3

Infinite Matroids

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The many different axiom systems for finite matroids given in Chapter 2 of White (1986) offer numerous possibilities when one is attempting to generalize the theory to structures over infinite sets. Some axiom systems that are equivalent when one has a finite ground set are no longer so when an infinite ground set is allowed. For this reason, there is no single class of structures that one calls infinite matroids. Rather, various authors with differing motivations have studied a variety of classes of matroid-like structures on infinite sets. Several of these classes differ quite markedly in the properties possessed by their members and, in some cases, the precise relationship between particular classes is still not known.

The purpose of this chapter is to discuss the main lines taken by research into infinite matroids and to indicate the links between several of the more frequently studied classes of infinite matroids.

There have been three main approaches to the study of infinite matroids, each of these being closely related to a particular definition of finite matroids. This chapter will discuss primarily the independent-set approach. Some details of the closure-operator approach will also be needed, but a far more complete treatment of this has been given by Klee (1971) and by Higgs (1969a, b, c). The third approach, via lattices, will not be considered here. This approach is taken by Maeda & Maeda (1970) and they develop it in considerable detail.

Throughout this chapter, the Axiom of Choice will be assumed.

3.1. Pre-independence Spaces and Independence Spaces

The first class of infinite matroids that we consider is obtained essentially by deleting the references to finite sets in the independence axioms (i1)–(i3) of Chapter 2 of White (1986). We note, however, that the finiteness of I_1 and I_2 in (i3), which is implicit when S is finite, is made explicit when $|S|$ is unrestricted.

A *pre-independence space* $M_P(S)$ is a set S together with a collection $\mathscr{I}$ of subsets of S (called *independent sets*) such that

(i1) $\mathscr{I} \neq \emptyset$.
(i2) A subset of an independent set is independent.
(i3′) (Finite augmentation) If I_1 and I_2 are finite members of $\mathscr{I}$ with $|I_2| > |I_1|$, then there exists x in $I_2 - I_1$ such that $I_1 \cup x \in \mathscr{I}$.

Generalizing the terminology of finite matroid theory, we call a subset X of S *dependent* if $X \notin \mathscr{I}$. A *circuit* of $M_P(S)$ is a minimal dependent set, and a *basis* of $M_P(S)$ is a maximal independent set. The notation $Y \subset\subset X$ indicates that Y is a finite subset of X.

Although pre-independence spaces seem to be natural objects for study, they have received little attention in their own right primarily because they fail to possess many of the fundamental properties of finite matroids.

3.1.1. Example. Let $S = \mathbb{R}$, the set of real numbers, and $\mathscr{I}$ be the set of all countable subsets of S. Then $\mathscr{I}$ is the collection of independent sets of a pre-independence space $M_P(S)$. However, $M_P(S)$ has no circuits and no bases.

In the face of such examples, it is natural to strengthen one's axiom system. As with finite matroids, a principal example of a pre-independence space is obtained from a vector space V. In this case, we let S be an arbitrary subset of V, and $\mathscr{I}$ be the collection of subsets of S that are linearly independent in V. Such a pre-independence space satisfies the following additional condition.

(I4) (Finite character) If $X \subseteq S$ and every finite subset of X is in $\mathscr{I}$, then X is in $\mathscr{I}$.

Much of the work done on infinite matroids has been algebraically motivated and for this reason (i1), (i2), and (i3′) have frequently been augmented by (I4). We shall call a pre-independence space satisfying (I4) an *independence space*. Such structures are also commonly referred to as *finitary matroids* (Bean, 1976; Higgs, 1969a; Klee, 1971). In this section we examine the properties of independence spaces. In the next section we shall add different conditions to (i1), (i2), and (i3′) with different consequences.

3.1.2. Example. It is easy to extend the relevant arguments from Chapter 4 of White (1987) to show that the set of partial transversals of an arbitrary family of subsets of a set S is the set of independent sets of a pre-independence space on S. However, this pre-independence space need not be an independence space. For instance, if $\mathscr{X} = (X_1, X_2, X_3, \ldots)$, and $X_i = \{1, i+1\}$ for all i, then every finite subset of $\mathbb{Z}^+$ is a partial transversal of $\mathscr{X}$, yet $\mathbb{Z}^+$ itself is not.

Two immediate consequences of (I4) are that every dependent set in an independence space contains a circuit and that this circuit is finite. Thus an independence space is uniquely determined by its collection of circuits. We leave it to the reader to show that the independence axioms, (i1), (i2), (i3′), and (I4), are cryptomorphic to each of the circuit axioms, the strong circuit axioms, and the basis axioms stated below.

3.1.3. Circuit axioms for independence spaces. *An independence space $M(S)$ is a set S together with a collection $\mathscr{C}$ of subsets* (*called circuits*) *such that $\mathscr{C}$ satisfies* (c1)–(c3) (*Chapter 2 of White, 1986*) *together with*
(C4) *Every circuit is finite.*

3.1.4. Strong circuit axioms for independence spaces. *These are the same as 3.1.3, except that* (c3) *is replaced by* (c3.1) *of Chapter 2 of White* (*1986*).

The bases of an independence space also behave similarly to their counterparts in a finite matroid. Indeed, an easy consequence of Zorn's Lemma is that every independent subset of an independence space is contained in a basis. Hence every independence space is determined by its collection of bases. To obtain a definition of an independence space in terms of bases, one augments the basis axioms (b1)–(b3) of Chapter 2 of White (1986) by the following form of the finite character condition.

(B4) If X is not contained in a basis, then some finite subset of X is not contained in a basis.

It is not difficult to extend the axioms (cl1)–(cl4) of White (1986) to give a closure-operator definition for independence spaces. We leave the reader to check the details of this (see Exercise 3.1).

3.1.5. Example. Let S be the set of edges of a graph Γ and let $\mathscr{C}(\Gamma)$ be the collection of edge-sets of cycles of Γ. It was noted in Chapters 1 and 6 of White (1986) that, when Γ is finite, $\mathscr{C}(\Gamma)$ is the set of circuits of a (finite) matroid on S. Since every cycle in an infinite graph is finite, we can extend this immediately to get that $\mathscr{C}(\Gamma)$ is the set of circuits of an independence space $M_\Gamma(S)$ regardless of whether Γ is finite or infinite.

If $M(S)$ is an independence space having $\mathscr{I}$ as its collection of independent sets, then for $X \subseteq S$, let $\mathscr{I}|X$ be defined, as for finite matroids, by

$$\mathscr{I}|X = \{Y \subseteq X\colon Y \in \mathscr{I}\}. \tag{3.1}$$

Clearly $\mathscr{I}|X$ is the collection of independent sets of an independence space $M(X)$ on X. We call $M(X)$ the *restriction* of $M(S)$ to X. One can also define the operation of contraction for independence spaces in the same way as for finite matroids. However, in order to establish that this operation is well defined and that it gives an independence space, we shall require three lemmas.

