THE STRUCTURE OF THE 3-SEPARATIONS OF 3-CONNECTED MATROIDS II

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ABSTRACT. The authors showed in an earlier paper that there is a tree that displays, up to a natural equivalence, all non-trivial 3-separations of a 3-connected matroid. The purpose of this paper is to show that if certain natural conditions are imposed on the tree, then it has a uniqueness property. In particular, suppose that, from every pair of edges that meet at a degree-2 vertex and have their other ends of degree at least three, one edge is contracted. Then the resulting tree is unique.

1. INTRODUCTION

Let M be a matroid with ground set E. A subset X of E is 3-*separating* if $r(X) + r(E - X) - r(M) \leq 2$. The partition $(X, E - X)$ is 3-separating if X is 3-separating. Furthermore, the partition $(X, E - X)$ is a 3-separation if it is 3-separating and $|X|, |E - X| \geq 3$. A 3-separating set X, or a 3-separating partition $(X, E - X)$, or a 3-separation $(X, E - X)$ is *exact* if $r(X) + r(E - X) - r(M) = 2$.

Let X be an exactly 3-separating set of M. If there is an ordering $(x_1, x_2, \ldots, x_n)$ of X such that, for all i in $\{1, 2, \ldots, n\}$, the set $\{x_1, x_2, \ldots, x_i\}$ is 3-separating, then X is *sequential*. An exactly 3-separating partition (X, Y) of M is *sequential* if either X or Y is a sequential 3-separating set.

For a set X of M, we say that X is *fully closed* if it is closed in both M and M^*, that is $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$, and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exact 3-separating partitions $\{A_1, B_1\}$ and $\{A_2, B_2\}$ of M are *equivalent* if $\{\mathrm{fcl}(A_1), \mathrm{fcl}(B_1)\} = \{\mathrm{fcl}(A_2), \mathrm{fcl}(B_2)\}$.

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The main theorem of [6, Theorem 9.1] says that every 3-connected matroid M with at least nine elements has a tree decomposition that displays, up to equivalence, all non-sequential 3-separations. From both an algorithmic and structural point of view, sequential and equivalent 3-separations are not problematic. Algorithmically, if one had a rank oracle, then listing all sequential 3-separations or listing all 3-separations equivalent to a given non-sequential 3-separation can be done so that each new item on the list is added in polynomial time. Structurally, such 3-separations can be characterized in terms of an extension of the usual matroid closure operator. Moreover, all of the possible structures that relate two equivalent non-sequential 3-separations as well as all of the possible structures that give rise to sequential 3-separations have been identified [3, 4]. We remark here that, in our first paper on this subject [6], we omitted mention of the important paper of Coullard, Gardner, and Wagner [1]. That paper contains precursors for graphs of many of the ideas that our paper developed for matroids.

This paper will make repeated reference to the results of [6]. In the next section, the main theorem of the paper is stated after the necessary background is introduced. In Sections 3–5, we develop properties of 3-separations and the particular trees we use to display them. The proof of the main result is given in Section 6, while Section 7 proves some useful consequences of the earlier results.

2. Main Result.

In this section, we state the main theorem of the paper together with the main result of [6]. The section begins by introducing the concepts and terminology needed to make these statements meaningful.

The first lemma is in constant use in our work on the structure of 3-separations in 3-connected matroids.

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

Flowers. One of the main difficulties in describing the behaviour of 3-separations in a 3-connected matroid is caused by the presence of crossing 3-separations, where two 3-separations (A_1, A_2) and (B_1, B_2) *cross* if each of the intersections $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, and $A_2 \cap B_2$ is non-empty. When each of these intersections contains at least two elements, Lemma 2.1 implies that each of these intersections is 3-separating. Of course, the union of any consecutive pair in the cyclic ordering $(A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_2, A_2 \cap B_1)$ is also 3-separating. This 4-tuple is an example of flower, a fundamental and particularly important structure in the study of the 3-separations of a 3-connected matroid.

Let n be a positive integer and let M be a 3-connected matroid. The partition $(P_1, P_2, \dots, P_n)$ of $E(M)$ is a *flower* Φ in M with *petals* $P_1, P_2, \dots, P_n$ if, for all i, we have $|P_i| \geq 2$, and both P_i and $P_i \cup P_{i+1}$ are 3-separating, where all subscripts are interpreted modulo n. We say that Φ *displays* a 3-separating partition (X, Y) of $E(M)$ if X is a union of petals of Φ. It is shown in [6, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \dots, P_n)$. Observe that when $n \leq 3$, the concepts of an anemone and a daisy coincide, but for $n \geq 4$, a flower cannot be both an anemone and a daisy.

Equivalent flowers and tight and loose petals. Let Φ_1 and Φ_2 be flowers of a 3-connected matroid M. A natural quasi ordering on the collection of flowers of M is obtained by setting $\Phi_1 \preceq \Phi_2$ whenever every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$, we say that Φ_1 and Φ_2 are *equivalent* flowers of M. Hence equivalent flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. The *order* of a flower Φ is the minimum number of petals in a flower equivalent to Φ.

Let Φ be a flower of M. An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ. An element that is not loose is *tight*. We say that a petal P_i is *loose* if all elements in P_i are loose. A *tight* petal is one that is not loose, that is one that contains at least one tight element. Lastly, if Φ has order at least three, then Φ is *tight* if all of its petals are tight; if Φ has order t where $t \in \{1, 2\}$, then Φ is tight if it has exactly t petals.

Local connectivity and flower types. The classes of anemones and daisies can be further refined using the concept of local connectivity. For sets X and Y in a matroid M, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined to be

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

When M is $\mathbb{F}$-representable and hence viewable as a subset of the vector space $V(r(M), \mathbb{F})$, the local connectivity $\sqcap(X, Y)$ is precisely the rank of the intersection of those subspaces in $V(r(M), \mathbb{F})$ that are spanned by X and Y.

For $n \geq 3$, an anemone $(P_1, P_2, \dots, P_n)$ is called

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in \{1, 2, \dots, n\}$;
(ii) a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct $i, j \in \{1, 2, \dots, n\}$; and
(iii) *spike-like* if $n \geq 4$, and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, \dots, n\}$.

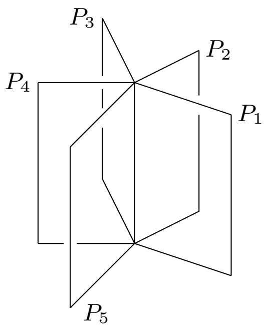

FIGURE 1. A representation of a rank-7 paddle.

Similarly, a daisy $(P_1, P_2, \ldots, P_n)$ is called

(i) *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and
(ii) *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\text{cl}(P_1) \cap \text{cl}(P_2) \cap \text{cl}(P_3)$ or $\text{cl}^*(P_1) \cap \text{cl}^*(P_2) \cap \text{cl}^*(P_3)$, and *swirl-like* otherwise. It is shown in [6] that every flower with at least three petals is one of these six different *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous.

To visualize a flower geometrically, it is useful to think of a collection of lines in projective space, where along these lines the petals of the flower are attached. For example, we can obtain a paddle by gluing the petals along a single common line. Figure 1 represents a 5-petal paddle in which each petal is a plane with enough structure to make the matroid 3-connected. The rank of this matroid is 7. Furthermore, Fig. 2 represents a 4-petal swirl-like flower. Again each petal is a plane. In that figure, the lines of attachment are the lines spanned by $\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_3, b_4\}$, and $\{b_4, b_1\}$, where $\{b_1, b_2, b_3, b_4\}$ is an independent set and each of the elements in this set may or may not be in the matroid. The rank of this matroid is 8.

Partial 3-trees. The type of tree used in the tree decomposition result in [6] is called a maximal partial 3-tree. In this subsection, we define such trees. Let π be a partition of a finite set E. Let T be a tree such that every member of π labels a vertex of T; some vertices may be unlabelled but no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*.

Let G be a subgraph of T having components $G_1, G_2, \ldots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \ldots, X_m$. In particular, if $V(G) = V(T)$, then

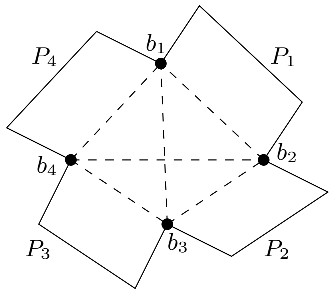

FIGURE 2. A representation of a rank-8 swirl-like flower.

$\{X_1, X_2, \ldots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T\backslash e$. In particular, if $e = v_1v_2$ for some vertices v_1 and v_2, then (Y_1, Y_2) is the *(ordered) partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T\backslash v_1v_2$ containing v_1. Let v be a vertex of T that is not a bag vertex. Then the *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v are in natural one-to-one correspondence with the components of $T - v$, and hence with the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \ldots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 3-connected matroid with ground set E. An *almost partial 3-tree* T for M is a π-labelled tree, where π is a partition of E such that the following conditions hold:

(i) For each edge e of T, the partition (X, Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 3-separation.
(ii) Every non-bag vertex v is labelled either D or A. Moreover, if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(iii) If a vertex v is labelled A, then the partition of E displayed by v is a tight maximal anemone of order at least 3.
(iv) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a tight maximal daisy of order at least 3.

By conditions (iii) and (iv), a vertex v labelled D or A corresponds to a flower of M. The 3-separations displayed by this flower are the 3-separations *displayed by v*. A vertex of a partial 3-tree is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 3-separation is *displayed by an almost partial 3-tree T* if it is displayed by some edge or some flower vertex of T.

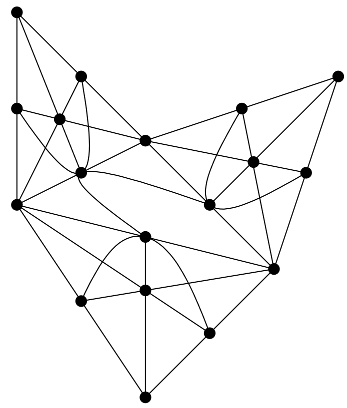

FIGURE 3. A rank-6 matroid containing a tight maximal flower of order 3.

A 3-separation (R, G) of M *conforms* with an almost partial 3-tree T if either (R, G) is equivalent to a 3-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a 3-separation (R', G') with the property that either R' or G' is contained in a bag of T.

An almost partial 3-tree for M is a *partial* 3-*tree* if

(v) every non-sequential 3-separation of M conforms with T.

