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A MATROID GENERALIZATION OF A RESULT OF DIRAC

JAMES OXLEY*

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This paper generalizes a theorem of Dirac for graphs by proving that if M is a 3-connected matroid, then, for all pairs $\{a,b\}$ of distinct elements of M and all cocircuits C^* of M, there is a circuit that contains $\{a,b\}$ and meets C^*. It is also shown that, although the converse of this result fails, the specified condition can be used to characterize 3-connected matroids.

1. Introduction

Dirac [2] proved that, for all $n \geq 2$, if G is a simple n-connected graph, then every two distinguished edges and every $n-2$ distinguished vertices lie in a common cycle. An immediate consequence of this is that, in an n-connected graph, there is a cycle through any specified set of n vertices. This paper shows that, for $n=3$, these results are special cases of a more general theorem for matroids.

The matroids for which every pair of distinct elements lie in some circuit are precisely the connected matroids. The problem of characterizing the matroids in which every triple of distinct elements lies in some circuit was solved by Seymour [4] in the binary case, but remains unsolved in general. Evidently, a matroid in which every 3-set lies in some circuit need not be 3-connected; for instance, consider the 2-sum of two copies of $U_{2,4}$. On the other hand, a 3-connected matroid can certainly have three elements that do not lie in a common circuit; for example, consider three edges meeting at a common vertex in a graph. The following theorem, the main result of the paper, asserts that all 3-connected matroids satisfy a condition intermediate between having every 2-set of elements in a circuit and having every 3-set of elements in a circuit.

(1.1) Theorem. *If M is a 3-connected matroid, then, for every pair $\{a,b\}$ of distinct elements of M and every cocircuit C^* of M, there is a circuit that contains $\{a,b\}$ and meets C^*.*

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The matroid terminology used here will follow Oxley [3]. In particular, a *triangle* in a matroid is a 3-element circuit, and a *triad* is a 3-element cocircuit. For a matroid M, the simple matroid and the cosimple matroid associated with M will be denoted by $\widetilde{M}$ and $\underset{\sim}{M}$, respectively.

The basic property of matroids that a circuit and a cocircuit cannot have exactly one common element will be referred to as *orthogonality*.

We now note two well-known results of Bixby [1] and Tutte [5] which will be used in the proof of the main theorem.

(1.2) Lemma. *Let e be an element of a 3-connected matroid M. Then either $\underset{\sim}{M\backslash e}$ or $\widetilde{M/e}$ is 3-connected.*

The next lemma is often called Tutte's Triangle Lemma.

(1.3) Lemma. *Let $\{e,f,g\}$ be a triangle of a 3-connected matroid M. If neither $M\backslash e$ nor $M\backslash f$ is 3-connected, then M has a triad containing e and exactly one of f and g.*

2. Proof and consequences

In this section, we prove the main result and present some straightforward corollaries of it.

Proof of Theorem 1.1. Assume that the theorem fails for a, b, and C^* in a minor-minimal counterexample M. Evidently

(2.1) $\{a,b\} \cap C^* = \emptyset$,
otherwise the fact that M is connected implies that M satisfies the theorem.

If $|E(M)| \leq 4$, then M is isomorphic to one of $U_{1,2}, U_{1,3}, U_{2,3}$, or $U_{2,4}$, and (2.1) fails in each of these cases. Thus we may assume that

(2.2) $|E(M)| > 4$.

Next we note that if $x \in C^* \cap \mathrm{cl}(\{a,b\})$, then, by (2.1), $x \in \mathrm{cl}(\{a,b\}) - \{a,b\}$, so $\{a,b,x\}$ is a circuit containing $\{a,b\}$ and meeting C^*; a contradiction. Thus

(2.3) $C^* \cap \mathrm{cl}(\{a,b\}) = \emptyset$.

Extending this, we now show that

(2.4) $\mathrm{cl}(\{a,b\}) \subsetneqq E(M) - C^*$.

Certainly $\mathrm{cl}(\{a,b\}) \subseteq E(M) - C^*$. Suppose equality holds here. Then $r(M) = 3$. As M is 3-connected, C^* contains a basis $\{u,v,w\}$ of M. Clearly, at least one of $\mathrm{cl}(\{u,v\})$, $\mathrm{cl}(\{u,w\})$, and $\mathrm{cl}(\{v,w\})$, say the first, avoids $\{a,b\}$. Then $\{u,v,a,b\}$ is a circuit and we have a contradiction that establishes (2.4).