3.1.6. Lemma. *Let B_1 and B_2 be bases of an independence space $M(S)$ and suppose that $x \in B_1 - B_2$. Then there is an element y of $B_2 - B_1$ such that $(B_1 - x) \cup y$ is a basis of $M(S)$.*

Proof. This result is an easy consequence of the proof of Proposition 2.1.1 of White (1986). □

3.1.7. Lemma. *Suppose that $M(S)$ is an independence space and $Y \subseteq X \subseteq S$. If B_1 and B_2 are bases of $M(S-X)$, then $B_1 \cup Y \in \mathscr{I}$ if and only if $B_2 \cup Y \in \mathscr{I}$.*

Proof. Suppose $B_1 \cup Y \in \mathscr{I}$, but $B_2 \cup Y \notin \mathscr{I}$. Consider $M((S-X)\cup Y)$. This is an independence space having $B_1 \cup Y$ as a basis. Moreover, $M((S-X)\cup Y)$ has a basis B such that $B_2 \subseteq B \subseteq B_2 \cup Y$. Now $B = B_2 \cup Y'$ where $Y' \subsetneqq Y$. Choose x in $Y - Y'$. Then $x \in (B_1 \cup Y) - B$ and so, by Lemma 3.1.6, there is an element y of $B - (B_1 \cup Y)$ such that $((B_1 \cup Y) - X) \cup y$ is a basis of $M((S-X)\cup Y)$. As $Y' \subseteq Y$, we have that $y \in B_2 - B_1$ and so $B_1 \subsetneqq B_1 \cup y \subseteq S - X$, and $B_1 \cup y$ is independent in $M(S-X)$. This contradicts the fact B_1 is a basis of $M(S-X)$ and thereby completes the proof of the lemma. □

We require one further lemma before defining contraction for independence spaces. If S and I are sets and $\mathscr{X} = (X_i\colon i \in I)$ is a family of subsets of S, then a *choice function* for $\mathscr{X}$ is a mapping $\phi\colon I \to S$ such that $\phi(i) \in X_i$ for all i in I. If $J \subseteq I$, then $\phi|_J$ denotes the mapping from J into S defined by $\phi|_J(j) = \phi(j)$ for all j in J.

3.1.8. Lemma. (*Rado's selection principle*) *Let $(X_i\colon i \in I)$ be a family of finite subsets of a set S. For each finite subset J of I, let ϕ_J be a choice function for $(X_i\colon i \in J)$. Then there is a choice function ϕ for $(X_i\colon i \in I)$ such that if $J \subset\subset I$, then there is a set K for which $J \subseteq K \subset\subset I$ and $\phi|_J = \phi_K|_J$.*

Mirsky's book (1971) contains several applications of this result together with a short proof of it using Tychonoff's theorem. We shall not reproduce these here.

3.1.9. Proposition. *Suppose that $M(S)$ is an independence space, $X \subseteq S$, and B is a basis of $M(S-X)$. Let*

$$\mathscr{I}.X = \{Y \subseteq X\colon Y \cup B \in \mathscr{I}\}. \tag{3.2}$$

Then $\mathscr{I}.X$ is the set of independent sets of an independence space $M.X$, the contraction of $M(S)$ to X.

Proof. By Lemma 3.1.7, $\mathscr{I}.X$ does not depend on the basis B chosen for $M(S-X)$. It is clear that $\mathscr{I}.X$ satisfies (i1) and (i2). Moreover, if $Y \subseteq X$ and every finite subset of Y is in $\mathscr{I}.X$, then every finite subset of $Y \cup B$ is in $\mathscr{I}$. Hence $Y \cup B \in \mathscr{I}$, and so $Y \in \mathscr{I}.X$. Thus, $\mathscr{I}.X$ satisfies (I4).

To show that $\mathcal{I}.X$ satisfies (i3′), we shall use Rado's selection principle. Suppose that U and T are finite members of $\mathcal{I}.X$ and $|T|>|U|$. Then $B\cup T$ and $B\cup U$ are in $\mathcal{I}$. Moreover, for every finite subset B_0 of B, we have that $M(B_0\cup T\cup U)$ is a finite matroid, and so, applying (i3) to $B_0\cup T$ and $B_0\cup U$, there is an element t of $T-U$ such that $B_0\cup U\cup t\in\mathcal{I}$. Thus the set $S_{B_0}=\{t\in T-U\colon B_0\cup U\cup t\in\mathcal{I}\}$ is finite and non-empty. Let I be the set of finite subsets of B and suppose that $J=\{B_1, B_2, \ldots, B_n\}\subset\subset I$. Let $B'=\bigcup_{i=1}^{n} B_i$ and choose an element t' from $S_{B'}$. We define a choice function ϕ_J for $(S_{B_1}, S_{B_2}, \ldots, S_{B_n})$ by $\phi_J(B_i)=t'$ for all i in $\{1, 2, \ldots, n\}$. Let ϕ be a choice function for $(S_{B_0}\colon B_0\in I)$ satisfying the conclusion of Lemma 3.1.8. We show next that ϕ maps every element of I to the same element t_0 of $T-U$. It will follow from this and (I4) that $B\cup U\cup t_0\in\mathcal{I}$, and hence that $U\cup t_0\in\mathcal{I}.X$, as required. To show that the image of ϕ is a single element of $T-U$, suppose that $B_0, B_0'\in I$, and let $J=\{B_0, B_0'\}$. Then there is a set K such that $J\subseteq K\subset\subset I$ and $\phi_K|_J=\phi|_J$. But $\phi_K(B_0)=\phi_K(B_0')$ and so $\phi(B_0)=\phi(B_0')$. □

It is well known that all bases of a vector space are equicardinal. This result was extended to independence spaces by Rado (1949).

3.1.10. Proposition. *If B_1 and B_2 are bases of an independence space $M(S)$, then $|B_1|=|B_2|$.*

The proof of this result uses the following infinite extension of Rado's theorem on independent transversals (see Chapter 4 of White, 1987). Note that, since every restriction of an independence space $M(S)$ to a *finite* set X is a finite matroid, one can define the *rank* $r(X)$ of X to be the rank of the matroid $M(X)$.

3.1.11. Proposition. *Let $M(S)$ be an independence space and $\mathscr{X}=(X_i\colon i\in I)$ be a family of finite subsets of S. The following statements are equivalent.*

(i) $r\left(\bigcup_{j\in J} X_j\right)\geqslant|J|$ *for every* $J\subset\subset I$.