We now define a quasi order on the set of partial 3-trees for M. Let T_1 and T_2 be two partial 3-trees for M. Then $T_1 \preceq T_2$ if all of the non-sequential 3-separations displayed by T_1 are displayed by T_2. If $T_1 \preceq T_2$ and $T_2 \preceq T_1$, then T_1 is *equivalent* to T_2. A partial 3-tree is *maximal* if it is maximal with respect to this quasi order. We shall sometimes use MP3T to abbreviate 'maximal partial 3-tree'.

Main results. The following theorem is the main result of [6, Theorem 9.1, Corollary 9.2].

Theorem 2.2. *Let M be a 3-connected matroid with $|E(M)| \geq 9$. Then M has a maximal partial 3-tree T. Moreover, every non-sequential 3-separation of M is equivalent to a 3-separation displayed by T.*

Our concern in [6] was to show that, up to equivalence, we could display all non-sequential 3-separations of a 3-connected matroid in a tree. Having shown that an MP3T succeeds in doing this, we did not consider the question of whether such an MP3T is unique. The purpose of this paper is to explore that question and our main result is a uniqueness theorem. Our initial investigation of this issue involve considering MP3T's having the minimum number of vertices. Subsequently, we looked at a structurally more natural class of MP3T's, which we define in the next paragraph. Before doing this, we note that we shall prove in Lemma 4.4 that, for every tight flower Φ of order at least four in a 3-connected matroid M and, for every MP3T T for M, there is a vertex of T that displays a flower equivalent to Φ. But an MP3T for M need not display a tight maximal flower of order three. For example, let M be the 3-connected matroid that is formed by taking three distinct triangles in $M(K_4)$ and, along each, attaching a copy of the Fano matroid via generalized parallel connection (see Figure 3). Then M has 18 elements and rank 6. One possible MP3T T_1 for M consists of a bag vertex that is labelled by the ground set of $M(K_4)$ and is adjacent to exactly three other bag vertices, each labelled by the elements of one of the copies of F_7 that are not in the initial $M(K_4)$. We can transform T_1 into another MP3T for M by moving each element of the initial $M(K_4)$ into one of the bags whose elements span it, and then relabelling the resulting empty degree-3 bag vertex as a degree-3 flower vertex.

An MP3T for a 3-connected matroid M is a 3-*tree* if

(I) for every tight maximal flower of M of order three, there is an equivalent flower that is displayed by a vertex of T; and
(II) if a vertex v is incident with two edges, e and f, that display equivalent 3-separating partitions, then the other ends of e and f are flower vertices, v has degree two, and v labels a non-empty bag.

We shall call two edges in a 3-tree *twins* if they are incident with a common vertex and display equivalent 3-separating partitions. Note that condition (II) above implies that if e and f are twins and f and g are twins, then $e = g$. We shall prove in Theorem 5.3 that every 3-connected matroid has an associated 3-tree.

Given a 3-tree T, the *reduction* $R(T)$ of T is the unlabelled tree that is obtained from T by contracting one edge from every pair of twins in T. If an edge of $R(T)$ results from such a contraction, we call it a *twin-edge*. Every other edge of $R(T)$ corresponds to a unique edge of T; such edges will be called *stationary*. For each edge e of $R(T)$, there is a *corresponding* set of edges of T consisting of a single edge if e is stationary, and a pair of twins if e is a twin-edge. Let v be a vertex of $R(T)$. If v meets only stationary edges, then there is a unique vertex *corresponding* to v; if v meets a twin-edge, then, by (II), there is a unique flower vertex *corresponding* to v. We shall identify each vertex of T with the corresponding vertex of $R(T)$. Thus $V(T)$ is the disjoint union of $V(R(T))$ and the set of degree-two bag vertices of T that meet a pair of twins.

The following is the main result of the paper.

Theorem 2.3. *Let T_1 and T_2 be 3-trees for a 3-connected matroid M with $|E(M)| \geq 9$. Then the reductions of T_1 and T_2 are isomorphic trees. Indeed, there is an isomorphism φ from $V(R(T_1))$ onto $V(R(T_2))$ such that*

(i) *φ maps the vertices of T_1 of degree at least three bijectively onto the vertices of T_2 of degree at least three so that each flower vertex is* *mapped to an equivalent one of the same type and each bag vertex is mapped to a bag vertex of the same degree; and*

(ii) *if φ maps an edge u_1v_1 of $R(T_1)$ to an edge v_2u_2 of $R(T_2)$, then the equivalent 3-separations displayed by the one or two edges of T_1 corresponding to u_1v_1 are equivalent to the 3-separations displayed by the one or two edges of T_2 corresponding to u_2v_2.*

In addition, if φ maps adjacent flower vertices u_1 and v_1 of T_1 onto non-adjacent vertices u_2 and v_2 of T_2, then every element in the bag vertex w_2 of T_2 that is adjacent to u_2 and v_2 is loose in the flower displayed by u_2 or in the flower displayed by v_2, and is also loose in the flower displayed by u_1 or the flower displayed by v_1.

We remark here that, with a slightly modified definition of '3-tree', a similar uniqueness result holds if we replace '3-tree' by 'MP3T with the minimum number of vertices.' The basic modification in the definition involves how one treats flowers of order three for which one of the petals is sequential. We give no further details of that alternative approach.

3. Some Useful Lemmas

Two *ordered* exact 3-separating partitions (C_1, D_1) and (C_2, D_2) are *equivalent* if $\mathrm{fcl}(C_1) = \mathrm{fcl}(C_2)$ and $\mathrm{fcl}(D_1) = \mathrm{fcl}(D_2)$. We remark that this terminology differs slightly from that used in [6] where (C_1, D_1) and (C_2, D_2) were defined to be equivalent if $\{\mathrm{fcl}(C_1), \mathrm{fcl}(D_1)\} = \{\mathrm{fcl}(C_2), \mathrm{fcl}(D_2)\}$. The modification described above will simplify the exposition here in a number of places.

The next two lemmas are used frequently. The first follows from [6, Lemma 3.1(i)] and the second is established in [6, Lemma 3.3].

Lemma 3.1. *Let X be an exactly 3-separating set of a matroid M. Then X is sequential if and only if* $\mathrm{fcl}(E(M) - X) = E(M)$.

Lemma 3.2. *Let (A_1, A_2) be a non-sequential 3-separation of a 3-connected matroid M and let (B_1, B_2) be a 3-separation of M. Then (A_1, A_2) is equivalent to (B_1, B_2) if and only if* $\mathrm{fcl}(A_1) = \mathrm{fcl}(B_1)$.

The elementary proofs of the next two lemmas are omitted.

Lemma 3.3. *Let (X_1, Y_1) and (X_2, Y_2) be non-sequential 3-separations of a matroid M. If* $\mathrm{fcl}(X_2) \supseteq X_1$ *and* $\mathrm{fcl}(Y_2) \supseteq Y_1$, *then (X_1, Y_1) and (X_2, Y_2) are equivalent.*

Lemma 3.4. *Let (X_1, Y_1) and (X_2, Y_2) be crossing 3-separations that are displayed in an MP3T T. Then T has a vertex at which each of (X_1, Y_1) and (X_2, Y_2) is displayed other than by an edge.*

We also omit the straightforward proof of the first part of the next lemma, although we do include the proof of the second part.

Lemma 3.5. *Let $X_1, X_2, Y_1,$ and Y_2 be subsets of the ground set of a 3-connected matroid M.*

(i) *If $\text{fcl}(X_i) = \text{fcl}(Y_i)$ for each i, then $\text{fcl}(X_1 \cup X_2) = \text{fcl}(Y_1 \cup Y_2)$.*
(ii) *If (X_i, X_i') and (Y_i, Y_i') are equivalent non-sequential 3-separations of M for each i in $\{1, 2\}$ such that $|X_1 \cap X_2| \geq 2$ and $\text{fcl}(X_1 \cup X_2) \neq E$, then $(X_1 \cup X_2, X_1' \cap X_2')$ and $(Y_1 \cup Y_2, Y_1' \cap Y_2')$ are equivalent non-sequential 3-separations of M.*

Proof. For (ii), we note that, by Lemma 2.1, $X_1 \cup X_2$ is 3-separating. Since $\text{fcl}(X_1 \cup X_2) \neq E$, we deduce that $(X_1 \cup X_2, X_1' \cap X_2')$ is a non-sequential 3-separation. Moreover, by (i) and Lemma 3.2, this 3-separation is equivalent to $(Y_1 \cup Y_2, Y_1' \cap Y_2')$. □

The following lemma, which will be used repeatedly, can be obtained immediately from [6, Lemma 5.9] by using [6, Corollary 5.10].

Lemma 3.6. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight flower in a 3-connected matroid, where $n \geq 3$.*

(i) *If $2 \leq j \leq n-2$, then $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n)$ is a non-sequential 3-separation.*
(ii) *If $1 \leq j \leq n-2$, then*

$$\text{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\text{fcl}(P_1) - P_1) \cup (\text{fcl}(P_j) - P_j)$$

and every element of $(\text{fcl}(P_1) - P_1) \cup (\text{fcl}(P_j) - P_j)$ is loose. In particular, if $j < i \leq n$, then $P_i \nsubseteq \text{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j)$ and $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1}, \ldots, P_n)$ is a tight flower.

The following observations may help the reader. Let Φ be a tight flower of degree at least 3. Lemma 3.6(i) says that, for every 3-separation (X, Y) displayed by Φ in which X contains at least two petals and Y contains at least two petals, (X, Y) is non-sequential. Moreover, Lemma 3.6(ii) implies that if (X_1, Y_1) and (X_2, Y_2) are two equivalent 3-separations displayed by Φ, then $(X_1, Y_1) = (X_2, Y_2)$. To see this, suppose that (X_1, Y_1) and (X_2, Y_2) are equivalent 3-separations displayed by Φ but $(X_1, Y_1) \neq (X_2, Y_2)$. Without loss of generality, we may assume that X_1 contains no more petals of Φ than either Y_1 or X_2, so Y_1 contains at least two petals. Since $(X_1, Y_1) \neq (X_2, Y_2)$, it follows that there is a petal P of Φ such that $P \subseteq X_2 - X_1$, and so $P \subseteq Y_1$. But $\text{fcl}(X_1) = \text{fcl}(X_2)$ and so $P \subseteq \text{fcl}(X_1)$, contradicting Lemma 3.6(ii).

The following is an immediate consequence of the last lemma.

