# Matroids and Graphs with Few Non-Essential Elements

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**Abstract.** An essential element of a 3-connected matroid $M$ is one for which neither the deletion nor the contraction is 3-connected. Tutte's Wheels and Whirls Theorem proves that the only 3-connected matroids in which every element is essential are the wheels and whirls. In an earlier paper, the authors showed that a 3-connected matroid with at least one non-essential element has at least two such elements. This paper completely determines all 3-connected matroids with exactly two non-essential elements. Furthermore, it is proved that every 3-connected matroid $M$ for which no single-element contraction is 3-connected can be constructed from a similar such matroid whose rank equals the rank in $M$ of the set of elements $e$ for which the deletion $M \backslash e$ is 3-connected.

## 1. Introduction

An element $e$ of a 3-connected matroid $M$ is *deletable* if the deletion $M \backslash e$ is 3-connected; $e$ is *contractible* if the contraction $M/e$ is 3-connected. If $e$ is neither deletable nor contractible, it is *essential*. Tutte's Wheels and Whirls Theorem [15] established that a 3-connected matroid has no non-essential elements if and only if it is a wheel or whirl and has rank at least three. In an earlier paper [11], the authors showed that a 3-connected matroid with at least one non-essential element must have at least two such elements. This paper determines all 3-connected matroids with exactly two non-essential elements.

A 3-connected matroid for which no single-element contraction is 3-connected is called *contraction-minimally* 3-*connected*. The duals of such matroids, which are called *minimally* 3-*connected matroids*, have been studied quite extensively (see, for example, [3, 5, 6, 7, 8, 15]). It will be shown that every contraction-minimally 3-connected matroid with exactly two non-essential elements can be constructed by beginning with a line $L$, attaching wheels to $L$ so that a 3-point line of each wheel is identified with three points of $L$ and, finally, deleting some subset of $L$ consisting of elements that are not spokes of the attached wheels. In addition, this result will be extended by showing that in a contraction-minimally 3-connected matroid $M$,

if the rank of the set of deletable elements is $d$ for some $d \geq 3$, then $M$ can be obtained by attaching wheels to some contraction-minimally 3-connected matroid $N_0$ of rank $d$ where $N_0$ contains and is spanned by the set of deletable elements of $M$.

The matroid terminology used here will follow Oxley [9]. For a matroid $M$, the simple matroid and the cosimple matroid associated with $M$ will be denoted by $\tilde{M}$ and $\underline{M}$, respectively. We call these matroids the *simplification* and the *cosimplification* of $M$. The basic property that a circuit and a cocircuit in a matroid cannot have exactly one common element is referred to as *orthogonality*. $M(\mathcal{W}_r)$ and $\mathcal{W}^r$ will denote, respectively, the cycle matroid of the wheel with $2r$ edges, and the whirl of rank $r$. Frequently, $M(\mathcal{W}_r)$ will be referred to just as the *rank-r wheel*. In this matroid, the elements can be partitioned into *spokes* and *rim elements* corresponding to the role these elements play in the wheel graph $\mathcal{W}_r$. When $r \geq 4$, this partition is unique, the spokes being precisely the elements that are in exactly two triangles. When $r = 3$, since $M(\mathcal{W}_r) \cong M(K_4)$, any one of the triangles of $\mathcal{W}_3$ can be viewed as its rim. In this case, we arbitrarily designate a triangle as the rim and take the complementary set of edges as the set of spokes.

This paper will rely heavily on [11]. In this section and the next, we briefly review the definitions and theorems from that paper that will be needed here.

Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid $M$ such that, for all $i$ in $\{1, 2, \ldots, k-1\}$,

(i)   in $\{T_i, T_{i+1}\}$ exactly one set is a triangle and exactly one set is a triad;
(ii)  $|T_i \cap T_{i+1}| = 2$; and
(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \cdots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of $M$ of *length* $k$ with *links* $T_1, T_2, \ldots, T_k$. Evidently $T_1, T_2, \ldots, T_k$ is a chain of $M$ if and only if it is a chain of $M^*$.

An easy induction argument using orthogonality shows that if $T_1, T_2, \ldots, T_k$ is a chain in a matroid $M$, then $M$ has $k + 2$ distinct elements $a_1, a_2, \ldots, a_{k+2}$ such that $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all $i$ in $\{1, 2, \ldots, k\}$. The following result was proved in [11] by extending Tutte's proof of the Wheels and Whirls Theorem.

**1.1 Lemma.** *Let $M$ be a 3-connected matroid with at least four elements and suppose that $M$ is not a wheel or a whirl. Let $T_1, T_2, \ldots, T_k$ be a maximal chain in $M$. Then the elements of $T_1 \cup T_2 \cup \cdots \cup T_k$ can be labelled so that neither $a_1$ nor $a_{k+2}$ is essential where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all $i$.*

Now suppose that $T_1, T_2, \ldots, T_k$ is a maximal chain of a 3-connected matroid $M$ where $M$ is not a wheel or a whirl. We call this maximal chain a *fan* of $M$ with *links* $T_1, T_2, \ldots, T_k$. Let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all $i$. Then $\{a_1, a_2, \ldots, a_{k+2}\}$ is the *ground set* of the fan, and $a_1, a_2, \ldots, a_{k+2}$ are the *elements* of the fan. For $k \geq 2$, Lemma 1.1 implies that there are exactly two non-essential elements in $T_1 \cup T_2 \cup \cdots \cup T_k$, namely $a_1$ and $a_{k+2}$, since each of $a_2, a_3, \ldots, a_{k+1}$ is in both a triangle and a triad. We call $a_1$ and $a_{k+2}$ the *ends* of the fan. When $k = 1$, the fan has $T_1$ as its ground set and contains either two or three non-essential elements of $M$. In the
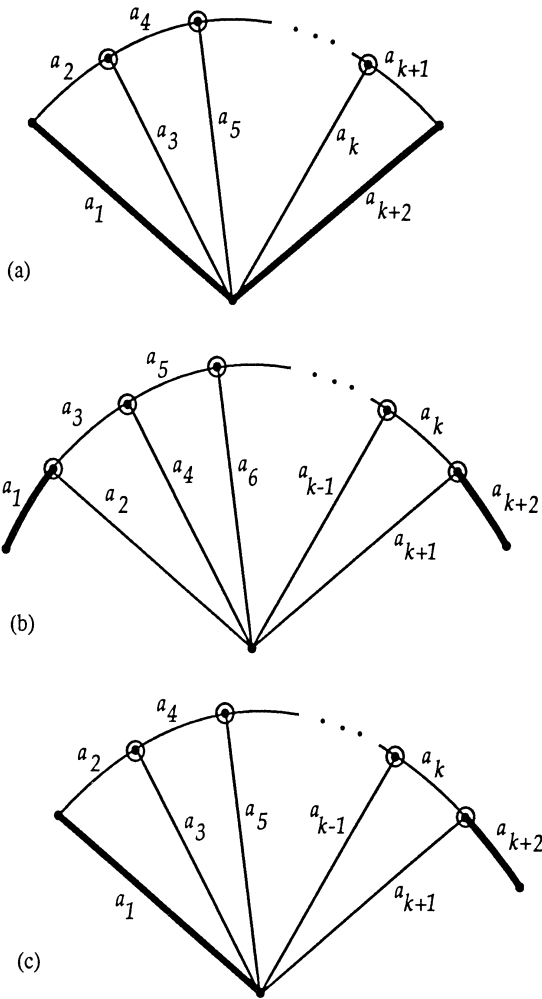
**Fig. 1.** The three types of chains

