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Unavoidable Minors of Large 3-Connected Matroids

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This paper proves that, for every integer n exceeding two, there is a number $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor that is isomorphic to one of the following matroids: an $(n+2)$-point line or its dual, the cycle or cocycle matroid of $K_{3,n}$, the cycle matroid of a wheel with n spokes, a whirl of rank n, or an n-spike. A matroid is of the last type if it has rank n and consists of n three-point lines through a common point such that, for all k in $\{1, 2, ..., n-1\}$, the union of every set of k of these lines has rank $k+1$.

Key Words: Matroid; 3-connected; unavoidable; minor; Ramsey theory.

1. INTRODUCTION

This paper is a continuation of [1]. In that paper, by building on some new Ramsey-theoretic results for matrices, we distinguished the unavoidable minors in large 3-connected binary matroids. The following is the main result of that paper where J_n denotes the $n \times n$ matrix of all ones.

1.1. Theorem. *For every integer n exceeding two, there is an integer $N(n)$ such that every 3-connected binary matroid with at least $N(n)$ elements has a minor isomorphic to the cycle matroid of $K_{3,n}$, its dual, the cycle matroid of the n-spoked wheel, or the vector matroid of the matrix $[I_n \mid J_n - I_n]$ over $GF(2)$.*

The purpose of this paper is to extend this theorem to the class of all 3-connected matroids. Precisely the same matrix results that were applied in [1] to prove Theorem 1.1 will be used here. However, in order to be able to apply these results to arbitrary 3-connected matroids instead of 3-connected binary matroids, we shall introduce the idea of a hamiltonian partial representation. This object is a matrix that can be associated with any matroid having a spanning circuit. We will show that this matrix

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captures enough information about the matroid to enable the main result of [1] to be extended from the relatively well-behaved class of binary matroids to the far-less-tractable class of arbitrary matroids.

This paper is already long, so we shall not repeat the interesting history of results of this type. This history, featuring most notably [3] and [5], may be found in [1]. The bulk of this paper will be concerned with developing the theory of hamiltonian partial representations and manipulating such representations to prove the main theorem. Next we prepare to state this theorem. If the all-ones column p is adjoined to the matrix $[I_n \mid J_n - I_n]$ over $GF(2)$, then the binary matroid of the resulting matrix has the following properties:

(i) the ground set is the union of n lines, $L_1, L_2, ..., L_n$, all having three points and passing through a common point p;

(ii) for all k in $\{1, 2, ..., n-1\}$, the union of any k of $L_1, L_2, ..., L_n$ has rank $k+1$; and

(iii) $r(L_1 \cup L_2 \cup \cdots \cup L_n) = n$.

For all $n \geqslant 3$, an arbitrary matroid satisfying these three conditions will be called an *n-spike* with *tip p*. It is not difficult to show that the vector matroid of the binary matrix $[I_n \mid J_n - I_n \mid \mathbf{1}]$, where $\mathbf{1}$ denotes the all-ones column, is the only n-spike that is binary. The main result of this paper is the following:

1.2. THEOREM. *For every integer n exceeding two, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$ the cycle matroid of a wheel with n spokes, the whirl of rank n, or an n-spike.*

In fact, we shall also prove a refinement of this theorem that restricts attention to certain special spikes, but the statement of this will require a more detailed consideration of spikes. The next result, whose straightforward proof is omitted, lists some elementary properties of spikes.

1.3. LEMMA. *For $n \geqslant 3$, let M be an n-spike with tip p. Let $I = \{1, 2, ..., n\}$ and, for all i in I, let $L_i = \{p, x_i, y_i\}$. Then*

(i) *L_i is a circuit of M for all i in I;*

(ii) *$(L_i \cup L_j) - p$ is a circuit of M for all distinct i and j in I;*

(iii) *every non-spanning circuit of M other than those listed in* (i) *and* (ii) *avoids p, contains a unique element from each of $\{x_1, y_1\}$, $\{x_2, y_2\}$, ..., $\{x_n, y_n\}$, and is also a hyperplane of M;*

(iv) *M/p can be obtained from an n-element circuit by replacing each element by two elements in parallel*; *and*

(v) *for all i in I, each of $M\backslash p/x_i$ and $(M\backslash p\backslash x_i)^*$ is an $(n-1)$-spike with tip y_i, and each of $M\backslash y_i/x_i$ and $M\backslash x_i/y_i$ is an $(n-1)$-spike with tip p.*

Part (iii) of this lemma is of most immediate interest here since it means that, to uniquely specify the n-spike M, it suffices to describe every circuit of M of the form listed there. For a subset J of I, let X_J and Y_J be, respectively, the sets $\{x_j : j \in J\}$ and $\{y_j : j \in J\}$. Then every circuit of M of the type listed in (iii) has the form $X_J \cup Y_{I-J}$ for some subset J of I.

An n-spike M is *uniform* if, whenever $X_J \cup Y_{I-J}$ is a circuit of M for some $J \subseteq I$, the set of circuits of M includes every set of the form $X_{J'} \cup Y_{I-J'}$ with $|J'| = |J|$. We shall show in Section 10 that every sufficiently large spike has a big uniform spike as a minor. Using this, we shall extend Theorem 1.2 as follows.

1.4. THEOREM. *For every integer n exceeding two, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to one of $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, $M(W_n)$, W^n, or a uniform n-spike.*

The matroid terminology used here will follow [4]. In particular, W_r denotes the r-spoked wheel graph and W^r denotes the whirl of rank r. Moreover, for a subset T of the ground set of a matroid M, we often write $M/(E(M)-T)$ as $M.T$. A spanning circuit of a matroid will often be called a *hamiltonian circuit*. A *hamiltonian matroid* is a matroid with a hamiltonian circuit, and a *hamiltonian minor* of a matroid is a minor that is also a hamiltonian matroid.

The proof of Theorem 1.2 follows the same general pattern as the proof of Theorem 1.1. In particular, we shall use a result from [1] that, up to duality, every sufficiently large 3-connected matroid has a big 3-connected hamiltonian minor. This result enables us to concentrate on hamiltonian matroids. In Section 2, we associate with such a matroid a hamiltonian partial representation and we discuss some of the basic properties of these matrices. Section 3 defines the crissing graph of such a representation, a close relative of the crossing graph discussed in [1], and a tool that will be used to extract crucial structural information about a matroid from a hamiltonian partial representation. Section 4 relates crissings and crossings with the aim of enabling the Ramsey results for matrices proved in [1] to be applied to hamiltonian partial representations.

Section 5 gives an overview of the proof of Theorem 1.2 and outlines the main steps in the proof. By exploiting a theorem from [1] that every sufficiently large connected graph has a big clique, a big star, or a big path as an induced subgraph, this section divides the main part of the proof into three cases. These cases are treated separately in the three subsequent sections. The longest and the most difficult of these is Section 7, the path case, since that case requires a very detailed analysis of hamiltonian partial representations to extract the desired conclusion. Section 9 completes the proof of Theorem 1.2 with quite a straightforward assembly of the pieces from earlier sections. Finally, Section 10, in addition to proving the existence of a big uniform spike as a minor of every sufficiently large spike, shows that Theorem 1.4 is the best-possible result of this type in the sense that the list of matroids given there contains no redundancy.

2. HAMILTONIAN PARTIAL REPRESENTATIONS

In this section, we introduce a new technique for dealing with arbitrary matroids having spanning circuits.

Let M be a matroid and let C be a hamiltonian circuit of M. For all e in $E(M)-C$, the matroid $M\mid(C\cup e)$ has corank 2 and has no coloops. Thus $[M\mid(C\cup e)]^*$ is a loopless rank-2 matroid. One parallel class of this line is $\{e\}$. The remaining parallel classes form a partition $\pi(e)$ of C. The next lemma summarizes some elementary properties of these partitions.

2.1. Lemma. *Let C be a hamiltonian circuit of a matroid M and suppose e is in $E(M)-C$.*

(i) *The set of circuits of $M\mid(C\cup e)$ is $\{(C\cup e)-P : P$ is a block of $\pi(e)$ or $P=\{e\}\}$.*

(ii) *$\pi(e)$ has just one block if and only if e is a loop.*

(iii) *For $f\in E(M)-(C\cup e)$, if e and f are parallel, then $\pi(e)=\pi(f)$.*

(iv) *For $f\in E(M)-(C\cup e)$, if $\pi(e)=\pi(f)$ and each has exactly two blocks, then e and f are parallel in M.*

Proof. The cocircuits of $[M\mid(C\cup e)]^*$ are all of the sets of the form $(C\cup e)-P$ where P is a parallel class of $[M\mid(C\cup e)]^*$. Part (i) follows immediately from this observation, and (ii) is a straightforward consequence of (i). Moreover, part (iii) is elementary.

To prove (iv), suppose that $\pi(e)$ and $\pi(f)$ are equal with each having P_1 and P_2 as their only blocks. Let $N=[M\mid(C\cup\{e,f\})]^*$. Then N/e has $\{f\}$, P_1, and P_2 as its only parallel classes, and N/f has $\{e\}$, P_1, and P_2 as its only parallel classes. Suppose x and y are parallel in N/e. Then $\{x, y\}$ or $\{x, y, e\}$ is a circuit of N. Since the only circuit of N/f contained in $\{x, y, e\}$ is $\{x, y\}$, we deduce that $\{x, y, e\}$ is not a circuit of N. Thus P_1

and P_2 are parallel classes of N. Hence $\{e, f\}$ is a cocircuit of N, that is, e and f are parallel in M. ▮

While (iv) in the last lemma is a partial converse of (iii), the additional hypothesis is needed to ensure the conclusion. For instance, suppose M_1 is a 5-point line with ground set $\{1, 2, 3, 4, 5\}$, and M_2 is obtained from a 4-point line with ground set $\{1, 2, 3, 4\}$ by adding 5 in parallel with 4. If $C = \{1, 2, 3\}$, then $\pi(4) = \pi(5)$ in both M_1 and M_2.

Let A be a matrix whose rows are indexed by the elements of the hamiltonian circuit C, and whose columns are indexed by the elements of $E(M) - C$. The entries of A are taken from some set X and are chosen so that the entries in rows i and j of column e are equal if and only if i and j lie in the same block of $\pi(e)$ or, equivalently, i and j are in the same series class of $M \mid (C \cup e)$. Such a matrix A is called a *hamiltonian partial representation* or *HPR* for M with respect to the hamiltonian circuit C. Every other such representation for M can be obtained from A by repeatedly applying the following operation: choose a particular column e of A and consider the set X' of distinct entries in e; let σ be a bijection from X' onto a set Y' and replace each entry x from X' by $\sigma(x)$ in column e, leaving all other columns unchanged. Evidently, equality of entries in an HPR is meaningful if and only if the entries lie in the same column.

From Lemma 2.1, when $|C| \geqslant 2$, the loops of M correspond precisely to the constant columns in an HPR. But parallel elements are harder to detect. Two equal columns certainly correspond to parallel elements if each column has exactly two distinct entries. But if the number of distinct entries exceeds two, then, as noted above, equality of the columns does not guarantee that the elements are parallel. On the other hand, while two parallel elements e and f will induce equal partitions $\pi(e)$ and $\pi(f)$ of C, our method of assigning entries to the columns e and f of A will not guarantee equality of these columns.

For readers familiar with partial representations of matroids or matroid representations over a field, we now describe briefly how these are related to hamiltonian partial representations. These remarks will not be needed in any proofs. Given an HPR A for M with respect to the circuit C, let $k \in C$. One can obtain a partial representation for M with respect to the basis $C - k$ as follows: for each column of A that does not have 0 as an entry, choose some symbol from that column and replace it by 0 throughout the column; permute the symbols used in each column so that, in the resulting matrix A', row k consists entirely of zeros; replace all non-zero entries in A' by ones and delete row k to produce a matrix A''. If $r(M) = r$, then $[I_r \mid A'' \mid \mathbf{1}]$ is a partial representation for M with respect to $C - k$ where $\mathbf{1}$ is a column of all ones, this column being labelled by the element k of C. Conversely, if one has $r + 1$ partial representations for M, one for each basis contained in C, then one may construct an HPR for M with respect to C.

Now suppose $[I_r \mid D \mid \mathbf{1}]$ is a representation for a matroid N over a field where the first r columns and the last column correspond to the basis B and the element e. Then an HPR for N with respect to the hamiltonian circuit $B \cup e$ can be obtained from D by adjoining a row of zeros, labelled by e.

We return now to our main discussion making some elementary observations about hamiltonian partial representations and the information that can be deduced from them. The straightforward proof of the first of these is omitted.

2.2. Lemma. *Let A be an HPR for a matroid M with respect to the circuit C where $|C| \geqslant 2$.*

(i) *If $e \in C$, then $C - e$ is a hamiltonian circuit of M/e, and an HPR for M/e with respect to this circuit can be obtained from A by deleting row e.*

(ii) *If $s \in E(M) - C$, then C is a hamiltonian circuit of $M \backslash s$, and an HPR for $M \backslash s$ with respect to this circuit can be obtained from A by deleting column s.*

This tells us that each submatrix of A is an HPR of a certain minor of M. The next lemma provides one of the main tools that we shall use for extracting information from hamiltonian partial representations.

2.3. Lemma. *Let A be an HPR for the matroid M with respect to the circuit C. For distinct elements i and j of C and $e \in E(M) - C$, the following are equivalent*:

(i) *the entries in rows i and j of column e of A are equal*;

(ii) $e \in \operatorname{cl}(C - \{i, j\})$; *and*

(iii) *e is not in the cocircuit $E(M) - \operatorname{cl}(C - \{i, j\})$.*

Proof. The equivalence of (ii) and (iii) is immediate. Now assume that (i) holds. Then $\{i, j\}$ is contained in some block P of $\pi(e)$. As $(C \cup e) - P$ is a circuit of M, it follows that $e \in \operatorname{cl}(C - \{i, j\})$, that is, (ii) holds; so (i) implies (ii). Finally, suppose that (i) fails. Then $(C \cup e) - \{i, j\}$ contains no circuits. Thus $e \notin \operatorname{cl}(C - \{i, j\})$. We conclude that (ii) implies (i).

2.4. Corollary. *Let A be an HPR for the matroid M with respect to the circuit C, and let i and j be distinct elements of C. Then i and j are in series in M if and only if rows i and j of A are equal.*

Proof. Assume rows i and j of A are equal. If $e \in E(M) - C$, then, by the last lemma, e is not in the cocircuit $E(M) - \mathrm{cl}(C - \{i, j\})$. It follows that this cocircuit equals $\{i, j\}$, that is, i and j are in series in M. The proof of the converse follows easily by reversing this argument. ∎

3. THE CRISSING GRAPH OF A FAMILY OF PARTITIONS

The crossing graph of a matrix was an important tool in the proof of the main result of [1]. In this section, we describe a closely related, but slightly different, object, the crissing graph of a family of partitions of a finite set. The relationship between the two concepts will be discussed in Section 4. The main result of this section, Theorem 3.10, establishes that a simple, cosimple, hamiltonian matroid is 3-connected if and only if its crissing graph is connected. This is an important step in the proof of Theorem 1.2.

Let $\Pi(C)$ be the lattice of partitions of a finite set C. Two members π_1 and π_2 of $\Pi(C)$ *do not criss* if π_1 and π_2 have blocks A_1 and A_2, respectively, such that $A_1 \cup A_2 = C$. When such blocks A_1 and A_2 are unique, we denote the pair (A_1, A_2) by $\mu(\pi_1, \pi_2)$. Thus $\mu(\pi_1, \pi_2)$ is well-defined unless either π_1 and π_2 are equal and have precisely two blocks, or exactly one of π_1 and π_2 equals $\{C\}$. For two partitions π_1 and π_2 in $\Pi(C)$, we write $\pi_1 \dagger \pi_2$ if π_1 and π_2 criss, and $\pi_1 \not\dagger \pi_2$ otherwise.

The following is an immediate consequence of the definition.

3.1. Lemma. *For π_1 and π_2 in $\Pi(C)$, $\pi_1 \dagger \pi_2$ if and only if the complement of every block of π_1 meets at least two blocks of π_2.* ∎

One natural way to derive a family of partitions of a finite set is from a matrix. Let A be a matrix and suppose that C is the set of row labels for A. If s labels a column of A, the natural partition $\pi(s)$ of C associated with A has two elements i and j of C in the same block of $\pi(s)$ if and only if the entries in rows i and j of column s are equal.

3.2. Lemma. *Let C be a hamiltonian circuit of a matroid M and e_1 and e_2 be distinct non-parallel elements of $E(M) - C$. Then $\pi(e_1)$ and $\pi(e_2)$ do not criss if and only if $C \cup \{e_1, e_2\}$ contains two disjoint circuits of M.*

Proof. Suppose first that $\pi(e_1)$ and $\pi(e_2)$ do not criss. If $\pi(e_1) = \pi(e_2)$, then $\pi(e_1)$ has either one or two blocks. In the first case, C and $\{e_1\}$ are disjoint circuits of M. In the second case, $(C - A_1) \cup e_1$ and $(C - A_2) \cup e_2$ are disjoint circuits of M where A_1 and A_2 are the blocks of $\pi(e_1)$. We may now assume that $\pi(e_1) \neq \pi(e_2)$. Then $(C - A_1) \cup e_1$ and $(C - A_2) \cup e_2$ are disjoint circuits of M where $\mu(\pi(e_1), \pi(e_2)) = (A_1, A_2)$.

Conversely, assume that $C \cup \{e_1, e_2\}$ contains two disjoint circuits C_1 and C_2 of M. If one of these circuits is C, then, since e_1 and e_2 are not parallel, we may assume the other is $\{e_1\}$. In that case, $\pi(e_1) = \{C\}$, so $\pi(e_1)$ and $\pi(e_2)$ do not criss. If neither C_1 nor C_2 is C, then we may assume that $e_1 \in C_1$ and $e_2 \in C_2$. Then $C - C_1$ and $C - C_2$ are blocks of $\pi(e_1)$ and $\pi(e_2)$, respectively, and $(C - C_1) \cup (C - C_2) = C$, so $\pi(e_1)$ and $\pi(e_2)$ do not criss. ▮

It follows from Lemma 3.2 that, for any two distinct elements e_1 and e_2 of $E(M) - C$, if $M \mid (C \cup \{e_1, e_2\}) = M(G)$ for some graph G, then $\pi(e_1)$ and $\pi(e_2)$ criss if and only if e_1 and e_2 are crossing chords of the cycle C of G, or, equivalently, G is a subdivision of K_4.

Next we shall derive some elementary properties of the relationship of crissing defined above. Recall that $\pi_1 \leqslant \pi_2$ in the partition lattice $\Pi(C)$ if every block of π_1 is contained in a block of π_2. Moreover, $\pi \wedge \pi'$ has as its blocks all non-empty sets of the form $B \cap B'$ where B is a block of π and B' is a block of π'.