Corollary 3.7. *In a 3-connected matroid M, let $(P_1, P_2, \ldots, P_n)$ be a tight flower for some $n \geq 4$. Then $(P_1 \cup P_2, P_3, \ldots, P_n)$ is also a tight flower in M.*

The process of taking unions of consecutive petals in a flower may be iterated to produce another flower. Any such flower obtained from $(P_1, P_2, \dots, P_n)$ is a *concatenation* of $(P_1, P_2, \dots, P_n)$. The next lemma is [6, Corollary 5.10].

Lemma 3.8. *If Φ is a flower in a 3-connected matroid, then the order of Φ is the number of petals in any tight flower equivalent to Φ.*

Recall that a flower of order one or two is tight if it has one or two petals, respectively, whereas a flower of order at least three is tight if all of its petals are tight. This definition means, for example, that a 3-petal flower Φ in which every petal is tight need not be a tight flower. In particular, if Φ has either two petals whose union is sequential, or three petals that are sequential, then Φ has order one. If Φ has exactly two sequential petals and their union is not sequential, then Φ has order two; and if Φ has at most one sequential petal, then it has order three and is tight. The next lemma shows that, for $k \geq 4$, a k-petal flower in which every petal is tight behaves much more predictably.

Lemma 3.9. *If Φ is a k-petal flower with $k \geq 4$ and every petal of Φ is tight, then Φ is a tight flower.*

Proof. If Φ has order at least three, then the result is an immediate consequence of [6, Lemma 5.8]. Now assume that Φ has order at most two. Then we may assume that, for some j with $2 \leq j \leq k-2$, the set $P_1 \cup P_2 \cup \cdots \cup P_j$ is sequential. Thus there is an ordering $(x_1, x_2, \dots, x_n)$ of $P_1 \cup P_2 \cup \cdots \cup P_j$ such that, for all i in $\{1, 2, \dots, n\}$, the set $\{x_1, x_2, \dots, x_i\}$ is 3-separating. Since both P_1 and $\{x_1, x_2, \dots, x_i\}$ are 3-separating, Lemma 2.1 implies that their intersection is 3-separating. It follows that we may assume that $(x_1, x_2, \dots, x_n)$ is ordered so that the first $|P_1|$ elements are in P_1. Repeating this argument using $P_1 \cup P_2$ instead of P_1, we may assume that, in $(x_1, x_2, \dots, x_n)$, the elements of P_1 are immediately followed by those of P_2. We deduce that $P_2 \subseteq \mathrm{fcl}(P_1)$ so P_2 is not tight; a contradiction. □

By definition, equivalent flowers display the same sets of non-sequential 3-separations, up to equivalence. The next lemma, which combines several results from [6], shows that, up to equivalence, equivalent tight flowers also display the same sets of sequential 3-separations.

Lemma 3.10. *Let $(P_1, P_2, \dots, P_n)$ be a tight maximal flower Φ in a 3-connected matroid M and let $(Q_1, Q_2, \dots, Q_m)$ be a tight maximal flower Ψ of M that is equivalent to Φ. Then $m = n$ and there is a permutation α of $\{1, 2, \dots, n\}$ such that $\mathrm{fcl}(P_i) = \mathrm{fcl}(Q_{\alpha(i)})$ for all i. Thus, for every 3-separation displayed by Φ, there is an equivalent 3-separation displayed by Ψ.*

Proof. Let Φ have order t. If $t \in \{1, 2\}$, then, since Ψ and Φ are equivalent and tight, $t = m = n$ and the lemma follows. Now suppose $t \geq 3$. Then

$t = m = n$ by the last lemma. By [6, Theorem 5.1, Lemmas 5.3 and 5.8], if τ is the set of tight elements of Φ, then τ is the set of tight elements of Ψ. Moreover, there is a permutation α of $\{1, 2, \ldots, n\}$ such that $\mathrm{fcl}(P_i) = \mathrm{fcl}(\tau \cap P_i) = \mathrm{fcl}(\tau \cap Q_{\alpha(i)}) = \mathrm{fcl}(Q_{\alpha(i)})$ for all i. □

The next two lemmas concern a non-sequential 3-separation that is displayed in an MP3T for a 3-connected matroid M. They show that if (X, Y) is displayed by an edge in T, then every other MP3T for M has an edge displaying a 3-separation equivalent to (X, Y); and, if (X, Y) is not displayed by an edge of T, then no other MP3T for M has an edge displaying a 3-separation equivalent to (X, Y).

Lemma 3.11. *Let T_1 and T_2 be maximal partial 3-trees for a 3-connected matroid M. If (X_1, Y_1) is a non-sequential 3-separation that is displayed by an edge of T_1, then there is an edge of T_2 that displays a 3-separation (X_2, Y_2) that is equivalent to (X_1, Y_1).*

Proof. Certainly T_2 displays a 3-separation (X_2, Y_2) that is equivalent to (X_1, Y_1). Assume that (X_2, Y_2) is not displayed by an edge of T_2. Then T_2 has a vertex that displays a tight flower $(P_1, P_2, \ldots, P_n)$ such that $X_2 = P_1 \cup P_2 \cup \cdots \cup P_j$ and $Y_2 = P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n$ for some j with $2 \leq j \leq n-2$. Let (Z_2, W_2) be a partition of $E(M)$ with $Z_2 = P_{j-s+1} \cup P_{j-s+2} \cup \cdots \cup P_{j+t}$, where s and t are non-negative integers, such that $P_j, P_{j+1} \subseteq Z_2$ and $P_1, P_n \subseteq W_2$. Then, by Lemma 3.6, (Z_2, W_2) is a non-sequential 3-separation of M and an equivalent 3-separation (Z_1, W_1) must be displayed in T_1. Then, as (X_1, Y_1) is displayed by an edge in T_1, without loss of generality, $Z_1 \subseteq X_1$ and $W_1 \supseteq Y_1$. Thus $\mathrm{fcl}(Z_2) = \mathrm{fcl}(Z_1) \subseteq \mathrm{fcl}(X_1) = \mathrm{fcl}(X_2)$. Hence $P_{j+1} \subseteq \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j)$; a contradiction to Lemma 3.6. □

Lemma 3.12. *Let T_1 and T_2 be maximal partial 3-trees for a 3-connected matroid M. If (X_1, Y_1) is a 3-separation that is displayed by a vertex of T_1 but not by an edge of T_1, then there is a unique 3-separation (X_2, Y_2) that is equivalent to (X_1, Y_1) and is displayed in T_2. Moreover, (X_2, Y_2) is displayed by a vertex and not by an edge.*

Proof. As T_2 is an MP3T, there is a 3-separation (X_2, Y_2) that is equivalent to (X_1, Y_1) and is displayed by T_2. By Lemma 3.11, if (X_2, Y_2) is displayed by an edge, then T_1 has an edge that displays a 3-separation (X_3, Y_3) equivalent to (X_2, Y_2) and hence to (X_1, Y_1). Then, by symmetry, we may assume that $X_1 \subseteq X_3$ or $Y_1 \subseteq X_3$. The latter implies the contradiction that $\mathrm{fcl}(X_3) \supseteq Y_1 \cup X_1 = E$. Hence $X_1 \subseteq X_3$. Now there is a petal P of the flower of T_1 that displays (X_1, Y_1) that is disjoint from both X_1 and Y_3. Since $\mathrm{fcl}(X_1) = \mathrm{fcl}(X_3)$, it follows that $P \subseteq \mathrm{fcl}(X_1)$, so, by Lemma 3.6, P is loose; a contradiction. We conclude that each 3-separation equivalent to (X_1, Y_1) that is displayed by T_2 is displayed by a vertex but not by an edge. If there is more than one such 3-separation, then a similar argument to the above yields a contradiction. □

From now on, we shall say that a 3-separating partition of a 3-connected matroid is *strictly displayed by a vertex v* of an MP3T T for M if it is displayed by v but not by an edge incident with v. Observe that this definition implies that when a 3-separating partition is strictly displayed by a vertex, that vertex must have degree at least four.

4. Flowers and Maximal Partial 3-Trees

In this section, we prove some properties of flowers and maximal partial 3-trees that will be used in the proof of the main result but are also of independent interest. Throughout the discussion here, whenever we refer to an MP3T or a 3-tree T, it will be implicit that T is an MP3T or a 3-tree for a 3-connected matroid M. When we say that a flower vertex is *tight*, we shall mean that the flower displayed by v is tight.

We are interested in how a tight maximal flower Φ in a 3-connected matroid M shows up in an MP3T for M. We begin with the case when Φ has order 3, which differs from the higher-order case.

Lemma 4.1. *Let (P_1, P_2, P_3) be a tight maximal flower Φ in a 3-connected matroid M where none of P_1, P_2, and P_3 is sequential. Let T be a maximal partial 3-tree for M. Then there is a degree-3 vertex of T at which 3-separations equivalent to each of $(P_1, E-P_1), (P_2, E-P_2)$, and $(P_3, E-P_3)$ are displayed.*

Proof. For each i in $\{1,2,3\}$, let $(P_i', E-P_i')$ be a 3-separation displayed by T that is equivalent to $(P_i, E-P_i)$. Let τ be the set of tight elements of Φ. Then, for distinct i and j, we have $\text{fcl}(P_i') = \text{fcl}(P_i) = \text{fcl}(P_i \cap \tau)$, and $P_j \cap \tau$ avoids $\text{fcl}(P_i')$. Thus $P_i \cap \tau \subseteq P_i'$.

Suppose $P_1' \cap P_2' \neq \emptyset$. Then P_1' and P_2' are displayed at a flower vertex v of T and $P_1' \cap P_2'$ is a union of petals of the corresponding flower Φ_v. Since $P_1' - P_2' \supseteq P_1 \cap \tau$, we have $P_1' \cap P_2' \subseteq \text{fcl}(P_1' - P_2')$. As $P_1' - P_2'$ is also a 3-separating set that is a union of petals of Φ_v, it follows by Lemma 3.6 that the petals in $P_1' \cap P_2'$ are loose in Φ_v; a contradiction. Thus $P_1' \cap P_2' = \emptyset$ and, by symmetry, we deduce that P_1', P_2', and P_3' are disjoint.

For each i in $\{1,2,3\}$, let v_i be a vertex of T at which a 3-separation equivalent to $(P_i, E-P_i)$ is displayed and choose these vertices so that the distance between v_1 and v_2 is minimized. Assume that $v_1 \neq v_2$ and let w be the vertex of the path from v_1 to v_2 that is the minimum distance in T from v_3. Without loss of generality, we may assume that $w \neq v_2$. Let e_2 be the edge that meets w and lies on the path from w to v_2. The partition $(W, E-W)$ displayed by e_2 has P_2' in one set and $P_1' \cup P_3'$ in the other. By Lemma 3.3, it follows that $(W, E-W)$ is equivalent to $(P_2, E-P_2)$ since $\text{fcl}(P_1' \cup P_3') = \text{fcl}(P_1 \cup P_3) = \text{fcl}(E-P_2)$. Hence the choice of v_2 is contradicted. We conclude that $v_1 = v_2$.

