e \in C_j - Y$. Consequently, $N_m \backslash e$, which equals $N_n \backslash (X \cup \{e\})/Y$, is binary for every $e \in C_j - Y$. But $|C_j - Y| \geqslant |C_j| - |Y| = 2n + 1 - 2(n-m) = 2m + 1 > 1$, contradicting the fact that there is only one element e in $E(N_m)$ for which $N_m \backslash e$ is binary. ∎

Proof of (2.3). We begin with a slightly informal description of a sequence of matroids $M_5, M_6, ...$ from which a bad sequence in $\mathcal{M}_{0,2,3}$ will be derived. This will be followed by a precise definition of the matroids

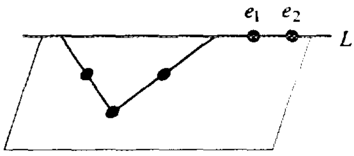

FIGURE 1

involved. Let r be an integer exceeding four and take $r-2$ five-element rank-3 matroids each isomophic to the matroid that is represented geometrically in Fig. 1. Now join these rank-3 matroids together along the line L somewhat like the pages of a book being joined along its spine with each new page adding one to the rank. Figure 2 indicates how these $r-2$ five-element matroids are fitted together, with the solid dots indicating the points of M_r. The labeling on these points makes the definition of M_r precise; $\{e_1, e_2, ..., e_r\}$ is the natural basis of the vector space $V(r, \mathbb{Q})$ over the rational numbers $\mathbb{Q}$; for all k in $\{3, 4, ..., r-1\}$, let

$$X_k = \{e_1 + 2^{k-2}e_2, e_1 + 2^{k-2}e_2 + 2^{2-k}e_k, e_k, \\ e_1 + 2^{k-1}e_2 + 2^{1-k}e_k, e_1 + 2^{k-1}e_2\}$$

and let

$$X_r = \{e_1 + 2^{r-2}e_2, e_1 + 2^{r-2}e_2 + 2^{2-r}e_r, e_r, e_1 + 2e_2 + 2^{-1}e_r, e_1 + 2e_2\}.$$

Then M_r is the vector matroid on the set $B \cup (\bigcup_{k=3}^{r} X_k)$. Hence, we can think of M_r as a restriction of $PG(r-1, \mathbb{Q})$. Clearly, for all k in $\{3, 4, ..., r\}$, the matroid $M_r | (X_k \cup \{e_1, e_2\})$ is isomorphic to the rank-3 matroid in Fig. 1. Now let $N_r = M_r \backslash (L - \{e_1, e_2\})$, where L is the line spanned by e_1 and e_2. We shall show that $N_5, N_6, ...$ is a bad sequence (with respect to the minor relation) in $\mathcal{M}_{0,2,3}$.

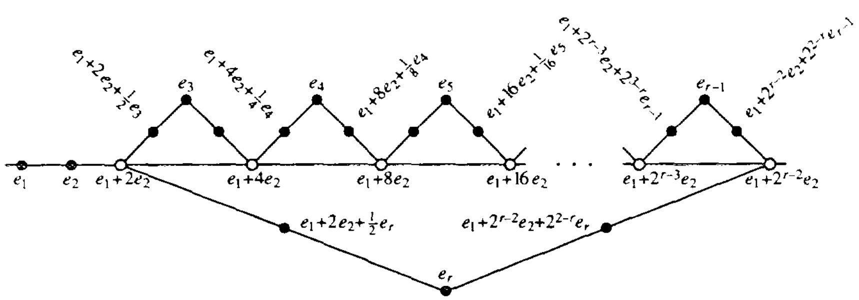

FIGURE 2

For k in $\{3, 4, ..., r\}$, let $Y_k = X_k - L$. Then Y_k is a three-element cocircuit of N_r. Moreover, in $N_r/\{e_1, e_2\}$, each of $Y_3, Y_4, ..., Y_r$ is the ground set of a component. Thus N_r is certainly in $\mathscr{M}_{0,2,3}$ for each r.

To show that $N_5, N_6, ...$ is a bad sequence, we shall look more closely at the structure of N_r. First we note that it is straightforward to prove that N_r/z is 3-connected unless z is in $\{e_1, e_2\}$. From this it follows immediately that

(1) *$N_r/\{x, y\}$ is disconnected if and only if $\{x, y\} = \{e_1, e_2\}$.*

Next we examine the circuits of N_r and prove the following.

(2) *If C is a circuit of N_r, then either*

(i) *C meets at most two of the sets $Y_3, Y_4, ..., Y_r$ and $|C| \leqslant 5$; or*

(ii) *C has exactly two elements in common with three of $Y_3, Y_4, ..., Y_r$, and $|C| = 6$.*

Since each Y_i is a cocircuit of N_r, if C meets Y_i, it must contain at least two elements of this set. Moreover, the union of m of the sets $Y_3, Y_4, ..., Y_r$ is easily seen to have rank equal to $m+2$. Thus if C meets m of the sets $Y_3, Y_4, ..., Y_r$, then $r(C) \leqslant m+2$. But $r(C) = |C| - 1 \geqslant 2m - 1$. Hence $m + 2 \geqslant 2m - 1$, so $m \leqslant 3$. Moreover, if equality holds in the last inequality, it must hold in the last four inequalities and (ii) must hold. On the other hand, if $m < 3$, then (i) must hold.