first case, we take the ends of the fan to be the non-essential elements in $T_1$; in the second case, we arbitrarily choose two of the elements of $T_1$ to be the ends of the fan. Figure 1 (a), (b), and (c) show the three types of chains, where a triangle in these graphs corresponds to a triangle in the chain, while a triad in the chain corresponds to a circled vertex. Maximal chains of these three types will be called *type-1*, *type-2*, and *type-3 fans*, respectively. In the figure, the non-essential elements of these fans have been marked in bold. Two fans are *equal* if they have the same sets of links. The *internal spokes* of a fan are the elements of the fan that belong to at least two triangles of the fan. A *trivial fan* is one with either three or four elements. In particular, a type-3 fan that is trivial has exactly four elements.
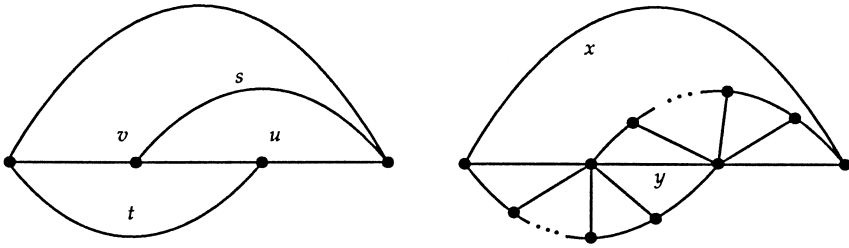
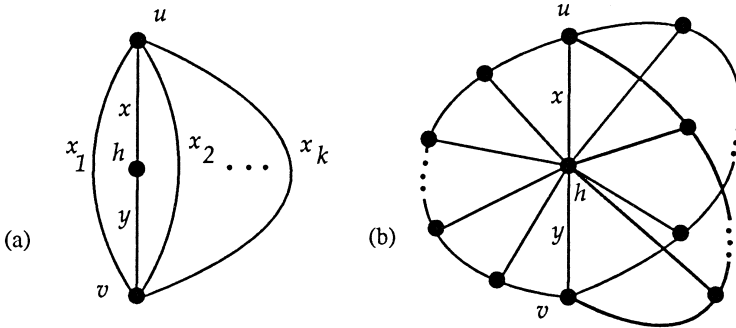**Fig. 2.** Construction of a twisted wheel from $K_4$

**Fig. 3.** Construction of a multidimensional wheel

We know from Lemma 1.1 that if $M$ is a 3-connected matroid other than a wheel or a whirl, then every essential element of $M$ is in a fan the ends of which are non-essential. Thus, if $M$ has exactly two non-essential elements, these two elements must occur as the ends of every fan. Therefore $M$ is formed by somehow attaching these fans together across the two non-essential elements. In what follows, we shall describe precisely how these attachments are done. If $M$ is graphic, it is not difficult to find some examples of such attachments. The right-hand graph in Figure 2 is called a *twisted wheel*. It can be obtained from $K_4$, drawn as in the left-hand graph, by subdividing the edges $s$ and $t$ by inserting at least one new vertex into each and then joining each of the newly created vertices to one of $u$ and $v$ as shown. Evidently $x$ and $y$ are the only non-essential elements in the cycle matroid of such a graph. It is clear that the cycle matroid of a twisted wheel can also be constructed by appropriately joining two type-3 fans with ends $x$ and $y$.

The graph in Figure 3(b) is an example of a 3-dimensional wheel. In general, for all $k \geq 2$, a *k-dimensional wheel* is constructed as follows: begin with the 3-vertex graph in Figure 3(a) in which $u$ and $v$ are joined by a path $u, h, v$ of length two and by $k$ parallel edges $x_1, x_2, \ldots, x_k$. Subdivide each of these parallel edges by inserting at least one new vertex into each and, finally, join each newly created vertex to $h$. When $k = 2$, the resulting graph is a wheel. When $k \geq 3$, it is clear that the cycle matroid of the resulting graph has $x$ and $y$ as its only non-essential elements. Moreover, this matroid can be obtained by appropriately joining $k$ type-1 fans with ends $x$ and $y$. Observe that if each of $x_1, x_2, \ldots, x_k$ is subdivided into

exactly two edges, the resulting $k$-dimensional wheel is isomorphic to the graph that is obtained from $K_{3,k}$ by adding two edges joining distinct pairs of vertices in the 3-vertex class of this bipartite graph. A *multidimensional wheel* is a $k$-dimensional wheel for some $k \geq 3$.

The next result, which follows easily from the main results of this paper, asserts that the only graphic matroids with exactly two non-essential elements are those described above.

**1.2 Theorem.** *Let $M$ be a 3-connected graphic matroid. Then $M$ has exactly two non-essential elements if and only if $M$ is the cycle matroid of a twisted wheel or a multidimensional wheel.*

In every 3-connected matroid $M$ with exactly two non-essential elements, every element is in a fan and so belongs to a triangle or a triad. Thus $M$ has no element that is both deletable and contractible. Hence, in $M$, we must have one of the following:

(i)   both non-essential elements are deletable but not contractible;
(ii)  both non-essential elements are contractible but not deletable; or
(iii) one non-essential element is deletable but not contractible and the other is contractible but not deletable.

Hence, for example, the cycle matroid of a multidimensional wheel satisfies (i), whereas the cycle matroid of a twisted wheel satisfies (iii). Evidently, in cases (i), (ii), and (iii), every fan of $M$ is of type-1, type-2, or type-3, respectively. Accordingly, in these three cases, we shall refer to $M$ itself as being of *type-1 type-2*, or *type-3*. Clearly the class of type-2 matroids coincides with the class of duals of type-1 matroids, so to specify all 3-connected matroids with exactly two non-essential elements, it suffices to describe the matroids of type-1 and those of type-3.