Next we prove the following:

(2.5) Lemma. *Suppose that* $x \in E(M) - C^* - \{a,b\}$. *Then*

(i) $M \backslash x$ *is not* 3*-connected; and*

(ii) *either* $\widetilde{M/x}$ *is not* 3*-connected, or* $\{a,b,x\}$ *is a circuit of* M.

Proof. To show (ii), suppose that $\widetilde{M/x}$ is 3-connected but that $\{a,b,x\}$ is not a circuit of M. Then we can label $\widetilde{M/x}$ so that its ground set contains $\{a,b\}$. Certainly C^* is a cocircuit of M/x. Hence C^* contains a cocircuit C_2^* of $\widetilde{M/x}$. Thus, by the choice of M, there is a circuit C_1 of $\widetilde{M/x}$ containing $\{a,b\}$ and meeting C_2^*. But C_1 or $C_1 \cup x$ is a circuit of M and so we have a contradiction, thereby establishing (ii). The proof of (i) is similar. ∎

We now show that

(2.6) $|\mathrm{cl}(\{a,b\})| \le 3$.

Suppose that x_1 and x_2 are distinct elements of $\mathrm{cl}(\{a,b\}) - \{a,b\}$. Then $M|\{x_1,x_2,a,b\} \cong U_{2,4}$. Moreover, by (2.4) and (2.5)(i), neither $M\backslash x_1$ nor $M\backslash x_2$ is 3-connected. Thus, by Tutte's Triangle Lemma applied to the triangle $\{x_1,x_2,a\}$, we deduce that M has a triad containing two elements of this triangle. By orthogonality, this triad is contained in $\{x_1,x_2,a,b\}$. Hence M has a triad that is also a triangle, so $M \cong U_{2,4}$; a contradiction to (2.2).

(2.7). *M has no triad containing* $\{a,b\}$.

To see this, assume that $\{a,b,d\}$ is a triad of M. Then $M\backslash d$ is connected and has $\{a,b\}$ as a cocircuit. Moreover, $C^* - d$ contains a cocircuit C_3^* of $M\backslash d$. Evidently $M\backslash d$ has a circuit that contains a and meets C_3^*. By orthogonality, this circuit must also contain b, and so we have a contradiction that establishes (2.7).

By (2.4), there is an element in $E(M) - C^* - \mathrm{cl}(\{a,b\})$. Moreover, by (2.5)(ii), for every such element z, the matroid $\widetilde{M/z}$ is not 3-connected. Thus, by Lemma 1.2, $\underset{\sim}{M\backslash z}$ is 3-connected. Moreover, by (2.7), $\underset{\sim}{M\backslash z}$ has both a and b as elements. We now show that

(2.8) $z \in \mathrm{cl}_{M^*}(C^*)$.

Assume the contrary. Then $M\backslash z$ has C^* as a cocircuit. Since C^* contains at most one element of each series class of $M\backslash z$, we may assume that $\underset{\sim}{M\backslash z}$ has C^* as a cocircuit. Thus, by the choice of M, there is a circuit C_1 of $\underset{\sim}{M\backslash z}$ that meets C^* and contains $\{a,b\}$. It follows that the choice of M is contradicted, and (2.8) holds.

By (2.8), there is a partition $\{X_1,X_2,\ldots,X_k\}$ of C^* such that $k \geq 2$ and $(C^*-X_i)\cup z$ is a cocircuit of M for all i. Next we show that

(2.9) $|C^*-X_i|=2$ *for all* i.

Suppose that $|C^*-X_i|>2$ for some i. Then we may assume that $\underset{\sim}{M\backslash z}$ has C^*-X_i as a cocircuit. But, by (2.7), $\underset{\sim}{M\backslash z}$ also contains $\{a,b\}$, and it is straightforward to check that the choice of M is contradicted. Thus (2.9) holds.