Now assume that $v_3 \neq v_1$, and consider the edge e_3 that meets v_1 and lies on the path from v_1 to v_3. The 3-separation $(E - P_3'', P_3'')$ that e_3 displays has $P_1' \cup P_2'$ contained in the first set and P_3' contained in the second so, by Lemma 3.3, it is equivalent to $(E - P_3', P_3')$ and we may replace P_3' by P_3''.

We may now assume that each of $(P_1', E - P_1'), (P_2', E - P_2')$, and $(P_3', E - P_3')$ is displayed at v_1. Suppose that v_1 is incident with an edge f displaying a 3-separation $(Z, E - Z)$ with $P_1' \cup P_2' \cup P_3' \subseteq Z$. Now

$$\mathrm{fcl}(P_1') \cup \mathrm{fcl}(P_2') \cup \mathrm{fcl}(P_3') \supseteq P_1 \cup P_2 \cup P_3 = E.$$

Hence $(Z, E - Z)$ is sequential, so f does not exist when v_1 is a bag vertex. Nor does it exist when v_1 is a flower vertex for, in that case, by applying Lemma 3.6 to each of $\mathrm{fcl}(P_1')$, $\mathrm{fcl}(P_2')$, and $\mathrm{fcl}(P_3')$, we deduce that $E - Z$ is a loose petal of this flower. We conclude that, when v_1 is a bag vertex, it has degree 3, and, when v_1 is a flower vertex, $P_1' \cup P_2' \cup P_3' = E$.

Assume that the flower Ψ displayed by v_1 is $(Q_1, Q_2, \ldots, Q_k)$ where $k \geq 4$. We may suppose that $P_1' = Q_1 \cup Q_2 \cup \cdots \cup Q_t$ and $P_2' = Q_{t+1} \cup Q_{t+2} \cup \cdots \cup Q_{t+s}$ for some $t \geq 2$. Then $\Phi \preceq \Psi$ but $(Q_t \cup Q_{t+1}, E - (Q_t \cup Q_{t+1}))$ is a non-sequential 3-separation that is displayed by Ψ but not by Φ, so Φ is not a maximal flower. This contradiction implies that $k = 3$. □

Lemma 4.2. *Let (P_1, P_2, P_3, P_4) be a tight flower in a 3-connected matroid M. Let (A, B) and (C, D) be 3-separations in M equivalent to $(P_1 \cup P_2, P_3 \cup P_4)$ and $(P_2 \cup P_3, P_4 \cup P_1)$, respectively. Then (A, B) and (C, D) cross.*

Proof. By Lemma 3.6, $\mathrm{fcl}(P_3 \cup P_4) = \mathrm{fcl}(P_3) \cup \mathrm{fcl}(P_4)$. Since $\mathrm{fcl}(P_3 \cup P_4) = \mathrm{fcl}(B)$, we deduce that the tight elements of P_1 are in A. Likewise, these tight elements are in D. Hence $A \cap D \neq \emptyset$. The lemma follows by symmetry. □

Lemma 4.3. *Let Ψ and Φ be tight flowers, $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_n)$, in a 3-connected matroid and suppose that $\Psi \preceq \Phi$.*

(i) *If $(P_1 \cup \cdots \cup P_j, P_{j+1} \cup \cdots \cup P_m)$ and $(Q_1 \cup \cdots \cup Q_k, Q_{k+1} \cup \cdots \cup Q_n)$ are equivalent, where $2 \leq j \leq m - 2$, then $j \leq k$ and $m - j \leq n - k$.*
(ii) *The order of Ψ is at most that of Φ.*

Proof. To prove (i), by symmetry, it suffices to show that $j \leq k$. We argue by induction on j. If $j = 2$, let $\Psi' = (P_1, P_2, P_3, P_4 \cup \cdots \cup P_n)$. Then $\Psi' \preceq \Psi$ and, by Corollary 3.7, Ψ' is tight. Now $\Psi' \preceq \Phi$, so there are 3-separations equivalent to $(P_1 \cup P_2, E - (P_1 \cup P_2))$ and $(P_2 \cup P_3, E - (P_2 \cup P_3))$ displayed by Φ and these must cross. Hence $P_1 \cup P_2$ is not displayed by a single petal of Φ so $j = 2 \leq k$. Now assume that if $2 \leq j < t \leq m - 2$, then $P_1 \cup P_2 \cup \cdots \cup P_j$ is displayed by at least j petals of Φ. Thus each of $P_1 \cup P_2 \cup \cdots \cup P_{t-1}$ and $P_2 \cup P_3 \cup \cdots \cup P_t$ is displayed by at least $t - 1$ petals of Φ. These sets of petals do not coincide otherwise $P_t \subseteq \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_{t-1})$ which, by Lemma 3.6, contradicts the fact that P_t is tight. We deduce that $P_1 \cup P_2 \cup \cdots \cup P_t$ is displayed by at least t petals of Φ, and (i) follows by induction.

For (ii), observe first that if Ψ has order one or two, then its order is at most that of Φ. If Ψ has order 3, then, by Lemma 3.8, $m = 3$ so Ψ displays at least two unordered non-sequential 3-separations. Hence so does Φ. Thus Φ has order at least three. If Ψ has order at least four, then we can apply (i) to get that $m \leq n$, which, by Lemma 3.8, implies the required result. □

Lemma 4.4. *Let T be an MP3T for a 3-connected matroid M and let Φ be a tight maximal flower of M of order at least four. Then there is a vertex v of T that displays a flower equivalent to Φ.*

Proof. Let Φ be the flower $(P_1, P_2, \ldots, P_n)$ and consider an arbitrary concatenation (Q_1, Q_2, Q_3, Q_4) of Φ to a 4-petal flower. By Corollary 3.7, this flower is tight. The MP3T T displays 3-separations (A, B) and (C, D) that are equivalent to $(Q_1 \cup Q_2, Q_3 \cup Q_4)$ and $(Q_2 \cup Q_3, Q_4 \cup Q_1)$, respectively. By Lemma 4.2, (A, B) and (C, D) cross and, by Lemma 3.4, T has a vertex at which each of (A, B) and (C, D) is strictly displayed. Moreover, by Lemma 3.12, the only 3-separations equivalent to (A, B) or (C, D) that are displayed by T are (A, B) and (C, D) themselves. In particular, if $(Q_1, Q_2, Q_3, Q_4) = (P_1, P_2, P_3, P_4 \cup \cdots \cup P_n)$, then 3-separations equivalent to $(P_1 \cup P_2, E - (P_1 \cup P_2))$ and $(P_2 \cup P_3, E - (P_2 \cup P_3))$ are strictly displayed at a common vertex v of T. Similarly, a 3-separation equivalent to $(P_3 \cup P_4, E - (P_3 \cup P_4))$ is strictly displayed at v. Extending this, we deduce that, for all distinct i and j such that $(P_i \cup P_j, E - (P_i \cup P_j))$ is a 3-separation of M, there is an equivalent 3-separation $(R_{ij}, E - R_{ij})$ strictly displayed at v. Provided that $\text{fcl}(P_1 \cup P_2 \cup P_3) \neq E$, it follows by Lemma 3.5 that $(R_{12} \cup R_{23}, E - (R_{12} \cup R_{23}))$ is a non-sequential 3-separation equivalent to $(P_1 \cup P_2 \cup P_3, E - (P_1 \cup P_2 \cup P_3))$. Since $R_{12} \cup R_{23}$ is a 3-separating union of petals of the flower Φ_v displayed by v, it follows that $(R_{12} \cup R_{23}, E - (R_{12} \cup R_{23}))$ is displayed at v, possibly by a single edge. By continuing in this way, we deduce that if $(P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_k}, E - (P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_k}))$ is a 3-separation of M where $2 \leq k \leq n - 2$, then there is an equivalent 3-separation displayed at v. Finally, if, for example, $(P_2 \cup P_3 \cup \cdots \cup P_n, P_1)$ is a non-sequential 3-separation, then, as there are 3-separations equivalent to each of $(P_2 \cup P_3 \cup \cdots \cup P_{n-1}, P_n \cup P_1)$ and $(P_3 \cup P_4 \cup \cdots \cup P_n, P_1 \cup P_2)$ displayed at v, Lemma 3.5 implies that there is a 3-separation equivalent to $(P_2 \cup P_3 \cup \cdots \cup P_n, P_1)$ displayed at v. We conclude that $\Phi \preceq \Phi_v$. But Φ is a maximal flower, so Φ and Φ_v are equivalent. □

Corollary 4.5. *Let T and T' be maximal partial 3-trees for a 3-connected matroid. Then there is a bijection φ between the flower vertices of T of degree at least four and the flower vertices of T' of degree at least four such that the flower displayed by v is equivalent to that displayed by $\varphi(v)$.*

Proof. If Φ is a tight maximal flower displayed at a vertex v of T of degree at least four, then, by Lemma 4.4, there is a vertex v' of T' that displays a tight maximal flower Φ' equivalent to Φ. By Lemma 3.8, Φ and Φ' have the same

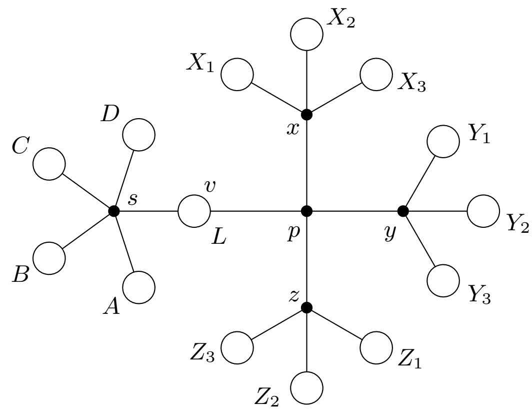

FIGURE 4. The 3-tree T_1.

number of petals, so v and v' have the same degree. Thus, corresponding to each flower vertex of T of degree at least four, there is a flower vertex of T' of the same degree displaying an equivalent flower. By symmetry, every flower vertex of T' of degree at least four has a corresponding flower vertex of T. The fact that this correspondence is one-to-one is an immediate consequence of Lemma 3.12. □

5. 3-TREES

The first theorem of this section shows that every 3-connected matroid has a corresponding 3-tree. But we begin the section with an example to show how 3-trees for a 3-connected matroid can differ from each other and from MP3T's with the minimum number of vertices. For $n \geq 3$ and $k \geq 2$, the *free* (n,k)*-swirl* is the matroid that is obtained by beginning with a basis $\{1,2,\ldots,n\}$, adding k points freely on each of the n lines spanned by $\{1,2\},\{2,3\},\ldots,\{n,1\}$, and then deleting $\{1,2,\ldots,n\}$. The usual free n-swirl coincides with the free $(n,2)$-swirl. We observe that, when $n+k > 5$, the free (n,k)-swirl can be viewed as a swirl-like flower whose n petals consist of the sets of k points that were freely placed on the n lines above. The *spine* of a paddle $(P_1,P_2,\ldots,P_n)$ is the set $\text{cl}(P_1)\cap\text{cl}(P_2)\cap\cdots\cap\text{cl}(P_n)$, which coincides with each of the sets $\text{cl}(P_i)\cap\text{cl}(P_j)$ with $1 \leq i < j \leq n$.