To state these results, we shall need to use a special case of the operation of generalized parallel connection introduced by Brylawski [2]. Let $M_1$ and $M_2$ be matroids such that $E(M_1) \cap E(M_2) = \Delta$ where $\Delta$ is a triangle of both $M_1$ and $M_2$. Assume that $M_1$ is binary. The *generalized parallel connection* $P_\Delta(M_1, M_2)$ of $M_1$ and $M_2$ across $\Delta$ is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets $X$ of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of $M_1$, and $X \cap E(M_2)$ is a flat of $M_2$. If both $M_1$ and $M_2$ are binary having more than six elements and $\Delta$ does not contain a cocircuit of $M_1$ or of $M_2$, then $P_\Delta(M_1, M_2) \backslash \Delta$ is what Seymour [13, p. 305] calls the 3-*sum* and Truemper [14, p. 183] calls the $\Delta$-*sum of* $M_1$ and $M_2$.

Let $G_1$ and $G_2$ be graphs whose sets of edge labels are disjoint except that each has a triangle $\Delta$ whose edges are labelled by $e, f$, and $g$. If $G$ is the graph that is obtained by identifying these triangles so that edges with the same labels coincide, then the cycle matroid of $G$ is precisely the matroid $P_\Delta(M(G_1), M(G_2))$. We remark that the graph $G \backslash \{e, f, g\}$ is what Robertson and Seymour [25] call the 3-*sum* of $G_1$ and $G_2$.

The next theorem describes how to construct all type-1 matroids. As an introduction to this result, it is instructive to consider a geometric construction for the cycle matroid of a multidimensional wheel. Begin with a 3-point line $\{x, y, z\}$ and

$k$ wheels for some $k \geq 3$. Let $\{x, y, z\}$ also label a triangle in each of these wheels with $x$ and $y$ being spokes. Attach the wheels to the line, one at a time, via generalized parallel connection. Finally, delete the element $z$ to obtain the desired matroid.

**1.3 Theorem.** *The class of 3-connected matroids that have exactly two non-essential elements, $x$ and $y$, each of which is deletable, coincides with the class of matroids $M$ that are constructed as follows.*

(i)   *Let $L$ be an $n$-point line for some $n \geq 3$, and $x$ and $y$ be two elements of $L$.*
(ii)  *Let $N_1, N_2, \ldots, N_k$ be a collection of wheels of rank at least three such that $E(L), E(N_1), E(N_2), \ldots, E(N_k)$ are disjoint and $k \geq 3$.*
(iii) *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be a collection of triangles whose union is $L$ such that each contains $\{x, y\}$, and let $\Delta_1', \Delta_2', \ldots, \Delta_k'$ be triangles in $N_1, N_2, \ldots, N_k$, respectively.*
(iv)  *For each $i$ in $\{1, 2, \ldots, k\}$, take a bijection from $\Delta_i'$ to $\Delta_i$ and relabel each element of $\Delta_i'$ by the corresponding element of $\Delta_i$ so that $x$ and $y$ label spokes of $N_i$.*
(v)   *Let $A_0 = L$ and, for all $i$ in $\{1, 2, \ldots, k\}$, let $A_i = P_{\Delta_i}(N_i, A_{i-1})$.*
(vi)  *Let $M = A_k \backslash (L - \{x, y\})$.*

It should be noted that the triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$, which appear in the above construction of type-1 matroids, need not be distinct. Like type-1 matroids, type-3 matroids are obtained by attaching wheels to a certain root matroid. This root matroid has a special form and, indeed, plays a fundamental role elsewhere [4]. For all $n \geq 3$, an *$n$-spike* with *tip $p$* is any matroid that satisfies the following three conditions:

(i)   the ground set is the union of $n$ lines, $L_1, L_2, \ldots, L_n$, all having three points and passing through a common point $p$;
(ii)  for all $k$ in $\{1, 2, \ldots, n-1\}$, the union of any $k$ of $L_1, L_2, \ldots, L_n$ has rank $k + 1$; and
(iii) $r(L_1 \cup L_2 \cup \cdots \cup L_n) = n$.

For example, it is not difficult to show that the only $n$-spike that is binary is the vector matroid of the binary matrix $[I_n | J_n - I_n | \mathbf{1}]$, where $J_n$ is the $n \times n$ matrix of all ones and $\mathbf{1}$ is the all-ones column. Moreover, every $n$-spike is 3-connected.

**1.4 Theorem.** *The class of 3-connected matroids that have exactly two non-essential elements, one of which is deletable and one of which is contractible, coincides with the non-wheels and non-whirls that are in the class of matroids $M$ that are constructed as follows.*

(i)   *Let $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ be the triangles of an $n$-spike $N$ that contain the tip $y$ of $N$ where $\Delta = \{y, x, z\}$ and $\Delta_i = \{y, x_i, z_i\}$ for all $i$.*
(ii)  *Let $N_0 = N \backslash z$ and, for some $t \leq n - 1$, let $N_1, N_2, \ldots, N_t$ be a collection of wheels of rank at least three such that $E(N_0), E(N_1), E(N_2), \ldots, E(N_t)$ are disjoint.*

(iii) *Let $\Delta'_1, \Delta'_2, \ldots, \Delta'_t$ be triangles in $N_1, N_2, \ldots, N_t$, respectively.*

(iv) *For each $i$ in $\{1, 2, \ldots, t\}$, take a bijection from $\Delta'_i$ to $\Delta_i$ and relabel each element of $\Delta'_i$ by the corresponding element of $\Delta_i$ so that $y$ labels a spoke and $z_i$ a rim element of $N_i$.*

(v) *Let $R_0 = N_0$ and, for all $i$ in $\{1, 2, \ldots, t\}$, let $R_i = P_{\Delta_i}(N_i, R_{i-1})$.*

(vi) *Let $M = R_t \backslash z_1, z_2, \ldots, z_t$.*

The last two theorems will be proved in Sections 3 and 4, respectively. Indeed, the first will be derived as a special case of a much more general result, Theorem 3.1, which specifies, for all $d \geq 2$, all contraction-minimally 3-connected matroids in which the set of deletable elements has rank $d$.

## 2. Preliminaries

In this section, we present several results that will be used in the proofs of our main results. We begin by stating the two main theorems of [11]. Tutte [15] proved that every essential element of a 3-connected matroid is in a chain. The first theorem determines precisely when such an element is in more than one maximal chain.

**2.1 Theorem.** *Let $M$ be a 3-connected matroid that is not a wheel or a whirl. Suppose that $e$ is an essential element of $M$. Then $e$ is in a fan, both ends of which are non-essential. Moreover, this fan is unique unless*

(a) *every fan containing $e$ consists of a single triangle and any two such triangles meet in $\{e\}$;*

(b) *every fan containing $e$ consists of a single triad and any two such triads meet in $\{e\}$;*

(c) *$e$ is in exactly three fans; these three fans are of the same type, each has five elements, together they contain a total of six elements; and, depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of $M$ to this set of six elements is isomorphic to $M(K_4)$.*

When a 3-connected matroid $M$ has a chain of odd length exceeding two, we can break off a wheel from $M$ to leave a certain 3-connected minor of $M$. The next theorem specifies precisely how this is done when the chain begins, and hence ends, with a triangle. If the chain begins with a triad, we apply the dual of this result.