(2.10) *Either*

(i) $C^*=\{x_1,x_2,x_3\}$ *where every 3-subset of* $C^*\cup z$ *is a triad of* M*; or*

(ii) $C^*=\{x_1,x_1',x_2,x_2'\}$ *where both* $\{x_1,x_1',z\}$ *and* $\{x_2,x_2',z\}$ *are triads of* M.

To see this, note that, by (2.9), $|X_2\cup X_3\cup\ldots\cup X_k|=2=|X_1\cup X_3\cup\ldots\cup X_k|$. Since each X_i is non-empty, $k\leq 3$. Moreover, if $k=3$, then (i) holds, while if $k=2$, then (ii) holds.

Now, z was arbitrarily chosen in $E(M)-C^*-\mathrm{cl}(\{a,b\})$. Thus, by (2.8),

(2.11) $\mathrm{cl}_{M^*}(C^*)\supseteq E(M)-\mathrm{cl}(\{a,b\})$.

Next we prove the following:

(2.12) *Either*

(i) $\mathrm{cl}(\{a,b\})=\{a,b\}$ *and* C^* *spans* M^**; or*

(ii) M *has an element* c *such that* $\mathrm{cl}(\{a,b\})=\{a,b,c\}$ *and* $r(M^*)=r_{M^*}(C^*)+1$.

By (2.6), $|\mathrm{cl}(\{a,b\})|\leq 3$. If $|\mathrm{cl}(\{a,b\})|=2$, then, as M^* has no 2-cocircuits, (2.11) implies that C^* spans M^*, so (i) holds. If $|\mathrm{cl}(\{a,b\})|=3$, let $\mathrm{cl}(\{a,b\})=\{a,b,c\}$. Then, by orthogonality, $c\notin \mathrm{cl}_{M^*}(C^*)$. Thus $r(M^*)\neq r_{M^*}(C^*)$ and (2.11) implies that (ii) holds.

On combining (2.10) and (2.12), we deduce that:

(2.13) *One of the following occurs:*

(i) $|C^*|=3$ *and* $r(M^*)=2$*;*

(ii) $|C^*|=3=r(M^*)$ *and* $\mathrm{cl}(\{a,b\})=\{a,b,c\}$*;*

(iii) $|C^*|=4$ *and* $r(M^*)=3$*;*

(iv) $|C^*|=4=r(M^*)$ *and* $\mathrm{cl}(\{a,b\})=\{a,b,c\}$.

We complete the proof of the theorem by showing that each of these possibilities yields a contradiction. In particular, if (i) holds, then M^* is a line and so, for $j\in C^*$, the set $E(M)-j$ is a circuit of M containing $\{a,b\}$ and meeting C^*; a contradiction. Thus (i) does not hold.

Next we note that if (ii) holds, then C^* spans a line in M^* whose complement is $\{a,b,c\}$. As $|C^*|=3$, for some e in C^*, the line in M^* through e and c avoids $\{a,b\}$. The complement of this line is a circuit of M that contains $\{a,b\}$ and meets C^*. This contradiction implies that (ii) does not hold.

Now suppose that (iii) holds. Then, for some 2-subset $\{u,v\}$ of C^*, the line L through u and v in M^* avoids $\{a,b\}$. Thus $E(M)-L$ is a circuit of M that contains $\{a,b\}$ and meets C^*. Hence (iii) does not hold.

We may now assume that (iv) holds. Then, by (2.10), M^* has a circuit $\{x_1,x_1',x_2,x_2'\}$ that spans the plane complementary to $\{a,b,c\}$. Consider M^*/c. We show next that:

(2.14) *At least one of the six lines of M^*/c that are spanned by the 2-subsets of $\{x_1,x_1',x_2,x_2'\}$ avoids $\{a,b\}$.*

To see this, note that an element d is on at most three of these lines with equality occurring only if d is parallel to one of x_1,x_1',x_2, or x_2'. Thus the required line exists unless each of a and b is parallel to one of x_1,x_1',x_2, and x_2'. Now a and b cannot both be parallel to the same element of $\{x_1,x_1',x_2,x_2'\}$ otherwise $\{a,b,c\}$ has rank two in M^* and so M^* is not 3-connected. If each of a and b is parallel to a different element of $\{x_1,x_1',x_2,x_2'\}$, then the other two members of this subset span a line that avoids $\{a,b\}$. Hence (2.14) holds.