(ii) *Every finite subfamily of $\mathscr{X}$ has an independent transversal.*

(iii) *$\mathscr{X}$ has an independent transversal.*

Proof. The fact that (i) implies (ii) follows from the finite case of Rado's theorem. Moreover, it is clear that (iii) implies (i). We shall complete the proof of Proposition 3.1.11 by using Rado's selection principle to show that (ii) implies (iii).

Suppose $L\subset\subset I$. Then, by (ii), $(X_i\colon i\in L)$ has an independent transversal. Thus there is an injective choice function ϕ_L for $(X_i\colon i\in L)$ such that $\phi_L(L)\in\mathcal{I}$.

Let ϕ be a choice function for $(X_i: i \in I)$ satisfying the conclusion of Lemma 3.1.8. Then if $i_1, i_2 \in I$ and $J' = \{i_1, i_2\}$, there is a finite subset K' of I such that $J' \subseteq K'$ and $\phi_{K'}|_{J'} = \phi|_{J'}$. As $\phi_{K'}(i_1) \neq \phi_{K'}(i_2)$, it follows that $\phi(i_1) \neq \phi(i_2)$, so ϕ is an injection and hence $\phi(I)$ is a transversal of $\mathscr{X}$. To see that $\phi(I) \in \mathscr{I}$, suppose that $Y \subset\subset \phi(I)$. Then $Y = \phi(J)$ for some $J \subset\subset I$. Now I has a finite subset K such that $K \supseteq J$ and $\phi_K|_J = \phi|_J$. Since $\phi_K(K) \in \mathscr{I}$, we have that $Y = \phi(J) = \phi_K(J) \in \mathscr{I}$. Thus every finite subset of $\phi(I)$ is in $\mathscr{I}$, and so $\phi(I) \in \mathscr{I}$, as required. □

Proof of Proposition 3.1.10. The required result will follow if we can show that for $U, T \in \mathscr{I}$ and $|U| < |T|$, there is an element t of $T - U$ such that $U \cup t \in \mathscr{I}$; that is, if we can show that the finiteness restriction on I_1 and I_2 in (i3′) can be dropped.

Suppose that $U, T \in \mathscr{I}$ and $|U| < |T|$, but $U \cup t \notin \mathscr{I}$ for all t in $T - U$. Then it is easy to show that for each element t of $T - U$ there is a unique circuit C_t such that $t \in C_t \subseteq U \cup t$. Now, if $t \in T - U$, let $X_t = C_t - t$, and if $t \in T \cap U$, then let $X_t = \{t\}$. We shall show next that the family $\mathscr{X} = (X_t: t \in T)$ satisfies Proposition 3.1.11(i). Suppose $T' \subset\subset T$. Then $T' \in \mathscr{I}$. Now let g be the closure operator of the finite matroid $N = M\left(T' \cup \left(\bigcup_{t \in T'} X_t\right)\right)$. Then, for all t in $T' - U$, the set C_t is a circuit of N and so $T' \subseteq g\left(\bigcup_{t \in T'} X_t\right)$. Hence $r\left(\bigcup_{t \in T'} X_t\right) \geqslant r(T') = |T'|$. Thus, by Proposition 3.1.11, $\mathscr{X}$ has a transversal; that is, there is an injection from T into a subset of U. Hence $|T| \leqslant |U|$; a contradiction. This completes the proof of Proposition 3.1.10. □

It follows from Proposition 3.1.10 that in an independence space $M(S)$ one can define the *rank* $r(X)$ of an *arbitrary* subset X of S to be the common cardinality of all bases of $M(X)$.

The preceding discussion has shown that a large number of fundamental properties of finite matroids are shared by independence spaces. In particular, the basic operations of restriction and contraction can be defined for independence spaces. Another important and powerful tool for finite matroids that one would naturally wish to extend to independence spaces is the operation of orthogonality. We shall show, however, that this cannot be done. Indeed, the lack of a satisfactory theory of orthogonality for independence spaces has been an important motivating factor in the study of other classes of infinite matroids.

Suppose that $M(S)$ is an independence space and let $\mathscr{I}^*$ be defined as follows.

$$\mathscr{I}^* = \{X: S - X \text{ contains a basis of } M(S)\}. \qquad (3.3)$$

We leave the reader to check that $\mathscr{I}^*$ is the set of independent sets of a pre-independence space $M^*(S)$. Evidently, every independent set of $M^*(S)$ is contained in a basis of $M^*(S)$. The term 'cofinitary matroid' has often been used to refer to such pre-independence spaces $M^*(S)$ (Bean, 1976; Klee, 1971). When S is finite $\mathscr{I}^*$, of course, is the collection of independent sets of the matroid orthogonal to $M(S)$. However, as the next example shows, when S is infinite, $M^*(S)$ may fail to satisfy the finite character condition.

3.1.12. Example. Let S be an infinite set and k be a positive integer. If $\mathscr{I}_k = \{X \subseteq S: |X| \leqslant k\}$, then $\mathscr{I}_k$ is the set of independent sets of an independence space $M^k(S)$. However, although every finite subset of S is in $\mathscr{I}_k^*$, the set S itself is not.

This example plays a central role in the proof of the next theorem. Let S be an arbitrary infinite set and $\mathscr{S}$ be the set of independence spaces on S. An *orthogonality function* Δ on $\mathscr{S}$ is a mapping from $\mathscr{S}$ into $\mathscr{S}$ such that for all $\mathscr{I}$ in $\mathscr{S}$ we have that Δ is an involution, that is,

$$\Delta(\Delta\mathscr{I}) = \mathscr{I} \tag{3.4}$$

and

$$(\Delta\mathscr{I})|X = (\mathscr{I}\cdot X)^* \quad \text{for all } X \subset\subset S. \tag{3.5}$$

The second of these conditions expresses agreement between Δ and the usual orthogonality for finite methods.

3.1.13. Theorem. *There is no orthogonality function on the collection $\mathscr{S}$ of independence spaces on an infinite set S.*

Proof. Assume that there is an orthogonality function Δ on $\mathscr{S}$. Now if k is a positive integer and $X \subset\subset S$, then $S - X$ is infinite, so $\mathscr{I}_k|(S-X)$ contains a basis of $M^k(S)$. Thus $\mathscr{I}_k.X = \{\varnothing\}$, hence $X \in (\mathscr{I}_k.X)^*$ and so, by (3.5), $X \in \Delta\mathscr{I}_k$. It follows that $\Delta\mathscr{I}_k$ contains all finite subsets of S and hence $\Delta\mathscr{I}_k = 2^S$. Therefore, if j and k are distinct positive integers, then, by (3.4), $\mathscr{I}_j = \Delta(\Delta\mathscr{I}_j) = \Delta(\Delta\mathscr{I}_k) = \mathscr{I}_k$; a contradiction. This completes the proof of Theorem 3.1.13. □

Rado (1966) raised the problem of developing a non-trivial theory of infinite matroids in which the finite character condition does not feature. The motivation for discarding this condition is increased by Example 3.1.12, which suggests that the problem with attempting to define orthogonality for independence spaces may arise because the class of independence spaces is too restricted. The rest of this chapter will be concerned with solving Rado's problem.