Begin with a free $(5,4)$-swirl $S = (A,B,C,D,L)$, where each of A, B, C, D, and L is a line of S. Now use L as the spine of a paddle P and attach three petals X, Y, and Z to this spine making each of X, Y, and Z a free $(4,4)$-swirl with $X = (X_1,X_2,X_3,L)$ and Y and Z defined similarly.

One choice for an MP3T for this matroid M is to begin with a bag vertex v for L adjacent to a flower vertex s corresponding to the swirl S, where s has degree 5 and has bag vertices labelled by A, B, C, D, and L as its neighbours. The vertex v is also adjacent to a flower vertex p corresponding to the paddle

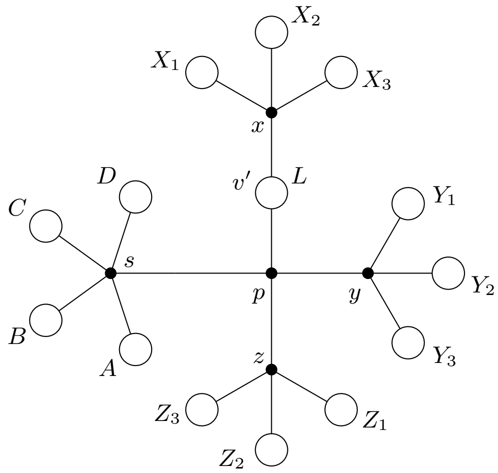

FIGURE 5. The 3-tree T_2.

P; and p is also adjacent to flower vertices x, y, and z corresponding to the swirls X, Y, and Z. Finally, x is adjacent to three bag vertices corresponding to the petals X_1, X_2, and X_3; and one has a similar configuration at each of y and z. The resulting MP3T is the tree T_1 shown in Fig. 4, where large open circles represent bag vertices. Clearly, T_1 is a 3-tree. It is not difficult to see that T_1 has the minimum number of vertices among possible MP3T's for M. Indeed, all edges of T_1 display inequivalent 3-separations except for the edges vs and vp. Moreover, the crossing 3-separations of M force each of the flower vertices of T_1 and the only loose elements in any of these flowers are the elements of L, which are loose in the paddle P. These elements cannot be placed in any of the bag vertices A, B, C, D, X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, or Z_3. Hence T_1 must have at least one additional bag vertex to accommodate the elements of L. We conclude that T_1 has the minimum number of vertices among MP3T's for M.

Now we can modify the tree T_1 and place the elements of L elsewhere in the tree. First replace the 2-edge path from p to s by a single edge. Then take the edge px and subdivide it inserting a new bag vertex v' labelled by L. This gives a new MP3T T_2 as shown in Fig. 5 with the same number of vertices as T_1. Moreover, T_2 is also a 3-tree for M so we have now shown that 3-trees need not be unique. However, observe that the reductions of T_1 and T_2 are isomorphic.

We can obtain another 3-tree for M by leaving one element of L in its original bag v and then adding new bags by subdividing each of px, py, and pz and putting one element of L in each of these new bags. This new tree T_3 is a 3-tree for M but it certainly does not have the minimum number of vertices among MP3T's for M.

Although MP3T's with the minimum number of vertices need not be 3-trees, they do satisfy (II) for a 3-tree. To verify this, we shall use the following preliminary result.

Lemma 5.1. *Let e and f be edges of a maximal partial 3-tree T, and let e and f both be incident with a vertex v of degree at least three. If e and f display the 3-separating partitions (X_e, Y_e) and (X_f, Y_f), respectively, where $X_f \subseteq Y_e$, then $\mathrm{fcl}(X_e) \not\supseteq X_f$.*

Proof. If v is a flower vertex, then v is tight, so $\mathrm{fcl}(X_e) \not\supseteq X_f$. Hence we may assume that v is a bag vertex. Let v_f be the end of f that is different from v. If v_f is a flower vertex, then v_f is tight having X_e contained in a petal and X_f as the union of the other petals; so $\mathrm{fcl}(X_e) \not\supseteq X_f$. If v_f is a bag vertex, then f displays a non-sequential 3-separation, so $\mathrm{fcl}(X_e) \not\supseteq X_f$. □

Lemma 5.2. *Let T be a maximal partial 3-tree for a 3-connected matroid M with the minimum number of vertices. Then T satisfies (II) for a 3-tree.*

Proof. Suppose that a vertex v of T is incident with a pair of twins e and f. Assume that v has degree at least three and let X_e, X_f, and X_g be, respectively, the subsets of E displayed by the components of $T\backslash e, T\backslash f$, and $T\backslash g$ that avoid v, where g is an edge of T incident with v that is different from e and f. Then $X_f \subseteq E - X_e$ so, by Lemma 5.1, $\mathrm{fcl}(X_e) \not\supseteq X_f$. Since e and f are twins, we deduce that $(X_e, E - X_e)$ is equivalent to $(E - X_f, X_f)$. Thus $\mathrm{fcl}(X_f) \supseteq E - X_e \supseteq X_g$. Since $X_g \subseteq E - X_f$, this contradicts Lemma 5.1. Hence v has degree two and so v labels a bag vertex.

If v labels an empty bag, then we can contract one of the edges incident with v to obtain an MP3T with fewer vertices than T. Hence v labels a non-empty bag. Now suppose that v_e, the end of e other than v, labels a bag vertex. Then by contracting e from T, and making the vertex that results from identifying v_e and v into a bag vertex labelled by the union of the labels on v_e and v, we obtain a π-labelled tree. Moreover, since e and f are twins, this π-labelled tree is also an MP3T for M. But this new MP3T has fewer vertices than T. Hence the ends of e and f different from v both label flower vertices. □

Theorem 5.3. *If M is a 3-connected matroid with at least nine elements, then M has a 3-tree.*

Proof. Let T be an MP3T for M satisfying (II). Such an MP3T exists by Lemma 5.2. By Lemma 4.4, T displays all tight maximal flowers of order at least four. Choose T so that it displays the maximum number of tight maximal flowers of M of order 3. Suppose that M has a tight maximal flower Φ for which no equivalent flower is displayed by T. Let $\Phi = (P_1, P_2, P_3)$.

Suppose first that all of P_1, P_2, and P_3 are non-sequential. Then, by Lemma 4.1, T has a degree-3 vertex v at which 3-separations equivalent to

$(P_1, E-P_1), (P_2, E-P_2)$, and $(P_3, E-P_3)$ are displayed. By assumption, v is not a flower vertex so it is a bag vertex. Let P_1', P_2', and P_3' be the unions of the bag labels in the three components of $T\backslash v$ where $(P_i', E-P_i')$ is equivalent to $(P_i, E-P_i)$. Let V label the bag vertex v. Then $V = E(M)-(P_1' \cup P_2' \cup P_3')$ and, since $E = P_1 \cup P_2 \cup P_3$ and $\text{fcl}(P_i) = \text{fcl}(P_i')$ for each i, each element of V is in $\text{fcl}(P_1') \cup \text{fcl}(P_2') \cup \text{fcl}(P_3')$. For each i, let $v_i v$ be the edge of T that displays $(P_i', E-P_i')$. Now modify T as follows. If the set $V \cap \text{fcl}(P_1')$ is non-empty, then subdivide the edge vv_1 adding a new bag vertex u_1 labelled by that set. If the set $[V \cap \text{fcl}(P_2')] - \text{fcl}(P_1')$ is non-empty, then subdivide the edge vv_2 adding a new bag vertex u_2 labelled by that set. Finally, if $[V \cap \text{fcl}(P_3')] - [\text{fcl}(P_1') \cup \text{fcl}(P_2')]$ is non-empty, then subdivide the edge vv_3 adding a new bag vertex u_3 labelled by that set. In the resulting π-labelled tree, relabel v as a flower vertex. The resulting π-labelled tree is an MP3T and v displays a flower equivalent to Φ. For each i such that u_i exists, the edges $v_i u_i$ and $u_i v$ display equivalent 3-separations. If v_i labels a bag vertex, then contract the edge $v_i u_i$ labelling the resulting composite vertex by the union of the labels on v_i and u_i. After these contractions, we obtain an MP3T T' that satisfies (II), displays all of the tight maximal flowers displayed by T and also displays a flower equivalent to Φ. Thus T' contradicts the choice of T.

We may now assume that P_1 and P_2 are non-sequential but P_3 is sequential. Then T has vertices v_1 and v_2 at which 3-separations $(P_1', E-P_1')$ and $(P_2', E-P_2')$ are displayed, where $(P_i', E-P_i')$ is equivalent to $(P_i, E-P_i)$. We choose these vertices so that the distance between v_1 and v_2 is minimized. Suppose first that $v_1 = v_2$. Assume that this vertex is a flower vertex v and Φ_v is the corresponding flower. Suppose that v has degree at least four. Then Φ_v certainly displays a pair of crossing 3-separations. Because $(P_1, E-P_1)$ and $(P_2, E-P_2)$ do not cross, it follows by Lemma 4.2 that Φ_v displays a non-sequential 3-separation that is not displayed by Φ, contradicting the maximality of Φ. Hence, when v is a flower vertex, it has degree 3. In that case, Φ_v is equivalent to Φ because the non-sequential 3-separations they display coincide up to equivalence since $\text{fcl}(P_1' \cup P_2') = \text{fcl}(P_1 \cup P_2) = E$. This contradiction implies that we may assume that v is a bag vertex. Then every edge incident with v displays a non-sequential 3-separation. Since $\text{fcl}(P_1' \cup P_2') = E$, we deduce that v has degree 2. Let v be labelled by V. Then V contains all the tight elements of P_3 so $|V| \geq 2$. In this case, we modify T by creating a new bag vertex v' labelled by V, adding an edge vv', and relabelling v as a flower vertex. The new π-labelled tree T' is easily seen to be an MP3T satisfying (II). Since a flower equivalent to Φ is displayed by v in T', we deduce that T' contradicts the choice of T.