**2.2 Theorem.** *Let $M$ be a 3-connected matroid and suppose that, for some non-negative integer $n$, the sequence*

$$\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \ldots, \{y_n, x_n, y_{n+1}\}$$

*is a chain in $M$ in which $\{y_0, x_0, y_1\}$ is a triangle. Then*

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1) \backslash z$$

*where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathcal{W}_{n+2}$ is labelled as in Figure 4; and $M_1$ is obtained from*

the matroid $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n$ by relabelling $x_n$ as $z$. Moreover, either

(i)  $M_1$ is 3-connected; or
(ii)  $z$ is in a unique 2-circuit $\{z, h\}$ of $M_1$, and $M_1 \backslash z$ is 3-connected.

In the latter case,

$$M = P_{\Delta_2}(M(\mathscr{W}_{n+2}), M_2)$$

where $\Delta_2 = \{y_0, y_{n+1}, h\}$; $\mathscr{W}_{n+2}$ is labelled as in Figure 4 with $z$ relabelled as $h$; and $M_2$ is $M_1 \backslash z$, which equals $M \backslash x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_n$.

We shall now describe how essential elements behave when a wheel is broken off as in Theorem 2.2. In that theorem, the resulting 3-connected matroid is $M_1$ or $M_2$. We shall first consider the latter [11, Proposition 4.4].

**2.3 Proposition.** *Let $M = P_{\Delta_2}(M(\mathscr{W}_{n+2}), M_2)$ where $n$ is a positive integer and $\Delta_2$ is a triangle. Suppose that $M_2$ is 3-connected having at least four elements, and let $e$ be an element of $M_2$. Then*

(a)  *$M_2/e$ is 3-connected if and only if either $M/e$ is 3-connected or $M_2 \cong U_{2,4}$;*
(b)  *$M \backslash e$ is 3-connected if and only if either $M_2 \backslash e$ is 3-connected or $e \in \Delta_2$.*

*Hence, if $e$ is essential in $M$, then $e$ is essential in $M_2$; and if $e$ is non-essential in $M$, then $e$ is non-essential in $M_2$ or $e \in \Delta_2$.*

The next result [11, Proposition 4.5] shows that, when the matroid $M_1$ in Theorem 2.2 is 3-connected, every essential element of $M$ that is in $M_1$ is also essential in $M_1$. However, the behaviour of the non-essential elements of $M$ is less straightforward.

**2.4 Proposition.** *Let $M$ be a 3-connected matroid that is not a whirl. Suppose that there is a positive integer $n$ such that*

$$M = P_{\Delta_1}(M(\mathscr{W}_{n+2}), M_1) \backslash z$$

*where $\Delta_1 = \{y_0, y_{n+1}, z\}$ and $\mathscr{W}_{n+2}$ is labelled as in Figure 4. Let $M_1$ be 3-connected and $e$ be an element of $E(M_1) - z$. Then*

(a)  *$M/e$ is 3-connected if and only if either $M_1/e$ is 3-connected; or $e \notin \Delta_1$, there is a unique triangle of $M_1$ containing $\{e, z\}$, and $M_1/e \backslash z$ is 3-connected;*
(b)  *$M \backslash e$ is 3-connected if and only if either $M_1 \backslash e$ is 3-connected; or $e \in \Delta_1$ and $e$ is not in a triad of $M$.*

*Hence, if $e$ is essential in $M$, then $e$ is essential in $M_1$. However, if $e$ is non-essential in $M$, then either $e$ is non-essential in $M_1$; or $e \in \Delta_1$ and $e$ is not in a triad of $M$; or $e \notin \Delta_1$, there is a unique triangle of $M_1$ containing $\{e, z\}$, and $M_1/e \backslash z$ is 3-connected.*

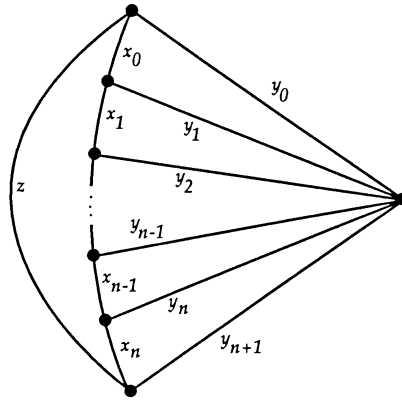The following result [10, p. 323] will be used in the case when $n = 3$.

**Fig. 4.** A labelled wheel.

**2.5 Lemma.** *Let n be an integer exceeding one and M be a matroid having no circuits with fewer than n elements. If $M|X$ and $M|Y$ are n-connected and the closures of X and Y have at least $n-1$ common elements, then $M|(X \cup Y)$ is n-connected.*

The last of our preliminaries is a very useful result of Lemos [5, Theorem 1].

**2.6 Theorem.** *Let M be a 3-connected matroid with at least four elements and C be a circuit of M. If, for all e in C, the matroid $M \backslash e$ is not 3-connected, then C meets at least two distinct triads of M.*

## 3. All Non-Essential Elements are Deletable

In this section, we prove a general result that describes all contraction-minimally 3-connected matroids. In such a matroid $M$, if the set of deletable elements has rank $d$ and $d \le 1$, then, by Theorem 2.1, $d = 0$ and $M$ is a wheel or a whirl. The main result of this section shows that, when $d \ge 2$, the matroid $M$ can be constructed by attaching fans to a rank-$d$ matroid $N_0$ that contains and is spanned by the set of deletable elements of $M$. Moreover, $N_0$ is either contraction-minimally 3-connected or is isomorphic to a 3- or 4-point line.

We shall first present a *constructive description* of a matroid $N$.

(I)     Begin with a 3-connected matroid $N_0$ of rank at least two. We call $N_0$ the *root matroid*.

(II)    Let $N_1, N_2, \ldots, N_k$ be a collection of wheels each having rank at least three such that $E(N_0), E(N_1), \ldots, E(N_k)$ are disjoint.

(III)   Let $\Delta'_1, \Delta'_2, \ldots, \Delta'_k$ be triangles in $N_0$ where these triangles are not necessarily distinct. Let $\Delta''_1, \Delta''_2, \ldots, \Delta''_k$ be triangles in $N_1, N_2, \ldots, N_k$, respectively.

(IV)    For each $i$ in $\{1, 2, \ldots, k\}$, take a bijection from $\Delta''_i$ to $\Delta'_i$ and relabel each element of $\Delta''_i$ by the corresponding element of $\Delta'_i$.