3.3. Lemma. *Let π_1, π_2, and π_3 be members of $\Pi(C)$. If $\pi_1 \leqslant \pi_2$ and $\pi_2 \dagger \pi_3$, then $\pi_1 \dagger \pi_3$.*

Proof. Suppose $\pi_1 \not\dagger \pi_3$. Then π_1 and π_3 have blocks A_1 and A_3, respectively, such that $A_1 \cup A_3 = C$. But π_2 has a block A_2 that contains A_1. Thus $A_2 \cup A_3 = C$ and $\pi_2 \not\dagger \pi_3$; a contradiction. ▮

3.4. Lemma. *Let π_1, π_2, and π_3 be members of $\Pi(C)$ other than $\{C\}$ such that $\pi_2 \notin \{\pi_1, \pi_3\}$. Suppose that π_2 crisses neither π_1 nor π_3. Also assume that A_2 and A_2' are distinct where $\mu(\pi_1, \pi_2) = (A_1, A_2)$ and $\mu(\pi_2, \pi_3) = (A_2', A_3)$. Then π_1 and π_3 are distinct and non-crissing, and $\mu(\pi_1, \pi_3) = (A_1, A_3)$.*

Proof. Clearly $A_1 \cup A_2 = C$ and $A_2' \cup A_3 = C$. Thus $A_1 \cup A_3 \supseteq (C - A_2) \cup (C - A_2') = C - (A_2 \cap A_2')$. As A_2 and A_2' are distinct blocks of π_2, it follows that $A_1 \cup A_3 = C$, so $\pi_1 \not\dagger \pi_3$. Moreover, provided π_1 and π_3 are distinct, $\mu(\pi_1, \pi_3) = (A_1, A_3)$. It remains to consider what happens when $\pi_1 = \pi_3$. In that case, $A_1 \neq A_3$ as $\pi_1 \neq \{C\}$. Thus $\pi_1 = \{A_1, A_3\}$. Therefore $A_2 \supseteq A_3$ and $A_2' \supseteq A_1$, so $A_2 \cup A_2' \supseteq C$. Hence $A_2 = A_3$ and $A_2' = A_1$, so $\pi_1 = \pi_2$; a contradiction. ▮

The following is an extension of the last lemma.

3.5. Lemma. *Let π_1, π_2, and π_3 be members of $\Pi(C)$ other than $\{C\}$ such that π_2 crisses neither π_1 nor π_3, but π_1 crisses π_3. Let $\mu(\pi_1, \pi_2) = (A_1, A_2)$ and $\mu(\pi_2, \pi_3) = (A_2', A_3)$. Then*

(i) $A_2 = A_2'$;

(ii) $A_1 \cup A_3 \neq C$;

(iii) $A_2 \cup (A_1 \cap A_3) = C$; *and*

(iv) $(\pi_1 \wedge \pi_3) \not\dagger \pi_2$ *and* $\mu(\pi_1 \wedge \pi_3, \pi_2) = (A_1 \cap A_3, A_2)$.

Proof. The assumptions on crissing imply that $\pi_2 \notin \{\pi_1, \pi_3\}$. Thus either $\pi_1 = \pi_3$, in which case (i) certainly holds; or $\pi_1 \neq \pi_3$ and (i) follows from Lemma 3.4. Part (ii) follows immediately from the fact that $\pi_1 \dagger \pi_3$. To see (iii), note that $A_2 \cup (A_1 \cap A_3) = (A_2 \cup A_1) \cap (A_2 \cup A_3) = C \cap C = C$. Thus, as $A_2 \neq C$, it follows that $A_1 \cap A_3$ is a non-empty block of $\pi_1 \wedge \pi_3$. Hence $(\pi_1 \wedge \pi_3) \not\dagger \pi_2$. Moreover, $\pi_1 \wedge \pi_3 \neq \pi_2$ otherwise $\pi_1 \geqslant \pi_2$ and Lemma 3.3 implies the contradiction that $\pi_2 \dagger \pi_3$. Hence $\mu(\pi_1 \wedge \pi_3, \pi_2)$ is well-defined and equals $(A_1 \cap A_3, A_2)$. ▮

For a family $(\pi_v : v \in V)$ of partitions in $\Pi(C)$, the *crissing graph* $\Gamma(\pi_v : v \in V)$ has vertex set V; an edge joins u and v in this graph if and only if u and v are distinct members of V for which $\pi_u \dagger \pi_v$.

3.6. LEMMA. *Suppose the graph $\Gamma(\pi_v : v \in V)$ is connected and $\pi \in \Pi(C)$. Then $\pi \dagger (\bigwedge_{v \in V} \pi_v)$ if and only if $\pi \dagger \pi_u$ for some u in V.*

Proof. Suppose $\pi \dagger \pi_u$ for some u in V. As $\bigwedge_{v \in V} \pi_v \leqslant \pi_u$, it follows by Lemma 3.3 that $\bigwedge_{v \in V} \pi_v \dagger \pi$.

Now suppose that, for all v in V, the partition π_v does not criss π but that $\pi \dagger (\bigwedge_{v \in V} \pi_v)$. Choose V so that $|V|$ is minimal subject to these conditions. Clearly $|V| \geqslant 2$. As $\Gamma(\pi_v : v \in V)$ is connected, none of the partitions π_v equals $\{C\}$. Let $\Gamma(\pi_v : v \in V) = \Gamma$. Choose a vertex u of Γ such that $\Gamma - u$ is connected and let x be a neighbor of u in Γ. Let $\pi' = \bigwedge_{v \in V - u} \pi_v$. By the minimality of $|V|$, it follows that $\pi \not\dagger \pi'$. But $\pi_u \dagger \pi_x$ and $\pi_x \geqslant \pi'$. Thus $\pi_u \dagger \pi'$ by Lemma 3.3. Hence π crisses neither π_u nor π', but π' crisses π_u. Thus, by Lemma 3.5, $\pi \not\dagger (\pi_u \wedge \pi')$. But $\pi_u \wedge \pi' = \bigwedge_{v \in V} \pi_v$ and $\pi \dagger \bigwedge_{v \in V} \pi_v$. This contradiction completes the proof of the lemma. ▮

For a matroid M having a hamiltonian circuit C, let $\Pi(M, C)$ be the family of partitions $(\pi(e) : e \in E(M) - C)$ considered in the last section. As in [1], we shall use connectivity properties of the matroid M to deduce properties of the crissing graph $\Gamma(\Pi(M, C))$.

3.7. LEMMA. *Let M be a loopless matroid with a hamiltonian circuit C. Then, for every 2-separation $\{X, Y\}$ of M, either*

(i) *X or Y contains a 2-circuit of M, or*

(ii) *both $X \cap C$ and $Y \cap C$ have at least two elements and these sets span X and Y, respectively.*

Proof. Since M is loopless having a spanning circuit, M is certainly connected. Let $\{X, Y\}$ be a 2-separation of M and suppose that neither X nor Y contains a 2-circuit. Then, as $r(X)+r(Y)-r(M)=1$ and $\min\{|X|, |Y|\} \geqslant 2$, it follows that $\min\{r(X), r(Y)\} \geqslant 2$ and so $\max\{r(X), r(Y)\} \leqslant r(M)-1$. Now $r(C-e)=r(M)$ for all e in C. Hence $|X \cap C| \geqslant 2$ and $|Y \cap C| \geqslant 2$. Moreover,

$$\begin{aligned} r(M)+1 &= |X \cap C| + |Y \cap C| \\ &= r(X \cap C) + r(Y \cap C) \\ &\leqslant r(X)+r(Y) \\ &= r(M)+1. \end{aligned}$$

Thus equality holds throughout the above and therefore (ii) holds. ∎

3.8. LEMMA. *Let C be a hamiltonian circuit of a matroid M and let e be an element of $E(M)-C$. The following statements are equivalent for a subset X of C:*

(i) $e \in \mathrm{cl}(X)$; *and*

(ii) $X \cup A = C$ *for some block A of $\pi(e)$.*

Proof. Observe that (i) holds if and only if M has a circuit in $X \cup e$ containing e. By Lemma 2.1(i), this is equivalent to (ii). ∎

We shall require just one more lemma to prepare for the main result of this section.

3.9. LEMMA. *Let C be a hamiltonian circuit of a matroid M and let V' be the vertex set of some component of $\Gamma(\Pi(M, C))$. If D is a block of $\bigwedge_{v \in V'} \pi(v)$ such that each of $|D|$ and $|C-D|$ is at least two, then $\{\mathrm{cl}(D), E(M)-\mathrm{cl}(D)\}$ is a 2-separation of M.*

Proof. For all v in V', the partition $\pi(v)$ has a block D_v containing D. Thus, by Lemma 3.8, $v \in \mathrm{cl}(C-D_v)$ and so $v \in \mathrm{cl}(C-D)$. Hence

(1) $V' \subseteq \mathrm{cl}(C-D)$.

We show next that

(2) $(E(M)-C)-V' \subseteq \mathrm{cl}(D) \cup \mathrm{cl}(C-D)$.

Assume that this fails, letting s be an element of $(E(M)-C)-V'$ that is in neither $\mathrm{cl}(D)$ nor $\mathrm{cl}(C-D)$. Then, by Lemma 3.8, neither $C-D$ nor D is contained in a block of $\pi(s)$. Thus $\pi(s)$ crisses the partition $\{D, C-D\}$. But $\{D, C-D\} \geqslant \bigwedge_{v \in V'} \pi(v)$. Hence, by Lemma 3.3, $\pi(s) \dagger (\bigwedge_{v \in V'} \pi(v))$. Thus, by Lemma 3.6, $\pi(s) \dagger \pi(t)$ for some $t \in V'$. Therefore s is in the same component as t in $\Gamma(\Pi(M, C))$. This contradiction to the choice of s completes the proof of (2).

On combining (1) and (2), we deduce that $\mathrm{cl}(D) \cup \mathrm{cl}(C-D) = E(M)$. As $r(M)+1 = |D| + |C-D| = r(\mathrm{cl}(D)) + r(\mathrm{cl}(C-D))$ and $\mathrm{cl}(C-D) \supseteq E(M) - \mathrm{cl}(D) \supseteq C-D$, we deduce that $\{\mathrm{cl}(D), E(M)-\mathrm{cl}(D)\}$ is a 2-separation of M. ▌

The next result, the main result of this section, identifies from the crissing graph when a hamiltonian matroid is 3-connected.

3.10. Theorem. *Let M be a matroid with a hamiltonian circuit C and suppose that $|E(M)| \geqslant 4$. Then the following two statements are equivalent.*

(i) *M is 3-connected.*

(ii) (*a*) *$\Gamma(\Pi(M, C))$ is connected*;

(b) *no two elements of C are in series*; *and*

(c) *no two elements of $E(M) - C$ are in parallel.*

Proof. We show first that if (i) fails, then so does (ii). Suppose that M is not 3-connected but that all of (a), (b), and (c) hold. First we show that

(1) $|E(M) - C| \geqslant 2$.

Assume the contrary. If $E(M) - C$ is empty, then C is a series class of M so (b) fails. Hence we may suppose that $E(M) - C = \{x\}$. Then M^* is a line containing at least four elements. As M^* is not 3-connected, it has a 2-circuit, which is contained in C. Thus (b) fails and (1) holds.

We show next that no element of $E(M) - C$ is a loop. If such a loop x exists, then $\pi(x) = \{C\}$ so $\pi(x)$ does not criss any other partition and therefore $\Gamma(\Pi(M, C))$ has just one vertex; a contradiction to (1). Thus no element of $E(M) - C$ is a loop and therefore, as C is hamiltonian and $|E(M)| \geqslant 4$, M is loopless.

Next we show that M has no 2-circuits. Assume the contrary, letting $\{x, y\}$ be a circuit. By (c), $\{x, y\}$ meets C. If $\{x, y\} \subseteq C$, then $\{x, y\} = C$ so $r(M) = 1$ and, by (c) again, $|E(M)| \leqslant 3$; a contradiction. Thus we may assume that $x \in E(M) - C$ and $y \in C$. Then $\pi(x)$ has two blocks, $\{y\}$ and $C - \{y\}$. Thus $\pi(x)$ crisses no $\pi(t)$ for $t \in (E(M) - C) - x$. Again, since $\Gamma(\Pi(M, C))$ is connected, it follows that x is the unique vertex of this graph. This contradiction to (1) completes the proof that no two elements of M are in parallel.

Since M has a hamiltonian circuit but no loops, it is certainly connected. As it is not 3-connected but is simple, Lemma 3.7 implies that M has a 2-separation $\{X, Y\}$ such that both $X \cap C$ and $Y \cap C$ have at least two elements and these sets span X and Y, respectively. Now $1 = r(X) + r(Y) - r(M) = r(X) + r^*(X) - |X|$. Thus if $C \supseteq X$, then, as $C \cap Y \neq \emptyset$, it follows that $r^*(X) = 1$ so the elements of X form a series class of M, violating (b). Therefore, C does not contain X. Similarly, it does not contain Y. Hence neither $X - C$ nor $Y - C$ is empty and so, as $\Gamma(\Pi(M, C))$ is connected, there are elements x and y in $X - C$ and $Y - C$, respectively, such that $\pi(x) \dagger \pi(y)$. Thus, by Lemma 3.8, since $X \cap C$ spans X, there is a block A_x of $\pi(x)$ such that $(X \cap C) \cup A_x = C$. Similarly, $(Y \cap C) \cup A_y = C$ for some block A_y of $\pi(y)$. Thus $A_x \cup A_y = C$ so $\pi(x) \not\dagger \pi(y)$. This contradiction completes the proof that (ii) implies (i).

Now suppose that (i) holds. Then, as $|E(M)| \geqslant 4$, M is loopless and has no non-trivial series or parallel classes. Thus (b) and (c) hold and it only remains to show that (a) holds. Suppose that it does not. Then, for some $k \geqslant 2$, the graph $\Gamma(\Pi(M, C))$ has k components, the vertex sets of which are $V_1, V_2, ..., V_k$, say. Since M is 3-connected, it follows by Lemma 3.9 that if $i \in \{1, 2, ..., k\}$ and D is a block of $\bigwedge_{s \in V_i} \pi(s)$, then $|D|$ or $|C - D|$ is less than two. But if $|C - D|$ is 0 or 1, then M has a loop or a 2-circuit; a contradiction. Thus, for all i, every block of $\bigwedge_{s \in V_i} \pi(s)$ is a singleton. Now take $t \in V_1$. Then, for all s in V_2, the vertices t and s are in different components so $\pi(t) \not\dagger \pi(s)$. Thus, by Lemma 3.6, $\pi(t) \not\dagger (\bigwedge_{s \in V_2} \pi(s))$. Therefore $\pi(t)$ and $\bigwedge_{s \in V_2} \pi(s)$ have blocks A_t and A_s, respectively, such that $A_t \cup A_s = C$. But, from above, A_s is a singleton. Thus $|A_t| \geqslant |C| - 1$. Hence $\pi(t)$ has a block with at least $|C| - 1$ elements. Therefore M has a 1- or 2-circuit; a contradiction. ▮

The next lemma will enable us to move from a 3-connected hamiltonian matroid whose crissing graph has a certain induced subgraph to a 3-connected hamiltonian minor of the matroid whose crissing graph is the special subgraph.

3.11. Lemma. *Suppose that, for a 3-connected matroid M with a hamiltonian circuit C, the crissing graph $\Gamma(\Pi(M, C))$ has a connected induced subgraph Γ_1 with vertex set V_1 of size at least two. Let $N = M | (V_1 \cup C)$. Then the cosimplification $\underset{\sim}{N}$ of N has a hamiltonian circuit $\tilde{C}$, the crissing graph $\Gamma(\Pi(\underset{\sim}{N}, \tilde{C}))$ equals Γ_1, and $\underset{\sim}{N}$ is 3-connected.*

Proof. Clearly, C is a hamiltonian circuit of N and N is simple. Let A be an HPR for N with respect to C. Suppose i and j are elements of C that are in series in N. Then, by Lemma 2.3, rows i and j of the matrix A are equal. Consider N/j. It has $C - j$ as a hamiltonian circuit. We show next that

(1) $\Gamma(\Pi(N, C)) = \Gamma(\Pi(N/j, C - j))$.

Evidently, both these crissing graphs have vertex set V_1. Moreover, if $x \in V_1$, then $\pi_{N/j}(x)$ is obtained from $\pi_N(x)$ by removing j from the unique block of the latter that contains it. Suppose $\pi_N(s) \not\dagger \pi_N(t)$ for some s and t in V_1. Then $\pi_N(s)$ and $\pi_N(t)$ have blocks A_s and A_t whose union is C. The union of the corresponding blocks of $\pi_{N/j}(s)$ and $\pi_{N/j}(t)$ is $C-j$. We conclude that if $\pi_N(s) \not\dagger \pi_N(t)$, then $\pi_{N/j}(s) \not\dagger \pi_{N/j}(t)$. Now suppose that $\pi_N(s) \dagger \pi_N(t)$ but $\pi_{N/j}(s) \not\dagger \pi_{N/j}(t)$. Then $\pi_{N/j}(s)$ and $\pi_{N/j}(t)$ have blocks A_s and A_t, respectively, such that $A_s \cup A_t = C-j$. Without loss of generality, we may assume that $i \in A_s$. Then, since rows i and j of the matrix A are equal, $A_s \cup j$ is a block of $\pi_N(s)$. Hence $\pi_N(s) \not\dagger \pi_N(t)$. This contradiction completes the proof of (1).

If two elements, v and w, of N/j are parallel, then, as v and w are not parallel in N, it follows that $\{v, w, j\}$ is a circuit of N. Also $i \notin \{v, w\}$ since if, say, $i = v$, then $\pi_{N/j}(w) = \{\{i\}, C - \{i, j\}\}$, which does not criss any partition, contradicting the fact that Γ_1 is connected. Thus the circuit $\{v, w, j\}$ meets the cocircuit $\{i, j\}$ in a single element. This contradiction establishes that no two elements of V_1 are parallel in N/j.

We may now repeat the above process of contracting, one at a time, the series elements of N until we obtain the cosimplification $\underset{\sim}{N}$. It has a hamiltonian circuit $\tilde{C}$, the crissing graph $\Gamma(\Pi(\underset{\sim}{N}, \tilde{C}))$ is Γ_1, no two elements of V_1 are parallel in $\underset{\sim}{N}$, and no two elements of $\tilde{C}$ are in series in $\underset{\sim}{N}$. Thus, by Theorem 3.10, $\underset{\sim}{N}$ is 3-connected. ∎

As in [1], we shall be interested here in reducing to the case when the crissing graph is a clique, a star, or a path. The next lemma identifies an important case that gives rise to the second of these possibilities.

3.12. PROPOSITION. *Let M be a* 3*-connected matroid having an element e such that, for some $k \geqslant 3$, the matroid $M \backslash e$ is a k-spoked fan* (*see Fig.* 5). *Let C be the unique hamiltonian circuit of $M \backslash e$. Then the crissing graph $\Gamma(\Pi(M, C))$ is isomorphic to $K_{1, k-2}$.*

Proof. If s and t are distinct internal spokes of the fan, then, by Lemma 3.2, $\pi(s)$ and $\pi(t)$ do not criss. But, since M is 3-connected, $\Gamma(\Pi(M, C))$ is connected and hence this graph is isomorphic to $K_{1, k-2}$ with the vertex e being adjacent to all other vertices. ∎

4. CRISSINGS AND CROSSINGS

In this paper, we have introduced hamiltonian partial representations of matroids and defined when two columns in such a matrix criss. In our earlier paper [1], matroids were treated using their more familiar matrix representations, and a related definition of crossing columns was used. To prove the main result of this paper, we shall use theorems from [1], so we now discuss the link between the old and new ideas.