We may now suppose that $v_1 \neq v_2$. Since $\text{fcl}(P_1' \cup P_2') = E$, there is an ordering $x_1, x_2, \ldots, x_m$ of $E - (P_1' \cup P_2')$ such that $P_1' \cup P_2' \cup \{x_1, x_2, \ldots, x_k\}$ is 3-separating for all k in $\{0, 1, \ldots, m\}$. Let $u_1 u_2$ be an edge on the path between v_1 and v_2, and let (U_1, U_2) be the partition of E displayed by $u_1 u_2$

where $P'_i \subseteq U_i$. As $\{x_{m-2}, x_{m-1}, x_m\}$ is a triangle or a triad, these last three elements can be reordered so that, without loss of generality, $\{x_{m-1}, x_m\} \subseteq U_2$. Then, since U_1 is 3-separating, we deduce, by repeated applications of Lemma 2.1, that each of $P'_1, P'_1 \cup x'_1, P'_1 \cup x'_1 \cup x'_2, \dots, P'_1 \cup x'_1 \cup x'_2 \cup \cdots \cup x'_n$ is 3-separating where $x'_1, x'_2, \dots, x'_n$ is the ordering induced on $U_1 - P'_1$ by $x_1, x_2, \dots, x_m$. We conclude that $U_1 \subseteq \mathrm{fcl}(P'_1)$. Hence u_1u_2 displays a 3-separation equivalent to $(P'_1, E - P'_1)$. By replacing v_1 by u_2, we obtain a contradiction. □

Next we show that two edges of a 3-tree that display equivalent 3-separating partitions must be twins.

Lemma 5.4. *Let e and f be distinct edges of a 3-tree T. If e and f display equivalent 3-separating partitions, then T has a degree-2 bag vertex that is incident with both e and f.*

Proof. Take a shortest path R in T that uses both e and f. Let R have ends v_e and v_f that are incident with e and f, respectively. Let $\{X_e, X_f, X_3\}$ be the partition of $E(M)$ displayed by $T\backslash\{e, f\}$, where v_e and v_f are in the components of this graph corresponding to X_e and X_f, respectively. Assume that the interior of R contains a vertex u of degree at least three. Let e' and f' be the edges on the v_eu- and v_fu-paths that are incident with u. Let g' be a third edge incident with u. Then, as the 3-separating partitions displayed by e and f are equivalent, $\mathrm{fcl}(X_e)$ contains those elements of E in the bags of the component of $T\backslash g'$ that does not contain u. This contradicts Lemma 5.1. Thus u and every other vertex in the interior of R has degree 2 and so is a bag vertex.

Suppose that u is not the unique interior vertex of R. Since the 3-separating partitions displayed by e and f are equivalent, $\mathrm{fcl}(X_e) \supseteq X_3$, and so the two edges incident with u display equivalent 3-separating partitions. But u is adjacent to at least one bag vertex, which contradicts condition (II) defining a 3-tree. We conclude that u is the unique vertex in the interior of R. □

Lemma 5.5. *If (X_1, Y_1) is a sequential 3-separating partition displayed in a 3-tree T and (X_2, Y_2) is an equivalent 3-separating partition displayed by T, then $(X_2, Y_2) = (X_1, Y_1)$.*

Proof. By Lemma 3.6, we may assume that (X_1, Y_1) is displayed by an edge, say e, of T because (X_1, Y_1) is sequential and every flower vertex of T is tight. We may also assume that X_1 labels a bag vertex and that it is joined by an edge e_1 to a flower vertex v_1 of T. Evidently (X_2, Y_2) must also be displayed by an edge, say e_2, of T. Since e_1 is not incident with a degree-2 bag vertex of T, it follows, by Lemma 5.4, that $e_2 = e_1$. □

The next lemma was proved in [4, Lemma 4.1].

Lemma 5.6. *Every 3-separation of a sequential matroid is sequential.*

Lemma 5.7. *If a 3-connected matroid M is sequential and T is a 3-tree for M, then T consists of a single bag vertex.*

Proof. By the last lemma, M has no non-sequential 3-separations. Thus T has no flower vertices of degree 3 and, by Lemma 3.6, none of degree 4 or more. Hence every edge of T joins two bag vertices. But such edges display non-sequential 3-separations. Thus T consists of a single bag vertex. □

Lemma 5.8. *Let v be a bag vertex of degree at least two in a 3-tree T. Then every 3-separation displayed by an edge incident with v is non-sequential.*

Proof. Let e be an edge of T incident with v and let u be the other end of e. If u labels a bag vertex, then e certainly displays a non-sequential 3-separation. Therefore suppose that u labels a flower vertex of T. Let (U, V) be the 3-separation displayed by uv. As u displays a tight flower, U is not sequential. If V is sequential, then pick an edge f incident with v but different from e. Arguing as above, we deduce that the other end of f must be a flower vertex and hence must have degree at least three. But, since V is sequential, this flower cannot be tight; a contradiction. □

Lemma 5.9. *Let (X_1, Y_1) and (X_2, Y_2) be equivalent 3-separating partitions that are displayed in a 3-tree T. Then either*

(i) *$X_1 = X_2$ and $Y_1 = Y_2$; or*
(ii) *(X_1, Y_1) and (X_2, Y_2) are displayed by edges that meet a common degree-2 bag vertex and whose other ends are flower vertices.*

Proof. Assume that neither (i) nor (ii) holds. By Lemma 5.5, we may assume that (X_1, Y_1) and (X_2, Y_2) are non-sequential otherwise (i) holds. Moreover, by Lemma 5.4, we may assume that one of (X_1, Y_1) and (X_2, Y_2), say the latter, is strictly displayed by a vertex. Then, applying Lemma 3.12 taking $T_1 = T_2 = T$, we deduce that $(X_2, Y_2) = (X_1, Y_1)$. □

Let T be a 3-tree for a 3-connected matroid M and let v be a bag vertex of T. If v has degree at least 3, then, by Lemmas 5.8 and 5.9, every edge incident with v displays a non-sequential 3-separation and no pair of such 3-separations are equivalent.

Lemma 5.10. *Let T be a 3-tree for a non-sequential 3-connected matroid M. Suppose that (X, Y) is a sequential 3-separation of M displayed by T. Then (X, Y) is displayed by a pendant edge of T that is incident with a flower vertex.*

Proof. By Lemma 3.6, if (X, Y) is displayed by a flower vertex of T, then (X, Y) is displayed by an edge incident with this vertex. We deduce that (X, Y) is indeed displayed by an edge, say e, of T. By Lemma 5.8, e is not incident with a bag vertex of degree at least two. Thus, if $e = uv$, then each of u and v is either a bag vertex of degree one, or a flower vertex. If both u and v are bag vertices, then (X, Y) is non-sequential; a contradiction. If both u and v are flower vertices, then, as (X, Y) is sequential, u or v is not tight; a contradiction. We conclude that the lemma holds. □

Recall that a 3-separating partition of a 3-connected matroid is strictly displayed by a vertex of an MP3T if it is displayed by a vertex but not by an incident edge.

Lemma 5.11. *Let T_1 and T_2 be 3-trees for a 3-connected matroid M. Let (X_1, Y_1) be a non-sequential 3-separation that is strictly displayed by a vertex v_1 of T_1. Let Φ be the flower at v_1, and let (W_1, Z_1) be a 3-separation displayed by v_1 that crosses (X_1, Y_1). Then there are unique 3-separations, (X_2, Y_2) and (W_2, Z_2), that are equivalent to (X_1, Y_1) and (W_1, Z_1), respectively, and are displayed by T_2. Moreover, (X_2, Y_2) and (W_2, Z_2) are strictly displayed by the same vertex of T_2.*

Proof. As (W_1, Z_1) crosses (X_1, Y_1), the former is strictly displayed by v_1. By Corollary 3.12, there are 3-separations (X_2, Y_2) and (W_2, Z_2) that are equivalent to (X_1, Y_1) and (W_1, Z_1), respectively, and are strictly displayed by vertices v_2 and v_3, respectively, of T_2. Assume that $v_2 \neq v_3$. Then, without loss of generality, $W_2 \subseteq X_2$ and $Z_2 \supseteq Y_2$. Now $W_1 \cap Y_1$ contains a petal P of Φ, so

$$P \subseteq \mathrm{fcl}(W_1) = \mathrm{fcl}(W_2) \subseteq \mathrm{fcl}(X_2) = \mathrm{fcl}(X_1).$$

Thus $P \subseteq \mathrm{fcl}(X_1) - X_1$, so, by Lemma 3.6, P is loose; a contradiction. □

Lemma 5.12. *Let T_1 and T_2 be 3-trees for a 3-connected matroid M. Let (X_1, Y_1) and (U_1, V_1) be inequivalent non-sequential 3-separations such that both are strictly displayed by the same vertex v_1 of T_1. Then there are unique 3-separations, (X_2, Y_2) and (U_2, V_2), that are equivalent to (X_1, Y_1) and (U_1, V_1), respectively, and are displayed by T_2. Moreover, (X_2, Y_2) and (U_2, V_2) are strictly displayed by the same vertex of T_2.*

Proof. By Corollary 3.12, there are 3-separations, (X_2, Y_2) and (U_2, V_2), that are equivalent to (X_1, Y_1) and (U_1, V_1), respectively, and are strictly displayed by vertices v_2 and v_3, respectively, of T_2. Moreover, (X_2, Y_2) and (U_2, V_2) are unique. Assume that $v_2 \neq v_3$.