(V)  Construct the sequence of matroids $A_0, A_1, \ldots, A_k$ as follows: let $A_0 = N_0$; and, for all $i$ in $\{1, 2, \ldots, k\}$, let $A_i = P_{A_i'}(N_i, A_{i-1})$.

(VI) Let $V$ be a subset of $\varDelta_1' \cup \varDelta_2' \cup \cdots \cup \varDelta_k'$ such that
   (a)  for all $i \geq 1$, if $N_i$ has rank at least four, then $V$ does not contain a spoke of $N_i$;
   (b)  for all $i \geq 1$, if $N_i$ has rank three, then $V$ contains at most one element of $N_i$;
   (c)  if $k \leq 1$, then $|E(N_0) - V| \geq 5 - k$.

(VII) Let $N = A_k \backslash V$.

The triangles $\varDelta_1', \varDelta_2', \ldots, \varDelta_k'$ of $N_0$ will be called *triangles of attachment*. We now define $\mathcal{N}$ to consist of all ordered pairs the first member of which is a matroid $N$ that has a constructive description, and the second member of which is such a constructive description of $N$. We shall denote a member of $\mathcal{N}$ by the associated matroid $N$ and assume the matroids and sets in its constructive description are labelled as in (I)–(VII).

   The next theorem, the main result of this section, identifies all contraction-minimally 3-connected matroids with a non-empty set of deletable elements. The characterization of all type-1 matroids in Theorem 1.3 will follow quite straightforwardly from this theorem.

**3.1 Theorem.** *A matroid $N$ is a contraction-minimally 3-connected matroid in which the set $D$ of deletable elements has rank $d$ for some $d \geq 2$ if and only if $N$ is a member of $\mathcal{N}$ for which the root matroid $N_0$ has rank $d$ and has the following properties*:

(i)   *$E(N_0)$ contains and is spanned by $D$*;
(ii)  *for every element $e$ of $N_0$,*
   (a)  *$e$ is in some triangle of attachment; or*
   (b)  *$e$ is in a triangle of $N_0$ that is contained in $E(N)$; or*
   (c)  *$N_0/e$ is not 3-connected*;
(iii) *for each $t$ in $\{2, 3\}$, every $t$-cocircuit $C^*$ of $N_0$ meets at least $4 - |C^* - V|$ of the triangles of attachment.*

   Before beginning the proof of this theorem, we first note that, in (iii), 2-cocircuits arise if and only if $N_0 \cong U_{2,3}$. This case is unavoidable and is indeed the only possibility for $N_0$ when $N$ is binary and $d = 2$. Next, we briefly outline the overall strategy of the proof. Suppose that one begins with a contraction-minimally 3-connected matroid $M$ in which the set $D$ of deletable elements has rank $d$. Every element $e$ of $M$ not in the closure of $D$ is essential and so is in a type-1 fan both ends of which are in $D$. We can shrink this fan to a single triangle quite easily: first delete every internal spoke of the fan thereby putting all the rim elements of the fan in series; then contract all but one of these rim elements. By repeating this process for every fan in $M$ with at least six elements and then simplifying the resulting matroid, we obtain a 3-connected matroid $N_0$ such that $E(N_0)$ contains and is spanned by $D$, and $N_0$ has rank 2 or is contraction-minimally 3-connected. It can be shown that $M$ is a member of $\mathcal{N}$ having $N_0$ as its

root. Thus the root is uniquely determined by the initial matroid $M$ up to a possible relabelling of some elements that results from choosing to retain a different element from a parallel or series class that arises in the above construction.

One attractive feature of the above decomposition is that the root $N_0$ has the same rank as the set $D$ of deletable elements of $M$. Now suppose that we alter perspective and begin with a contraction-minimally 3-connected matroid $N_0$ as the root in the construction of a member $N$ of $\mathcal{N}$. To ensure that $N$ is contraction-minimally 3-connected, we need the root $N_0$ to obey (ii) of Theorem 3.1. Every element of $E(N) - E(N_0)$ can be shown to be essential in $N$. But, to guarantee that the set of deletable elements of $N$ contains a basis of $N_0$, we need to ensure that the set $V$ of elements that is deleted in the last step of the construction of $N$ must obey (iii) of Theorem 3.1.

The proof of Theorem 3.1 will require several preliminaries. In particular, we begin with a technical lemma involving the matroid $A_k$ that appears in the construction of a member of $\mathcal{N}$. We shall then move on to an investigation of the properties of the members of $\mathcal{N}$ including the identification of exactly which members of this class are contraction-minimally 3-connected.

**3.2 Lemma.** *For all $m$ in $\{0, 1, 2, \ldots, k\}$,*

$$r(A_m) = \sum_{i=0}^{m} r(N_i) - 2m.$$

*Moreover, the flats of $A_m$ are precisely those subsets $F$ of $\bigcup_{i=0}^{m} E(N_i)$ such that $F \cap E(N_j)$ is a flat of $N_j$ for all $j$ in $\{0, 1, 2, \ldots, m\}$.*

The proof of this lemma follows by a routine induction argument and so is omitted. An immediate consequence of the lemma is that the matroid $A_k$ is independent of the order in which the wheels $N_1, N_2, \ldots, N_k$ are attached to the root matroid $N_0$.

**3.3 Theorem.** *Let $N$ be a member of $\mathcal{N}$. Then*

(i)   *$N$ is 3-connected;*
(ii)  *all elements of $E(N) - E(N_0)$ are essential;*
(iii) *$r(N) = r(A_k)$; and*
(iv)  *$N_0$ is isomorphic to a minor of $N$.*

*Proof.* We show first that $r(N) = r(A_k)$. If $r(N) < r(A_k)$, then $V$ contains a cocircuit $C^*$ of $A_k$. Now $C^*$ meets $\Delta_i'$ for some $i$, but $C^*$ contains at most one element of $\Delta_i'$; a contradiction. Hence we do indeed have $r(N) = r(A_k)$.

To establish that $A_k \backslash V$ is 3-connected, we first partition the set $N_1, N_2, \ldots, N_k$ of wheels so that two such wheels belong to the same class exactly when they have the same triangle of attachment. Now arbitrarily order the classes in this partition. Each element of $V$ belongs to some triangle of attachment and no such triangle contains more than one element of $V$. Associate each element $v$ of $V$ with the last class in the imposed order for which $v$ is in the corresponding triangle of attachment.