Following [1], we call a matrix an *F-matrix* if all of its entries are in some set F where F contains 0 and exactly $q-1$ other elements for some $q \geqslant 2$. Let A be an F-matrix $(a_{i,j})$. Column s of A *dominates* column t if the columns are equal or if, for some non-zero α, the entry $a_{i,s}$ equals α whenever $a_{i,t}$ is non-zero. Columns s and t *cross* if neither dominates the other, and A has a row in which both $a_{i,s}$ and $a_{i,t}$ are non-zero. Thus identical columns do not cross. Moreover, non-identical columns s and t do not cross if either

(1) $0 \in \{a_{i,s}, a_{i,t}\}$ for all i; or

(2) $|\{a_{i,s}: a_{i,t} \neq 0\}| \leqslant 1$; or

(3) $|\{a_{i,t}: a_{i,s} \neq 0\} \leqslant 1$.

4.1. Lemma. *Suppose that columns s and t of an F-matrix A do not cross. Adjoin a zero row to A, let C be the set of row labels of the new matrix A^+, and let $\pi(s)$ and $\pi(t)$ be the partitions of C associated with columns s and t of A^+. Then either*

(i) *$\pi(s)$ and $\pi(t)$ do not criss*; *or*

(ii) *$\pi(s) = \pi(t)$ and each has at least three blocks.*

Proof. Suppose s and t are non-identical. Then, since s and t do not cross, it follows from (1)–(3) that the union of the block of $\pi(s)$ or $\pi(t)$ corresponding to the entry 0 with a block of $\pi(t)$ or $\pi(s)$ associated with some α is C. Thus $\pi(s) \not\chi \pi(t)$.

Now suppose that s and t are identical. Then $\pi(s) = \pi(t)$ and so $\pi(s)$ and $\pi(t)$ do not criss unless each has at least three blocks. ∎

We now know that adjoining a zero row to two non-crossing columns induces two non-crissing partitions provided these partitions are distinct. The next lemma, whose statement is somewhat technical, implies that if we have two non-crissing partitions associated with columns of a matrix and both columns are zero in some row, then, in the matrix obtained by deleting that row, the associated columns are non-crossing.

4.2. Lemma. *Let A be an F-matrix $(a_{i,j})$ whose rows are indexed by the set C. Let s and t be columns of A, and let $\pi(s)$ and $\pi(t)$ be the associated partitions of C. Suppose that $\pi(s)$ does not criss $\pi(t)$, let k be an element of C, and let σ_s and σ_t be permutations of F mapping $a_{k,s}$ and $a_{k,t}$, respectively, to 0. Let A' be obtained from A by applying σ_s and σ_t to the entries* *of columns s and t, respectively, and then deleting row k. Then columns s and t of A' do not cross.*

Proof. Let $A' = (a'_{i,j})$. Since $\pi(s)$ does not criss $\pi(t)$, these partitions have blocks X_s and Y_t, say, whose union is C. Without loss of generality, we may assume that $k \in X_s \cap Y_t$ or $X_s - Y_t$. In the first case, it is clear that, for all i in $C-k$, at least one of $a'_{i,s}$ and $a'_{i,t}$ is zero. Hence columns s and t of A' do not cross. We may now assume that $k \in X_s - Y_t$. In that case, if $a'_{i,s}$ is non-zero, then $i \notin X_s$, so $i \in Y_t$. Indeed, $|\{a'_{i,t}: a'_{i,s} \neq 0\}| = 1$. Hence columns s and t of A' do not cross. ∎

A hamiltonian partial representation A of a matroid M with respect to the circuit C will be said to be in *reducible form* if the last row of A is zero. It is clear that if A is an F-matrix that is an HPR for M, then, by applying a permutation of F to each individual column of A, we can obtain an HPR of M in reducible form. A *reduced hamiltonian partial representation D* for a matroid M is a matrix that can be obtained from an HPR for M that is in reducible form by deleting the last row. The next result follows immediately on combining Lemmas 4.2 and 4.1.

4.3. PROPOSITION. *Let M be a 3-connected matroid having a hamiltonian circuit C and suppose $|E(M)| \geqslant 4$. Let A be an F-matrix that is an HPR for M with respect to C. Suppose that A is in reducible form, and let A' be the corresponding reduced HPR for M. Let s and t be elements of $E(M) - C$ such that $\pi(s) \neq \pi(t)$. Then $\pi(s) \dagger \pi(t)$ if and only if s crosses t.*

There are two matrix theorems from [1] that will be needed in the proof of our main theorem. For completeness, we state both of these below. Before giving these statements, we shall also need to recall some further definitions from [1]. The matrix B is a *row-permuted submatrix* of the matrix A if B can be obtained from some submatrix of A by permuting its rows. Now suppose α, β, and γ are elements of F that are not all equal. A square F-matrix $A = (a_{i,j})$ is (α, β, γ)-*diagonal* if

$$a_{i,j} = \begin{cases} \alpha, & \text{if } i < j; \\ \beta, & \text{if } i = j; \\ \gamma, & \text{if } i > j. \end{cases}$$

Now let α and β be arbitrary non-zero elements of F. An F-matrix is (α, β)-*complete* if the rows of A consist of all of the $\binom{n}{2}$ distinct vectors of length n that have exactly two non-zero entries, the first being α and the second β.

The next two results are slight restatements of Theorems 2.8 and 2.7, respectively, of [1].

4.4. THEOREM. *There is a function g_6 with the following property: Suppose h is an integer greater than two, $|F| = q$, and A is an F-matrix with at least $g_6(h, q)$ columns such that no two columns of A are identical, and two distinct columns j' and j'' cross if and only if $1 \in \{j', j''\}$. Then A contains a row-permuted submatrix B that satisfies one of the following conditions:*

(i) *B is obtained from an $(\alpha, \alpha, 0)$-diagonal matrix with h rows by replacing its first column by a column of the form $(\beta, \delta, \delta, ..., \delta)^T$ for some $\beta \neq \delta$, and then adjoining, to the bottom of the matrix, a new row of the form $(\gamma, 0, 0, ..., 0)$ for some $\gamma \neq 0$.*

(ii) *B is obtained from a $(0, \alpha, \alpha)$-diagonal matrix with h rows by deleting its last column, adjoining to the beginning of the matrix a new column of the form $(\delta, \delta, ..., \delta, \beta)^T$ for some $\beta \neq \delta$, and then adjoining, to the bottom of the matrix a new row of the form $(\gamma, 0, 0, ..., 0)$ for some $\gamma \neq 0$.*

(iii) *B is obtained by putting a $(0, \alpha, 0)$-diagonal matrix with h rows above a $(0, \beta, 0)$-diagonal matrix with h rows and then adjoining, to the beginning of the matrix, a new column in which the first h entries all equal some non-zero δ and the last h entries all equal some $\gamma \neq \delta$.* ▌

4.5. THEOREM. *There is a function g_5 with the following property: If h is an integer exceeding one, $|F| = q$, and A is an F-matrix with at least $g_5(h, q)$ columns such that every two columns of A cross, then A contains a row-permuted submatrix B that has h columns and satisfies one of the following conditions:*

(i) *B is obtained from a $(0, \alpha, 0)$-diagonal matrix by adjoining a new row every entry of which is equal to some non-zero β.*

(ii) *B is obtained from an $(\alpha, \alpha, 0)$-diagonal matrix or a $(0, \alpha, \alpha)$-diagonal matrix by adjoining a new row all of whose entries are equal to some β from $F - \{0, \alpha\}$.*

(iii) *B is (α, β, γ)-diagonal where α and γ are both non-zero and $\alpha \neq \beta$.*

(iv) *B is obtained by putting a $(0, \alpha, \alpha)$-diagonal matrix above an $(\alpha, \alpha, 0)$-diagonal matrix.*

(v) *B is (α, β)-complete where α and β are both non-zero.* ▌

5. AN OVERVIEW OF THE PROOF

In earlier sections, we have introduced the new tools that we shall use to prove our main theorem. This proof is long and has many technical details. In view of this, it seems useful to outline in this section how the proof will proceed. Some parts of this outline will be relatively complete; others will be sketchy with the full details to follow. It may be helpful for the reader to recognize that, in spite of significantly more technical details here, to a large extent the proof of our main theorem follows the same lines as the proof of the main result of [1].

The first important step in proving the main theorem is to show that every 3-connected matroid with a huge number of elements has either a big circuit or a big cocircuit. This follows from the following result of Lovász, Schrijver, and Seymour (see [4]), which was strengthened by Reid [5].

5.1. THEOREM. *Let n be an integer greater than one. Every connected matroid with more than 4^n elements has a circuit or cocircuit with more than n elements.*

By passing to the dual if necessary, we may now assume that our huge 3-connected matroid M has a big circuit. The next important step is to establish that M has a big 3-connected hamiltonian minor. This is accomplished using the following immediate consequence of Theorem 3.1 of [1].

5.2. THEOREM. *Let C be a maximum-sized circuit of a 3-connected matroid M. Then M has a 3-connected minor M' in which C is a hamiltonian circuit.* ∎

We are now able to focus attention on a 3-connected matroid with a hamiltonian cycle C where $|C|$ is big. Not surprisingly, we now consider an HPR for such a matroid. The next result shows that either we obtain a minor of one of the desired types, or every column in an HPR has a bounded number of distinct entries.

5.3. LEMMA. *Let A be an HPR of a matroid M with respect to a circuit C. If A has a column with more that q distinct entries, then M has a $U_{q,q+2}$-minor.*

Proof. Let e be a column of A that has at least $q+1$ distinct entries and consider $M_1 = M \mid (C \cup e)$. Now we choose elements $i_1, i_2, ..., i_{q+1}$ of C such that the entries in the corresponding rows of column e are all distinct. Let $M_2 = M_1/(C - \{i_1, i_2, ..., i_{q+1})$. Then M_2 has $\{i_1, i_2, ..., i_{q+1}\}$ as a hamiltonian circuit and has an HPR with respect to this circuit that consists of a single column with $q+1$ distinct entries. Evidently $r(M_2) = q$ and $r^*(M_2) = 2$. Moreover, by Corollary 2.4, M_2 has no 2-cocircuits. Thus $M_2 \cong U_{q,q+2}$. ∎

In view of this lemma, when we consider hamiltonian partial representations in what follows, we shall generally assume that all the entries of such a matrix are taken from a q-element set F that contains 0.

Next we transfer attention to the crissing graph of an HPR of a 3-connected matroid. The vertices of such a graph are the elements of the matroid that are not in the hamiltonian circuit. We shall want the number of these vertices to be large. The next lemma enables us to get a bound on this number.

5.4. LEMMA. *Let $|F| = q$ and let A be an F-matrix that is an HPR for a 3-connected matroid M with respect to the circuit C. Then A has a set X of at least $\log_q |C|$ columns such that every two columns in X induce different partitions of C, and $M \mid (X \cup C)$ is 3-connected.*

Proof. Consider the crissing graph $\Gamma = \Gamma(\Pi(M, C))$. As M is 3-connected, Theorem 3.10 implies that this graph is connected. Suppose e and f are columns of A such that $\pi(e) = \pi(f)$. Then e and f, as vertices of Γ, have precisely the same neighbors in $V(\Gamma) - \{e, f\}$. Hence $\Gamma - \{f\}$ is certainly connected. Moreover, this graph equals $\Gamma(\Pi(M \backslash f, C))$. We should like to assert that $M \backslash f$ is 3-connected. By Theorem 3.10, this is so unless, in $M \backslash f$, two elements of C are in series. But in the exceptional case, the corresponding rows of the matrix obtained from A by deleting column f are equal. Hence the corresponding rows of A are equal and so two elements of C are in series in M; a contradiction. We conclude that $M \backslash f$ is 3-connected. By repeating this procedure of deleting columns from A one at a time while A retains another column that yields the same partition of C, we eventually obtain a 3-connected restriction M_1 of M containing C such that, in the corresponding HPR A_1, say, every two columns induce distinct partitions of C. In particular, every two columns of A_1 are distinct. Since no two elements of C are in series, Corollary 2.4 implies that the rows of A_1 are distinct. Thus, if A_1 has k distinct columns, then $|C| \leqslant |F|^k = q^k$. The lemma follows immediately. ∎

By Theorem 3.10, a 3-connected hamiltonian matroid has a connected crissing graph. Next we recall Theorem 5.3 from [1] which guarantees, in every sufficiently large simple connected graph, a large induced subgraph that is a star, a path, or a clique. We shall also need to recall some notation: P_m will denote a path with m vertices; $R(x, y)$ will denote the least positive integer k such that, in every edge-coloring of a k-clique with y colors, there is a monochromatic x-clique.

5.5. THEOREM. *Let m be a positive integer and let G be a simple connected graph on $(R(m, 2))^m$ vertices. Then G has an induced subgraph isomorphic to P_m, $K_{1,m}$, or K_m.*

By applying this theorem to the crissing graph and using Lemma 3.11 and Lemma 5.4, we are able to reduce to the case of a large 3-connected hamiltonian matroid whose crissing graph is a star, a path, or a clique. These three cases will be treated separately in the next three sections. The first and third of these sections will rely heavily on significant theorems from [1] and so will be relatively short. The second section is long and involves making very detailed use of hamiltonian partial representations. After these three sections are complete, we shall be able to finish the proof of the main theorem and this will be done in Section 9.

6. THE STAR CASE

In this section, we consider the case when the crissing graph of a 3-connected hamiltonian matroid is a star. The main result of this section is the following:

6.1. THEOREM. *Let m be an integer exceeding two and M be a* 3-*connected matroid with a hamiltonian circuit C such that M has an F-matrix A as an HPR with respect to C. If $d \geqslant g_6((m+2)^8, q)$ and $\Gamma(\Pi(M, C))$ is isomorphic to $K_{1,d}$, then M has a minor isomorphic to one of $M(W_m)$, W^m, $U_{m,m+2}$, or $M^*(K_{3,m})$.*

The proof of this theorem relies heavily on Theorem 4.4. With the help of a straightforward preliminary result, Lemma 6.9, we shall show that cases (i) and (ii) of that theorem yield a wheel or whirl minor of M. First, however, we consider the case when (iii) of Theorem 4.4 arises. In that case, it is more difficult to produce a minor of one of the desired types. The next theorem will be used to accomplish this goal. First, however, we introduce another class of matroids. Let h be an integer exceeding one. An *h-raft* is a matroid of rank $2h-2$ whose ground set is the union of h disjoint triangles such that, for all $k<h$, the union of every set of k of these triangles has rank $2k$.

6.2. THEOREM. *Suppose that m is an integer exceeding two and h is an integer such that $h \geqslant (m+2)^8$. If M is an h-raft, then M has either $U_{m,m+2}$ or $M^*(K_{3,m})$ as a minor.*

In order to prove this theorem, we now develop some structure theory for rafts. Hence suppose M is an h-raft and let the h distinguished triangles of M be $L_1, L_2, \ldots, L_h$ where $L_i = \{x_i, y_i, z_i\}$. Let $H = \{1, 2, \ldots, h\}$ and, for all subsets I of H, let $X(I) = \{x_i : i \in I\}$. Define $Y(I)$ and $Z(I)$ similarly, and abbreviate $X(H)$, $Y(H)$, and $Z(H)$ to simply X, Y, and Z. Finally, let $L(I) = X(I) \cup Y(I) \cup Z(I)$.

The following is a straightforward consequence of the definition of a raft.

6.3. Lemma. *For all subsets I of H with $|I| \geqslant 2$, the matroid $M.L(I)$ is an $|I|$-raft.* ∎

For all distinct i and j in $\{1, 2, ..., h\}$, let

$$N_{i,j} = M.(L_i \cup L_j).$$

Then $N_{i,j}$ is a 2-raft. In particular, it is a loopless rank-2 matroid having both $\{x_i, y_i, z_i\}$ and $\{x_j, y_j, z_j\}$ as circuits.

These 2-rafts are interrelated as follows:

6.4. Lemma. *If a_i is parallel to b_j in $N_{i,j}$, and b_j is parallel to c_k in $N_{j,k}$, then a_i is parallel to c_k in $N_{i,k}$.*

Proof. Consider the matroid $N_{i,j,k} = M.(L_i \cup L_j \cup L_k)$. This matroid has rank 4 and has L_i, L_j, and L_k as 3-circuits. Moreover, since, for example, $N_{i,j} = N_{i,j,k}/\{x_k, y_k\}\backslash z_k$, it follows that $N_{i,j,k}$ has circuits contained in $\{a_i, b_j, x_k, y_k\}$ and $\{b_j, c_k, x_i, y_i\}$ that contain $\{a_i, b_j\}$ and $\{b_j, c_k\}$, respectively. Thus, taking ranks in $N_{i,j,k}$, we have

$$r(\{a_i, b_j, x_k, y_k, z_k\}) = 3 = r(\{b_j, c_k, x_i, y_i, z_i\}).$$

By submodularity, since $\{a_i, b_j, x_k, y_k, z_k\} \cup \{b_j, c_k, x_i, y_i, z_i\}$ has rank four, the rank of $\{a_i, b_j, x_k, y_k, z_k\} \cap \{b_j, c_k, x_i, y_i, z_i\}$ is at most two. Thus $\{a_i, b_j, c_k\}$ has rank equal to two and this set is a circuit of $N_{i,j,k}$. The lemma follows now without difficulty. ∎

Now associate an auxiliary graph $G(M)$ with the h-raft M. The vertex set of $G(M)$ is $E(M)$, and $G(M)$ has an edge joining elements a_i of L_i and b_j of L_j if and only if $\{a_i, b_j\}$ is a circuit of $N_{i,j}$.

6.5. Lemma. *The graph $G(M)$ uniquely determines the h-raft M.*

Proof. M has rank $2h-2$ and it is not difficult to see that its set of bases is composed of:

(i) for all i in H, every set that consists of exactly two elements from each of the $h-1$ distinguished triangles L_k such that $k \in H - i$; and

(ii) for all distinct i and j in H, every set B that consists of exactly two elements from each of the $h-2$ distinguished triangles L_k with $k \in H - \{i, j\}$ along with exactly one element from each of L_i and L_j such that $B \cap (L_i \cup L_j)$ is independent in $N_{i,j}$.

Evidently the bases of type (ii) correspond to pairs of non-adjacent vertices a_i and b_j in $G(M)$, and the lemma follows easily. ∎

By Lemma 6.4, the graph $G(M)$ is a vertex-disjoint union of cliques. Moreover, we may permute the labels in each L_i so that each of these cliques has its vertex set contained in one of X, Y, or Z. From now on, we shall assume that this permutation has been done. For all subsets I of H, let $G_{X(I)}$, $G_{Y(I)}$, and $G_{Z(I)}$ be the subgraphs of $G(M)$ induced by $X(I)$, $Y(I)$, and $Z(I)$, respectively.