Without loss of generality, $U_2 \subseteq X_2$ and $V_2 \supseteq Y_2$. Then $\mathrm{fcl}(U_1) = \mathrm{fcl}(U_2) \subseteq \mathrm{fcl}(X_2) = \mathrm{fcl}(X_1)$. If U_1 contains a petal of the flower Φ at v_1 that is not in X_1, then that petal is loose; a contradiction to Lemma 3.6. Hence $U_1 \subseteq X_1$ and so $Y_1 \subseteq V_1$. As (X_1, Y_1) and (U_1, V_1) are inequivalent, $U_1 \subsetneqq X_1$ and $Y_1 \subsetneqq V_1$. Thus there is an ordering $(P_1, P_2, \ldots, P_n)$ of the petals of Φ such that $X_1 = P_1 \cup P_2 \cup \cdots \cup P_j$ and $U_1 = P_s \cup P_{s+1} \cup \cdots \cup P_t$, where $2 \leq j \leq n-2$ and $1 \leq s \leq t-1 \leq j-1$. As $U_1 \neq X_1$, we may assume that $s \geq 2$. Now $(P_n \cup P_1 \cup \cdots \cup P_s, E - (P_n \cup P_1 \cup \cdots \cup P_s))$ is a non-sequential 3-separation of M, and so there is an equivalent 3-separation that is strictly displayed by a vertex v_4 of T_2.

By Lemma 5.11 applied to $(P_n \cup P_1 \cup \cdots \cup P_s, E - (P_n \cup P_1 \cup \cdots \cup P_s))$ and (X_1, Y_1), we deduce that $v_4 = v_2$. Applying the same lemma to $(P_n \cup P_1 \cup \cdots \cup P_s, E - (P_n \cup P_1 \cup \cdots \cup P_s))$ and (U_1, V_1), we deduce that $v_3 = v_4$. We conclude that $v_2 = v_3$ and this contradiction completes the proof. □

Lemma 5.13. *Let e and f be edges of a 3-tree T for a 3-connected matroid M that display non-sequential 3-separations (X_e, Y_e) and (X_f, Y_f), respectively. Assume that $X_f \subseteq X_e$. Let (X'_e, Y'_e) and (X'_f, Y'_f) be 3-separations that are equivalent to (X_e, Y_e) and (X_f, Y_f), respectively, and are displayed in a 3-tree T' for M. Then either*

(i) *$X'_f \subseteq X'_e$; or*
(ii) *(X_e, Y_e) and (X_f, Y_f) are equivalent.*

Proof. By Lemmas 3.11 and 5.9, (X'_e, Y'_e) and (X'_f, Y'_f) are both displayed by edges of T'. There are four possibilities:

(a) $X'_f \subseteq X'_e$ and $Y'_f \supseteq Y'_e$;
(b) $Y'_f \subseteq X'_e$ and $X'_f \supseteq Y'_e$;
(c) $X'_f \subseteq Y'_e$ and $Y'_f \supseteq X'_e$;
(d) $Y'_f \subseteq Y'_e$ and $X'_f \supseteq X'_e$.

If (b) holds, then

$$\mathrm{fcl}(X_e) \supseteq \mathrm{fcl}(X_f) = \mathrm{fcl}(X'_f) \supseteq \mathrm{fcl}(Y'_e) = \mathrm{fcl}(Y_e) \supseteq Y_e,$$

so (X_e, Y_e) is sequential; a contradiction. By symmetry, (c) does not hold either. If (d) holds, then

$$\mathrm{fcl}(X_e) \supseteq \mathrm{fcl}(X_f) = \mathrm{fcl}(X'_f) \supseteq \mathrm{fcl}(X'_e) = \mathrm{fcl}(X_e).$$

Hence $\mathrm{fcl}(X_e) = \mathrm{fcl}(X_f)$, and so (X_e, Y_e) and (X_f, Y_f) are equivalent. Finally, if (a) holds, then so does (i). □

Lemma 5.14. *Let e and f be adjacent edges in a 3-tree T for a 3-connected matroid M. Assume that e and f display inequivalent non-sequential 3-separations (X_e, Y_e) and (X_f, Y_f). If T' is also a 3-tree for M, then it has adjacent edges e' and f' that display 3-separations (X'_e, Y'_e) and (X'_f, Y'_f) that are equivalent to (X_e, Y_e) and (X_f, Y_f), respectively.*

Proof. Without loss of generality, we may assume that $X_f \subseteq X_e$. Let v be the common vertex of e and f. Let $\{X_f, Y_e, Z\}$ be the partition of $E(M)$ induced by $T\backslash\{e, f\}$ where v is in the component corresponding to Z, while X_f and Y_e correspond to the components containing the ends of f and e, respectively, that are different from v.

By Lemma 3.11, T' has edges e' and f' that display 3-separations (X'_e, Y'_e) and (X'_f, Y'_f) that are equivalent to (X_e, Y_e) and (X_f, Y_f), respectively. Choose such edges so that the shortest path R containing e' and f' has minimum length. We may assume that this length is at least three. By Lemma 5.13, since $X_f \subseteq X_e$ and $Y_f \supseteq Y_e$, we have $X'_f \subseteq X'_e$ and $Y'_f \supseteq Y'_e$. By the choice of R, no edge of $E(R) - \{e', f'\}$ displays a 3-separation that is equivalent to either (X'_e, Y'_e) or (X'_f, Y'_f). Let g' be the edge of R that is adjacent to f'. Let (X'_g, Y'_g) be the 3-separation displayed by g', where $X'_f \subseteq X'_g \subseteq X'_e$. By Lemma 5.10, (X'_g, Y'_g) is non-sequential. Hence there is a 3-separation (X_g, Y_g) equivalent to (X'_g, Y'_g) that is displayed by an edge g of T and, by Lemma 5.13, $X_f \subseteq X_g \subseteq X_e$. Since f and e are adjacent, it follows that $g \in \{f, e\}$. Thus (X_g, Y_g) is (X_f, Y_f) or (X_e, Y_e), so (X'_g, Y'_g) is equivalent to (X'_f, Y'_f) or (X'_e, Y'_e); a contradiction. □

Lemma 5.15. *For some $k \geq 2$, let $e_1, e_2, \ldots, e_k$ be the edges incident with a vertex v in a 3-tree T for a 3-connected matroid M such that every e_i displays a non-sequential 3-separation and no two such edges display equivalent 3-separations. For each i, let X_i be the union of the bag labels in the component of $T \backslash e_i$ avoiding v. Let T' be another 3-tree for M. Then there is a degree-k vertex v' in T' with incident edges $e'_1, e'_2, \ldots, e'_k$ such that, for all i, if X'_i is the union of the bag labels in the component of $T' \backslash e'_i$ avoiding v', then $(X'_i, E - X'_i)$ is equivalent to $(X_i, E - X_i)$. Moreover, if v is a bag vertex, then v' is also a bag vertex.*

Proof. By Lemma 5.14, there are adjacent edges e'_1 and e'_2 of T' that display 3-separations $(X'_1, E - X'_1)$ and $(X'_2, E - X'_2)$ that are equivalent to $(X_1, E - X_1)$ and $(X_2, E - X_2)$, respectively. Moreover, by Lemma 5.13, $X'_2 \subseteq E - X'_1$. Let v' be the vertex that is common to e'_1 and e'_2. If v is a flower vertex, then, by (I) or Lemma 4.2, T' has a vertex w' that displays a flower equivalent to that displayed by v. If $w' \neq v'$, then both w' and v' display 3-separations equivalent to $(X'_1, E - X'_1)$ and $(X'_2, E - X'_2)$. This leads to a contradiction to Lemma 5.9. Hence $w' = v'$ and the lemma holds. A similar argument establishes the lemma if v' is a flower vertex. We may now assume that both v and v' are bag vertices.

Assume that $k = 2$. If v' has degree greater than two, then, by Lemma 5.8, there is a non-sequential 3-separation $(X'_3, E - X'_3)$ displayed at v' such that $E - X'_3 \supseteq X'_1 \cup X'_2$. Then

$$\text{fcl}(E - X'_3) \supseteq \text{fcl}(X'_1 \cup X'_2) \supseteq X_1 \cup X_2.$$

Now T displays a 3-separation $(X_3, E - X_3)$ that is equivalent to $(X'_3, E - X'_3)$. As $k = 2$, without loss of generality, $X_3 \subseteq X_1$ or $E - X_3 \subseteq X_1$. In the first case, $E - X_3 \supseteq E - X_1$ so

$$\text{fcl}(E - X_3) = \text{fcl}(E - X'_3) \supseteq (X_1 \cup X_2) \cup (E - X_1) = E;$$

a contradiction to the fact that $(X_3, E-X_3)$ is non-sequential. In the second case,

$$\mathrm{fcl}(E - X_2) \supseteq \mathrm{fcl}(X_1) \supseteq \mathrm{fcl}(E - X_3) = \mathrm{fcl}(E - X_3') \supseteq X_2,$$

so $(X_2, E - X_2)$ is sequential; a contradiction. We conclude that if $k = 2$, then v' has degree exactly two and the lemma holds.

We may now assume that $k \geq 3$. Then T' has edges f_{31} and f_{32}, which may be equal, that both display 3-separations equivalent to $(X_3, E - X_3)$ such that f_{3i} is adjacent to an edge that displays a 3-separation equivalent to $(X_i, E-X_i)$. Either $f_{31} = f_{32}$ or, by Lemma 5.9, these edges are distinct and meet at a degree-2 bag vertex. Since the only edge of T' other than e_i' that can display a 3-separation equivalent to $(X_i, E - X_i)$ must share a degree-2 vertex with e_i', it follows that $f_{31} = f_{32}$ and this edge, which we relabel e_3', is incident with v'. Similarly, there are edges $e_4', e_5', \ldots, e_k'$ incident with v' that display 3-separations equivalent to $(X_4, E-X_4), (X_5, E-X_5), \ldots, (X_k, E-X_k)$. Thus $k' \geq k$, where k' is the degree of v'. If $k' > k$, then v' is incident with an edge e_{k+1}' that is not in $\{e_1', e_2', \ldots, e_k'\}$. By Lemma 5.8, the 3-separation $(X_{k+1}', E - X_{k+1}')$ that is displayed by e_{k+1}' is non-sequential, and so there is an edge e_{k+1} of T that displays a 3-separation equivalent to $(X_{k+1}', E - X_{k+1}')$. By Lemmas 5.14 and 5.9, e_{k+1} must be incident with v but distinct from $e_1, e_2, \ldots, e_k$. This contradiction implies that $k' = k$ and thereby completes the proof. □

6. Proof of the Main Theorem

In this section, we prove the main result of the paper.

Proof of Theorem 2.3. By Lemma 5.7, the theorem holds if M is sequential. We may thus assume that M has at least one non-sequential 3-separation. If T_1 has no vertices of degree more than two and every degree-two vertex is incident with a pair of twins, then, up to equivalence, M has exactly one non-sequential 3-separation. In this case, both T_1 and T_2 consist of a single edge and the theorem holds.