Let L be one of the lines whose existence was established by (2.14). Then $L\cup c$ is a hyperplane of M^*. Its complement is a circuit of M containing $\{a,b\}$ and meeting C^*. This contradiction completes the proof that (iv) cannot occur and thereby finishes the proof of the theorem. ∎

The next result is an immediate consequence of Theorem 1.1.

(2.15) Corollary. *Let M be a 3-connected matroid with at least two elements. If C_1^*,C_2^*, and C_3^* are cocircuits of M, then M has a circuit that meets all three of these cocircuits.*

Applying (1.1) and (2.15) to graphs, we get the following results of Dirac [2].

(2.16) Corollary. *Let G be a 3-connected simple graph. If a and b are edges and v is a vertex of G, then a, b, and v all lie on some common cycle of G.*

(2.17) Corollary. *Let G be a 3-connected graph. If u,v, and w are vertices of G, then u, v, and w all lie on some common cycle of G.*

The last two corollaries are special cases of results that hold for n-connected graphs for all $n\geq 2$. It is not known whether Theorem 1.1 can be extended to n-connected matroids for $n\geq 4$. When Theorem 1.1 and Corollary 2.15 are applied to cographic matroids, they seem to produce new graph results. We state just the first of these.

(2.18) Corollary. *Let G be a 3-connected graph. If a and b are edges and C is a cycle of G, then G has a minimal edge cut that contains a and b and meets C.*

3. The converse

The matroid $U_{2,4}\oplus_2 U_{2,4}$, the 2-sum of two 4-point lines, shows that the converse of Theorem 1.1 fails. Nevertheless, the condition in that theorem can be used to characterize 3-connected matroids. In this section, we state and prove such a result.

A matroid N is a *non-trivial extension* of the matroid M if N has an element e such that $N\backslash e = M$ and e is neither a loop nor a coloop of N and e is not in a 2-circuit of N.

(3.1) Theorem. *Let M be a matroid having rank and corank at least two. Then M is 3-connected if and only if M and all its non-trivial extensions have the property that, for every pair of elements a and b and every cocircuit C^*, there is a circuit that contains $\{a,b\}$ and meets C^*.*

Proof. Suppose M is 3-connected. Then every non-trivial extension M^+ of M is also 3-connected and so, by Theorem 1.1, M and every such M^+ satisfy the specified condition.

Now suppose that M is not 3-connected. We may assume that M is simple, otherwise M has elements a and b and a cocircuit C^* avoiding $\{a,b\}$ such that $\{a,b\}$ contains a circuit; hence M does not satisfy the specified condition. Suppose that M is disconnected. Then M has a component M_1 with at least two elements. Let a and b be elements of M_1, and C^* be a cocircuit of a component of M other than M_1. Then M does not satisfy the specified condition. Thus we may assume that M is connected. Hence $M = M_1 \oplus_2 M_2 = P(M_1, M_2)\backslash p$ where each of M_1 and M_2 has at least three elements and is isomorphic to a proper minor of M, and p is the basepoint of the parallel connection.

Suppose that $P(M_1, M_2)$ is simple. Then M_2 has a cocircuit C^* avoiding p, and C^* is a cocircuit of $P(M_1, M_2)$, a non-trivial extension of M. Letting $a = p$ and b be an element of $E(M_1) - p$, we see that $P(M_1, M_2)$ has no circuit that contains $\{a,b\}$ and meets C^*; a contradiction.

We may now assume that $P(M_1, M_2)$ is non-simple. Then p is parallel to some element q of M. Without loss of generality, we may assume that $q \in E(M_2)$, and so $\{p,q\}$ is a circuit of M_2. As M is simple, $r(M_2) \geq 2$ and $M \cong P(M_1, M_2\backslash q)$. Now let C^* be a cocircuit of $M_2\backslash q$ avoiding p, take $a = p$, and choose $b \in E(M_1) - p$. Then $P(M_1, M_2\backslash q)$ has no circuit that contains $\{a,b\}$ and meets C^*; a contradiction. ∎

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James Oxley

Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803-4918
USA
`oxley@math.lsu.edu`