(3) *Suppose that i and j are distinct members of $\{3, 4, ..., r\}$ and $i < j$. If $j = i+1$ or $(i, j) = (3, r)$, then $N_r | (\{e_1, e_2\} \cup Y_i \cup Y_j)$ is isomorphic to the matroid that is represented geometrically in Fig. 3(a); otherwise $N_r | (\{e_1, e_2\} \cup Y_i \cup Y_j)$ is isomorphic to the matroid in Fig. 3(b).*

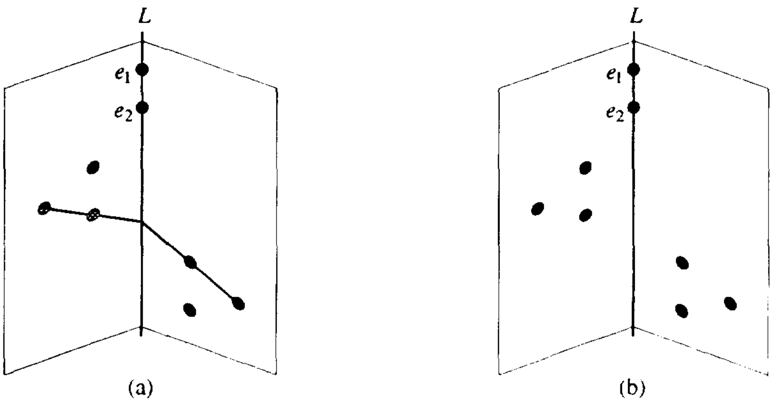

FIGURE 3

To show this, we first note that, in $PG(r-1, \mathbb{Q})$, the line through the two points $e_1 + 2^{k-2}e_2 + 2^{2-k}e_k$ and $e_1 + 2^{k-1}e_2 + 2^{1-k}e_k$ of $Y_k - \{e_k\}$ meets the line L in the point $e_1 + 3 \cdot 2^{k-2}e_2$. Similarly, the line through the points $e_1 + 2^{r-2}e_2 + 2^{2-r}e_r$ and $e_1 + 2e_2 + 2^{-1}e_r$ of $Y_r - \{e_r\}$ meets L in the point $e_1 + 2(2^{r-3}+1)\, e_2$. Since the points

$$e_1, e_2, e_1 + 2e_2, e_1 + 4e_2, ..., e_1 + 2^{r-2}e_2,$$
$$e_1 + 3 \cdot 2e_2, e_1 + 3 \cdot 4e_2, ..., e_1 + 3 \cdot 2^{r-2}e_2, e_1 + 2(2^{r-3}+1)\, e_2$$

are distinct, $N_r | (\{e_1, e_2\} \cup Y_i \cup Y_j)$ has no 3- or 4-circuits other than the 4-circuits implied by the appropriate part of Fig. 3.

The next observation is a straightforward consequence of (2) and (3).

(4) *$N_r \backslash \{e_1, e_2\}$ has exactly $r-2$ four-element circuits. Moreover, these circuits may be labeled $C_1, C_2, ..., C_{r-2}$ so that if i and j are distinct elements of $\{1, 2, ..., r-2\}$, then $|C_i \cap C_j|$ is 1 when $j = i \pm 1$ and is 0 otherwise, where all subscripts are read modulo $r-2$.*

Now suppose that, for some $t < r$, the matroid N_t is isomorphic to a minor of N_r. Then $E(N_r)$ has disjoint subsets X and Y such that $N_t \cong N_r \backslash X/Y$, where $|Y| = r-t$ and $|X| = 2(r-t)$.

(5) *Y does not meet $\{e_1, e_2\}$ and contains at most one element of each Y_i.*

To see this, we note that, on contracting one of e_1 and e_2, or two elements from some Y_i, we obtain a matroid N having an element e such that no component of N/e has more than three elements. The 3-connected matroid N_t cannot be isomophic to a minor of such a matroid N.

By (5), Y contains exactly one element from each of $r-t$ of the sets $Y_3, Y_4, ..., Y_r$. In N_r/Y, the line L spanned by $\{e_1, e_2\}$ contains at least $2|Y|+2$ elements. Since N_t has no circuits of size less than 4 and $N_r/Y \backslash X \cong N_t$, the line L contains at most two elements in $N_r \backslash X/Y$. But $|X| = 2|Y|$, so L contains exactly two elements, say x and y, in $N_r \backslash X/Y$. Moreover, the contraction of $\{x, y\}$ from $N_r \backslash X/Y$ produces a disconnected matroid. Thus, by (1), in the isomorphism between $N_r \backslash X/Y$ and N_t, the elements x and y of $N_r \backslash X/Y$ must correspond to the elements e_1 and e_2 of N_t. Therefore, by (4), $N_r \backslash X/Y \backslash \{x, y\}$ has exactly $t-2$ four-element circuits, which can be labeled $D_1, D_2, ..., D_{t-2}$ so that $|D_i \cap D_{i+1}| = 1$ for all i in $\{1, 2, ..., t-2\}$, where subscripts are read modulo $t-2$. But the only 4-circuits of N_r avoiding L are $C_1, C_2, ..., C_{r-2}$. If C is a 4-circuit of N_r/Y that is not a 4-circuit of N_r, then it follows by (2) that, for some subset Y' of Y with $1 \leqslant |Y'| \leqslant 2$, the set $C \cup Y'$ is a circuit of N_r. But, by (2) and (3), the 4-circuit C of N_r/Y must meet L so C is not a circuit of $N_r \backslash X/Y \backslash \{x, y\}$. We conclude that $N_r \backslash X/Y \backslash \{x, y\}$ does not have the required collection of 4-circuits to be isomorphic to N_t, and this contradiction completes the proof of (2.3). ∎