3.2. B-matroids

In this section we discuss the properties of basis-matroids (B-matroids), a class of pre-independence spaces introduced by Higgs (1969a) whose members have many of the properties of finite matroids. It is shown that this class contains the class of independence spaces and is closed under restriction and contraction. Moreover, unlike the class of independence spaces, the class of B-matroids is closed under the natural orthogonality function.

If $\mathscr{L}$ is a collection of subsets of a set S, then an *$\mathscr{L}$-subset* of S is a subset of S that is a member of $\mathscr{L}$.

A *B-matroid* $M_B(S)$ is a set S together with a collection $\mathscr{I}$ of subsets of S such that $\mathscr{I}$ satisfies (i1) and (i2) together with

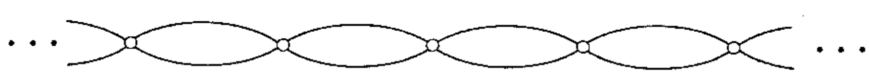
(I_B1) If $T \subseteq X \subseteq S$ and $T \in \mathscr{I}$, then there is a maximal $\mathscr{I}$-subset of X containing T.

(I_B2) For all $X \subseteq S$, if B_1 and B_2 are maximal $\mathscr{I}$-subsets of X and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - x) \cup y$ is a maximal $\mathscr{I}$-subset of X.

It is easy to check that (I_B2) implies (i3) and hence that every B-matroid is a pre-independence space. Thus, if $M_B(S)$ is a B-matroid, the members of $\mathscr{I}$ are called independent sets, and a maximal member of $\mathscr{I}$ is called a basis. An immediate consequence of Lemma 3.1.6 is that every independence space is a B-matroid. But, as the following example shows, not every B-matroid is an independence space.

3.2.1. Example. Let S be the edge set of the infinite graph Γ shown in Figure 3.1 and let $\mathscr{I}$ consist of those subsets of S that do not contain the edge set of any cycle or two-way infinite path in Γ. It is straightforward to check that $\mathscr{I}$ satisfies (i1), (i2), (I_B1), and (I_B2). Hence $\mathscr{I}$ is the collection of independent sets of a B-matroid $M_B(S)$ on S. However, $\mathscr{I}$ does not satisfy (I4) because every finite subset of a two-way infinite path is in $\mathscr{I}$, yet the path itself is not. Therefore, $M_B(S)$ is not an independence space.

Figure 3.1.



This example is one member of a class of B-matroids introduced by Higgs (1969c) (see Exercise 3.19).

If $M_B(S)$ is a B-matroid having $\mathscr{I}$ as its collection of independent sets, then for $X \subseteq S$, let $\mathscr{I}|X$ be defined as in (3.1). It is easy to check that $\mathscr{I}|X$ is the set of independent sets of a B-matroid, $M_B(X)$, the restriction of $M_B(S)$ to X.

We show next that the operation of contraction can be defined for

B-matroids as in (3.2). First note that the argument that was used to prove Lemma 3.1.7 can be applied in this case to give that $\mathscr{I}.X$ does not depend on the basis chosen for $\mathscr{I}|(S-X)$.

3.2.2. Proposition. *If $M_B(S)$ is a B-matroid and $X \subseteq S$, then $\mathscr{I}.X$ is the set of independent sets of a B-matroid $M_B.X$ on X.*

Proof. Let B be a basis of $M_B(S-X)$. Clearly $\mathscr{I}.X$ satisfies (i1) and (i2). Now suppose $Z \subseteq Y \subseteq X$ and Z is an $\mathscr{I}.X$-subset. Then $Z \cup B \in \mathscr{I}$ and so $Z \cup B$ is contained in a maximal $\mathscr{I}$-subset B' of $S-(X-Y)$. Now $B' = Z' \cup B$ where $Z' \supseteq Z$ and $Z' \cap (S-Y) = \varnothing$. Evidently Z' is a maximal $\mathscr{I}.X$-subset of Y containing Z. Thus $\mathscr{I}.X$ satisfies (I_B1).

We now check that $\mathscr{I}.X$ satisfies (I_B2). Let B_1 and B_2 be maximal $\mathscr{I}.X$ subsets of a subset Y of X and suppose $x \in B_1 - B_2$. Then $B_1 \cup B$ and $B_2 \cup B$ are maximal $\mathscr{I}$-subsets of $S-(X-Y)$. Since $x \in (B_1 \cup B)-(B_2 \cup B)$, there is an element y of $(B_2 \cup B)-(B_1 \cup B)$ such that $((B_1 \cup B)-x) \cup y$ is a maximal $\mathscr{I}$-subset of $S-(X-Y)$. Clearly $y \in B_2 - B_1$. Moreover, $((B_1 \cup B)-x) \cup y = ((B_1 - x) \cup y) \cup B$, and so $(B_1 - x) \cup y$ is a maximal $\mathscr{I}.X$-subset of Y. Hence $\mathscr{I}.X$ satisfies (I_B2) and the proposition is proved. □

In the preceding section it was suggested that one of the problems with attempting to define orthogonality for independence spaces may be that the class of independence spaces is too restricted. In this section we shall confirm this by showing that (3.3) is an involution on the class of B-matroids. In order to achieve this objective we shall consider certain aspects of the closure-operator approach to infinite matroids as developed by Klee (1971). We defer to the exercises consideration of an equivalent approach that was adopted by Higgs (1969a, b, c).

An *operator* f on a set S is a function from 2^S to 2^S satisfying

(cl1) $X \subseteq f(X)$ for all $X \subseteq S$.
(cl2) $X \subseteq Y \subseteq S$ implies $f(X) \subseteq f(Y)$.

We shall concentrate primarily on operators satisfying the following two additional conditions.

(cl3) $f(f(X)) = f(X)$ for all $X \subseteq S$.
(cl4′) If $X,\ Y \subseteq S$ and $p \in f(Y)-f(Y-X)$, then $x \in f((Y-x) \cup p)$ for some x in X.

Such operators will be called *idempotent-exchange-* or *IE-operators.*

It is straightforward to check that (cl1)–(cl3) and (cl4′) provide another cryptomorphic axiom system for finite matroids to add to the many presented in White (1986).