We now build up $A_k \backslash V$ as follows. Beginning with the first class of wheels sharing a common triangle of attachment, attach these wheels one by one to $N_0$ until one wheel in the class remains unattached. By Lemma 2.5, the matroid that one has at this stage is 3-connected. Finally, attach the last wheel in the class and, if there is one, delete the element of $V$ associated with the class. By Proposition 2.3(b), the resulting matroid, $Q$ say, is still 3-connected unless the class contains just one wheel, $|E(N_0)| = 3$, and there is an element of $V$ associated with the class. But, in the exceptional case, it follows that every triangle of attachment of $N$ equals $E(N_0)$. Thus $k = 1$ and so, by (VI)(c), $|E(N_0) - V| \geq 4$; a contradiction. We conclude that $Q$ is indeed 3-connected. We may now repeat this process, one by one attaching to $Q$ all the wheels in the second class and, after attaching the last wheel in the class, deleting the element of $V$ associated with the class, if there is one. Doing this for each of the classes in order, we maintain a 3-connected matroid. We conclude that the matroid obtained at the end of this process, namely $A_k \backslash V$, is 3-connected.

Each element of $E(A_k) - E(N_0)$ is in both a triangle and a triad of $A_k$. Since $N = A_k \backslash V$ and $r(N) = r(A_k)$, each of these triads must avoid $V$ otherwise $N$ is not cosimple and so is not 3-connected. Moreover, the restrictions governing the set $V$ guarantee that each element of $E(A_k) - E(N_0)$ is in a triangle of $N$. We conclude that every element of $E(N) - E(N_0)$ is essential in $N$.

Finally, we observe that $N_0$ is certainly a minor of $A_k$. Moreover, $V$ contains at most one element of each $\Delta_i'$. If $v \in V \cap \Delta_k'$, then we may view it as a rim element of $N_k$. Moreover, since $A_k = P_{\Delta_k'}(N_k, A_{k-1})$, by deleting spokes and contracting rim elements from $N_k$, it is easy to see that $A_k \backslash v$ has a minor isomorphic to $A_{k-1}$. By repeatedly applying this idea, we deduce that $N_0$ is isomorphic to a minor of $N$.                                                                                                        □

The next result identifies exactly which members of $\mathcal{N}$ are contraction-minimally 3-connected.

**3.4 Proposition.** *Let $N$ be a member of $\mathcal{N}$. Then $N$ is contraction-minimally 3-connected if and only if, for every element $e$ of the 3-connected root matroid $N_0$,*

(i)   *$e \in \Delta_i'$ for some $i$; or*
(ii)  *$e$ is in a triangle of $N_0$ avoiding $V$; or*
(iii) *$N_0/e$ is not 3-connected.*

*Proof.* Suppose that, for every element $e$ of $E(N_0)$, one of (i)–(iii) holds. By Theorem 3.3, every element of $E(N) - E(N_0)$ is essential in $N$. All remaining elements of $N$ lie in $E(N_0) - V$. Take such an element $e$. If (i) holds, then $N/e$ breaks up as a 2-sum and so is not 3-connected. If (ii) holds, then $N/e$ has a 2-circuit and so is not 3-connected. Finally, suppose that (iii) holds. In this case, showing that $N/e$ is not 3-connected will require more effort. Assume the contrary, that $N/e$ is 3-connected. Then neither (i) nor (ii) holds. It follows that, for all $i$, the element $e$ is not in $\mathrm{cl}_{N_0}(\Delta_i')$, so $\Delta_i'$ is a triangle of $N_0/e$. Now $N/e = (A_k \backslash V)/e = (A_k/e)\backslash V$. Moreover, by the definition of $A_0, A_1, \ldots, A_k$ and the properties of the generalized parallel connection, we deduce that, for all $i$ in $\{1, 2, \ldots, k\}$, the contraction $A_i/e$

equals $P_{\Delta_i'}(N_i, A_{i-1}/e)$ where $A_0 = N_0$. Since $\widetilde{N_0/e}$ is not 3-connected, $N_0/e$ has a vertical 2-separation $\{S, T\}$. Let $K = \{1, 2, \ldots, k\}$. Clearly, for every $i$ in $K$, exactly one of $|\Delta_i' \cap S|$ and $|\Delta_i' \cap T|$ exceeds one. Let $K_S = \{i \in K : |\Delta_i' \cap S| > 1\}$ and $K_T = K - K_S$. Also let $S' = S \cup [\bigcup_{i \in K_S}(E(N_i) - E(N_0))]$ and $T' = T \cup [\bigcup_{i \in K_T}(E(N_i) - E(N_0))]$. Then $r_{A_k/e}(S') = r_{A_k/e}(S) + \sum_{i \in K_S}[r(N_i) - 2]$ and $r_{A_k/e}(T') = r_{A_k/e}(T) + \sum_{i \in K_T}[r(N_i) - 2]$. Hence

$$r_{A_k/e}(S') + r_{A_k/e}(T') = r_{A_k/e}(S) + r_{A_k/e}(T) + \sum_{i=1}^{k}[r(N_i) - 2].$$

But, by assumption, $r_{N_0/e}(S) + r_{N_0/e}(T) = r(N_0/e) + 1$. Moreover, $N_0/e$ is a restriction of $A_k/e$ and $r(A_k/e) = r(N_0/e) + \sum_{i=1}^{k}[r(N_i) - 2]$. Thus $\{S', T'\}$ is a vertical 2-separation of $A_k/e$. Hence the simplification of $A_k/e$ is not 3-connected. Since $N/e$ is 3-connected, $A_k/e \backslash V$ is 3-connected. As $r(A_k/e \backslash V) = r(A_k/e)$, we can adjoin elements of $V$ to $A_k/e \backslash V$ until we get a matroid isomorphic to the simplification of $A_k/e$. This matroid must be 3-connected; a contradiction. We conclude that, in case (iii), $N/e$ is not 3-connected. This completes the proof that no element of $N$ is contractible, that is, $N$ is contraction-minimally 3-connected.

Conversely, suppose that $N$ is contraction-minimally 3-connected and let $e$ be an element of $N_0$ for which neither (i) nor (ii) holds. Then $e \notin \bigcup_{i=1}^{k}\text{cl}_{N_0}(\Delta_i')$. Hence $\Delta_i'$ is a triangle of $N_0/e$ for all $i$ in $\{1, 2, \ldots, k\}$. We need to show that $\widetilde{N_0/e}$ is not 3-connected. Assume the contrary. By assumption, $N_0/e$ has no 2-circuits avoiding $V$. Let $V'$ be a minimal subset of $V$ for which $N_0/e \backslash V'$ is simple. Then this matroid is certainly 3-connected. For all $t$ in $\{1, 2, \ldots, k\}$, if $v \in V' \cap \Delta_t'$, then relabel $v$ in $N_t$ by the element of $N_0/e \backslash V'$ to which it is parallel in $N_0/e$. Now $N/e$ can be constructed as follows: for all $i$ in $\{1, 2, \ldots, k\}$, let $A_i/e \backslash V' = P_{\Delta_i'}(N_i, A_{i-1}/e \backslash V')$, and let $N/e = (A_k/e \backslash V') \backslash (V - V')$. By Theorem 3.3(i), since $N_0/e \backslash V'$ is 3-connected and (vi) holds with $N_0/e \backslash V'$ and $V - V'$ replacing $N_0$ and $V$, respectively, we deduce that $N/e$ is in $\mathcal{N}$ and hence is 3-connected; a contradiction. We conclude that $\widetilde{N_0/e}$ is not 3-connected, thereby completing the proof of Proposition 3.4.                                                                                    □

The following result is an immediate consequence of the last proposition.