Proof of (2.4). It is clear from (1.6) that we only need to consider the class $\mathcal{M}$ of connected matroids in $\mathcal{M}_{0,1,c}$. The case when $c=0$ is trivial and thus we assume that $c \geq 1$. Let $\mathcal{N}$ be the class of connected matroids having at most $c+1$ elements. Then, for each M in $\mathcal{M}-\mathcal{N}$, we conclude from the definition of $\mathcal{M}_{0,1,c}$ that there is an element e of M such that each of the connected components $M_1, M_2, ..., M_k$ of M/e has at most c elements. Let $N_i = M|(E(M_i) \cup \{e\})$ for all i in $\{1, 2, ..., k\}$ and let $N_0 \cong U_{1,2}$ such that $E(N_0) \cap E(M) = \{e\}$. It is clear that each N_i belongs to $\mathcal{N}$ and that $M \cong PS(N_0, N_1, ..., N_k)$. Since $\mathcal{M}$ is closed under isomorphism, it follows that $\mathcal{M} \subseteq PS(\mathcal{N})$. Observe that $\mathcal{N}$ is the union of finitely many classes of matroids each of which consists of pairwise isomorphic matroids. Thus we conclude by (1.5) and (1.4) that $(\mathcal{N}(Q), \leq_m)$ is a wqo for every wqo $(Q, \leq)$. Therefore the result follows from (1.10). ∎

Remark. It is not difficult to see from the above proof that, actually, $\mathcal{M}_{0,1,c}(Q)$ is well-quasi-ordered by the deletion-minor relation and $\mathcal{M}_{1,0,c}(Q)$ is well-quasi-ordered by the contraction-minor relation.

Proof of (2.5). Let $(Q, \leq)$ be a wqo. It follows from (1.6) and (1.8) that we need only consider the class $\mathcal{M}$ of simple connected members of $\mathcal{M}_{2,0,2}$. Observe that there is no simple connected matroid with exactly two elements. Thus $\mathcal{M} \subseteq \mathcal{M}_{2,0,1}$. Hence, from (1.7), it suffices to show that $(\mathcal{M}_{0,2,1}(Q), \leq_m)$ is a wqo. Let $\mathcal{N}$ be the class of simple connected members of $\mathcal{M}_{0,2,1}$. By (1.6) and (1.8) again, we need only show that $(\mathcal{N}(Q), \leq_m)$ is a wqo. But this is clear because it is not difficult to see that $\mathcal{N} = \{U_{2,n} : n \geq 3\}$. ∎

The next two results will be used in proving (2.6).

(2.7) LEMMA. *If* $(Q, \leq)$ *is a wqo, then so is* $(\mathcal{M}_{2,1,1}(Q), \leq_m)$.

Proof. From (1.6) and (1.8), we need only consider the class $\mathcal{M}$ of simple connected matroids in $\mathcal{M}_{2,1,1}$. We shall prove that $\mathcal{M} \subseteq \mathcal{M}_{2,0,1}$ for the result will then follow from (2.5). Let M be a member of $\mathcal{M}$. Then there are disjoint subsets X and Y of $E(M)$ such that all the connected components of $M\backslash X/Y$ have just one element, $|X| \leq 2$, and $|Y| \leq 1$. If $|Y| = 0$, then M is in $\mathcal{M}_{2,0,1}$, as we wanted. If $Y = \{y\}$, let N be the connected component of $M\backslash X$ that contains y. Since all the connected components of N/y have a single element, we must have $r(N) \leq 1$. Therefore, since M is simple, we conclude that $E(N) = \{y\}$. It follows that all the connected components of $M\backslash X$ have a single element and so M is in $\mathcal{M}_{2,0,1}$. Hence $\mathcal{M} \subseteq \mathcal{M}_{2,0,1}$, as required. ∎

(2.8) LEMMA. *Let $\mathcal{M}$ be the class of connected matroids that have a cobasis of rank two. Then $(\mathcal{M}(Q), \leqslant_m)$ is a wqo for every wqo $(Q, \leqslant)$.*

Proof. This proof is the core of the proof of (2.6). Since it is long, certain significant steps have been identified as statements (1)–(10).

For every member M of $\mathcal{M}$, we call a partition (A, B) of $E(M)$ *good* if A is a cobasis of M with $r(A) = 2$. Since, in such a good partition, B is a basis of M and $r^*(B) = |B| - r(M) + r(A) = r(A) = 2$, it follows that

(1) *if (A, B) is a good partition of a member M of $\mathcal{M}$, then $\mathcal{M}^*$ is a member of $\mathcal{M}$ with a good partition (B, A).*

We also have the following observation.

(2) *Let M be a member of $\mathcal{M} - \mathcal{M}_{1,2,1}$ and let (A, B) be a good partition of M. If x and y are parallel in M, then both elements are in A.*

Suppose that (2) does not hold. Then B meets $\{x, y\}$. But B is independent, so it must contain exactly one of x and y, say x. It follows, since $r^*(B) = 2 = r(A)$, that the sets $B - \{x\}$ and $A - \{y\}$ contain elements x' and y', respectively, such that $\{x, x'\}$ and $\{y, y'\}$ are independent in M^* and M, respectively. Thus $\{x, x'\}$ spans B in M^*, and $\{y, y'\}$ spans A in M. Therefore, every connected component of $M\backslash\{x, x'\}/\{y, y'\}$ is either a loop or a coloop. Since x is a loop of M/y and hence is a loop of $M\backslash x'/\{y, y'\}$, we conclude that every connected component of $M\backslash x'/\{y, y'\}$ is either a loop or a coloop. Thus M belongs to $\mathcal{M}_{1,2,1}$. This contradiction implies that (2) must hold.