Next we prove the following:

6.6. Lemma. *Let $h_1 = \lfloor \sqrt{h} \rfloor$, $h_2 = \lfloor \sqrt{h_1} \rfloor$, and $h_3 = \lfloor \sqrt{h_2} \rfloor - 2$. Then either*

(i) *G_X or G_Y or G_Z has a stable set with $h_3 + 2$ vertices; or*

(ii) *for some h_3-element subset $\hat{I}$ of H, the graphs $G_{X(\hat{I})}$, $G_{Y(\hat{I})}$, and $G_{Z(\hat{I})}$ are all cliques.*

Proof. Assume that (i) fails. The graph G_X has either

(X)(i) a stable set with h_1 vertices; or

(X)(ii) a clique with h_1 vertices.

Since (i) fails, (X)(i) also fails. Thus (X)(ii) holds. Let $X(I')$ be the vertex set of such a clique. Clearly $G_{Y(I')}$ is, like G_Y, a disjoint union of cliques. Therefore $G_{Y(I')}$ has either

(Y)(i) a stable set with h_2 vertices; or

(Y)(ii) a clique with h_2 vertices.

Since (Y)(ii) must hold, we let $Y(I'')$ be the vertex set of such a clique. Evidently $G_{Z(I'')}$ has either

(Z)(i) a stable set with $h_3 + 2$ vertices; or

(Z)(ii) a clique with h_3 vertices.

Since (Z)(ii) must hold, if we let $Z(\hat{I})$ be the vertex set of such a clique, then we deduce that (ii) of the lemma must hold. ∎

6.7. Lemma. *Let $Z(I)$ be a stable set in G_Z of size $p+2$ where $p \geqslant 2$. Then $M.(X(I) \cup Y(I)) \mid X(I) \cong U_{p,\, p+2}$.*

Proof. By Lemma 6.3, $M.L(I)$ is a $(p+2)$-raft. Because $Z(I)$ is a stable set of G_Z, this set is independent in $M.L(I)$. Moreover, from the description of the bases of a raft in the proof of Lemma 6.5, we have that, for all p-element subsets I' of I, the set $Z(I) \cup X(I')$ is a basis of $M.L(I)$. Hence $X(I')$ is a basis of $M.L(I)/Z(I)$. But the last matroid is $M.(X(I) \cup Y(I))$ and so, on restricting this matroid to $X(I)$, we have a $(p+2)$-element matroid of rank p in which every p-element subset is a basis. This matroid is isomorphic to $U_{p,\,p+2}$. ∎

6.8. LEMMA. *Let $\hat{I}$ be a p-element subset of H such that $p \geqslant 3$ and all of the graphs $G_{X(\hat{I})}$, $G_{Y(\hat{I})}$, and $G_{Z(\hat{I})}$ are cliques. Then $M.L(\hat{I}) \cong M^*(K_{3,\,p})$.*

Proof. Label the graph $K_{3,\,p}$ so that the three vertices of degree p are incident with the sets $\{x_1, x_2, ..., x_p\}$, $\{y_1, y_2, ..., y_p\}$, and $\{z_1, z_2, ..., z_p\}$, and the p vertices of degree three are incident with the sets $\{x_1, y_1, z_1\}$, $\{x_2, y_2, z_2\}$, ..., $\{x_p, y_p, z_p\}$. It is easy to see that $M^*(K_{3,\,p})$ is a p-raft with distinguished triangles $\{x_1, y_1, z_1\}$, $\{x_2, y_2, z_2\}$, ..., $\{x_p, y_p, z_p\}$. Also, $\{x_1, x_2, y_1, y_2, z_3, z_4, ..., z_p\}$ is a not a spanning tree of $K_{3,\,p}$, so its complement, $\{x_3, x_4, ..., x_p, y_3, y_4, ..., y_p, z_1, z_2\}$, is not a basis of $M^*(K_{3,\,p})$. Thus, by the proof of Lemma 6.5, z_1 and z_2 are adjacent in $G(M^*(K_{3,\,p}))$. By symmetry, each of $\{x_1, x_2, ..., x_p\}$, $\{y_1, y_2, ..., y_p\}$, and $\{z_1, z_2, ..., z_p\}$ induces a clique in $G(M^*(K_{3,\,p}))$. The lemma now follows by Lemma 6.5. ∎

To prove Theorem 6.2, we only need to combine some of the preceding lemmas.

Proof of Theorem 6.2. As $h \geqslant (m+2)^8$, in Lemma 6.6, $h_1 \geqslant (m+2)^4$, so $h_2 \geqslant (m+2)^2$, and $h_3 \geqslant m$. If (i) of Lemma 6.6 occurs, then Lemma 6.7 implies that M has a $U_{m,\,m+2}$-minor; if (ii) of Lemma 6.6 occurs, then Lemma 6.8 implies that M has an $M^*(K_{3,\,m})$-minor. ∎

The next lemma, which is relatively straightforward, will also be used in the proof of Theorem 6.1.

6.9. LEMMA. *Let k be an integer exceeding one, and let N be a matroid whose ground set is $\{x_1, x_2, ..., x_k, y_1, y_2, ..., y_k\}$ such that $\{x_1, x_2, ..., x_k\}$ is a basis and, for all i in $\{1, 2, ..., k\}$, the set $\{x_i, y_i, x_{i+1}\}$ is a circuit where $x_{k+1} = x_1$. Then N is a wheel or whirl of rank k.*

Proof. Let $X = \{x_1, x_2, ..., x_k\}$ and let $Y = \{y_1, y_2, ..., y_k\}$. For all y_i in Y, the matroid $N \backslash y_i$ is isomorphic to the cycle matroid of a fan for which X is the set of spokes. Hence every circuit of $N \backslash y_i$ contains exactly two elements of X. Moreover, we know all the circuits of N except for those that contain Y. Moreover, $Y - y_i$ is independent in N for all i. Now either Y is dependent, or Y is independent. In the former case, Y is a circuit of N and it follows that N is a k-spoked wheel. Thus we may assume that Y is independent, so Y is a basis of N. Take x_i in X and consider the fundamental circuit $C(x_i, Y)$. If this circuit avoids some y_j, then $C(x_i, Y)$ is a circuit of the k-spoked fan $N \backslash y_j$ that contains just one element of X; a contradiction. Thus $C(x_i, Y) = Y \cup x_i$ for all i and it follows that N is a whirl of rank k. ∎

Proof of Theorem 6.1. We may assume that A is in reducible form and let A' be the associated reduced HPR for M. Because $\Gamma(\Pi(M, C))$ is a star, no two columns of A are identical. Thus, by Proposition 4.3, no two columns of A' are identical and we may assume that two distinct columns j' and j'' of A' cross if and only if $1 \in \{j', j''\}$. Let $h = (m+2)^8$. Since A' has at least $g_6(h, q)$ columns, Theorem 4.4 implies that A' has a row-permuted submatrix B' satisfying (i), (ii), or (iii) of that theorem. But every matrix satisfying (ii) can be transformed into one satisfying (i) by a sequence of row and column permutations. We conclude that we may assume that M has a minor having a reduced HPR that satisfies (i) or (iii) of Theorem 4.4.

In the first case, we may assume M has a hamiltonian minor N for which the HPR with respect to the circuit $C_N = \{0, 1, 2, ..., h+1\}$ is the matrix shown in Fig. 1 where $\alpha \neq 0$; $\beta \neq \delta$; and $\gamma \neq 0$. We now distinguish two subcases: (a) $\delta \neq 0$ and (b) $\delta = 0$.

Assume that (a) holds and consider $N_1 = N/h \backslash e_h$. An HPR for this matroid with respect to the circuit $C_N - \{h\}$ is obtained by deleting the second last row and the last column of the matrix in Fig. 1. By applying Lemma 2.3, we deduce that the cocircuits of N_1 include $\{0, 1, e_1\}$, $\{1, 2, e_2\}$, ..., $\{h-2, h-1, e_{h-1}\}$, $\{h-1, h+1, e_1\}$. Moreover, N_1 has rank h. Consider $\{e_1, 1, 2, ..., h-1\}$. This is a basis for N_1^* if and only if $\{0, e_2, e_3, ..., e_{h-1}, h+1\}$ is a basis for N_1. But the last set has a unique element in each of the triads of N_1 noted above. Thus $\{0, e_2, e_3, ..., e_{h-1}, h+1\}$ is, indeed, a basis for N_1 and so $\{e_1, 1, 2, ..., h-1\}$ is a basis for N_1^*. We may now apply Lemma 6.9 to N_1^* using the triads noted for N_1, which are triangles for N_1^*. From this lemma, it follows that N_1^*, and hence N_1, is a wheel or a whirl of rank h.

Now assume that (b) holds, that is, $\delta = 0$ in Fig. 1. In that case, we know that none of α, β, or γ is zero. Let $N_1 = N/(h+1) \backslash e_h$. Then, as in (a), all of $\{0, 1, e_1\}$, $\{1, 2, e_2\}$, ..., $\{h-2, h-1, e_{h-1}\}$, $\{h-1, h, e_1\}$ are triads

$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & \cdots & e_{h-1} & e_h \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ h-2 \\ h-1 \\ h \\ h+1 \end{array} &
\begin{pmatrix}
\beta & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\
\delta & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\
\delta & 0 & \alpha & \alpha & \cdots & \alpha & \alpha \\
\delta & 0 & 0 & \alpha & \cdots & \alpha & \alpha \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta & 0 & 0 & 0 & \cdots & \alpha & \alpha \\
\delta & 0 & 0 & 0 & \cdots & 0 & \alpha \\
\gamma & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 1

of N_1. Moreover, if $Y = \{0, e_2, e_3, ..., e_{h-1}, h\}$, then Y contains exactly one element of each of these cocircuits and, as $|Y| = r(N_1)$, it follows that Y is a basis of N_1. Hence $E(N_1) - Y$ is a basis of N_1^* and, by applying Lemma 6.9 to the last matroid, we deduce that N_1^*, and hence N_1, is a wheel or whirl of rank h.

It remains to consider the case when M has a minor having a reduced HPR satisfying (iii) of Theorem 4.4. Thus we may assume that M has a hamiltonian minor N for which the HPR with respect to the circuit $C_N = \{1, 2, ..., 2h, 2h+1\}$ is the matrix shown in Fig. 2 where $0 \notin \{\alpha, \beta, \gamma\}$ and $\delta \neq \gamma$.

Consider $N/(2h+1)$. It has an HPR that is obtained by deleting the last row of the matrix in Fig. 2. This HPR has no two identical rows and its associated crissing graph is a star. Thus, by Theorem 3.10, $N/(2h+1)$ is 3-connected. By Lemma 2.1(i), for all i in $\{1, 2, ..., h\}$, since $\pi_{N/(2h+1)}(e_i)$ has $(C_N - \{2h+1\}) - \{i, h+i\}$ as a block, it follows that $\{i, h+i, e_i\}$ is a circuit of $N/(2h+1)$. Let $N_1 = N/(2h+1)/e_0$. To complete the proof of Theorem 6.1, we shall use Theorem 6.2 to prove the next result.

6.10. LEMMA. *N_1 has a minor isomorphic to $U_{m,m+2}$ or $M^*(K_{3,m})$.*

Proof. We show first that

(1) $\{i, h+i, e_i\}$ is a triangle of N_1 for all i in $\{1, 2, ..., h\}$.

Suppose that, for some i, this set is not a triangle of N_1. Then, since $N/(2h+1)$ is 3-connected, it follows that every 3-element subset of $\{i, h+i, e_i, e_0\}$ and, in particular, $\{i, e_0, e_i\}$ is a triangle of $N/(2h+1)$. Now take j in $\{1, 2, ..., h\} - \{i\}$. The cocircuit of $N/(2h+1)$ that is the complement of the closure of $(C_N - \{2h+1\}) - \{j, j+h\}$ contains e_0, but

$$
\begin{array}{c}
\\ 1 \\ 2 \\ \vdots \\ h-1 \\ h \\ h+1 \\ h+2 \\ \vdots \\ 2h-1 \\ 2h \\ 2h+1
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_0 & e_1 & e_2 & \cdots & e_{h-1} & e_h \end{array} \\
\begin{pmatrix}
\delta & \alpha & 0 & \cdots & 0 & 0 \\
\delta & 0 & \alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta & 0 & 0 & \cdots & \alpha & 0 \\
\delta & 0 & 0 & \cdots & 0 & \alpha \\
\gamma & \beta & 0 & \cdots & 0 & 0 \\
\gamma & 0 & \beta & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & 0 & 0 & \cdots & \beta & 0 \\
\gamma & 0 & 0 & \cdots & 0 & \beta \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 2

neither e_i nor i. Thus $N/(2h+1)$ has a cocircuit and a circuit with exactly one common element. This contradiction completes the proof of (1).

Evidently, N_1 has rank $2h-2$. Therefore, by Theorem 6.2, the lemma will follow if we can show that

(2) $\{1, 2, ..., 2h\} - \{i, h+i\}$ is independent in N_1 for all i in $\{1, 2, ..., h\}$.

The last assertion holds if, for all such i, the set $\{e_0, 1, 2, ..., 2h, 2h+1\} - \{i, h+i\}$ is independent in N. But, since the partition of C_N induced by e_0 in N has exactly three blocks, $\{2h+1\}$, $\{1, 2, ..., h\}$, and $\{h+1, h+2, ..., 2h\}$, it follows, by Lemma 2.1, that $N \mid (C_N \cup e_0)$ contains exactly four circuits: C_N, $\{e_0\} \cup \{1, 2, ..., h, 2h+1\}$, $\{e_0\} \cup \{h+1, h+2, ..., 2h, 2h+1\}$, and $\{e_0\} \cup \{1, 2, ..., 2h\}$. As $\{i, h+i\}$ meets each of these circuits, $\{e_0, 1, 2, ..., 2h\} - \{i, h+i\}$ is independent in $N/(2h+1)$, so (2) holds. This completes the proof of Lemma 6.10 and thereby finishes the proof of Theorem 6.1. ∎

7. THE PATH CASE

In this section, we analyze the structure of a 3-connected matroid for which the crissing graph with respect to some hamiltonian circuit is a path.

The main result of this section is the following:

7.1. THEOREM. *Let m be an integer exceeding two, and let M be a 3-connected matroid with a hamiltonian circuit C such that the crissing graph $\Gamma(\Pi(M, C))$ is isomorphic to a path of length r for some $r \geqslant m^2+1$. Then either*

(i) *M has a $U_{2,m}$-minor; or*

(ii) *M has a 3-connected minor that is a single-element extension of an $(m+1)$-spoked fan.*

The proof of this theorem is long. The first main step is to prove the next theorem, which describes the structure of an HPR of a 3-connected matroid for which the crissing graph with respect to some circuit is a path. The proof of Theorem 7.2 is broken into a sequence of lemmas and concludes with the proof of Lemma 7.12. The specific form of the HPR of M, which is shown in Fig. 3 below, will then be analyzed. Another auxiliary graph will be associated with this HPR and the cases when this auxiliary graph has a big stable set and a big clique will then be treated. From the first of these, (i) of Theorem 7.1 will follow fairly easily, the proof of this being completed in Lemma 7.17. The other case, which will require much more effort to produce a minor of one of the desired types, concludes with the proof of Lemma 7.29.

Theorem 7.1 will rely heavily on the next result.

7.2. THEOREM. *For some fixed integer n exceeding three, let* $1, 2, ..., n$ *be an n-vertex path P. Let M be a* 3*-connected matroid that is minor-minimal with the property that the crissing graph with respect to some hamiltonian circuit C is P. Then, for all i in* $\{1, 2, ..., n-2\}$, *the partitions* $\pi(1) \wedge \pi(2) \wedge \cdots \wedge \pi(i)$ *and* $\pi(i+2) \wedge \pi(i+3) \wedge \cdots \wedge \pi(n)$ *have blocks* A_i *and* B_{i+2}, *respectively, such that*

(i) $A_1 \supsetneqq A_2 \supsetneqq \cdots \supsetneqq A_{n-2}$;

(ii) $B_3 \subsetneqq B_4 \subsetneqq \cdots \subsetneqq B_n$;

(iii) $A_i \cup B_{i+2} = C$;

(iv) $A_i \cap B_{i+2} = \emptyset$;

(v) $|C - A_1| = 2 = |C - B_n|$; *and*

(vi) $|A_j - A_{j+1}| = |B_{j+3} - B_{j+2}|$ *for all j in* $\{1, 2, ..., n-3\}$.

Proof. For all i and j in $\{1, 2, ..., n\}$, let $S_i = \{1, 2, ..., i\}$ and $T_j = \{j, j+1, ..., n\}$. For all such i and j, define $\pi(S_i) = \bigwedge_{s \in S_i} \pi(s)$ and $\pi(T_j) = \bigwedge_{t \in T_j} \pi(t)$. If $t \geqslant i+2 \geqslant 3$, then $\pi(t)$ crisses none of $\pi(1), \pi(2), ..., \pi(i)$. Hence, by Lemma 3.6, $\pi(t) \not\dagger \pi(S_i)$. Similarly, if $s \leqslant j-2 \leqslant n-2$, then $\pi(s)$ crisses none of $\pi(j), \pi(j+1), ..., \pi(n)$, so $\pi(s) \not\dagger \pi(T_j)$.

Now suppose that $n \geqslant j \geqslant i+2 \geqslant 3$. Then none of $\pi(j), \pi(j+1), ..., \pi(n)$ crisses $\pi(S_i)$. Hence, by Lemma 3.6 again, $\pi(T_j) \not\dagger \pi(S_i)$. We show next that, for all such i and j,

(1) $\mu(\pi(S_i), \pi(T_j))$ is well-defined.

First, we note that, since M is loopless, none of $\pi(1), \pi(2), ..., \pi(n)$ has a unique block. Thus $\mu(\pi(S_i), \pi(T_j))$ is well-defined unless $\pi(S_i)$ and $\pi(T_j)$ are equal and have exactly two parts. In the exceptional case, $\pi(1) = \pi(2) = \cdots = \pi(i) = \pi(j) = \pi(j+1) = \cdots = \pi(n)$ and this common partition has exactly two blocks; so $\pi(1) = \pi(n)$ and, by Lemma 2.1, 1 and n are parallel; a contradiction. We conclude that (1) holds.

For all i and j with $n \geqslant j \geqslant i+2 \geqslant 3$, define $\mu(\pi(S_i), \pi(T_j)) = (A_{i,j}, B_{i,j})$. Next we prove the following:

7.3. LEMMA. *For all i and j such that* $3 \leqslant i+2 \leqslant j \leqslant n-1$,

(i) $A_{i,j} = A_{i,j+1}$; *and*

(ii) $B_{i,j} \subseteq B_{i,j+1}$.

Proof. For all such i and j, we know that $\pi(j) \dagger \pi(j+1)$. Thus, by Lemma 3.6, $\pi(j) \dagger \pi(T_{j+1})$. Moreover, $\pi(T_j) \leqslant \pi(j)$ so, by Lemma 3.3, $\pi(T_j) \dagger \pi(T_{j+1})$. But $\pi(S_i)$ crosses neither $\pi(T_j)$ nor $\pi(T_{j+1})$, and none of these partitions equals $\{C\}$. Thus, by Lemma 3.5, since $\mu(\pi(S_i), \pi(T_j)) = (A_{i,j}, B_{i,j})$ and $\mu(\pi(S_i), \pi(T_{j+1})) = (A_{i,j+1}, B_{i,j+1})$, we have $A_{i,j} = A_{i,j+1}$. Moreover, $\mu(\pi(T_j) \wedge \pi(T_{j+1}), \pi(S_i)) = (B_{i,j} \cap B_{i,j+1}, A_{i,j})$. But $\pi(T_j) \wedge \pi(T_{j+1}) = \pi(T_j)$ and, as $\mu(\pi(T_j), \pi(S_i)) = (B_{i,j}, A_{i,j})$, we deduce that $B_{i,j} \cap B_{i,j+1} = B_{i,j}$, and (ii) follows immediately. ▌

The next lemma follows by the same argument that was used to prove the last result.