For an operator f on a set S, let f^* be defined, for all $X \subseteq S$, by

$$f^*(X) = X \cup \{x\colon x \notin f(S-(X \cup x))\}.$$

3.2.3. Proposition. *If f is an operator on a set S, then f^* is an operator on S and $(f^*)^* = f$. Moreover, if f is an IE-operator, so is f^*.*

The proof of this is left as an exercise.

Again, for f an operator on S and $X \subseteq S$, the *restriction* f_X and *contraction* f^X of f to X are defined, for all $Y \subseteq X$, by

$$f_X(Y) = f(Y) \cap X, \text{ and } f^X(Y) = f(Y \cup (S-X)) \cap X \text{ respectively.}$$

3.2.4. Proposition. *If f is an operator on a set S and $X \subseteq S$, then both f_X and f^X are operators on X, and $(f^*)_X = (f^X)^*$ and $(f_X)^* = (f^*)^X$. Furthermore, if f is an IE-operator, so are both f_X and f^X.*

Proof. It is not difficult to check that both f_X and f^X are operators on X and that if f is an IE-operator on S, then both f_X and f^X are IE-operators on X. Now, if $Y \subseteq X$, then

$$\begin{aligned}(f^*)_X(Y) = f^*(Y) \cap X &= (Y \cup \{y \in S\colon y \notin f(S-(Y \cup y))\}) \cap X \\ &= Y \cup \{y \in X\colon y \notin f((S-X) \cup (X-(Y \cup y)))\} \\ &= Y \cup \{y \in X\colon y \notin f^X(X-(Y \cup y))\} = (f^X)^*(Y).\end{aligned}$$

Thus $(f^*)_X = (f^X)^*$. If we replace f by f^* in this equation, then, by Proposition 3.2.3 we get $f_X = ((f^*)^X)^*$. Hence, by Proposition 3.2.3 again, $(f_X)^* = (f^*)^X$, and the proof of the proposition is complete. □

Let f be an operator on a set S and suppose $X \subseteq S$. Then X is *independent* if $x \notin f(X-x)$ for all x in X; otherwise X is *dependent*. We call X *spanning* or *non-spanning* according as $f(X) = S$ or $f(X) \subsetneqq S$; and X is a *basis* if X is both independent and spanning. A minimal dependent set is a *circuit* and a maximal non-spanning set is a *hyperplane*. Clearly when S is finite and f is an IE-operator, f is the closure operator of a (finite) matroid on S and then the definitions above are consistent with the usage of these terms in White (1986). We leave the routine proofs of the next two results as exercises.

3.2.5. Proposition. *If f is an IE-operator on S, then the sets of bases, maximal independent sets, and minimal spanning sets are identical.*

3.2.6. Proposition. *If f is an operator, then*

(i) *the f-independent sets are precisely the complements of the f^*-spanning sets;*
(ii) *the f-circuits are precisely the complements of the f^*-hyperplanes; and*
(iii) *the f-bases are precisely the complements of the f^*-bases.*

The next proposition shows that B-matroids can be axiomatized in this operator framework. This important fact will be used in the discussion of orthogonality for B-matroids. We require the following lemma.

3.2.7. Lemma. *Let f be an IE-operator on a set S and $\mathscr{I}$ be the collection of f-independent sets. If $X \subseteq S$ and B is a maximal $\mathscr{I}$-subset of X, then $f(B) = f(X)$. Moreover, if $I \subseteq S$ and $I \in \mathscr{I}$, then $f(I) = I \cup \{x\colon I \cup x \notin \mathscr{I}\}$.*

Proof. As B is a maximal $\mathscr{I}$-subset of X, by Propositions 3.2.4 and 3.2.5, $f_X(B) = X$. Thus $X \subseteq f(B)$ and so, by (cl2) and (cl3), $f(X) \subseteq f(f(B)) = f(B)$. But, by (cl2), $f(B) \subseteq f(X)$, hence $f(B) = f(X)$ as required. If $I \in \mathscr{I}$ and $I \cup x \notin \mathscr{I}$, then $y \in f((I \cup x) - y)$ for some element y of $I \cup x$. If $y = x$, then $x \in f(I)$. But if $y \neq x$, we still get $x \in f(I)$ by (cl4′) because $y \notin f(I - y)$. Therefore, $f(I) \supseteq I \cup \{x\colon I \cup x \notin \mathscr{I}\}$. Now, if $x \in f(I) - I$, then $I \cup x$ is not f-independent; that is, $I \cup x \notin \mathscr{I}$. Hence $f(I) \subseteq I \cup \{x\colon I \cup x \notin \mathscr{I}\}$, and the required result follows. □

3.2.8. Proposition. *Suppose that $M_B(S)$ is a B-matroid having $\mathscr{I}$ as its collection of independent sets and let f be defined, for all subsets X of S, by*

$$f(X) = X \cup \{x\colon I \cup x \notin \mathscr{I} \text{ for some } I \subseteq X \text{ such that } I \in \mathscr{I}\}. \tag{3.6}$$

Then f is the unique IE-operator on S having $\mathscr{I}$ as its collection of independent sets, and, since $M_B(S)$ is a B-matroid, $\mathscr{I}$ satisfies (I_B1). Conversely, if f is an IE-operator on S such that the set $\mathscr{I}$ of f-independent sets satisfies (I_B1), then $\mathscr{I}$ is the set of independent sets of a B-matroid on S.

Proof. Suppose that $M_B(S)$ is a B-matroid and that f is defined as in (3.6). Then clearly f satisfies (cl1) and (cl2). We show next that for all $X \subseteq S$,

$$f(X) = \begin{cases} X \cup \{x\colon X \cup x \notin \mathscr{I}\}, & \text{if } X \in \mathscr{I}, \\ f(I_X), & \text{if } I_X \text{ is a maximal } \mathscr{I}\text{-subset of } X. \end{cases} \tag{3.7}$$

If $X \in \mathscr{I}$ and $I \cup x \notin \mathscr{I}$ for some $I \subseteq X$, then $X \cup x \notin \mathscr{I}$. Thus (3.6) and (3.7) agree when $X \in \mathscr{I}$. If $X \notin \mathscr{I}$ and I_X is a maximal $\mathscr{I}$-subset of X, then, by (cl2), $f(X) \supseteq f(I_X)$. We now show that the reverse inclusion also holds. If $x \in X - I_X$, then $I_X \cup x \notin \mathscr{I}$, so $x \in f(I_X)$ and hence, $f(I_X) \supseteq X$. Next assume that $x \in f(X) - X$. Then $I \cup x \notin \mathscr{I}$ for some $\mathscr{I}$-subset I of X. If $I_X \cup x \notin \mathscr{I}$, then $x \in f(I_X)$, so suppose that $I_X \cup x \in \mathscr{I}$. Then $I_X \cup x$ is a maximal $\mathscr{I}$-subset of $X \cup x$. Moreover, I is contained in a maximal $\mathscr{I}$-subset B of $X \cup x$, and $x \notin B$. Thus $x \in (I_X \cup x) - B$, and so, by (I_B2), there is an element y of $B - (I_X \cup x)$ such that $((I_X \cup x) - x) \cup y = I_X \cup y$ is a maximal $\mathscr{I}$-subset of $X \cup x$. But $I_X \cup y \subseteq X$ and so the choice of I_X is contradicted. Thus (3.7) is established.