Let $\mathcal{N}_1$ be the class of members of $\mathcal{M}$ that are both simple and cosimple. For a matroid M, let M^d denote the simple matroid that is associated with the dual of $\tilde{M}$. We show next that

(3) *if M is in $\mathcal{M} - \mathcal{M}_{1,2,1}$, then either $\tilde{M}$ or M^d is in $\mathcal{N}_1 \cup \mathcal{M}_{2,1,1}$.*

Let M be a member of $\mathcal{M} - \mathcal{M}_{1,2,1}$ and let (A_1, B_1) be a good partition of M. We shall view $\tilde{M}$ as a deletion minor of M rather than as a matroid on the set of parallel classes of M. Then $E(\tilde{M})$ is a subset of $E(M)$. It follows from (2) that B_1 is a subset of $E(\tilde{M})$. Let $A_2 = E(\tilde{M}) - B_1$. It is easy to verify that (A_2, B_1) is a good partition of $\tilde{M}$ and thus that $\tilde{M}$ is a member of $\mathcal{M}$. If $\tilde{M} \in \mathcal{N}_1$, then (3) holds. If $\tilde{M} \notin \mathcal{N}_1$, then it is clear that $(\tilde{M})^*$ is not simple.

Let N denote $(\tilde{M})^*$. Then $\tilde{N} = M^d$. Moreover, by (1), N is a member of $\mathcal{M}$ with a good partition (B_1, A_2). If $N \in \mathcal{M}_{1,2,1}$, then $\tilde{M} \in \mathcal{M}_{2,1,1}$ and thus (3) holds. Hence we may assume that $N \notin \mathcal{M}_{1,2,1}$. Then, from (2), we conclude that every non-trivial parallel class of N is a subset of B_1. Let $B_2 = E(\tilde{N}) - A_2$. It is clear that $\tilde{N}$ is a member of $\mathcal{M}$ with a good partition (B_2, A_2). If $\tilde{N}$ is cosimple, then $\tilde{N}$, which is M^d, is in $\mathcal{N}_1$ and (3) holds.

We may now suppose that $\tilde{N}$ is not cosimple. Then, since $N \notin \mathcal{M}_{1,2,1}$, the rank of N exceeds one. Thus, as N is connected, $\tilde{N}$ has no coloops. Hence there must be two elements x and y that are in series in $\tilde{N}$. Since x and y are not in series in N, we conclude that at least one of x and y, say x, is contained in a non-trivial parallel class of N. It follows, by (2), that x is in B_1. Hence x is in B_2. However, by (1), $(\tilde{N})^*$ is a member of $\mathcal{M}$ with a good partition (A_2, B_2). Therefore, we conclude, by (2), that $(\tilde{N})^* \in \mathcal{M}_{1,2,1}$ and so $\tilde{N} \in \mathcal{M}_{2,1,1}$. Thus the proof of (3) is completed.

Let $\mathcal{N}$ be the class of members of $\mathcal{N}_1$ whose rank and corank both exceed two. Clearly $\mathcal{N}_0 - \mathcal{N} \subseteq \mathcal{M}_{2,0,1} \cup \mathcal{M}_{0,2,1}$. Thus (3) can be restated as follows.

(4) *For every M in $\mathcal{M}$, either $\tilde{M}$ or M^d is in $\mathcal{N} \cup \mathcal{M}_{1,2,1} \cup \mathcal{M}_{2,1,1}$.*

Let $\mathcal{L} = \mathcal{N} \cup \mathcal{M}_{1,2,1} \cup \mathcal{M}_{2,1,1}$. From (4), it follows that $\mathcal{M}$ is contained in the union of $\overline{\mathcal{L}}$ and $\overline{(\mathcal{L})^*}$. Thus, by (1.5), (1.7), (1.8), and (2.7), we need only show that $(\mathcal{N}(Q), \leqslant_m)$ is a wqo.

Next we describe the set of circuits of a member M of $\mathcal{N}$. Let (A, B) be a good partition of M. Since B is a basis of M, the rank of M is $|B|$, and so M is uniquely determined by its set of non-spanning circuits. As B is independent in M, it follows that

(5) *for each a in A, there is at most one b in B such that $(B - \{b\}) \cup \{a\}$ is a circuit of M.*

Let $b_1, b_2, ..., b_k$ be the elements of B such that $(B - \{b_i\}) \cup \{a\}$ is a circuit of M for some a in A. It follows from (5) that we may choose $a_1, a_2, ..., a_k$ from A in such a way that, for every i in $\{1, 2, ..., k\}$, the set $(B - \{b_i\}) \cup \{a_i\}$ is a circuit of M. We shall show next that

(6) *the non-spanning circuits of M are*

(i) $\{(B - \{b_i\}) \cup \{a_i\} : i = 1, 2, ..., k\}$ *and*

(ii) $\{C : C \subseteq A$ *and* $|C| = 3\}$.

To prove this, we first observe that, from the choice of $a_1, b_1, a_2, b_2, ..., a_k, b_k$, all the sets of type (i) are circuits of M. Moreover, since M is simple and $r(A) = 2$, all the sets of type (ii) are also circuits of M; and

(7) *if C is a circuit of M with $C \subseteq A$, then C is of type* (ii).

To complete the proof of (6), it remains to show that

(8) *if C is a non-spanning circuit of M and $C \cap B \neq \emptyset$, then C is of type* (i).

To prove this, we shall first prove that

(9) *if* C *is a non-spanning circuit of* M *and* $C \cap B \neq \emptyset$, *then* $|C \cap B| = |B| - 1$.

Since B is a basis of M, it is clear that $|C \cap B| \leqslant |B| - 1$. On the other hand, since $C \cap B \neq \emptyset$, it follows that $C \cap B$ contains a circuit of M/A. Since M^* is simple and $r^*(B) = 2$, it is clear that M/A is a uniform matroid of rank $|B| - 2$. Hence all the circuits of M/A have $|B| - 1$ elements. Therefore $|C \cap B| \geqslant |B| - 1$ and thus (9) is proved.