7.4. LEMMA. *For all i and j such that $4 \leqslant i+2 \leqslant j \leqslant n$,*

(i) $B_{i-1,j} = B_{i,j}$; *and*

(ii) $A_{i-1,j} \supseteq A_{i,j}$.

Now, for all i in $\{1, 2, ..., n-2\}$, we define A_i to be the set that equals all of $A_{i,i+2}, A_{i,i+3}, ..., A_{i,n}$. Likewise, for all j in $\{3, 4, ..., n\}$, let B_j be the set that equals all of $B_{j-2,j}, B_{j-3,j}, ..., B_{1,j}$.

On combining the last two lemmas, we deduce that

(2) $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-2}$; and

(3) $B_3 \subseteq B_4 \subseteq \cdots \subseteq B_n$.

By definition, for all i and j with $n \geqslant j \geqslant i+2 \geqslant 3$, we have $A_{i,j} \cup B_{i,j} = C$. Thus $A_i \cup B_j = C$ for all such i and j, that is, (iii) of Theorem 7.2 holds. To complete the proofs of (i) and (ii) of the theorem, it remains to show that all the inclusions in (2) and (3) are proper.

Suppose that, for some i, we have $B_{i+1} = B_{i+2}$. Then $A_i \cup B_{i+1} = C$ so $\pi(S_i) \not\dagger \pi(T_{i+1})$. But $\pi(i) \dagger \pi(i+1)$ and $\pi(S_i) \leqslant \pi(i)$ so, by Lemma 3.3, $\pi(S_i) \dagger \pi(i+1)$. Thus, as $\pi(i+1) \geqslant \pi(T_{i+1})$, Lemma 3.3 implies that $\pi(S_i) \dagger \pi(T_{i+1})$. This contradiction implies that strict inequality holds throughout (2) and (3). Hence (i) and (ii) hold.

By definition, A_1 is a block of $\pi(1)$ and B_n is a block of $\pi(n)$. Moreover, the following lemma is straightforward and its proof is omitted.

7.5. LEMMA. *For all i in $\{2, 3, ..., n-2\}$ and all j in $\{3, 4, ..., n-1\}$, the partitions $\pi(i)$ and $\pi(j)$ have unique blocks A'_i and B'_j such that $A_i = A'_i \cap A_{i-1}$ and $B_j = B'_j \cap B_{j+1}$.*

The proofs of (iv)–(vi) of the theorem will be contained in the following sequence of seven lemmas.

7.6. LEMMA. $|C - A_1| \geqslant 2$ *and* $|C - B_n| \geqslant 2$.

Proof. $\pi(1)$ crisses $\pi(2)$, so $|C - A_1| \neq 1$. Similarly, $\pi(n-1)$ crisses $\pi(n)$, so $|C - B_n| \neq 1$. ▌

Next we consider the effect on non-crissing elements of contracting an element of C.

7.7. LEMMA. *If $e \in C$, then $C-e$ is a hamiltonian circuit of M/e. Moreover, for all positive integers i and all k in $\{2, 3, ..., n-i\}$, the partitions $\pi_{M/e}(i)$ and $\pi_{M/e}(i+k)$ do not criss.*

Proof. It is clear that $C-e$ is a hamiltonian circuit of M/e. Moreover, for all i and k satisfying the specified conditions, $A_i \cup B_{i+k} = C$. As $\pi_{M/e}(i)$ and $\pi_{M/e}(i+k)$ have blocks containing $A_i - e$ and $B_{i+k} - e$, we deduce that $\pi_{M/e}(i) \not\# \pi_{M/e}(i+k)$. ∎

We know, then, that in an HPR of M, non-crissing columns remain non-crissing on deleting some row. Next, we consider what happens to crissing columns upon deleting a row. First, observe that if $e \in C$ and $i \in E(M) - C$, then the blocks of $\pi_{M/e}(i)$ consist of the blocks of $\pi_M(i)$ that avoid e along with $B(e) - e$, provided it is non-empty, where $B(e)$ is the block of $\pi_M(i)$ containing e.

7.8. LEMMA. *Suppose $e \in C$ and $k \in \{1, 2, ..., n-1\}$. If none of* (i)–(v) *below holds, then the partitions $\pi_{M/e}(k)$ and $\pi_{M/e}(k+1)$ criss.*

(i) $B_{k+1} - B_k = \{e\}$ *and* $k \geqslant 3$.

(ii) $A_k - A_{k+1} = \{e\}$ *and* $k \leqslant n-3$.

(iii) $e \in C - (A_k \cup B_{k+1})$ *and* $2 \leqslant k \leqslant n-2$.

(iv) $e \in C - A_1$ *and* $k \in \{1, 2\}$.

(v) $e \in C - B_n$ *and* $k \in \{n-2, n-1\}$.

Proof. Suppose $\pi_{M/e}(k) \not\# \pi_{M/e}(k+1)$. Then $\pi_{M/e}(k)$ and $\pi_{M/e}(k+1)$ have blocks X_k and Y_{k+1}, respectively, such that $X_k \cup Y_{k+1} = C - e$. These blocks are also blocks of $\pi_M(k)$ and $\pi_M(k+1)$, respectively. We shall show that one of (i)–(v) must hold.

First suppose that $k \geqslant 3$ and $e \in B_{k+1}$. Then, as $B_{k+1} \supseteq B_3 \supseteq C - A_1$, Lemma 7.6 implies that $|B_{k+1}| \geqslant 2$. Clearly, either (a) $Y_{k+1} \supseteq B_{k+1} - e$, or (b) $Y_{k+1} \not\supseteq B_{k+1} - e$. In the first case, since the block of $\pi_M(k+1)$ containing $B_{k+1} - e$ must also contain e, it follows that $Y_{k+1} \cup e$ is a block of $\pi_M(k+1)$, so $\pi_M(k) \not\# \pi_M(k+1)$; a contradiction. Thus (b) holds. Hence, as $\pi_{M/e}(k+1)$ has a block containing all of $B_{k+1} - e$, it follows that Y_{k+1} must avoid $B_{k+1} - e$, and so $X_k \supseteq B_{k+1} - e$. Therefore $X_k \supseteq B_k - e$. Hence $e \notin B_k$, otherwise $X_k \cup e$ is a block of $\pi_M(k)$ and so $\pi_M(k) \not\#$ $\pi_M(k+1)$; a contradiction. It follows that $B_{k+1} - e = B_k$ since all of $B_{k+1} - e$ is in the same block X_k of $\pi_M(k)$ and this block contains B_k, and so equals B'_k. Thus (i) holds.

Next suppose that $k \leqslant n-3$ and $e \in A_k$. Then, by mimicking the above argument, we deduce that $A_k - A_{k+1} = \{e\}$, that is, (ii) holds.

If $k=1$, then (ii) or (iv) holds; and if $k=n-1$, then (i) or (v) holds. Thus we may assume that $2 \leqslant k \leqslant n-2$. Then we may suppose that $e \in A_k \cup B_{k+1}$, otherwise (iii) holds. We shall suppose that $e \in A_k$, for a similar argument treats the case when $e \in B_{k+1}$. We may also assume that $k=n-2$, otherwise (ii) holds. Moreover, $e \notin B_{n-1}$, otherwise (i) holds. Now $A_{n-2} - e \nsubseteq X_{n-2}$, otherwise $\pi_M(n-2) \dagger \pi_M(n-1)$. Thus X_{n-2} avoids $A_{n-2} - e$ and so $A_{n-2} - e \subseteq Y_{n-1}$. As $A_{n-2} \cup B_n = C$, it follows that $e \in C - B_n$, otherwise $\pi_M(n-1) \not\dagger \pi_M(n)$. Hence (v) holds and the lemma is proved. ∎

7.9. Lemma. *Suppose* $e \in A_i \cap B_{i+2}$ *for some* i *in* $\{1, 2, ..., n-2\}$. *Then* $\Gamma(\Pi(M/e, C-e)) = P$ *and* M/e *is* 3-*connected.*

Proof. Clearly $\Gamma(\Pi(M/e, C-e))$ has the same vertex set as $\Gamma(\Pi(M, C))$, namely P. Moreover, by Lemma 7.7, two members of P that are non-adjacent in $\Gamma(\Pi(M, C))$ are still non-adjacent in $\Gamma(\Pi(M/e, C-e))$.

Now suppose that $\pi_{M/e}(k) \not\dagger \pi_{M/e}(k+1)$ for some k in $\{1, 2, ..., n-1\}$. Then one of (i)–(v) in Lemma 7.8 must hold. But $e \in A_i$, so e is in all of $A_1, A_2, ..., A_i$. Likewise, the fact that $e \in B_{i+2}$ implies that e is in all of $B_{i+2}, B_{i+3}, ..., B_n$. In particular, neither (iv) nor (v) of Lemma 7.8 holds. Moreover, if (iii) holds, then e is in both $C - A_k$ and $C - B_{k+1}$, so $k \geqslant i+1$ and $k+1 \leqslant i+1$. This contradiction eliminates the possibility of (iii) occurring.

Suppose (i) occurs. As $B_{k+1} - B_k = \{e\}$ and $e \in B_{i+2}$, we have $k < i+2 \leqslant k+1$, that is, $k = i+1$. Hence $B_{i+2} = B_{i+1} \cup e$. But $e \in A_i$ and $A_i \cup B_{i+2} = C$, so $A_i \cup B_{i+1} = C$, that is, $\pi_M(i) \not\dagger \pi_M(i+1)$; a contradiction. Hence (i) fails and a similar argument establishes that (ii) fails. We conclude that $\Gamma(\Pi(M/e, C-e)) = P$.

Certainly, $\Gamma(\Pi(M/e, C-e))$ is connected and has no two elements of $C-e$ in series. Moreover, by Lemma 2.1, no two elements of P are parallel. Theorem 3.10 now implies immediately that M/e is 3-connected. ∎

Part (iv) of Theorem 7.2 follows easily by combining Lemma 7.9 with the fact that M is minor-minimal with the specified properties.

The following lemma is a straightforward combination of (iii) and (iv) of the theorem.

7.10. Lemma. (i) $B_{i+2} = C - A_i$ *for all* i *in* $\{1, 2, ..., n-2\}$; *and*

(ii) $A_j - A_{j+1} = B_{j+3} - B_{j+2}$ *for all* j *in* $\{1, 2, ..., n-3\}$.

Next we prove (vi) of the theorem. It will suffice to establish the first equality in that assertion since the second can be obtained by combining the first with (ii) of Lemma 7.10.

7.11. LEMMA. *If* $|A_i - A_{i+1}| \geqslant 2$ *for some* i *in* $\{1, 2, ..., n-3\}$, *then, for some* e *in* $A_i - A_{i+1}$, *the crissing graph* $\Gamma(\Pi(M/e, C-e))$ *is* P *and* M/e *is* 3-*connected.*

Proof. Suppose $e \in A_i - A_{i+1}$. Then $\Gamma(\Pi(M/e, C-e))$ has vertex set P. Moreover, by Lemma 7.7, two members of P that are non-adjacent in $\Gamma(\Pi(M, C))$ remain non-adjacent in $\Gamma(\Pi(M/e, C-e))$.

Suppose that $\pi_{M/e}(k) \not\asymp \pi_{M/e}(k+1)$ for some k in $\{1, 2, ..., n-1\}$. Then one of (i)–(v) of Lemma 7.8 must hold. As $e \in A_i - A_{i+1}$, we deduce that e is in all of $A_1, A_2, ..., A_i, B_{i+3}, B_{i+4}, ..., B_n$. Thus neither (iv) nor (v) of Lemma 7.8 holds. Moreover, since e is in none of $A_{i+1}, A_{i+2}, ..., A_{n-2}$, and $|A_i - A_{i+1}| \geqslant 2$, part (ii) of Lemma 7.8 cannot hold. By Lemma 7.10(ii), $A_i - A_{i+1} = B_{i+3} - B_{i+2}$. Therefore, as (ii) of Lemma 7.8 cannot hold, neither can (i).

Finally, if (iii) of Lemma 7.8 holds, then e is in both $C - A_k$ and $C - B_{k+1}$. The first assertion implies that $k \geqslant i+1$, and the second that $k+1 \leqslant i+2$. Hence $k = i+1$. Thus $\pi_{M/e}(i+1) \not\asymp \pi_{M/e}(i+2)$. Therefore $\pi_{M/e}(i+1)$ and $\pi_{M/e}(i+2)$ have blocks X_{i+1} and Y_{i+2}, respectively, such that $X_{i+1} \cup Y_{i+2} = C - e$. Moreover, since $\pi_M(i+1)$ crisses $\pi_M(i+2)$, the sets X_{i+1} and Y_{i+2} are also blocks of $\pi_M(i+1)$ and $\pi_M(i+2)$. If $X_{i+1} \not\supseteq A_{i+1}$, then X_{i+1} avoids A_{i+1} and so $Y_{i+2} \supseteq A_{i+1}$. But $A_{i+1} \cup B_{i+3} = C$, so $Y_{i+2} \cup B_{i+3} = C$ and, therefore, $\pi_M(i+2) \not\asymp \pi_M(i+3)$. This contradiction implies that $X_{i+1} \supseteq A_{i+1}$. But, by Lemma 7.5, the block A'_{i+1} of $\pi_M(i+1)$ that contains A_{i+1} avoids $A_i - A_{i+1}$. Hence $Y_{i+2} \supseteq (A_i - A_{i+1}) - e$. Moreover, $X_{i+1} \not\supseteq B_{i+2}$, otherwise $\pi_M(i) \not\asymp \pi_M(i+1)$. Thus Y_{i+2} meets B_{i+2} and so $Y_{i+2} \supseteq B_{i+2}$. We conclude that $\pi_M(i+2)$ has a block that contains $[(A_i - A_{i+1}) - e] \cup B_{i+2}$.

Now choose e' in $(A_i - A_{i+1}) - e$. Then, arguing as above, either $\Gamma(\Pi(M/e', C-e')) = P$, or $\pi_M(i+2)$ has a block containing $[(A_i - A_{i+1}) - e'] \cup B_{i+2}$. In the latter case, the relevant block of $\pi_M(i+2)$ must also contain $[(A_i - A_{i+1}) - e] \cup B_{i+2}$ and so contains all of $(A_i - A_{i+1}) \cup B_{i+2}$. Since $A_{i+1} \cup [(A_i - A_{i+1}) \cup B_{i+2}] = C$, it follows that $\pi_M(i+1) \not\asymp \pi_M(i+2)$; a contradiction. We conclude that $A_i - A_{i+1}$ does indeed have an element e for which $\Gamma(\Pi(M/e, C-e)) = P$ and thus, by Lemma 2.1 and Theorem 3.10, M/e is 3-connected. ∎

On combining the last lemma with the fact that M is minor-minimal with the specified properties, we deduce immediately that (vi) of Theorem 7.2 holds.

Finally, (v) of the theorem will follow from the minimality of M, the next lemma, and the symmetric argument that must hold for $C - B_n$.

7.12. LEMMA. *If* $|C - A_1| \geqslant 3$, *then for some* e *in* $C - A_1$, *the crissing graph* $\Gamma(\Pi(M/e, C-e)) = P$ *and* M/e *is* 3-*connected.*

Proof. Choose e in $C-A_1$. Then, by arguing as in previous lemmas, we deduce that this lemma holds unless $\pi_{M/e}(k) \not\to \pi_{M/e}(k+1)$ for some k in $\{1, 2, ..., n-1\}$. Thus one of (i)–(v) of Lemma 7.8 must hold. Clearly (ii) cannot hold; nor, by Lemma 7.10, can (i). Evidently $A_1 \cup B_n = C$, so (v) cannot hold. Moreover, since $A_1 \cup B_3 = C$, it follows that e is in all of $B_3, B_4, ..., B_n$ and therefore (iii) cannot hold. We conclude that (iv) holds, so $k \in \{1, 2\}$.

Now $\pi_{M/e}(k)$ and $\pi_{M/e}(k+1)$ have blocks X_k and Y_{k+1} such that $X_k \cup Y_{k+1} = C-e$. These blocks are also blocks of $\pi_M(k)$ and $\pi_M(k+1)$, respectively. We show next that we may assume that $k=1$. Thus suppose $k=2$. Then $X_2 \cup Y_3 = C-e$. As $e \in B_3 - Y_3$, it follows that B_3 avoids Y_3, so $B_3 - e \subseteq X_2$. Since $A_1 \cup B_3 = C$, it follows that $A_1 \cup X_2 = C-e$. Thus $\pi_{M/e}(1) \not\to \pi_{M/e}(2)$, so we may, indeed, assume that $k=1$.

By Lemma 7.5, no block of $\pi_M(2)$ contains A_1. Hence X_1 meets A_1, so $X_1 \supseteq A_1$. But A_1 is a block of $\pi_M(1)$ so $X_1 = A_1$. Thus $Y_2 \supseteq (C-A_1)-e$.

Now choose $e' \in (C-A_1)-e$. Then we may assume that $\pi_M(2)$ has a block containing $(C-A_1)-e'$, otherwise the lemma holds with e' replacing e. Thus $\pi_M(2)$ has blocks containing each of $(C-A_1)-e$ and $(C-A_1)-e'$. Since $|C-A_1| \geqslant 3$, these blocks are equal, so $\pi_M(2)$ has a block containing $C-A_1$. Therefore $\pi_M(2) \not\to \pi_M(1)$; a contradiction. This completes the proof of Lemma 7.12 and thereby finishes the proof of Theorem 7.2. ∎

Theorem 7.2 enables us to give precise information about the structure of an HPR of a minor-minimal 3-connected matroid whose crissing graph is a fixed path. Recall that, in an HPR, equality of entries is only meaningful if they occur in the same column. An $(r+1)\times r$ matrix will be called *good* if it has the form shown in Fig. 3 where, for all i, the elements α_i and β_i are distinct; γ_i and δ_i are distinct; $\beta_1 \neq \delta_1 \neq \gamma_1$; and $\gamma_r \neq \alpha_r \neq \beta_r$.

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ r-3 \\ r-2 \\ r-1 \\ r
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & \cdots & s_{r-1} & s_r \end{array} \\
\begin{pmatrix}
\beta_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\gamma_1 & \beta_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\delta_1 & \gamma_2 & \beta_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\delta_1 & \delta_2 & \gamma_3 & \beta_4 & \alpha_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\delta_1 & \delta_2 & \delta_3 & \gamma_4 & \beta_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \alpha_{r-1} & \alpha_r \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \beta_{r-1} & \alpha_r \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \gamma_{r-1} & \beta_r \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \delta_{r-1} & \gamma_r
\end{pmatrix}
\end{array}
$$

Figure 3

The next result, a straightforward consequence of Theorem 7.2, will enable us to focus on good matrices for the rest of this section.