From (3.7), it is easy to see that f satisfies (cl3). To show that f satisfies (cl4′), suppose $X \subseteq Y$ and $p \in f(Y) - X$ and $p \notin f(Y - X)$. Let I_{Y-X} be a maximal $\mathscr{I}$-subset of $Y-X$. Then I_{Y-X} is contained in a maximal $\mathscr{I}$-subset I_Y of Y, and, as $p\in f(Y)=f(I_Y)$, we have $I_Y\cup p\notin\mathscr{I}$ and so I_Y is a maximal $\mathscr{I}$-subset of $Y\cup p$. Now $p\notin f(Y-X)=f(I_{Y-X})$, hence $I_{Y-X}\cup p\in\mathscr{I}$, and so $I_{Y-X}\cup p\subseteq I'_Y$, a maximal $\mathscr{I}$-subset of $Y\cup p$. If there is an element x of X such that $x\notin I'_Y$, then $x\in f(Y)\subseteq f(Y\cup p)=f(I'_Y)=f((Y-x)\cup p)$; that is, (cl4′) holds. Thus assume that $X\subseteq I'_Y$. Then $I'_Y=I_{Y-X}\cup X\cup p\supsetneqq I_{Y-X}\cup X\supseteq I_Y$. This is a contradiction since both I_Y and I'_Y are maximal $\mathscr{I}$-subsets of $Y\cup p$. We conclude that f satisfies (cl4′).

The fact that $\mathscr{I}$ is precisely the collection of f-independent sets follows without difficulty from (3.7). Moreover, by Lemma 3.2.7, f is the unique IE-operator on S having $\mathscr{I}$ as its collection of independent sets.

To prove the converse, let f be an IE-operator on S having $\mathscr{I}$ as its collection of independent sets and suppose that $\mathscr{I}$ satisfies (I_B1). Evidently $\mathscr{I}$ satisfies (i1) and (i2). Therefore to show $\mathscr{I}$ is the collection of independent sets of a B-matroid, it remains only to check that (I_B2) holds. Note that, by Lemma 3.2.7 and (I_B1), f is defined as in (3.7) for all subsets of S. Suppose $X\subseteq S$ and let B_1 and B_2 be maximal $\mathscr{I}$-subsets of X. Assume that $x\in B_1-B_2$ and, for all y in B_2-B_1, the set $(B_1-x)\cup y\notin\mathscr{I}$. Then, by (3.7), $B_2-B_1\subseteq f(B_1-x)$. Hence $f(B_1-x)=f((B_1-x)\cup B_2)\supseteq f(B_2)\supseteq X$. Hence $x\in f(B_1-x)$; a contradiction of the fact that B_1 is f-independent. This completes the proof of Proposition 3.2.8. □

If $M_B(S)$ is a B-matroid and f is defined as in (3.6), then we call f the *closure operator* of $M_B(S)$.

3.2.9. Proposition. *Let f be the closure operator of a B-matroid $M_B(S)$ having $\mathscr{I}$ as its collection of independent sets. Then f_X and f^X are the closure operators of $M_B(X)$ and $M_B.X$ respectively.*

Proof. It is clear that $\mathscr{I}|X$ is the collection of f_X-independent sets, hence by Proposition 3.2.8, f_X is the closure operator of $M_B(X)$. Now suppose that $Y\in\mathscr{I}.X$, but Y is not f^X-independent. Then $Y\cup B\in\mathscr{I}$ for some basis B of $M_B(S-X)$, and $y\in f^X(Y-y)$ for some y in Y. Thus $y\in f((Y-y)\cup(S-X))$ and, as $S-X\subseteq f(B)\subseteq f((Y-y)\cup B)$, it follows by (cl2) and (cl3) that $f((Y-y)\cup(S-X))\subseteq f((Y-y)\cup B)$. Therefore $y\in f((Y-y)\cup B)$. This contradicts the fact that $Y\cup B\in\mathscr{I}$. Thus, if $Y\in\mathscr{I}.X$, then Y is f^X-independent. On the other hand, if Y is f^X-independent and $Y\notin\mathscr{I}.X$, then $Y\cup B\notin\mathscr{I}$ where B is a basis of $M_B(S-X)$. As $B\in\mathscr{I}$, by (I_B1), $Y\cup B$ contains a maximal $\mathscr{I}$-subset $Y'\cup B$ where $Y'\subsetneqq Y$. By Lemma 3.2.7, $f(Y'\cup B)=f(Y\cup B)$. Thus, if $y\in Y-Y'$, then $y\in f(Y'\cup B)\subseteq f((Y-y)\cup B)\subseteq f((Y-y)\cup(S-X))$; that is, Y is not f^X-independent. This contradiction implies that if Y is f^X-independent, then $Y\in\mathscr{I}.X$. We conclude that the set of f^X-independent sets equals $\mathscr{I}.X$ and hence that f^X is the closure operator of $M_B.X$. This completes the proof of Proposition 3.2.9. □

We are now in a position to prove that the class of B-matroids is closed under the natural orthogonality function (3.3).

3.2.10. Theorem. *Suppose that $M_B(S)$ is a B-matroid having f as its closure operator. Then f^* is the closure operator of a B-matroid on S and its collection $\mathscr{I}^*$ of independent sets is given by $\mathscr{I}^* = \{X: S - X$ contains a basis of $M_B(S)\}$.*

Proof. By Proposition 3.2.3, f^* is an IE-operator on S. Now, by Propositions 3.2.3 and 3.2.6(i), the f^*-independent sets are precisely the complements of the f-spanning sets. In addition, by (3.7), a set is f-spanning if and only if it contains an f-basis. Therefore $\mathscr{I}^*$ is as given. It follows from Proposition 3.2.8 that to complete the proof we need to show that $\mathscr{I}^*$ satisfies (I_B1). Suppose that $Y \subseteq X \subseteq S$ and $Y \in \mathscr{I}^*$. Then $S - Y$ is f-spanning and so $X - Y$ is f^X-spanning. But, by Proposition 3.2.9, f^X is the closure operator of the B-matroid $M_B.X$ and so, by (3.7), $X - Y$ contains an f^X-basis B. Thus $X - B \supseteq Y$ and, by Proposition 3.2.6(iii), $X - B$ is an $(f^X)^*$-basis. Therefore, since $(f^X)^* = (f^*)_X$ by Proposition 3.2.4, it follows that $X - B$ is a maximal f^*-independent subset of X containing Y. Thus $\mathscr{I}^*$ satisfies (I_B1), and so, by Proposition 3.2.8, Theorem 3.2.10 is proved. □

The attention in this chapter has been concentrated on two classes of infinite matroids, the classes of independence spaces and of B-matroids. The motivation for the study of independence spaces was algebraic. The last result of this chapter provides additional motivation for the study of B-matroids, showing that this is a particularly natural class of infinite matroids to examine.