We now complete the proof of (8). Since $C \not\subseteq A$, it follows, by (9), that there is an element b in B such that $C \cap B = B - \{b\}$. But $B - \{b\}$ is independent in M and C is a circuit of M of cardinality at most $|B|$. Therefore there is an element a of A such that $C = (B - \{b\}) \cup \{a\}$. Thus $b = b_i$ for some i. To finish proving (8), we must show that $a = a_i$. If $a \neq a_i$, let $C_i = (B - \{b_i\}) \cup \{a_i\}$ and let d be an element of $C_i \cap C$. Then there is a circuit C' of M such that $C' \subseteq (C_i \cup C) - \{d\}$. Clearly $C' \subseteq (B - \{b_i, d\}) \cup \{a, a_i\}$ so C' is non-spanning. Moreover, since $C_i \cap C = B - \{b_i\}$, it follows that $|C' \cap B| \leqslant |B| - 2$. By applying (9) to C', we conclude that $C' \subseteq A$. Then, since $|(C_i \cup C) \cap A| = 2$, we have $|C'| \leqslant 2$, contradicting (7). Thus (8) and hence (6) is proved.

Now we finish the proof of (2.8) by showing that $(\mathcal{N}(Q), \leqslant_m)$ is a wqo. Let M be a member of $\mathcal{N}$, and let $A, B, a_1, b_1, a_2, b_2, ..., a_k, b_k$ be as in (6). Let A_1 be the set of elements $a_{k+1}, a_{k+2}, ..., a_m$ of $A - \{a_1, a_2, ..., a_k\}$, and let B_1 be the set of elements $b_{k+1}, b_{k+2}, ..., b_n$ of $B - \{b_1, b_2, ..., b_k\}$. We encode M as $([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1, B_1)$. Then it is not difficult to verify the following:

(10) (i) *If* $k \geqslant 1$ *and* $\min\{|A|, |B|\} \geqslant 4$, *then, for all i in* $\{1, 2, ..., k\}$,

$$([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1, B_1) \backslash a_i / b_i$$
$$= ([(a_1, b_1), (a_2, b_2), ..., (a_{i-1}, b_{i-1}),$$
$$(a_{i+1}, b_{i+1}), ..., (a_k, b_k)], A_1, B_1).$$

(ii) *If* $|A| \geqslant 4$, *then, for all i in* $\{k+1, k+2, ..., m\}$, $([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1, B_1) \backslash a_i = ([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1 - \{a_i\}, B_1)$.

(iii) *If* $|B| \geqslant 4$, *then, for all i in* $\{k+1, k+2, ..., n\}$, $([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1, B_1) / b_i = ([(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)], A_1, B_1 - \{b_i\})$.

Let f be a Q-labeling of M. We define $p(M, f)$ to be

$$([(f(a_1), f(b_1)), (f(a_2), f(b_2)), ..., (f(a_k), f(b_k))],$$
$$[f(a_{k+1}), ..., f(a_m)], [f(b_{k+1}), ..., f(b_n)]).$$

Clearly, if we denote $(Q \times Q)^{<\omega} \times Q^{<\omega} \times Q^{<\omega}$ by $\mathcal{Q}$, then $p(M)$ is a member of $\mathcal{Q}$. Let $\preccurlyeq$ be the binary relation $(\leqslant \times \leqslant)^{<\omega} \times \leqslant^{<\omega} \times \leqslant^{<\omega}$. It follows from (1.3) and (1.4) that $(\mathcal{Q}, \preccurlyeq)$ is a wqo. Moreover, by (10), it is straightforward to check that, for any two members (M, f) and (N, g) of $\mathcal{N}(Q)$, if $p(N, g) \preccurlyeq p(M, f)$, then $(N, g) \leqslant_m (M, f)$. Thus (2.8) follows.

Proof of (2.6). Let $\mathcal{M}$ be as in (2.8) and let $\mathcal{N}$ be the class of matroids whose rank or corank is 2. Since $\mathcal{N} \subseteq \mathcal{M}_{2,0,1} \cup \mathcal{M}_{0,2,1}$, it follows from (2.5) and (1.5) that $(\mathcal{N}(Q), \leqslant_m)$ is a wqo for every wqo $(Q, \leqslant)$. Thus, to prove (2.6), it follows, by (2.7), (2.8), (1.5), (1.6), (1.7), and (1.10), that we need only show that every connected matroid in $\mathcal{M}_{2,2,1}$ is in $\mathcal{M}_{1,2,1}$, $\mathcal{M}_{2,1,1}$, $\mathcal{M}$, or $PS(\mathcal{N})$.

Let M be a connected matroid in $\mathcal{M}_{2,2,1} - (\mathcal{M}_{1,2,1} \cup \mathcal{M}_{2,1,1})$ and let X and Y be disjoint subsets of $E(M)$ with $|X| = |Y| = 2$ such that every connected component of $M \backslash X/Y$ is either a loop or a coloop. Since $M \notin \mathcal{M}_{1,2,1} \cup \mathcal{M}_{2,1,1}$, it follows that X and Y are independent in M^* and M, respectively. Let A be the set of elements a of M such that either a is in X, or a is a coloop of $M \backslash X/Y$. Let $B = E(M) - A$. Then B is the set of elements b of M such that either b is in Y, or b is a loop of $M \backslash X/Y$. It follows that every element of B is spanned by Y and thus $r(B) = 2$. Similarly, $r^*(A) = 2$. Thus $r(M) = |A| + r(B) - r^*(A) = |A|$. If A is independent in M, then it is a basis of M and so B is a cobasis of M. In that case, it follows that M is in $\mathcal{M}$. Therefore we may assume that $r(A) < |A|$. Since M is connected, we must have $r(A) + r(B) > r(M)$. Then, from the last two inequalities and the fact that $r(B) = 2$, we conclude that $r(A) + r(B) = r(M) + 1$. Hence M is a 2-sum of a rank-2 and a corank-2 matroid. Thus M is in $PS(\mathcal{N})$ and the proof of (2.6) is complete. ∎

3. Matroids Representable over a Finite Field

For a field F and a class of matroids $\mathcal{M}$, let $\mathcal{M}^F$ denote the class of matroids in $\mathcal{M}$ that are representable over F. We know from (1.2) that not every class $\mathcal{M}_{a,b,c}$ is well-quasi-ordered by the minor relation. However, from the main result of this section, it follows that, for every finite field F and all non-negative integers a, b, and c, the class $\mathcal{M}^F_{a,b,c}$ is well-quasi-ordered.