7.13. COROLLARY. *Let M be a 3-connected matroid that is minor-minimal with the property that its crissing graph with respect to some hamiltonian circuit is a path of length r for some fixed $r \geqslant 4$. Then M has an HPR that is good.* ▮

Frequent use will be made throughout the rest of this section of Lemma 2.3. We begin by applying that lemma to an arbitrary hamiltonian matroid having a good HPR.

7.14. COROLLARY. *Let M_0 be a hamiltonian matroid for which the HPR with respect to the hamiltonian circuit C_0 is the good matrix labelled as in Fig. 3. Then, for all i and j in $\{0, 1, ..., r\}$ with $i < j$, the set $E(M_0) - \mathrm{cl}(C_0 - \{i, j\})$ is a cocircuit $C^*_{i,j}$ of M_0 meeting C_0 in $\{i, j\}$. Moreover:*

(i) *$C^*_{i,j}$ avoids both $\{s_1, s_2 ..., s_{i-1}\}$ and $\{s_{j+2}, s_{j+3} ..., s_r\}$.*

(ii) *If $i \geqslant 1$, then $C^*_{i,j}$ contains s_i.*

(iii) *If $j \leqslant r-1$, then $C^*_{i,j}$ contains s_{j+1}.*

(iv) *If $j \geqslant 2$, then $C^*_{0,j}$ contains s_1.*

(v) *If $i \leqslant r-2$, then $C^*_{i,r}$ contains s_r.*

(vi) *If $1 \leqslant i \leqslant r-2$, then $C^*_{i,i+1} \cap \{s_1, s_2, ..., s_r\}$ is $\{s_i, s_{i+2}\}$ or $\{s_i, s_{i+1}, s_{i+2}\}$.*

(vii) *$C^*_{0,1} \cap \{s_1, s_2, ..., s_r\}$ is $\{s_2\}$ or $\{s_1, s_2\}$.*

(viii) *$C^*_{r-1,r} \cap \{s_1, s_2, ..., s_r\}$ is $\{s_{r-1}\}$ or $\{s_{r-1}, s_r\}$.*

7.15. LEMMA. *Let M_0 be a hamiltonian matroid for which an HPR is the good matrix labelled as in Fig. 3. For all i and j such that $1 \leqslant i < j \leqslant r-1$, let $Y_{i,j} = \{i, j\} \cup \{s_2, s_3, ..., s_{r-1}\}$ and $X_{i,j} = \{i, s_{i+1}, s_{i+2}, ..., s_j, j\}$. Then every circuit of M_0 contained in $Y_{i,j}$ contains $\{i, j\}$, and $Y_{i,j}$ contains at most one circuit of M_0. Moreover, such a circuit that is contained in $X_{i,j}$ contains $\{i, j, s_{i+1}, s_j\}$ and is denoted by $C_{i,j}$.*

Proof. Suppose that $Y_{i,j} - j$ contains a circuit D. Then certainly $D \neq \{i\}$. Let $k = \max\{t : s_t \in D\}$. Then $1 \leqslant k \leqslant r-1$. By Corollary 7.14, $C^*_{k,k+1}$ contains s_k but avoids $s_1, s_2, ..., s_{k-1}$. Hence $C^*_{k,k+1}$ meets the circuit D in a single element. This contradiction implies that $Y_{i,j} - j$ is independent. A symmetric argument establishes that $Y_{i,j} - i$ is independent. Thus if $Y_{i,j}$ contains a circuit, then such a circuit contains $\{i, j\}$ and so, by circuit elimination, is unique.

By Corollary 7.14, $C^*_{i-1,i}$ meets $X_{i,j}$ in $\{i, s_{i+1}\}$. Thus if $C_{i,j}$ exists, then, since it contains i, it must also contain s_{i+1}. Again a symmetric argument establishes that $s_j \in C_{i,j}$. ∎

On combining the last lemma with the circuit elimination axiom, one obtains the following:

7.16. Corollary. *If* $1 \leqslant i < j < k \leqslant r-1$ *and two of* $X_{i,j}$, $X_{j,k}$, *and* $X_{i,k}$ *are dependent, then all three are.*

We now begin the proof of the main result of this section.

Proof of Theorem 7.1. We may assume that M is a minor-minimal matroid satisfying the hypotheses of the theorem. Then, by Corollary 7.13, we may also assume that the HPR of M with respect to C is as shown in Fig. 3.

Now form an auxiliary graph G by taking $\{1, 2, ..., r-1\}$ as its vertex set and joining distinct vertices i and j by an edge if and only if $X_{i,j}$ is dependent. An immediate consequence of Corollary 7.16 is that each component of G is a clique. Then clearly either

(1) G has an m-vertex stable set; or

(2) G has a K_m-subgraph.

Assume first that (1) holds letting $\{i_1, i_2, ..., i_m\}$ be a stable set Z of vertices of G where $1 \leqslant i_1 < i_2 < ... < i_m \leqslant r-1$. The next lemma establishes that, in this case, (i) of Theorem 7.1 must hold.

7.17. Lemma. *Let* $Y = \{s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ *and* $Z = \{i_1, i_2, ..., i_m\}$. *Then* $[M \mid (Z \cup Y)]/Y \cong U_{2,m}$.

Proof. First we show that, for all j and k with $1 \leqslant j < k \leqslant m$,

(1) $\{i_j, i_k\} \cup Y$ is independent.

Since $Y \subseteq X_{i_1, i_m} - \{i_1, i_m\}$, Lemma 7.15 implies that Y is independent. Moreover, since i_j and i_k are non-adjacent in G, the set X_{i_j, i_k} is independent. Assume that $\{i_j, i_k\} \cup Y$ contains a circuit D. Then D must contain s_p for some p satisfying $i_k < p \leqslant r-1$ or some p satisfying $2 \leqslant p < i_j + 1$. In these two cases, take such an s_p with the highest or lowest index, respectively. Then D meets $C^*_{p,p+1}$ or $C^*_{p-2,p-1}$, respectively, in $\{s_p\}$. This contradiction establishes (1).

To complete the proof of the lemma, we shall now show that

(2) $r(Y \cup \{i_1, i_2, ..., i_m\}) = |Y| + 2$.

To see this, we first let $Z' = \{i_1, i_1+1, i_1+2, ..., i_m\}$. Note that $|Z'| = |Y| + 1$. Now Z' is a proper subset of C and so is an independent set in M. Consider M/Z'. This matroid has an HPR with the following submatrix:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i_{i-1} \\ i_{m+1} \\ i_{m+2} \\ \vdots \\ r \end{array}
\begin{array}{c} \begin{array}{cccc} s_{i_1+1} & s_{i_1+2} & \cdots & s_{i_m} \end{array} \\
\begin{pmatrix}
\alpha_{i_1+1} & \alpha_{i_1+2} & \cdots & \alpha_{i_m} \\
\alpha_{i_1+1} & \alpha_{i_1+2} & \cdots & \alpha_{i_m} \\
\alpha_{i_1+1} & \alpha_{i_1+2} & \cdots & \alpha_{i_m} \\
\vdots & \vdots & \cdots & \vdots \\
\alpha_{i_1+1} & \alpha_{i_1+2} & \cdots & \alpha_{i_m} \\
\delta_{i_1+1} & \delta_{i_1+2} & \cdots & \delta_{i_m} \\
\delta_{i_1+1} & \delta_{i_1+2} & \cdots & \delta_{i_m} \\
\vdots & \vdots & \cdots & \vdots \\
\delta_{i_1+1} & \delta_{i_1+2} & \cdots & \delta_{i_m}
\end{pmatrix} \end{array}.$$

By Lemma 2.1, an element s_k of Y is a loop in M/Z' if and only if $\alpha_k = \delta_k$. Moreover, all non-loops are parallel to each other. Hence $r(Y \cup Z) \leqslant r(Y \cup Z') \leqslant |Y| + 2$. But, by (1), equality holds here. Hence (2) holds. On combining (1) and (2), we deduce immediately that $[M \mid (Z \cup Y)]/Y \cong U_{2,m}$. ▮

This completes the proof that if (1) occurs, then M has a $U_{2,m}$-minor. We now assume that (2) occurs, this time letting $\{i_1, i_2, ..., i_m\}$ be the vertex set of a clique in G where $1 \leqslant i_1 < i_2 < ... < i_m \leqslant r-1$.

7.18. PROPOSITION. *M has a hamiltonian minor M' that, after possibly some relabelling involving the first and last rows and columns, has a good HPR of the following form*:

$$\begin{array}{c} \\ 0 \\ i_1 \\ i_i+1 \\ i_i+2 \\ i_i+3 \\ \vdots \\ i_m-1 \\ i_m \\ r \end{array}
\begin{array}{c} \begin{array}{cccccccc} s_1 & s_{i_1+1} & s_{i_1+2} & s_{i_1+3} & s_{i_1+4} & \cdots & s_{i_m} & s_r \end{array} \\
\begin{pmatrix}
\beta_1 & \alpha_{i_1+1} & \alpha_{i_1+2} & \alpha_{i_1+3} & \alpha_{i_1+4} & \cdots & \alpha_{i_m} & \alpha_r \\
\gamma_1 & \beta_{i_1+1} & \alpha_{i_1+2} & \alpha_{i_1+3} & \alpha_{i_1+4} & \cdots & \alpha_{i_m} & \alpha_r \\
\delta_1 & \gamma_{i_1+1} & \beta_{i_1+2} & \alpha_{i_1+3} & \alpha_{i_1+4} & \cdots & \alpha_{i_m} & \alpha_r \\
\delta_1 & \delta_{i_1+1} & \gamma_{i_1+2} & \beta_{i_1+3} & \alpha_{i_1+4} & \cdots & \alpha_{i_m} & \alpha_r \\
\delta_1 & \delta_{i_1+1} & \delta_{i_1+2} & \gamma_{i_1+3} & \beta_{i_1+4} & \cdots & \alpha_{i_m} & \alpha_r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_1 & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \delta_{i_1+4} & \cdots & \beta_{i_m} & \alpha_r \\
\delta_1 & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \delta_{i_1+4} & \cdots & \gamma_{i_m} & \beta_r \\
\delta_1 & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \delta_{i_1+4} & \cdots & \delta_{i_m} & \gamma_r
\end{pmatrix} \end{array}$$

where $\alpha_t \neq \beta_t$ *and* $\gamma_t \neq \delta_t$ *for all* t; *and* $\beta_1 \neq \delta_1 \neq \gamma_1$ *and* $\gamma_r \neq \alpha_r \neq \gamma_r$. *Moreover, for all* k *in* $\{1, 2, ..., m-1\}$, *the circuit* $C_{i_k, i_{k+1}}$ *of* M *is also a circuit of* M'.

Proof. Let $T_1 = \{1, 2, ..., i_1 - 2\} \cup \{i_m + 1, i_m + 2, ..., r-2\}$ and let $S_1 = \{s_{i_1}, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}, s_{i_m+1}\}$. If $M_1 = (M/T_1) \mid [(C - T_1) \cup S_1]$, then M_1 is certainly hamiltonian, having $C - T_1$ as a hamiltonian circuit. Moreover, this matroid has the following HPR:

$$\begin{array}{c c}
 & \begin{array}{ccccccc} s_{i_1} & s_{i_1+1} & s_{i_1+2} & s_{i_1+3} & \cdots & s_{i_m} & s_{i_m+1} \end{array} \\
\begin{array}{c} 0 \\ i_1 - 1 \\ i_1 \\ i_i + 1 \\ i_1 + 2 \\ \vdots \\ i_m - 1 \\ i_m \\ i_m + 1 \\ r \end{array} &
\begin{pmatrix}
\alpha_{i_1} & \alpha_{i_1+1} & \alpha_{i_1+2} & \alpha_{i_1+3} & \cdots & \alpha_{i_m} & \alpha_{i_m+1} \\
\beta_{i_1} & \alpha_{i_1+1} & \alpha_{i_1+2} & \alpha_{i_1+3} & \cdots & \alpha_{i_m} & \alpha_{i_m+1} \\
\gamma_{i_1} & \beta_{i_1+1} & \alpha_{i_1+2} & \alpha_{i_1+3} & \cdots & \alpha_{i_m} & \alpha_{i_m+1} \\
\delta_{i_1} & \gamma_{i_1+1} & \beta_{i_1+2} & \alpha_{i_1+3} & \cdots & \alpha_{i_m} & \alpha_{i_m+1} \\
\delta_{i_1} & \delta_{i_1+1} & \gamma_{i_1+2} & \beta_{i_1+3} & \cdots & \alpha_{i_m} & \alpha_{i_m+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_{i_1} & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \cdots & \beta_{i_m} & \alpha_{i_m+1} \\
\delta_{i_1} & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \cdots & \gamma_{i_m} & \beta_{i_m+1} \\
\delta_{i_1} & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \cdots & \delta_{i_m} & \gamma_{i_m+1} \\
\delta_{i_1} & \delta_{i_1+1} & \delta_{i_1+2} & \delta_{i_1+3} & \cdots & \delta_{i_m} & \delta_{i_m+1}
\end{pmatrix}
\end{array}$$

where, if $i_1 - 1 = 0$, the first row is omitted; and if $i_m + 1 = r$, the last row is omitted. In particular, if $i_1 = 1$ and $i_m = r - 1$, then the first part of the proposition holds. Now suppose that $i_1 > 1$. Then either (a) $\alpha_{i_1} \neq \delta_{i_1}$, or (b) $\alpha_{i_1} = \delta_{i_1}$. In case (b), since $\alpha_{i_1} \neq \beta_{i_1}$, it follows that $\delta_{i_1} \neq \beta_{i_1}$. In this case, letting $M_2 = M_1/0$ and relabelling $i_1 - 1$ as 0 and s_{i_1} as s_1, we obtain a hamiltonian minor of M in which the first column of the HPR has the desired form. In case (a), we let $M_2 = M_1/\{i_1 - 1\}$. After relabelling s_{i_1} as s_1 and α_{i_1}, γ_{i_1}, and δ_{i_1} as β_1, γ_1, and δ_1, respectively, we again obtain a hamiltonian minor of M in which the first column of the HPR has the desired form.

If $i_m < r - 1$, then a similar argument ensures that by contracting r or $i_m + 1$ and relabelling, we obtain a hamiltonian minor in which the last column of the HPR has the desired form. The fact that M has a hamiltonian minor M' with a good HPR labelled as specified now follows without difficulty.

Now, for all k in $\{1, 2, ..., m-1\}$, the circuit $C_{i_k, i_{k+1}}$ of M is a subset of $\{i_k, s_{i_k+1}, s_{i_k+2}, ..., s_{i_{k+1}}, i_{k+1}\}$ that contains $\{i_k, i_{k+1}\}$. Moreover, none of the elements of $C_{i_k, i_{k+1}}$ is relabelled in forming M'. Thus $C_{i_k, i_{k+1}} \subseteq E(M')$, so $C_{i_k, i_{k+1}}$ is a union of circuits of M'. But Lemma 7.15 implies that every circuit of M' contained in $C_{i_k, i_{k+1}}$ must contain $\{i_k, i_{k+1}\}$. From circuit elimination, it now follows that $C_{i_k, i_{k+1}}$ is a circuit of M'. ∎

Since M' has a good HPR, we may apply Corollary 7.14 to this matroid to deduce that M' has a collection of cocircuits $C^*_{i,j}(M')$ with the various properties described there. We shall now concentrate primarily on the matroid M' whose existence was proved in Proposition 7.18. Let $C' = \{0, i_1, i_1 + 1, ..., i_m, r\}$. Then C' is the distinguished hamiltonian circuit of M'. Clearly $r(M') = i_m - i_1 + 2$. Throughout the discussion to follow, we shall abbreviate $C^*_{i,j}(M')$ to $C^*_{i,j}$.

Let $Z = \{s_1, s_{i_1+1}, s_{i_2+1}, ..., s_{i_{m-1}+1}, s_r\}$ and let $W = \{0, i_1, i_2, ..., i_m, r\}$. Let the matroid $N = M' \backslash [C' - W]/[E(M') - (C' \cup Z)]$. Evidently $E(N) = Z \cup W$. The remainder of this section will be devoted to proving the following result, which will complete the proof of Theorem 7.1.

7.19. PROPOSITION. *The matroid N has a 3-connected single-element deletion that is a single-element extension of an $(m+1)$-spoked fan.*

The proof of this proposition is broken into steps, which are contained in the next ten lemmas. The length of the argument here arises from the fact that, in forming N, we *deleted* elements from the hamiltonian circuit and *contracted* elements from outside this circuit. For such a minor, we can no longer find an HPR simply by looking at an appropriate submatrix of the original HPR.

7.20. LEMMA. (i) *$\{0, s_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ and $\{s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}, s_r, r\}$ are both bases of M'.*

(ii) $r(N) = m + 1$.

Proof. Consider the set $\{0, s_1, s_{i_1+1}, s_{i_2+1}, ..., s_{i_m}\}$. If it is dependent in M', then take a circuit contained in this set and let s_k be the member of this circuit with largest index. Then, by Corollary 7.14, $C^*_{k,r}$ meets this circuit in a single element; a contradiction. We conclude that $\{0, s_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ is independent in M', and a symmetric argument establishes that $\{s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}, s_r, r\}$ is independent in M'. Since $r(M') = i_m - i_1 + 2$, it follows that each of the independent sets just considered is a basis of M'. Moreover, both of these bases contain $E(M') - (C' \cup Z)$. Thus

$$\begin{aligned} r(M'/[E(M') - (C' \cup Z)]) &= r(M') - |E(M') - C'| + |Z| \\ &= (i_m - i_1 + 2) - (i_m - i_1 + 2) + m + 1 \\ &= m + 1. \end{aligned}$$

But, since $C' - W$ avoids the spanning set $\{0, s_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ of M', it is coindependent in M'. Hence

$$\begin{aligned} r(N) &= r(M'/[E(M') - (C' \cup Z)] \backslash [C - W]) \\ &= r(M'/[E(M') - (C' \cup Z)]) \\ &= m + 1. \quad \blacksquare \end{aligned}$$

7.21. Lemma. *For all k in $\{1, 2, ..., m-1\}$, the set $\{i_k, i_{k+1}\}$ is independent in N. Moreover, $\{i_k, s_{i_k+1}, i_{k+1}\}$ is a triangle of N.*

Proof. By Lemma 7.15, the circuit $C_{i_k, i_{k+1}}$ of M' contains $\{i_k, i_{k+1}, s_{i_k+1}, s_{i_{k+1}}\}$ and is contained in $\{i_k, s_{i_k+1}, s_{i_k+2}, ..., s_{i_{k+1}}, i_{k+1}\}$. Thus $C_{i_k, i_{k+1}} \cap E(N) = \{i_k, i_{k+1}, s_{i_k+1}\}$ and no element of $C_{i_k, i_{k+1}}$ is deleted in forming N. Therefore $\{i_k, i_{k+1}, s_{i_k+1}\}$ is a union of circuits of N. Hence both assertions of the lemma will follow if we can establish the first.