Let S be an arbitrary infinite set and suppose that on every subset W of S we have a distinguished class $\mathscr{D}_W$ of pre-independence spaces so that

(3.1) and (3.2) give well-defined operations of restriction and contraction on $\mathscr{D}_W$ such that if $Z \subseteq W$, the restriction or contraction of a member of $\mathscr{D}_W$ is in $\mathscr{D}_Z$; (3.8)

and

the function defined by (3.3) is an involution on $\mathscr{D}_W$. (3.9)

Evidently, if for all $W \subseteq S$, we let $\mathscr{D}_W$ be the set of B-matroids on W, then (3.8) and (3.9) hold. In fact, for every choice of $\mathscr{D}_W$ subject to these conditions, the members of $\mathscr{D}_W$ are B-matroids.

3.2.11. Theorem. *If S is an infinite set, then the largest class of pre-independence spaces defined on S and all its subsets such that* (3.8) *and* (3.9) *are satisfied is the class of B-matroids.*

For the proof of this result the reader is referred to Oxley (1978a). Note, however, that there is an error in Oxley (1978a) in that only Theorem 3.2.11 above is proved, although a stronger result is stated there. Whether this stronger result is true is an unsolved problem (see Exercise 3.22).

Exercises

The more difficult exercises are marked with an asterisk.

Section 3.1

3.1. (a) Extend the closure axioms (cl1)–(cl4) of Chapter 2 of White (1986) to give a closure-operator definition of an independence space.

(b) Show that, by adding (B4) to (b1)–(b3) of White (1986), one gets an axiom system that is cryptomorphic to the system (i1), (i2), (i3′), and (I4).

(c) Show that (b) holds with (b3) replaced by (b3.1) of White (1986).

3.2. (a) Let I be a set and $\{S_i\colon i \in I\}$ be a collection of pairwise disjoint sets. For all i in I, let $M_i(S_i)$ be an independence space having $\mathscr{I}_i$ as its collection of independent sets. Show that $\left\{\bigcup_{i \in I} U_i\colon U_i \in \mathscr{I}_i\right\}$ is the collection of independent sets of an independence space on $\bigcup_{i \in I} S_i$. This independence space is called the *direct sum* of the independence spaces $M_i(S_i)$ $(i \in I)$.

*(b) (Las Vergnas, 1971; Mason, 1970; Bean, 1976) Let $M(S)$ be an independence space for which $M^*(S)$ is also an independence space. Show that $M(S)$ is the direct sum of a collection of finite matroids.

3.3. Let $M(S)$ be an independence space and suppose that $X \subset\subset S$. If the restriction of a pre-independence space is defined as in (3.1), show that $\mathscr{I}.X = (\mathscr{I}^* | X)^*$.

3.4. (Las Vergnas, 1971) Let $M(S)$ be an independence space and define $\mathscr{I}^+$ to be the collection of subsets X of S such that for all $Y \subset\subset X$ there is a basis of $M(S)$ disjoint from Y.

(a) Show that $\mathscr{I}^+$ is the collection of independent sets of an independence space on S.

(b) Prove that $((\mathscr{I}^+)^+)^+ = \mathscr{I}^+$, and $\mathscr{I}^+ | X = (\mathscr{I}.X)^*$ for all $X \subset\subset S$.

(c) Give an example to show that it is not necessarily the case that $(\mathscr{I}^+)^+ = \mathscr{I}$.

*(d) Let f be the closure operator of $M(S)$; that is, for all $X \subseteq S$,

$$f(X) = X \cup \{x\colon I \cup x \notin \mathscr{I} \text{ for some } I \subseteq X \text{ such that } I \in \mathscr{I}\}.$$

Show that $(\mathscr{I}^+)^+ = \mathscr{I}$ if and only if for all finite subsets X of S and all elements x of $S - f(X)$, there is a set Y containing X such that $S - Y$ is finite and $x \notin f(Y)$.

*3.5. (Piff, 1971) Let $\mathscr{X}$ be a family of subsets of a set S and suppose that every element of S is in finitely many members of $\mathscr{X}$. Use Rado's selection principle to show that the set of partial transversals of $\mathscr{X}$ is the collection of independent sets of an independence space on S.

*3.6. (Piff, 1971) Prove that if $M(S)$ is an independence space such that for all $X \subset\subset S$ the matroid $M(X)$ is graphic, then there is a graph Γ such that $M(S)$ is isomorphic to M_Γ.

*3.7. Show that the following are equivalent.
  (a) The Axiom of Choice.
  (b) (Zorn's lemma) If every chain in a non-empty partially ordered set possesses an upper bound, then the set has at least one maximal element.
  (c) (Tukey's lemma) If $\mathscr{X}$ is a non-empty collection of sets satisfying (i2) and (I4), then $\mathscr{X}$ has a maximal member.

Section 3.2

3.8. Higgs's approach to infinite matroids (Higgs, 1969a) uses the idea of a derived set operator. The series of problems below links this approach to that taken above. For an operator f on a set S, we define ∂_f: $2^S \to 2^S$ by

$$\partial_f(X) = \{x: x \in f(X - x)\}.$$

  (a) Show that ∂_f satisfies the following conditions.

   (D1) If $X \subseteq Y \subseteq S$, then $\partial_f(X) \subseteq \partial_f(Y)$.
   (D2) If $X \subseteq S$ and $x \in \partial_f(X)$, then $x \in \partial_f(X - x)$.

  (b) Show that for all $X \subseteq S$, $f(X) = X \cup \partial_f(X)$.
  (c) Let ∂ be a function on 2^S satisfying (D1) and (D2). Define f: $2^S \to 2^S$ by $f(X) = X \cup \partial(X)$. Show that f is an operator on S and that $\partial_f = \partial$.
  (d) Characterize IE-operators f in terms of properties of the corresponding derived set operators ∂_f.
  (e) If $X \subseteq S$ and f is an operator on S, find ∂_{f^*}, ∂_{f_X}, and ∂_{f^X} in terms of ∂_f.
  (f) For an operator f, characterize f-independent sets, f-spanning sets and f-bases in terms of ∂_f.