**3.5 Corollary.** *If a matroid $N$ in $\mathcal{N}$ is contraction-minimally 3-connected, then the root matroid $N_0$ is contraction-minimally 3-connected or has rank 2.*

We remark that, as can be deduced from Proposition 3.4, the converse of the last result is not true. For example, let $N_0$ be the 7-element rank-3 matroid shown in Figure 5, let $N_1$ be a 3-wheel with a triangle labelled by $\{1, 2, 3\}$, let $\Delta_1' = \{1, 2, 3\}$, and let $V = \{2\}$. Then, for $N = P_{\Delta_1'}(N_1, N_0) \backslash V$, the matroid $N/4$ is 3-connected, so $N$ is not contraction-minimally 3-connected although $N_0$ is.

Next we consider the problem of identifying the deletable elements in a member $N$ of $\mathcal{N}$. Theorem 3.3 implies that the set of deletable elements is contained in $E(N_0)$. We should like to ensure that the set of deletable elements spans $E(N_0) \cap E(N)$ so that each contraction-minimally 3-connected member of $\mathcal{N}$ can be built
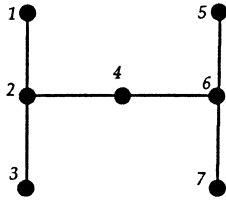
**Fig. 5.** $N_0$

from a unique root. This will involve imposing an additional condition on $N_0$. This condition has the effect of guaranteeing that distinct wheels that are attached to $N_0$ produce distinct fans and never coalesce into a single, larger fan. The following elementary lemma will be used in the proof of the next result.

**3.6 Lemma.** *If $t \in \{1, 2, \ldots, k\}$ and $A_k$ has a cocircuit $C^*$ that meets $\Delta_t'$, then $C^*$ meets $E(N_t) - \Delta_t'$.*

*Proof.* Since $C^* \cap \Delta_t'$ is non-empty, $|C^* \cap \Delta_t'| \geq 2$. Moreover, it is easy to see that some triangle of the wheel $N_t$ contains exactly one element of $C^* \cap \Delta_t'$. This triangle, which is also a triangle of $A_k$, must contain another element of $C^*$.        □

**3.7 Lemma.** *Let $N$ be a matroid in $\mathcal{N}$ and suppose that the root matroid $N_0$ has the property that, for each $t$ in $\{2, 3\}$, every $t$-cocircuit $C^*$ of $N_0$ meets at least $4 - |C^* - V|$ of the triangles of attachment. If $w \in \Delta_i' - V$ for some $i$ in $\{1, 2, \ldots, k\}$, then $N \backslash w$ is 3-connected.*

*Proof.* Assume that $N \backslash w$ is not 3-connected. As $w \in \Delta_i'$, either

(i)  $\widetilde{N/w}$ is not 3-connected; or
(ii) $k = 1$ and $r(N_0) = 2$.

But, in the latter case, since $|E(N_0) - V| \geq 4$, the matroid $N_0 \backslash V$ is a line with at least four points and so $N \backslash w$ is 3-connected; a contradiction.

We may now assume that (ii) does not hold. Thus (i) holds and so, by a result of Bixby [1] (see also [9, Proposition 8.4.6]), the cosimplification of $N \backslash w$ is 3-connected. Since $N \backslash w$ is not 3-connected, $w$ must be in some triad $T^*$ of $N$. But $N = A_k \backslash V$. Thus $A_k$ has a cocircuit $C^*$ that contains $T^*$ and is contained in $T^* \cup V$. Clearly $C^*$ meets $\Delta_i'$ so, by Lemma 3.6, $C^* - w$ and hence $T^* - w$ meets $E(N_i) - \Delta_i'$. Now either

(a) $(T^* - w) \cap (\Delta_i' - V)$ is non-empty; or
(b) $|V \cap \Delta_i'| = 1$.

In case (a), let $u$ be an element of $(T^* - w) \cap (\Delta_i' - V)$. Then the cosimplification of $N \backslash w$ equals the cosimplification of $N \backslash w / u$ and, since $u \in \Delta_i'$, is not 3-connected; a contradiction. Thus we may assume that (a) fails and that $|V \cap \Delta_i'| = 1$. Let $V \cap \Delta_i' = \{v_i\}$.

Consider the number $\tau(w)$ of indices $j$ such that $\Delta_j'$ contains $w$. As $T^* - w$ must

contain an element of $E(N_j) - \varDelta'_j$ for each such $j$, it follows that $\tau(w) \leq 2$. The rest of the proof of this lemma will consider separately the cases when $\tau(w) = 1$ and when $\tau(w) = 2$.

Suppose first that $\tau(w) = 1$. Then $i$ is the unique index $j$ such that $w \in \varDelta'_j$. By Theorem 3.3, $A_k \backslash (V - \{v_i\}) \backslash (E(N_i) - \varDelta'_i)$ is 3-connected. This matroid has a cocircuit $T_1^*$ contained in $(T^* \cup \{v_i\}) - (E(N_i) - \varDelta'_i)$. But $T^* - w$ meets $E(N_i) - \varDelta'_i$. Thus $T_1^*$ has size at most three and so it is a triad of the 3-connected matroid $A_k \backslash (V - \{v_i\}) \backslash (E(N_i) - \varDelta'_i)$. Therefore $T_1^* \cup (V - \{v_i\})$ contains a cocircuit $C^*$ of $A_k \backslash (E(N_i) - \varDelta'_i)$ containing $T_1^*$.

Suppose that $C^*$ contains an element $v'$ of $V - \{v_i\}$. Then $v'$ is in some $\varDelta'_p$ and $\varDelta'_p \neq \varDelta'_i$. But $V - \{v_i\}$ contains at most one element of $\varDelta'_p$, yet $C^*$ must contain at least two elements of $\varDelta'_p$. Hence $\varDelta'_p$ meets $T_1^*$. But $w \notin \varDelta'_p$ by assumption, and $v_i \notin \varDelta'_p$. Hence $y \in \varDelta'_p$ where $T_1^* = \{w, v_i, y\}$. But $y$ is in a triangle of $N_p \backslash v'$. This triangle is also a triangle of $A_k \backslash (V - \{v_i\}) \backslash (E(N_i) - \varDelta'_i)$, but it meets the triad $T_1^*$ in just one element; a contradiction. We conclude that $C^*$ avoids $V - \{v_i\}$. Hence $C^* = T_1^*$.