To show that $\{i_k, i_{k+1}\}$ is independent in N, we shall establish that $\{i_k, i_{k+1}\} \cup [\{s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\} - \{s_{i_k+1}\}]$ is independent in M'. But, in the notation of Lemma 7.15, this set is $Y_{i_k, i_{k+1}} - s_{i_k+1}$ and that lemma showed that $Y_{i_k, i_{k+1}}$ contains a unique circuit of M'. Since we know then that $X_{i_k, i_{k+1}}$ is dependent, the unique circuit contained in $Y_{i_k, i_{k+1}}$ is $C_{i_k, i_{k+1}}$. As this set contains s_{i_k+1}, we deduce that $Y_{i_k, i_{k+1}} - s_{i_k+1}$ is, indeed, independent in M'. $\blacksquare$

7.22. Lemma. *Both $\{0, s_1, i_1\}$ and $\{i_m, s_r, r\}$ are triangles of N.*

Proof. By symmetry, it suffices to establish the first of these assertions. In M', the partition of C' associated with s_1 has $C' - \{0, i_1\}$ as a single block. Thus $\{0, s_1, i_1\}$ is a triangle of M'. Since this set is contained in $E(N)$, it is a union of circuits of N. To complete the proof, we shall establish that

(1) $\{0, i_1\} \cup \{s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ is independent in M'.

By Lemma 7.15, $\{i_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ is certainly independent in M'. Thus (1) holds unless $\{0, i_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$ contains a unique circuit of M' and this circuit D contains 0. By Lemma 7.20, $i_1 \in D$. Assume that D contains s_j for some $j \geqslant i_1 + 1$. Take the largest such j. Then $C^*_{j,r} \cap D = \{s_j\}$; a contradiction. Thus $D = \{0, i_1\}$. But this set is a proper subset of the circuit C', so we have a contradiction. Hence (1) holds and the lemma is proved. $\blacksquare$

7.23. Lemma. *Both $\{0, i_1, i_2, ..., i_m\}$ and $\{i_1, i_2, ..., i_m, r\}$ are bases of N.*

Proof. By symmetry and the fact that $r(N) = m+1$, it suffices to show that the first set, I_1, spans N. Lemmas 7.21 and 7.22 certainly imply that the set I_1 spans $\{s_1, s_{i_1+1}, s_{i_2+1}, ..., s_{i_{m-1}+1}\}$, which is $Z - s_r$. Thus, in M', the set $I_1 \cup [E(M') - (C' \cup Z)]$, the union of I_1 with the elements that are contracted in forming N from M', spans $\{0, s_1, s_{i_1+1}, s_{i_1+2}, ..., s_{i_m}\}$. By Lemma 7.20, the last set is a basis for M'. Thus $I_1 \cup [E(M') - (C' \cup Z)]$ spans M' and so I_1 spans N. ▮

7.24. LEMMA. *The restriction of the matroid N to each of* $\{0, i_1, i_2, ..., i_m\} \cup \{s_1, s_{i_1+1}, s_{i_2+1}, ..., s_{i_{m-1}+1}\}$ *and* $\{i_1, i_2, ..., i_m, r\} \cup \{s_{i_1+1}, s_{i_2+1}, ..., s_{i_{m-1}+1}, s_r\}$ *is isomorphic to the cycle matroid of an* $(m+1)$*-spoked fan.*

Proof. The restriction to the first of these sets has $\{0, i_1, i_2, ..., i_m\}$ as a basis. The presence of the triangles, whose existence was shown in Lemmas 7.21 and 7.22, completes the proof that the first restriction is indeed isomorphic to the cycle matroid of an $(m+1)$-spoked wheel. The fact that the same conclusion holds for the second restriction follows by a symmetric argument. ▮

The two restrictions of N considered in the last lemma coincide with $N \backslash \{r, s_r\}$ and $N \backslash \{0, s_1\}$. The fans whose cycle matroids they equal are labelled as in Fig. 4.

7.25. LEMMA. $\{0, s_1, s_r, r\}$ *is a cocircuit of N.*

Proof. Clearly, $\{0, s_1\}$ is a cocircuit of $N \backslash \{r, s_r\}$. Thus $\{0, s_1, s_r, r\}$ contains a cocircuit C^* of N containing $\{0, s_1\}$. The circuit $\{i_m, s_r, r\}$

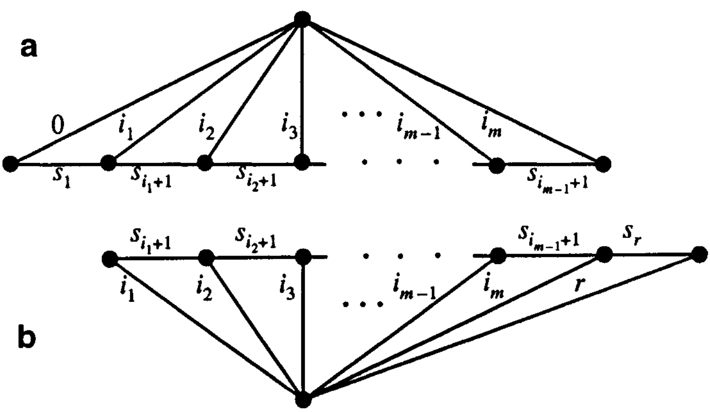

Figure 4

implies that C^* is $\{0, s_1\}$ or $\{0, s_1, s_r, r\}$. But $\{i_1, i_2, ..., i_m, r\}$ is a basis of N avoiding $\{0, s_1\}$. Hence $C^* = \{0, s_1, s_r, r, \}$. ▮

7.26. Lemma. *If N is non-simple, then N has a 3-connected single-element deletion N_1 such that $N_1 \backslash x$ is an $(m+1)$-spoked fan for some x in $E(N_1)$.*

Proof. Both $N\backslash\{r, s_r\}$ and $N\backslash\{0, s_1\}$ are $(m+1)$-spoked fans. Hence both of these matroids are simple. Suppose N itself is non-simple. Then 0 or s_1 is parallel to r or s_r. Since the proof of this lemma will use only these facts, we may view 0 and s_1 as interchangeable, and r and s_r as interchangeable within this proof. By this symmetry, we may assume that $\{0, r\}$ is a 2-circuit of N.

Let $N_1 = N\backslash 0$. Then $\{i_1, i_2, ..., i_m, r\}$ is certainly a basis for N_1. Moreover, as $\{0, i_1, s_1\}$ and $\{0, r\}$ are circuits of N, it follows that $\{r, i_1, s_1\}$ is a triangle of N_1. This triangle, along with the m other triangles shown in Fig. 4(b), implies, by Lemma 6.9, that N_1 is a wheel or whirl of rank $m+1$, and the lemma follows. ▮

By this lemma, we may now assume that N is simple.

7.27. Lemma. *Let $\{J, K\}$ be a 2-separation of $M(G_k)$ where G_k is labelled as in Fig. 5. Then, for some $j \leqslant k-2$ and up to a reversal of the subscript labels, J or K is $\{x_1, y_1, x_2, y_2, ..., x_j, y_j, x_{j+1}\}$.*

Proof. It is well known that the 2-separations of a graphic matroid are associated in a natural way with 2-vertex cuts of the graph. The lemma follows easily from this observation. ▮

7.28. Lemma. *The matroid N is 3-connected.*

Proof. As $N\backslash\{r, s_r\}$ and $N\backslash\{0, s_1\}$ are both connected, N is certainly connected. Assume that N has an exact 2-separation $\{X, Y\}$. Then $r(X) + r(Y) = r(N) + 1 = m+2$ and $\min\{|X|, |Y|\} \geqslant 2$. Since N is simple,

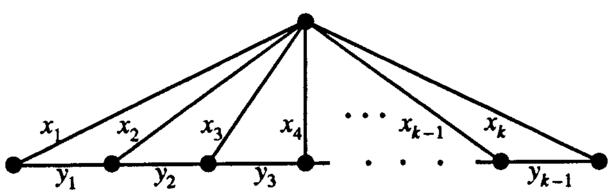

Fig. 5. A k-spoked fan.

$\min\{r(X), r(Y)\} \geqslant 2$ so $\max\{r(X), r(Y)\} < r(N)$. Now, as $N\backslash\{r, s_r\}$ is connected having the same rank as N, neither X nor Y is $\{r, s_r\}$. Moreover,

$$\begin{aligned} r(N)+1 &\leqslant r(X-\{r, s_r\}) + r(Y-\{r, s_r\}) \\ &\leqslant r(X) + r(Y) \\ &\leqslant r(N)+1. \end{aligned} \tag{1}$$

Hence equality holds throughout (1), so $r(X-\{r, s_r\}) = r(X)$. Without loss of generality, we may assume that $X \cap \{r, s_r\} \neq \emptyset$. Then, as $\{0, s_1, r, s_r\}$ is a cocircuit of N, the fact that $r(X-\{r, s_r\}) = r(X)$ implies that $X \cap \{0, s_1\} \neq \emptyset$. As $\max\{r(X-\{r, s_r\}), r(Y-\{r, s_r\})\} \leqslant r(N)-1$, (1) implies that $\min\{r(X-\{r, s_r\}), r(Y-\{r, s_r\})\} \geqslant 2$. Thus $\{X-\{r, s_r\}, Y-\{r, s_r\}\}$ is a 2-separation of the fan $N\backslash\{r, s_r\}$. Therefore, by Lemma 7.27, since X meets $\{0, s_1\}$, it contains this set. Thus X contains the cocircuit $\{0, s_1, r, s_r\}$ and so $r(Y-\{r, s_r\}) < r(Y)$ contradicting (1). ∎

7.29. LEMMA. *$N\backslash r$ or $N\backslash s_r$ is a 3-connected matroid for which some single-element deletion is an $(m+1)$-spoked fan.*

Proof. It suffices to show that $N\backslash r$ or $N\backslash s_r$ is 3-connected. Suppose that neither is. Since $\{i_m, r, s_r\}$ is a triangle of N, Tutte's Triangle Lemma [6] (or see [4, Lemma 8.4.9]) implies that N has a triad T^* containing r and exactly one of s_r and i_m. Since $N\backslash\{r, s_r\}$ has no coloops, T^* does not contain s_r and so must contain i_m. Moreover, $T^* - r$ is a union of cocircuits of $N\backslash\{r, s_r\}$. The only cocircuit of this matroid of size at most two containing i_m is $\{i_m, s_{i_{m-1}+1}\}$. Thus $T^* = \{r, i_m, s_{i_{m-1}+1}\}$. Similarly, N has a triad containing s_r and exactly one of r and i_m, and it follows as above that this triad is $\{s_r, i_m, s_{i_m-1}+1\}$. Hence, using these two triads and elimination, we deduce that $\{r, s_r, i_m\}$ is a triad of N. But this set is also a triangle of N. This is a contradiction (see [14, Proposition 8.1.7]) and so Lemma 7.29 holds. Hence so too do Proposition 7.19 and Theorem 7.1. ∎

8. THE CLIQUE CASE

This section deals with the case when the crissing graph of a 3-connected hamiltonian matroid is a clique. The argument here is much shorter than in the star and path cases and follows without too much difficulty from Theorem 4.5. The main result of this section is the following:

8.1. THEOREM. *Suppose that m is an integer exceeding two and M is a 3-connected matroid with a hamiltonian circuit C such that M has an*

F-matrix A as an HPR with respect to C. Assume that if x and y are distinct elements of $E(M)-C$, then $\pi(x)$ and $\pi(y)$ are different partitions of C. If $a \geqslant g_5(2m+2, q)$ and $\Gamma(\Pi(M, C))$ is isomorphic to K_a, then either

(i) *M has as a minor a wheel or a whirl of rank m or an m-spike; or*

(ii) *M or M^* has a 3-connected minor that is a single-element extension of an m'-spoked fan for some $m' \geqslant m$.*

Proof. We may assume that A is in reducible form with A' being the associated reduced HPR for M. Then, by Proposition 4.3, every two columns of A' cross. Let $h=2m+2$. Then, by Theorem 4.5, as A' has at least $g_5(h, q)$ columns, A' has a row-permuted submatrix B' of one of the five types specified there. These five cases will be treated, in order, below.

Corresponding to (i) of Theorem 4.5, we assume first that M has a hamiltonian minor N for which the HPR with respect to the circuit $C_N = \{0, 1, ..., h+1\}$ is as shown in Fig. 6 where α and β are both non-zero. Let $N_1 = N/(h+1)$. Then $\pi_{N_1}(e_i)$ has $(C_N - \{h+1\}) - \{i-1, h\}$ as a block for all i in $\{1, 2, ..., h\}$. Thus, for all such i, Lemma 2.1(i) implies that N_1 has $\{h, i-1, e_i\}$ as a circuit. Also $\{0, 1, 2, ..., h\} - \{i-1\}$ is a basis of N_1. Hence N_1 is an h-spike with tip h. Thus M certainly has an m-spike as a minor.

The two subtypes of (ii) of Theorem 4.5 are the same to within row and column permutations. Thus we shall assume next that some minor N of M has the HPR shown in Fig. 7 where α and β are distinct and non-zero. Then, by Theorem 3.10, N is 3-connected.

Let $N_1 = N/h$. Then, by Lemma 2.3, the following sets are triads of N_1: $\{0, 1, e_1\}$, $\{1, 2, e_2\}$, ..., $\{h-2, h-1, e_{h-1}\}$, $\{h-1, h+1, e_h\}$. Moreover, since the set $\{e_1, e_2, ..., e_h\}$ contains exactly one element of each of these triads, it is independent and hence is a basis of N_1. Thus $\{0, 1, 2, ..., h-1, h+1\}$ is a basis for N_1^*. It follows easily that N_1^* is an h-spoked fan. Thus N^* is a single-element extension of an h-spoked fan. Moreover, since N^* is 3-connected, the theorem holds in this case.

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ \vdots \\ h-2 \\ h-1 \\ h \\ h+1
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & \cdots & e_{h-1} & e_h \end{array} \\
\begin{pmatrix}
\alpha & 0 & 0 & \cdots & 0 & 0 \\
0 & \alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & \alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha & 0 \\
0 & 0 & 0 & \cdots & 0 & \alpha \\
\beta & \beta & \beta & \cdots & \beta & \beta \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 6

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ h-2 \\ h-1 \\ h \\ h+1 \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & \cdots & e_{h-1} & e_h \end{array} \\
\left(\begin{array}{cccccc}
\alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\
0 & \alpha & \alpha & \cdots & \alpha & \alpha \\
0 & 0 & \alpha & \cdots & \alpha & \alpha \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha & \alpha \\
0 & 0 & 0 & \cdots & 0 & \alpha \\
\beta & \beta & \beta & \cdots & \beta & \beta \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right) \end{array}$$

Figure 7

The third possibility to be considered in this proof corresponds to (iii) of Theorem 4.5. Thus we assume that some minor N of M has the HPR shown in Fig. 8 where neither α nor γ is zero, and $\alpha \neq \beta$. We now distinguish two subcases: (a) $\alpha \neq \gamma$; and (b) $\alpha = \gamma$. In case (a), the matroid $N_1 = N/\{1, 3, ..., h-1\} \backslash \{e_1, e_3, ..., e_{h-1}\}$ has an HPR that is obtained from the matrix in Fig. 8 by deleting the rows labelled by $1, 3, ..., h-1$, and the columns labelled by $e_1, e_3, ..., e_{h-1}$. Thus N_1 has the following HPR:

$$\begin{array}{c} \\ 0 \\ 2 \\ 4 \\ \vdots \\ h-4 \\ h-2 \\ h \end{array}
\begin{array}{c} \begin{array}{cccccc} e_2 & e_4 & e_6 & \cdots & e_{h-2} & e_h \end{array} \\
\left(\begin{array}{cccccc}
\alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\
\gamma & \alpha & \alpha & \cdots & \alpha & \alpha \\
\gamma & \gamma & \alpha & \cdots & \alpha & \alpha \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & \gamma & \gamma & \cdots & \alpha & \alpha \\
\gamma & \gamma & \gamma & \cdots & \gamma & \alpha \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right) \end{array}.$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ h-2 \\ h-1 \\ h \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & \cdots & e_{h-1} & e_h \end{array} \\
\left(\begin{array}{cccccc}
\beta & \alpha & \alpha & \cdots & \alpha & \alpha \\
\gamma & \beta & \alpha & \cdots & \alpha & \alpha \\
\gamma & \gamma & \beta & \cdots & \alpha & \alpha \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & \gamma & \gamma & \cdots & \beta & \alpha \\
\gamma & \gamma & \gamma & \cdots & \gamma & \beta \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right) \end{array}$$

Figure 8

Since α, γ, and 0 are distinct, we may interchange γ and 0 in this matrix to obtain another HPR for N_1. Then, deleting the last column, we deduce that $N_1 \backslash e_h$ has as an HPR the following matrix where α, γ, and 0 are distinct:

$$\begin{array}{c} \\ 0 \\ 2 \\ 4 \\ \vdots \\ h-4 \\ h-2 \\ h \end{array} \begin{array}{c} \begin{array}{cccccc} e_2 & e_4 & e_6 & \cdots & e_{h-4} & e_{h-2} \end{array} \\ \begin{pmatrix} \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\ 0 & \alpha & \alpha & \cdots & \alpha & \alpha \\ 0 & 0 & \alpha & \cdots & \alpha & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \gamma & \gamma & \gamma & \cdots & \gamma & \gamma \end{pmatrix} \end{array}$$

On interchanging the last two rows here, we obtain a matrix of the form in Fig. 7. From the argument given there, we deduce that N_1^* has a 3-connected minor that is a single-element extension of a fan with $(h-2)/2$ spokes. Since $(h-2)/2 = m$, the theorem holds in this case.

In case (b) associated with Fig. 8, we have $\alpha = \gamma \neq 0$. Applying the permutation $(\beta, \alpha, 0)$ to the labels produces a matrix of the form shown in Fig. 6 with the last row deleted. Thus M has an h-spike as a minor, and so M certainly has an m-spike as a minor.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ h-1 \\ h \\ h+1 \\ h+2 \\ h+3 \\ \vdots \\ 2h-1 \\ 2h \\ 2h+1 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & \cdots & e_{h-1} & e_h \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 \\ \alpha & \alpha & 0 & \cdots & 0 & 0 \\ \alpha & \alpha & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha & \alpha & \alpha & \cdots & \alpha & 0 \\ \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\ 0 & \alpha & \alpha & \cdots & \alpha & \alpha \\ 0 & 0 & \alpha & \cdots & \alpha & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \alpha \\ 0 & 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

Figure 9

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ h-2 \\ h-1 \\ h \end{array} \begin{array}{c} \begin{array}{ccccc} e_2 & e_3 & e_4 & \cdots & e_{h-1} \end{array} \\ \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 \\ \alpha & \beta & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta \\ 0 & 0 & 0 & \cdots & \alpha \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Figure 10

Next, corresponding to (iv) of Theorem 4.5, we may assume that some minor N of M has the HPR shown in Fig. 9 where $\alpha \neq 0$. Let $N_1 = N/\{h+1, h+2, ..., 2h-1\}$. Then N_1 has the following among its cocircuits: $\{1, 2, e_2\}$, $\{2, 3, e_3\}$, ..., $\{h-2, h-1, e_{h-1}\}$, $\{h-1, e_h, h\}$, $\{e_h, 2h+1, 2h\}$, $\{2h+1, 1, e_1\}$. Moreover, since $\{e_1, e_2, ..., e_{h-1}, h, 2h\}$ contains exactly one element of each of these cocircuits, it is independent and so, as $r(N_1) = h+1$, it is a basis of N_1. Thus N_1^* has $\{1, 2, ..., h-1, e_h, 2h+1\}$ as a basis. Therefore, by Lemma 6.9, N_1^*, and hence N_1, is a wheel or a whirl of rank $h+1$. Thus M certainly has, as a minor, a wheel or a whirl of rank m.