The next theorem is the main result of this section. It uses a new concept, that of the *type* $t(M)$ of a matroid M, which is defined inductively as follows. If $E(M) = \emptyset$, then $t(M) = 0$. If $E(M) \neq \emptyset$ and M is connected, then $t(M) = 1 + \min\{t(M \backslash e), t(M/e) : e \in E(M)\}$. If M is disconnected, then $t(M) = \max\{t(M_i)\}$, where the maximum is taken over all the connected components M_i of M. It is clear from the definition that $t(M^*) = t(M)$ for all matroids M.

(3.1) THEOREM. *Let F be a finite field, k be a non-negative integer, and* $\mathcal{M}$ *be the class of F-representable matroids of type at most k. Then* $(\mathcal{M}(Q), \leqslant_m)$ *is a wqo for every wqo* $(Q, \leqslant)$.

Before proving this theorem, we make the following remarks.

Remark 1. Let a, b, and c be non-negative integers and let M be a matroid in $\mathcal{M}_{a,b,c}$. It is clear that $t(M) \leqslant a+b+c$ and hence that the class of matroids of type at most t contains all classes $\mathcal{M}_{a,b,c}$ for which $a+b+c \leqslant t$. However, there are matroids of small type that are not members of any $\mathcal{M}_{a,b,c}$ for which $\max\{a, b, c\}$ is small. In particular, if D_n is the disjoint union of $2n+1$ cycles each with $n+1$ edges, then it is easy to verify that $M(D_n)$ has type two, but is not a member of $\mathcal{M}_{n,n,n}$.

On combining Theorem 3.1 with the first part of the last remark, we immediately obtain the following.

(3.2) COROLLARY. *Let* a, b, *and* c *be non-negative integers. Then* $(\mathcal{M}^F_{a,b,c}(Q), \leqslant_m)$ *is a wqo for every finite field F and every wqo* $(Q, \leqslant)$.

We note here that, although the matroids used to prove (2.1)–(2.3) are all representable over some finite field, there is no common finite field over which all those matroids can be represented. Moreover, there is no integer m such that all those matroids can be represented over some field with at most m elements. Indeed, the following is an immediate consequence of (3.2) and (1.5).

(3.3) COROLLARY. *Let a, b, and c be non-negative integers, let m be an integer exceeding one, and let* $\mathcal{N}$ *be the class of matroids in* $\mathcal{M}_{a,b,c}$ *that are representable over some field with at most m elements. Then* $(\mathcal{N}(Q), \leqslant_m)$ *is a wqo for every wqo* $(Q, \leqslant)$.

Seymour [11] has proved the following result.

(3.4) THEOREM. *If C is a largest circuit of a connected matroid M, then all the circuits of M/C have at most* $|C|-1$ *elements.*

It follows from this theorem that if M has no circuit of cardinality exceeding a positive integer k, then $t(M) \leqslant k(k+1)/2$. Thus (3.1) and (3.4) imply the following.

(3.5) COROLLARY. *Let F be a finite field and let k be a positive integer. The class of matroids that are representable over F and have no circuit of cardinality exceeding k is well-quasi-ordered by the minor relation.*

Remark 2. Statement (2.1) implies that the representability condition in (3.5) cannot be dropped.

Proof of (3.4). Suppose that M/C has a circuit D such that $|D| = |C|$. Since M is connected, it has a circuit J meeting both C and D. Choose this circuit so that $|J-(C \cup D)|$ is as small as possible. Clearly $M|(J \cup C \cup D)$ is connected. If $J-(C \cup D)$ is empty, then in $(M|(C \cup D))/C$, the set $J-C$ is a proper subset of D, so D is not a circuit of M/C; a contradiction.

We may therefore assume that $J-(C \cup D)$ is non-empty. Choose c in $C \cap J$ and d in $D \cap J$. Then, by the strong elimination axiom, M has a circuit J_1 such that $d \in J_1 \subseteq (J \cup C) - \{c\}$. Clearly J_1 meets both C and D. Thus, by the choice of J, it follows that $J_1 - (C \cup D) = J - (C \cup D)$.

Now choose x in $(J \cap J_1) - (C \cup D)$. Suppose that $J_1 \cap D \neq J \cap D$ and choose e in $(J - J_1) \cap D$. By elimination, M has a circuit J_2 that contains e and is contained in $(J \cup J_1) - \{x\}$. This circuit must meet C since otherwise it would be contained in J. But now J_2 contradicts the choice of J. We conclude that $J_1 \cap D = J \cap D$.

Recall that x is in $(J \cap J_1) - (C \cup D)$ and that c is in $(C \cap J) - J_1$. By elimination, M has a circuit J_3 containing c and contained in $(J \cup J_1) - \{x\}$. By the choice of J, it follows that J_3 avoids D. Suppose that $J_3 \neq C$. Then there is an element y in $(J_3 \cap J) - (C \cup D)$. Since d is in $J - J_3$ and y is in $J_3 \cap J$, there is a circuit of M that contains d and is contained in $(J \cup J_3) - \{y\}$. This circuit must meet C since otherwise it would be contained in J. Therefore this circuit contradicts the choice of J. We conclude that $J_3 = C$ and hence that $J \cup J_1 \supseteq C$.