We may now assume T_1 has a vertex v_1 of degree k such that it is either a bag vertex of degree 2 that is not incident with a pair of twins, or a bag or flower vertex of degree at least 3. If v_1 is a bag vertex, then, by Lemma 5.15, there is a degree-k bag vertex v_2 of T_2 such that if we label the sets displayed by $T_1 - v_1$ by $P_{11}, P_{12}, \ldots, P_{1k}$ and those displayed by $T_2 - v_2$ by $P_{21}, P_{22}, \ldots, P_{2k}$, then $(P_{1j}, E - P_{1j})$ is equivalent to $(P_{2j}, E - P_{2j})$ for all j.

If v_1 is a flower vertex, then there is a degree-k flower vertex v_2 of T_2 such that if $(P_{i1}, P_{i2}, \ldots, P_{ik})$ is the flower Φ_i displayed at v_i for each i, then, by (I) or Lemma 4.4, Φ_1 is equivalent to Φ_2, and so they have the same type. By Lemma 3.10, we may again assume that Φ_2 is labelled so that $(P_{1j}, E - P_{1j})$ is equivalent to $(P_{2j}, E - P_{2j})$ for all j.

Our isomorphism between $R(T_1)$ and $R(T_2)$ will map v_1 to v_2. Let the edges incident with v_i be $v_i w_{ij}$ for $j = 1, 2, \dots, k$ where P_{ij} is the union of the bag labels in the component of $T_i \backslash v_i w_{ij}$ containing w_{ij}. Let T_{ij} be the subtree obtained from this component by adjoining the edge $v_i w_{ij}$. By Lemma 5.13, for a non-sequential 3-separation $(X_{ij}, E - X_{ij})$ that is displayed by an edge with $X_{ij} \subseteq P_{ij}$, if $(X_{(i+1)j}, E - X_{(i+1)j})$ is equivalent to $(X_{ij}, E - X_{ij})$, then $X_{(i+1)j} \subseteq P_{(i+1)j}$, where $i+1$ is calculated modulo 2.

Now consider w_{11} and w_{21}. If both have degree one, then we map w_{11} to w_{21}, and the 3-separations displayed by the edges incident with w_{11} and w_{21} are equivalent. Now assume that w_{11}, say, has degree more than one. If w_{11} is incident with twins, let w'_{11} be the other neighbour of w_{11} apart from v_1. Otherwise, let $w'_{11} = w_{11}$. Then w'_{11} is a flower vertex of degree m for some $m \geq 3$ or a bag vertex that is not incident with a pair of twins and has degree m for some $m \geq 2$.

By Lemma 5.15, (I), or Lemma 4.4, there is a degree-m vertex w'_{21} of T_2 such that the 3-separations displayed by edges incident with w'_{21} or by the vertex w'_{21} and its incident edges coincide, up to equivalence, with the 3-separations displayed by edges incident with w'_{11} or by w'_{11} itself and its incident edges. By Lemma 5.13, w'_{21} is a vertex of T_{21}. The first edge on the path in T_2 from w'_{21} to v_2 displays a non-sequential 3-separation (W_2, Z_2) that is equivalent to one of the 3-separations (W_1, Z_1) displayed by an edge incident with w'_{11}. By two applications of Lemma 5.13, we deduce that (W_1, Z_1) is equivalent to the 3-separation displayed by $v_1 w_{11}$, namely $(P_{11}, E - P_{11})$. But $(P_{11}, E - P_{11})$ is equivalent to the 3-separation displayed by $v_2 w_{21}$. It follows by Lemma 5.9 that either $w'_{21} = w_{21}$, or w'_{21} is adjacent to w_{21} and w_{21} is a degree-2 bag vertex for which the two incident edges are twins. In each case, we map w'_{11} to w'_{21} and note that the 3-separations displayed by edges incident with w'_{11} or by w'_{11} coincide, up to equivalence, with the 3-separations displayed by edges incident with w'_{21} or by w'_{21} itself. Thus we can iterate the above process working outward from the vertices v_1 and v_2. It follows that $R(T_1)$ is isomorphic to $R(T_2)$ and that there is such an isomorphism φ satisfying (i) and (ii).

Finally, assume that φ maps adjacent flower vertices u_1 and v_1 of T_1 onto non-adjacent vertices u_2 and v_2 of T_2. Let w_2 be the bag vertex of T_2 that is adjacent to both u_2 and v_2, and let w_2 be labelled by W. Observe that $u_2 w_2$ and $w_2 v_2$ are twins. Let the partition of $T_2 \backslash \{u_2 w_2, w_2 v_2\}$ be $\{U_2, W, V_2\}$ where U_2 and V_2 correspond to the components containing u_2 and v_2, respectively. Let (U_1, V_1) be the partition displayed by the edge $u_1 v_1$ of T_1. Then $W \subseteq \mathrm{fcl}(U_2) \cap \mathrm{fcl}(V_2) = \mathrm{fcl}(U_1) \cap \mathrm{fcl}(V_1)$. Now, in T_1, every element of W must lie either in one of the petals contained in U_1 of the flower at u_1, or in one of the petals contained in V_1 of the flower at v_1. Since each element of U_1 is in $\mathrm{fcl}(U_2)$ and each element of V_1 is in $\mathrm{fcl}(V_2)$, we deduce that every element of W is, indeed, loose in the flower displayed by u_2 or is loose in the flower displayed by v_2. Hence, because φ maps flowers onto equivalent flowers, every element of W is also loose in the flower displayed by u_1 or in that displayed by v_1. □

7. Consequences

We conclude the paper by noting some additional useful properties of flowers and 3-trees. The main result of the section is Proposition 7.3, which describes a partition of the non-sequential 3-separations in a 3-connected matroid M into three classes and indicates how membership of these classes can be determined from any 3-tree for M.

Lemma 7.1. *Let Φ be a flower of order at least three in a 3-connected matroid M. Then, up to equivalence, there is a unique tight maximal flower Ψ such that $\Phi \preceq \Psi$.*

Proof. Let Ψ_1 and Ψ_2 be inequivalent tight maximal flowers such that $\Phi \preceq \Psi_1$ and $\Phi \preceq \Psi_2$. Take a 3-tree T for M. Then, by (I) and Lemma 4.4, T has distinct vertices v_1 and v_2 that display flowers equivalent to Ψ_1 and Ψ_2, respectively. Since Φ has order at least three, it displays at least two non-sequential 3-separations, so 3-separations equivalent to both of these are displayed at both v_1 and v_2. But this contradicts Lemma 5.9. □

Lemma 7.2. *Let T be a 3-tree for a 3-connected matroid M. Let e and f be edges of T that display inequivalent 3-separations and are incident with a common vertex v. Let (L, C, R) be the partition of $E(M)$ that refines the partitions displayed by both e and f. Then C is not 3-separating if and only if either v is a bag vertex, or v displays a daisy in which the petals displayed by e and f are non-consecutive.*

Proof. Suppose that v is a bag vertex and C is 3-separating. Then (L, C, R) is a tight flower Φ. Thus there is a tight maximal flower Ψ for which $\Phi \preceq \Psi$. By (I) or Lemma 4.4, T has a vertex w that displays a flower equivalent to Ψ. Thus there are 3-separations equivalent to both $(L, C \cup R)$ and $(L \cup C, R)$ that are displayed at both w and v. Since $w \neq v$, we get a contradiction to Lemma 5.9. We conclude that C is not 3-separating. The lemma now follows without difficulty. □

Proposition 7.3. *Let T be a 3-tree for a 3-connected matroid M and (A, B) be a non-sequential 3-separation of M. Let (A', B') be a 3-separation equivalent to (A, B) that is displayed in T. Then*

(i) *A is displayed by a pair of petals in a tight flower of M of order four if and only if (A', B') is displayed in T by a vertex and not by an edge.*

(ii) *A is displayed by a petal of a maximal flower of M of order at least three if and only if (A', B') is displayed in T by a vertex and an edge.*
(iii) *(A, B) is not displayed in a flower of M of order at least three if and only if (A', B') is displayed in T by an edge and not by a vertex.*

Proof. Consider (i). Assume that A is displayed by a pair of petals in a tight flower (P_1, P_2, P_3, P_4), say $A = P_1 \cup P_2$. Then $(P_1, P_2, P_3, P_4) \preceq \Phi$, a tight maximal flower of M. By Lemma 4.4, there is a vertex v of T that displays a flower Φ_v equivalent to Φ. Now Φ_v displays a 3-separation $(X, E - X)$ equivalent to $(P_1 \cup P_2, E - (P_1 \cup P_2)$ and, by Lemma 4.3, each of X and $E - X$ is a union of at least two petals of Φ_v. By Lemma 5.9, $(X, E - X) = (A', B')$ so (A', B') is displayed by a vertex and not an edge of T. The converse follows without difficulty using Corollary 3.7.

For (ii), suppose that A is displayed by a petal P of a maximal flower Ψ of M of order at least three. Then, by [6, Lemmas 5.3 and 5.7], Ψ is equivalent to a tight maximal flower Φ, which equals $(P_1, P_2, \ldots, P_k)$ say. Moreover, as P is a tight petal of Ψ, it follows by [6, Lemma 5.8] that Φ has a petal, say P_2, such that $(P_2, E - P_2)$ is equivalent to (A, B). Now T has a vertex v that displays a tight maximal flower $(Q_1, Q_2, \ldots, Q_j)$ equivalent to Φ. By Lemma 3.10, $j = k$ and there is a permutation α of $\{1, 2, \ldots, k\}$ such that $\mathrm{fcl}(P_i) = \mathrm{fcl}(Q_{\alpha(i)})$ for all i. Thus $(P_2, E - P_2)$, and hence (A', B'), is equivalent to $(Q_{\alpha(2)}, E - Q_{\alpha(2)})$. By Lemma 5.9, either $(A', B') = (Q_{\alpha(2)}, E - Q_{\alpha(2)})$, or T has a degree-2 bag vertex u and a flower vertex w such that wu and uv are edges of T and wu displays (A', B'). Thus (A', B') is displayed either by the vertex v and the edge uv, or by the vertex w and the edge wu. Again the straightforward proof of the converse is omitted.

For (iii), we note that if (A, B) is not displayed by a flower of M of order at least three, then T has an edge e that displays a 3-separation (A'', B'') that is equivalent to (A, B) and hence to (A', B'). Moreover, e joins two bag vertices. It follows by Lemma 5.9 that $(A'', B'') = (A', B')$. For the converse, we note that if (A, B) is displayed in a flower Φ of M of order at least three, then a 3-separation equivalent to (A, B) is displayed by a tight maximal flower equivalent to Φ and the result follows by (i) and (ii). □

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