3.9. Axiomatize B-matroids in terms of their collections of bases.

3.10. Show that Proposition 3.2.5 does not hold for arbitrary operators by giving an example of an operator whose sets of bases, maximal independent sets, and minimal spanning sets are all distinct.

3.11. Show that an IE-operator is not uniquely determined by the pair consisting of its collection of independent sets and its collection of spanning sets.

3.12. Consider the following conditions on an operator f on a set S.

  (cl3′) If X, $Y \subseteq S$ and $X \subseteq f(Y)$, then $f(X \cup Y) \subseteq f(Y)$.
  (C) If $Y \subseteq S$ and $p \in f(Y)$, then there is a minimal subset U of Y such that $p \in f(U)$ and U is independent.
  (H) If $Y \subseteq S$ and $p \in S - f(Y)$, then there is a maximal superset V of Y such that $p \notin f(V)$ and $V \cup p$ is spanning.

  Show that
  (a) f satisfies (cl3′) if and only if f satisfies (cl3);
  (b) f satisfies (C) if and only if f^* satisfies (H);
  (c) if f satisfies (cl4′) and p, Y, and U are as in (C), then $U \cup p$ is a circuit;
  (d) if f satisfies (cl3) and p, Y, and V are as in (H), then V is a hyperplane.

3.13. Find an example of a B-matroid $M_B(S)$ and an operator g on S so that g is not the closure operator of $M_B(S)$, yet $M_B(S)$ and g have the same collections of independent sets.

3.14. If f is the closure operator of a B-matroid, show that f satisfies (C) and (H).

3.15. Let f be an IE-operator on a set S and suppose that f satisfies (C). Show that the collection $\mathscr{C}$ of f-circuits satisfies (c1), (c2), and (c3.1) and that, for all $X \subseteq S$, $f(X) = X \cup \{x\colon x \in C \subseteq X \cup x \text{ for some } C \text{ in } \mathscr{C}\}$.

3.16. (Higgs, 1969a) Let f be an IE-operator on a set S and suppose that f satisfies both (C) and (H).

(a) (Unsolved) Is f the closure operator of a B-matroid on S?

(b) (Unsolved) If $X \subseteq S$, does f_X satisfy (H)?

(c) If $X \subseteq S$, does f_X satisfy (C)?

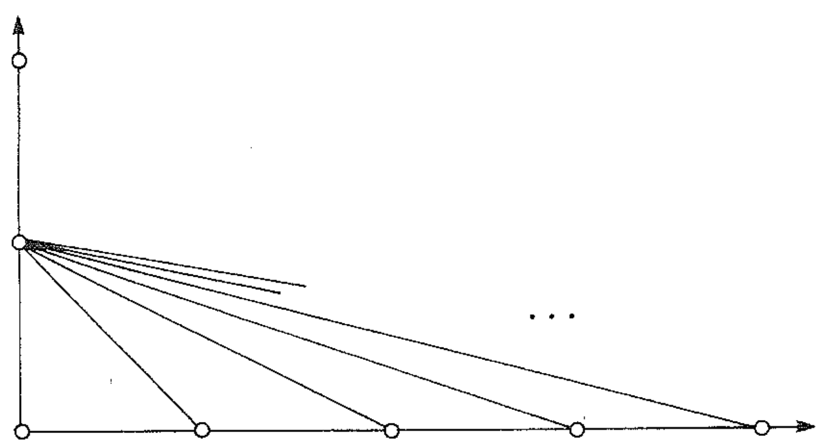
(d) Show that if (a) holds, then so does (b).

3.17. Let f be an IE-operator on a set S. Suppose f satisfies (C) and B is an f-basis. If $x \in S - B$, use the result of Exercise 3.15 to prove that there is a unique circuit C such that $x \in C \subseteq B \cup x$, and that, if $y \in B$, then $(B \cup x) - y$ is a basis if and only if $y \in C$.

*3.18. (Oxley, 1978a) Use the results of Exercises 3.14, 3.15, and 3.17 to show that if B_1 and B_2 are bases of a B-matroid and $b_1 \in B_1 - B_2$, then there is an element b_2 of $B_2 - B_1$ such that both $(B_1 - b_1) \cup b_2$ and $(B_2 - b_2) \cup b_1$ are bases.

*3.19. (Higgs, 1969c) Let Γ be a graph that may be finite or infinite, and let $\mathscr{K}(\Gamma)$ consist of all finite cycles of Γ together with all two-way infinite paths in Γ. Use the results of Exercises 3.14 and 3.15 to show that $\mathscr{K}(\Gamma)$ is the set of circuits of a B-matroid on the edge set of Γ if and only if Γ has no subgraph homeomorphic to the graph in Figure 3.2.

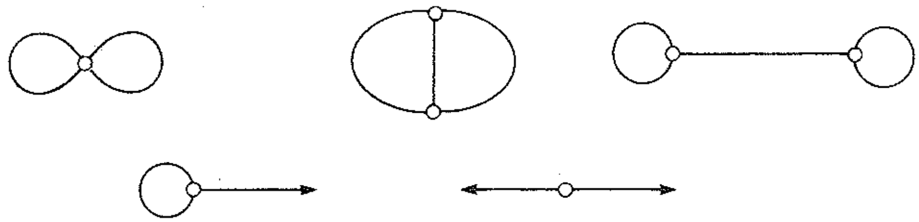
Figure 3.2.



*3.20. (Matthews and Oxley, 1977) Let Γ be a graph that may be finite or infinite, and let $\mathscr{N}(\Gamma)$ be the set of subgraphs of Γ that are homeomorphic to one of the five graphs shown in Figure 3.3 (where an arrow denotes a one-way infinite path). Use the results of Exercises 3.14 and 3.15 to show that $\mathscr{N}(\Gamma)$ is the set of circuits of a B-matroid on the set of edges of Γ. A detailed discussion of the properties of the finite matroids that arise in this way may be found in Chapter 4 of this volume.

*3.21. (Higgs, 1969b) Show that if the Generalized Continuum Hypothesis holds, then all bases of a B-matroid are equicardinal.

Figure 3.3.



3.22. Let S be an infinite set and $\mathscr{D}$ be a class of pre-independence spaces defined on S and all its subsets such that (3.8) and (3.9) are satisfied.
(a) Show that $\mathscr{D}$ need not equal the class of B-matroids defined on all subsets of S.
(b) (Unsolved) Does (a) remain true if we insist that $\mathscr{D}$ contains the class of all those independence spaces that are defined on some subset of S?

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