By interchanging the roles of C and D in the above argument, we deduce that M has a circuit J_4 such that $J_4 - (C \cup D) = J - (C \cup D)$; $J_4 \cap C = J \cap C$; and $J_4 \cup J \supseteq D$. Since $C \subseteq J \cup J_1$ and $D \subseteq J \cup J_4$, we conclude that

$$|J_1 \cap C| + |J \cap C| \geqslant |C|, \qquad |J_4 \cap D| + |J \cap D| \geqslant |D|.$$

Adding the above inequalities and using the fact that $|C| = |D|$, we get

$$|J \cap C| + |J_1 \cap C| + |J \cap D| + |J_4 \cap D| \geqslant 2\,|C|.$$

But $J \cap D = J_1 \cap D$ and $J \cap C = J_4 \cap C$. Thus

$$|J_1 \cap C| + |J_4 \cap C| + |J_4 \cap D| + |J_1 \cap D| \geqslant 2\,|C|.$$

Clearly $|J_i| = |J_i \cap C| + |J_i \cap D| + |J - (C \cup D)|$ for $i = 1, 4$. Thus

$$|J_1| - |J - (C \cup D)| + |J_4| - |J - (C \cup D)| \geqslant 2\,|C|,$$

and so

$$|J_1| + |J_4| \geqslant 2(|C| + |J-(C \cup D)|) > 2\,|C|.$$

Hence either $|J_1|$ or $|J_4|$ exceeds $|C|$, contradicting the choice of C. ∎

In our proof of (3.1) we shall, informally speaking, establish the well-quasi-ordering of matrix representations of matroids, rather than of the matroids themselves. However, we shall not consider all matrix representations of a matroid, but only those corresponding to type-compatible bases, which are defined inductively below. Let M be a matroid of type k. If $k=0$, then the empty set is the unique basis of M; it is *type-compatible*. Suppose now that k is positive and that the type-compatible bases have been defined for all matroids of type at most $k-1$. If the type-k matroid M is connected, then a basis B of M is *type-compatible* if there is an element e of M such that one of the following holds:

(i) The element e is in B, the type of M/e is $k-1$, and $B-\{e\}$ is a type-compatible basis of M/e.

(ii) The element e is not in B, the type of $M\backslash e$ is $k-1$, and B is a type-compatible basis of $M\backslash e$.

A basis of a disconnected matroid is *type-compatible* if it is the union of type-compatible bases of the connected components of the matroid. From the definition of type, it follows that every connected matroid M of positive type has an element e such that $\min\{t(M\backslash e), t(M/e)\} = t(M)-1$. Using this fact, it is easy to verify that every matroid has a type-compatible basis, and if B is such a basis of M, then $E(M)-B$ is a type-compatible basis of M^*.

Let M be an F-representable matroid of positive type and let B be a type-compatible basis of M. It is well known that if both $r(M)$ and $r^*(M)$ are positive, then M can be represented by a matrix $[I_B \mid A]$ over F, where the columns of I_B and A correspond to the elements of B and $E(M)-B$, respectively, and I_B is the $r(M) \times r(M)$ identity matrix. Let us index the rows of $[I_B \mid A]$ by the elements of B such that, for every e in B, the entry in row e and column e is 1. We call A a *type-compatible F-representation* of the pair (M, B). Note that, in general, (M, B) may have more than one type-compatible F-representation. For convenience, if $r(M)=0$ or $r^*(M)=0$, we define the $0 \times |E(M)|$ matrix and the $|E(M)| \times 0$ matrix, respectively, to be the only type-compatible representations of (M, B).

Let $(Q, \leqslant)$ be a quasi-ordering and let f be a Q-labeling of M. Let $L(A, f)$ denote the matrix obtained from A by adding a new last row and a new last column as follows. In the new row, the entry in column e is $f(e)$ for every e in $E(M)-B$. In the new column, the entry in row e is $f(e)$ for every e in B. The common entry of the new column and the new row is arbitrarily chosen subject to the condition that it not lie in Q. Clearly M, B, and f are uniquely determined by the matrix $L(A, f)$.

Suppose that A_1 and A_2 are type-compatible F-representations of (M_1, B_1) and (M_2, B_2). We write $L(A_1, f_1) \leqslant_L L(A_2, f_2)$ if $L(A_1, f_1)$ can be obtained from $L(A_2, f_2)$ by a sequence of the following operations:

(i) deleting a row other than the last row;

(ii) deleting a column other than the last column; and

(iii) replacing an element q_2 of Q that is in the last row or column by some q_1 in Q with $q_1 \leqslant q_2$. It is not difficult to verify that if $L(A_1, f_1) \leqslant_L L(A_2, f_2)$, then $(M_1, f_1) \leqslant_m (M_2, f_2)$. Let $\mathscr{L}(F, k, Q)$ be the set of matrices $L(A, f)$ that are obtained from type-compatible F-representations of pairs (M, B), where B is a type-compatible basis of a matroid M of type at most k and f is a Q-labeling of M. Then, to prove (3.1), it suffices to prove the following.

(3.6) LEMMA. *Let F be a finite field and let k be a non-negative integer. Then $(\mathscr{L}(F, k, Q), \leqslant_L)$ is a wqo for every wqo $(Q, \leqslant)$.*

Proof. We proceed by induction on k. The result holds trivially if $k = 0$, so we assume that $k > 0$ and that (3.6) holds for all smaller values of k. It follows from (1.3) that we need only consider the subset $\mathscr{L}_0$ of $\mathscr{L}(F, k, Q)$ that corresponds to connected matroids. Let $\mathscr{L}_c$ be the set of matrices in $\mathscr{L}_0$ that are of the form $L(A, f)$, where A is a type-compatible F-representation of (M, B) and B contains an element e for which $t(M/e) = k - 1$. Observe that if a matrix A is a type-compatible F-representation of (M, B), then the transpose A^t of A is a type-compatible F-representation of $(M^*, E(M) - B)$. Hence, for every matrix $L(A, f)$ in $\mathscr{L}_0 - \mathscr{L}_c$, the matrix $L(A^t, f)$ belongs to $\mathscr{L}_c$. Thus, by (1.5), it suffices to show that $(\mathscr{L}_c, \leqslant_L)$ is a wqo.