The final case that needs to be considered here is associated with (v) of Theorem 4.5, that is, when the reduced HPR is (α, β)-complete. In this case, M has a minor N having an HPR that equals the matrix in Fig. 10 where α and β are both non-zero.

By Lemma 2.1(i), the following sets are triangles of N: $\{1, 2, e_2\}$, $\{2, 3, e_3\}$, ..., $\{h-2, h-1, e_{h-1}\}$. Moreover, $\{1, 2, ..., h-1\}$ is a basis of N. Thus $N \backslash h$ is an $(h-1)$-spoked fan. Moreover, since it follows easily by Theorem 3.10 that N is 3-connected, the theorem also holds in this case and so its proof is complete. ▮

9. PROOF OF THE MAIN THEOREM

Having assembled the tools we need in the preceding sections, we are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. In addition to the functions g_5, g_6, and R from Theorems 4.5, 4.4, and 5.5, the proof will use the following numbers:

$$d = g_6((n+2)^8, n),$$

$$c = \max\{(d+1)^2 + 2, g_5(2d+6, n)\},$$

$$b = (R(c, 2))^c,$$

and

$$a = n^b.$$

Let M be a 3-connected matroid with more than 4^a elements. Then, by Theorem 5.1, we may choose some M_0 in $\{M, M^*\}$ such that M_0 has a circuit with more than a elements. By Theorem 5.2, if C is a maximum-sized circuit of M_0, then M_0 has a 3-connected minor M' in which C is a hamiltonian circuit. Let A be an HPR of M' with respect to C. Then A has more than a rows. Now either A has a column with more than n distinct entries, or every column of A has at most n distinct entries. In the first case, by Lemma 5.3, M_0 has a $U_{n, n+2}$-minor and the theorem holds. Thus we may assume that the second case holds and hence that A is an F-matrix where $|F| = n$. Then, as $a = n^b$, Lemma 5.4 implies that A has a set X of at least b columns such that $M' \mid (X \cup C)$ is 3-connected and every two members of X induce distinct partitions of C.

Next let $M'_X = M' \mid (X \cup C)$, let A_X be the submatrix of A consisting of those columns in X, and consider the crissing graph $\Gamma(\Pi(M'_C, C))$. Since M'_X is 3-connected, this graph is certainly connected. As it has at least $(R(c, 2))^c$ vertices, Theorem 5.5 implies that it has an induced subgraph isomorphic to $K_{1, c}$, P_c, or K_c. This subgraph has some subset Y of X as its vertex set. Consider $M' \mid (Y \cup C)$ and let A_Y be the corresponding submatrix of A_X. By Lemma 3.11, the cosimplification of $M' \mid (Y \cup C)$ is a 3-connected matroid M_1 having a hamiltonian circuit C_1 such that the associated crissing graph is isomorphic to $K_{1, c}$, P_c, or K_c. Moreover, an HPR A_1 for M_1 can be obtained from A_Y by deleting repeated rows. Since every two columns of A_Y induce distinct partitions of C, it follows that every two columns of A_1 induce distinct partitions of C_1.

The three possibilities for the crissing graph of M_1 will essentially be reduced to one using the following result.

9.1. Lemma. *Either M_0 has as a minor a matroid isomorphic to $M(W_n)$, W^n, $U_{2, n+2}$, or $(n+2)$-spike, or M_1 or M_1^* has a 3-connected minor having a hamiltonian circuit such that the associated crissing graph is isomorphic to $K_{1, d}$.*

Proof. Since $c = \max\{(d+1)^2 + 2, g_5(2d+6, n)\}$, by applying Lemma 3.11 again, we deduce that the lemma holds if $\Gamma(\Pi(M_1, C_1)) \cong K_{1, c}$. Now suppose that $\Gamma(\Pi(M_1, C_1)) \cong P_c$. Then, by Theorem 7.1, M_1 has either a $U_{2, d+1}$-minor, or a 3-connected minor that is a single-element extension of a $(d+2)$-spoked fan. In the first case, as $d \geqslant n+1$, it follows that M_1 and hence M_0 has a $U_{2, n+2}$-minor. In the second case, by Proposition 3.12, M_0 has a 3-connected minor with a hamiltonian circuit such that the associated crissing graph is $K_{1, d}$.

Finally, suppose that $\Gamma(\Pi(M_1, C_1)) \cong K_c$. Then, by Theorem 8.1, either (a) M_1 has as a minor a wheel or whirl of rank $d+2$ or a $(d+2)$-spike, or (b) M_1 or its dual has a 3-connected minor that is a single-element extension of a $(d'+2)$-spoked fan for some $d' \geqslant d$. In case (a), since $d \geqslant n$, it follows that M_1 has a minor isomorphic to one of $M(W_n)$, W^n, or an $(n+2)$-spike. In case (b), by Proposition 3.12 and Lemma 3.11, it follows that the lemma holds. ∎

If the first possibility of Lemma 9.1 holds, then, as M_0 is M or M^*, we deduce that M or M^* has a minor isomorphic to one of $M(W_n)$, W^n, $U_{2,n+2}$, or an $(n+2)$-spike. Since the dual of an $(n+2)$-spike has an n-spike as a minor, it follows that M has a minor isomorphic to one of $M(W_n)$, W^n, $U_{2,n+2}$, $U_{n,n+2}$, or an n-spike.

We may now assume that the second possibility in Lemma 9.1 holds. Let N be a 3-connected minor of M or M^* having a hamiltonian circuit C_N such that the associated crissing graph is isomorphic to $K_{1,d}$. Let A be an HPR of N with respect to C_N. Then we may assume that A is an F-matrix with $|F| = n$ otherwise, by Lemma 5.3, N has a $U_{n,n+2}$-minor. As $d = g_6((n+2)^8, n)$, it follows immediately from Theorem 6.1 that N has a minor isomorphic to one of $M(W_n)$, W^n, $U_{n,n+2}$, or $M^*(K_{3,n})$. The theorem now follows easily. ∎

10. UNIFORM SPIKES AND UNAVOIDABILITY

The purpose of this section is to complete the proof of Theorem 1.4 by showing that every sufficiently large spike contains a big uniform spike as a minor. In addition, the notion of unavoidability will be precisely defined and used to show that the list of matroids in Theorem 1.4 has no redundancy.

The desired result on spikes will be obtained as a consequence of a result for hypergraphs. For an n-element set I, a 2-*colored complete hypergraph* H on I is a hypergraph with vertex set I and edge set 2^I with every edge colored red or blue. For $I' \subseteq I$, every member of $2^{I'}$ is also a member of 2^I. We let $H \mid I'$, the *restriction* of H to I', be the 2-colored complete hypergraph with vertex set I' and edge set $2^{I'}$ in which every edge has the color it was given in H.

For precision in what follows, we extend the definition of an n-spike as follows: for an n-element set I with $n \geqslant 3$, an I-*spike* is a matroid M with ground set $\{p\} \cup \{x_i : i \in I\} \cup \{y_i : i \in I\}$ such that

(i) if $i \in I$ and $L_i = \{p, x_i, y_i\}$, then L_i is a circuit of M;

(ii) if J is a proper subset of I, then $r(\bigcup_{j \in J} L_j) = j + 1$; and

(iii) $r(M) = n$.

In particular, a $\{1, 2, ..., n\}$-spike is an n-spike as previously defined. Indeed, every I-spike is isomorphic to an n-spike, and we shall often refer to an arbitrary I-spike with $|I| = n$ as just an n-spike.

Now let M be an I-spike with tip p. We associate with M the 2-colored complete hypergraph $H(M)$ with vertex set I in which a subset J of I is colored red if and only if $X_J \cup Y_{I-J}$ is a circuit of M.

10.1. LEMMA. *Let M be an I-spike. If $I' \subseteq I$ and $|I'| \geqslant 3$, then $H(M) \mid I' = H(M \backslash X_{I-I'}/Y_{I-I'})$.*

Proof. It suffices to prove this when $|I - I'| = 1$. Let $I - I' = \{k\}$. Then $M \backslash x_k/y_k$ is an I'-spike by Lemma 1.3(v). Moreover, $X_J \cup Y_{I'-J}$ is a circuit of $M \backslash x_k/y_k$ if and only if $X_J \cup Y_{I-J}$ is a circuit of M. The lemma follows without difficulty. ∎

For an I-spike M and a subset A of $\{0, 1, ..., n\}$, the hypergraph $H(M)$ is *A-coherently colored* and M is *A-uniform* if, for all m in A, all edges of $H(M)$ of size m have the same color. In particular, if $A = \{0, 1, ..., n\}$, an A-coherently colored hypergraph is called simply *coherently colored*, and an A-uniform I-spike is called a *uniform I-spike*. This usage is clearly consistent with that introduced in Section 1.

The next result is a special case of a celebrated theorem of Ramsey (see, for example, [2]).

10.2. THEOREM. *For all non-negative integers k and l, there is a least integer $R_k(l)$ such that, for every 2-colored complete hypergraph H on I with $|I| \geqslant R_k(l)$, there is an l-element subset J of I for which $H \mid J$ is $\{k\}$-coherently colored.*

The next result is obtained by repeatedly applying the last theorem.

10.3. THEOREM. *Let n be a non-negative integer. Then there is a number $R'(n)$ such that, for every 2-colored complete hypergraph H on I with $|I| \geqslant R'(n)$, there is an n-element subset J of I for which $H \mid J$ is coherently colored.*

Proof. Define $m_0, m_1, ..., m_n$ inductively as follows: let $m_n = n$; for all k in $\{0, 1, ..., n-1\}$, let $m_k = R_{k+1}(m_{k+1})$, and let $R'(n) = m_0$. We prove, by induction on k, that every 2-colored complete hypergraph on a set of m_0 vertices has a $\{0, 1, ..., k\}$-coherently colored restriction on m_k vertices. This is certainly true for $k = 0$. Assume it true for $k = k'$. Then H has a $\{0, 1, ..., k'\}$-coherently colored restriction H' on $m_{k'}$ vertices. But $m_{k'} = R_{k'+1}(m_{k'+1})$, so, by Theorem 10.2, H' has a restriction on $m_{k'+1}$ vertices that is $\{k'+1\}$-coherently colored. But this restriction, as a restriction of H', is also $\{0, 1, ..., k'\}$-coherently colored, and hence it is $\{0, 1, ..., k'+1\}$-coherently colored. The assertion now follows by induction and, taking $k=n$, we get the theorem. ▮

On combining the last theorem with Lemma 10.1, we immediately deduce the desired result for uniform spikes.

10.4. COROLLARY. *For every integer n exceeding two, every $R'(n)$-spike has a uniform n-spike as a minor.*

Theorem 1.4 is now an immediate consequence of this corollary and Theorem 1.2.

In Theorem 1.4, we refined Theorem 1.2 by trimming down the list of unavoidable matroids. To conclude the paper, we shall show that no further trimming of this list can be done. Thus Theorem 1.4 is the best-possible result of this type. To achieve this end, we shall formally define unavoidable matroids and show that Theorem 1.4 essentially identifies all such matroids.

A 3-connected matroid M_0 will be called *unavoidable* if there is a minor-closed class $\mathscr{M}$ of matroids such that $\mathscr{M}$ contains infinitely many 3-connected members but only finitely many of these have no minor isomorphic to M_0. For example, let $\mathscr{M}$ consist of all matroids that are minors of members of $\{M(K_{3,k}): k \geqslant 3\}$. Then $\mathscr{M}$ certainly contains infinitely many 3-connected members. But, for a fixed $n \geqslant 3$, the matroid $M(K_{3,n})$ is unavoidable since the only 3-connected members of $\mathscr{M}$ with no $M(K_{3,n})$-minor have at most eight elements or have the form $M(K_{3,k})$, $M(K'_{3,k})$, $M(K'_{3,k})$, $M(K'''_{3,k})$ for some k in $\{3, 4, ..., n-1\}$. Here, if $K_{3,k}$ has vertex classes X and Y with $|X| = 3$, then $K'_{3,k}$, $K''_{3,k}$, and $K'''_{3,k}$ are simple graphs obtained from $K_{3,k}$ by adding one, two, or three edges, respectively, joining distinct pairs of vertices in X.

It is not difficult to check that every 3-connected minor of an unavoidable matroid is unavoidable. Thus, for instance, each of $M(K'_{3,n})$, $M(K''_{3,n})$, and $M(K'''_{3,n})$ is unavoidable for all $n \geqslant 3$.

The substance of Theorem 1.4 is that if a matroid is unavoidable, then it is isomorphic to a 3-connected minor of one of the matroids listed in that theorem. The next result asserts that the converse of this is also true.

10.5. THEOREM. *A 3-connected matroid is unavoidable if and only if, for some $n \geqslant 3$, it is isomorphic to a 3-connected minor of one of $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, $M(W_n)$, W^n, or a uniform n-spike.*

Before proving this theorem, we make some further observations concerning uniform spikes. Let M be a uniform n-spike with tip p. Then $n \geqslant 3$, $E(M) = \{p\} \cup X_I \cup Y_I$, and M is uniquely determined by the $(n+1)$-tuple

$(t_0, t_1, ..., t_n)$ of zeros and ones where $t_{|J|} = 1$ if and only if $X_J \cup Y_{I-J}$ is a circuit of M. If $\mathbf{t} = (t_0, t_1, ..., t_n)$, we shall write $S_n(\mathbf{t})$ for the uniform n-spike associated with this $(n+1)$-tuple. It follows easily that $t_i t_{i-1} = 0$ for all i in $\{1, 2, ..., n\}$. Moreover, it is not difficult to check that, for all $n \geqslant 3$ and every $(n+1)$-tuple $\mathbf{t}$ subject to the last restriction, the matroid $S_n(\mathbf{t})$ does indeed exist.

This new notation for uniform spikes makes it particularly easy to describe the 3-connected minors of $S_n(\mathbf{t})$. First consider contractions. By Lemma 1.3, $S_n(\mathbf{t})/p$ is the cycle matroid of the graph obtained from an n-cycle by doubling every edge. Thus $S_n(\mathbf{t})/p$ has no 3-connected minors except for trivial ones with fewer than four elements. Next, when $n \geqslant 4$, look at $S_n(\mathbf{t})/x_i$ and $S_n(\mathbf{t})/y_i$ for some i in I. These matroids have the elements y_i and x_i, respectively, in parallel with p. To produce 3-connected minors with at least four elements, we may assume that these elements y_i and x_i are deleted. Moreover, $S_n(t_0, t_1, ..., t_n)/x_i \backslash y_i \cong S_{n-1}(t_1, t_2, ..., t_n)$ and $S_n(t_0, t_1, ..., t_n)/y_i \backslash x_i \cong S_{n-1}(t_0, t_1, ..., t_{n-1})$. This establishes that a 3-connected minor of $S_n(\mathbf{t})$ with more than four elements is isomorphic to a deletion of $S_m(\mathbf{u})$ for some $m \leqslant n$ and some subsequence $\mathbf{u}$ of $\mathbf{t}$ of the form $(t_i, t_{i+1}, ..., t_{i+m})$ where $0 \leqslant i \leqslant n-m$. Because $\{x_i, x_j, y_i, y_j\}$ is a cocircuit of $S_n(\mathbf{t})$ for all 2-element subsets $\{i, j\}$ of I, if $S_n(\mathbf{t}) \backslash Z$ is 3-connected having at least four elements, then $|Z \cap (X_I \cup Y_I)| \leqslant 1$. Thus the potential 3-connected restrictions of $S_n(\mathbf{t})$ are $S_n(\mathbf{t})$, $S_n(\mathbf{t}) \backslash p$, $S_n(\mathbf{t}) \backslash x_i$, $S_n(\mathbf{t}) \backslash y_i$, $S_n(\mathbf{t}) \backslash \{p, x_i\}$, and $S_n(\mathbf{t}) \backslash \{p, y_i\}$. When $n \geqslant 4$, it is not difficult to check that each of these is 3-connected. Moreover, when $n = 3$, the first four but not the last two are 3-connected. We also note here that, when $n \geqslant 5$, each of the 3-connected restrictions of $S_n(\mathbf{t})$ noted has a minor of the form $S_{n-2}(\mathbf{u})$ where $\mathbf{u}$ is a consecutive subsequence of $\mathbf{t}$.

Proof of Theorem 10.5. Let $\mathscr{K}$ be the class of all 3-connected matroids that are minors of one of the matroids listed in the theorem. Assume that M_0 is an unavoidable 3-connected matroid and let $\mathscr{M}$ be an arbitrary minor-closed class of matroids that contains infinitely many 3-connected members. Then $\mathscr{M}$ contains arbitrarily large 3-connected matroids and therefore, by Theorem 1.4, contains infinitely many members of $\mathscr{K}$. Since the only 3-connected minors of members of $\mathscr{K}$ are also in $\mathscr{K}$ and M_0 is unavoidable, it follows that M_0 is in $\mathscr{K}$.

It remains to show that every member of $\mathscr{K}$ is unavoidable. We showed above that $M(K_{3,n})$ is unavoidable; duality implies that $M^*(K_{3,n})$ is too; and it is straightforward to check that each of $U_{n,n+2}$, $U_{2,n+2}$, $M(W_n)$, and W^n is unavoidable. Now consider $S_n(\mathbf{t})$. As this is a minor of $S_{n+1}(\mathbf{t}, 0)$, it will suffice to show that $S_n(\mathbf{t})$ is unavoidable when $t_n = 0$. Let $\mathscr{M}$ be the class of minors of matroids of the form $S_{k(n+1)-1}(\mathbf{t}, \mathbf{t}, ..., \mathbf{t})$ where $n+1$ copies of $\mathbf{t}$ are repeated in $(\mathbf{t}, \mathbf{t}, ..., \mathbf{t})$ and k ranges over all positive integers. Evidently $\mathscr{M}$ has infinitely many 3-connected members. But the longest subsequence of consecutive terms in $(\mathbf{t}, \mathbf{t}, ..., \mathbf{t})$ that does not have $\mathbf{t}$ as a subsequence has length at most $2n$. It follows from the discussion of 3-connected minors of matroids of the form $S_m(\mathbf{u})$ that only finitely many 3-connected members of $\mathscr{M}$ fail to have $S_n(\mathbf{t})$ as a minor. Hence $S_n(\mathbf{t})$ is unavoidable. ∎

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REFERENCES

1. G. Ding, B. Oporowski, J. Oxley, and D. Vertigan, Unavoidable minors of large 3-connected binary matroids, *J. Combin. Theory Ser. B* **66** (1996), 334–360.
2. R. L. Graham, B. L. Rothschild, and J. H. Spencer, “Ramsey Theory,” Wiley, New York, 1990.
3. B. Oporowski, J. Oxley, and R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* **57** (1993), 239–257.
4. J. Oxley, “Matroid Theory,” Oxford Univ. Press, New York, 1992.
5. T. J. Reid, Ramsey numbers for matroids, *Europ. J. Combin.* **18** (1997), 589–595.
6. W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.