Let $L(A, f)$ be a matrix in $\mathscr{L}_c$, such that A is a type-compatible F-representation of (M, B). Then there is an element x in B such that $t(M/x) = k - 1$. Let A' be the matrix obtained from A by deleting row x. Then it is clear that A' is a type-compatible F-representation of $(M/x, B - \{x\})$. Let $Q_1 = Q \cup F$ and let $\leqslant_1$ be the binary relation on Q_1 such that $p \leqslant_1 q$ if and only if either both p and q are in Q with $p \leqslant q$, or both p and q are in F with $p = q$. Since F is finite, it follows that $(Q_1, \leqslant_1)$ is a wqo and thus it follows from (1.4) that $(Q \times Q_1, \leqslant \times \leqslant_1)$ is also wqo. Let us denote $Q \times Q_1$ by Q'. Then we define a Q'-labeling f' of M/x as follows: For every element e in $B - \{x\}$, let $f'(e) = (f(e), f(x))$; for every element e in $E(M) - B$, let $f'(e) = (f(e), a_{xe})$, where a_{xe} is the entry of A in row x and column e. Clearly A' is a type-compatible F-repesentation of $(M/x, B - \{x\})$ and, by the choice of x, the type of M/e is $k - 1$. Thus

$L(A', f')$ is a member of $\mathscr{L}(F, k-1, Q')$, which is well-quasi-ordered by the relation $\leqslant_{\mathrm{L}}$, according to the induction hypothesis. It is easy to verify that if $L(A'_1, f'_1) \leqslant_{\mathrm{L}} L(A'_2, f'_2)$, then $L(A_1, f_1) \leqslant_{\mathrm{L}} L(A_2, f_2)$. Hence the result follows. ∎

An alternative proof of (3.1) can be obtained by first translating the matrix problem into one for bipartite graphs, and then using an edge-labeled version of Theorem 2.1 of [4].

4. Concluding Remarks

Let $\mathscr{M}$ be a class of matroids and let $SS(\mathscr{M}) = (PS(\mathscr{M}^*))^*$. From (1.7) and (1.10), we immediately obtain the following.

(4.1) Corollary. *If $(\mathscr{M}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$, then $(SS(\mathscr{M})(Q), \leqslant_{\mathrm{m}})$ is also a wqo for every wqo $(Q, \leqslant)$.*

The next result is an easy consequence of (1.7) and (1.8).

(4.2) Corollary. *If $(\mathscr{M}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$, then $((\overline{\mathscr{M}^*})^* (Q), \leqslant_{\mathrm{m}})$ is also a wqo for every wqo $(Q, \leqslant)$.*

Let $\mathscr{M}$ be a class of matroids and let k be a non-negative integer. The class of matroids $\mathscr{H}^k_{\mathscr{M}}$ is defined inductively as follows: let $\mathscr{H}^0_{\mathscr{M}} = \mathscr{M}$; for $k > 0$, let $\mathscr{N} = \mathscr{H}^{k-1}_{\mathscr{M}}$ and let $\mathscr{H}^k_{\mathscr{M}}$ be the union of $PS(\mathscr{N})$, $SS(\mathscr{N})$, $\overline{\mathscr{N}}$, and $(\overline{\mathscr{N}^*})^*$. From (1.5), (1.8), (1.10), (4.1), and (4.2), we deduce the following.

(4.3) Theorem. *Suppose that $\mathscr{M}$ is a class of matroids such that $(\mathscr{M}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$. Then $(\mathscr{H}^k_{\mathscr{M}}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$ and every non-negative integer k.*

Let m and k be non-negative integers and suppose that $m \geqslant 2$. We denote by $\mathscr{M}(m, k)$ the class of matroids of type at most k that are representable over some field with at most m elements. On combining (4.3) with (3.1) and (1.5), we obtain the following generalization of (3.1).

(4.4) Theorem. *Let m and k be non-negative integers and suppose that $m \geqslant 2$. Then $(\mathscr{H}^k_{\mathscr{M}(m,k)}(Q), \leqslant_{\mathrm{m}})$ is a wqo for every wqo $(Q, \leqslant)$.*

Let n be a positive integer and let G_n be the graph illustrated in Fig. 4. It is clear that, for every integer m exceeding 1 and every non-negative integer k, there is a positive integer n depending only on m and k such that $M(G_n)$ does not belong to $\mathscr{H}^k_{\mathscr{M}(m,k)}$. We do not know the answer to the following question: if a minor-closed class $\mathscr{M}$ of matroids does not contain

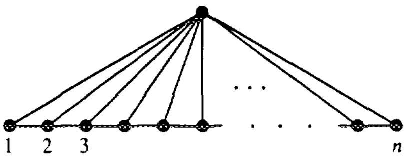

FIG. 4. G_n.

some fixed $M(G_n)$, is $\mathcal{M}$ a subset of some $\mathcal{H}^k_{\mathcal{M}(m,k)}$? One way to answer this question affirmatively would be to give an affirmative answer to the following.

(4.5) *Question.* Is there a function $f(F, n)$ with the following property: if M is a 3-connected matroid representable over the finite field F and if n is the largest integer such that M has an $M(G_n)$-minor, then there are disjoint subsets X and Y of $E(M)$ such that $|X| + |Y| \leqslant f(F, n)$ and $M \backslash X/Y$ has no $M(G_n)$-minor?

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