Now $N_0 = A_k \backslash [\bigcup_{j=1}^k (E(N_j) - \varDelta'_j)]$, so $T_1^*$ is a union of cocircuits of $N_0$. Thus either $T_1^*$ is a triad of $N_0$, or $N_0 \cong U_{2,3}$. But, in the latter case, we obtain that $r(N_0) = 2$ and $k = 1$ contradicting the fact that (ii) does not hold. Therefore $T_1^*$ is a triad of $N_0$. We know that $w$ is in exactly one triangle of attachment, namely $\varDelta'_i$. It follows that $\varDelta'_i$ is the only triangle of attachment to meet $T_1^*$ since, for every triangle $\varDelta'_j$ meeting $T_1^*$, there is an element of $T_1^*$ in $E(N_j) - \varDelta'_j$. Thus we have a contradiction to the condition that every triad $C^*$ of $N_0$ meets at least $4 - |C^* - V|$ triangles of attachment.

Next we consider the case when $\tau(w) = 2$. We know that $w \in \varDelta'_i$. Let $\varDelta'_h$ be the other triangle of attachment containing $w$. Since (b) above must hold for the triangle $\varDelta'_h$, we may suppose that $V \cap \varDelta'_h = \{v_h\}$, where $v_h$ and $v_i$, like $\varDelta'_h$ and $\varDelta'_i$, may or may not be distinct. By orthogonality, $T^* \cup \{v_i, v_h\}$ is a cocircuit of $A_k \backslash (V - \{v_i, v_h\})$. Thus $A_k$ has a cocircuit $C^*$ that contains $T^* \cup \{v_i, v_h\}$ and is contained in $T^* \cup V$. But, since $T^*$ avoids $E(N_t) - \varDelta'_t$ unless $t \in \{i, h\}$, Lemma 3.6 implies that $C^*$ avoids $V - \{v_i, v_h\}$. Hence $C^* = T^* \cup \{v_i, v_h\}$. On deleting $\bigcup_{j=1}^k (E(N_i) - \varDelta'_i)$ from $A_k$, we obtain $N_0$. Moreover, since $T^*$ meets each of $E(N_i) - \varDelta'_i$ and $E(N_h) - \varDelta'_h$, it follows that $|C^* \cap E(N_0)| = |C^*| - 2$. But $C^* \cap E(N_0)$ is a union of cocircuits of $N_0$. Thus $N_0$ has a cocircuit $C_0^*$ of size at most three that is contained in $C^* \cap E(N_0)$ and contains $w$ such that $|C_0^* - V| = 1$ and $C_0^*$ meets only two triangles of attachment. This cocircuit violates the condition on the cocircuits of $N_0$.  $\square$

**3.8 Proposition.** *Let $N$ be a matroid in $\mathcal{N}$ and suppose that the root matroid $N_0$ has the property that, for each $t$ in $\{2, 3\}$, every $t$-cocircuit $C^*$ of $N_0$ meets at least $4 - |C^* - V|$ of the triangles of attachment. Then the set of deletable elements of $N$ is a subset of $E(N_0)$ of rank $r(N_0)$.*

*Proof.* By Lemma 3.7, for all $i$ in $\{1, 2, \ldots, k\}$, every element of $\varDelta'_i - V$ is deletable. Hence the set of deletable elements of $N$ certainly spans $\mathrm{cl}_N(\bigcup_{i=1}^k (\varDelta'_i - V))$.

By Theorem 3.3, every element of $E(N) - E(N_0)$ is essential in $N$. To complete

the proof of the theorem, it suffices to show that every cocircuit $D^*$ of $N_0$ that is contained in $E(N_0) - \mathrm{cl}_{N_0}(\bigcup_{i=1}^{k} \Delta_i')$ contains a deletable element of $N$. If $D^* \cap V$ is non-empty, then, by orthogonality in $N_0$, the set $D^*$ meets $\Delta_i' - V$ for some $i$. Thus we may assume that $D^* \cap V = \varnothing$. Hence $D^* \subseteq E(N)$.

If $d \in D^*$ and $N_0 \backslash d$ is 3-connected, then $N \backslash d \in \mathcal{N}$ and so, by Theorem 3.3, $N \backslash d$ is 3-connected. Thus we may assume that $N_0 \backslash d$ is not 3-connected for all $d$ in $D^*$. As $N$ is contraction-minimally 3-connected, Proposition 3.4 implies that, for every element $d$ of $D^*$, either

(i)   $d$ is in a triangle of $N_0$ avoiding $V$; or
(ii)  $\widetilde{N_0/d}$ is not 3-connected.

If (ii) holds for all $d$ in $D^*$, then, for all such elements $d$, the matroid $N_0/d$ is not 3-connected. Thus, by the dual of Theorem 2.6, $N_0$ has a triangle meeting $D^*$. Since this also holds if (i) occurs, we may now assume that $N_0$ has a triangle $T$ that meets $D^*$. Then $|T \cap D^*| \geq 2$, so we may choose distinct elements $d_1$ and $d_2$ of $D^*$ that are also in $T$. Let $w$ be the third element of $T$. Since neither $N_0 \backslash d_1$ nor $N_0 \backslash d_2$ is 3-connected, Tutte's Triangle Lemma [15] (see also [9, Lemma 8.4.9]) implies that $N_0$ has a triad containing $d_1$ and exactly one of $d_2$ and $w$, and $N_0$ has a triad containing $d_2$ and exactly one of $d_1$ and $w$. If $N_0$ has a triad containing both $d_1$ and $d_2$, then the cocircuit condition on $N_0$ implies that this triad must meet $\Delta_i'$ for some $i$. By orthogonality, $d_1$ or $d_2$ is in $\Delta_i'$; a contradiction. Thus $N_0$ has distinct triads $T_1^*$ and $T_2^*$ meeting $T$ in $\{d_1, w\}$ and $\{d_2, w\}$, respectively. For each $i$ in $\{1, 2\}$, let $T_i^* - T = \{e_i\}$. By the cocircuit condition on $N_0$, the triad $T_1^*$ meets some $\Delta_j'$. Hence $\Delta_j'$ contains $\{w, e_1\}$. Thus, as $w \in \Delta_j'$ but $d_2 \notin \Delta_j'$, we must have that $e_2 \in \Delta_j'$. Hence $\Delta_j' = \{e_1, e_2, w\}$. Let $X = \{d_1, d_2, e_1, e_2, w\}$. Then, in $N_0$, we have $r(X) + r^*(X) - |X| \leq 1$. It follows, since $N_0$ is 3-connected, that $|E(N_0)| = 6$. Hence $|D^*| = 3$ and so $D^*$ meets some triangle of attachment; a contradiction.   $\square$

We are now ready to prove the main result of this section.

*Proof of Theorem 3.1.* First suppose that $N$ is a member of $\mathcal{N}$ for which the root matroid $N_0$ has rank $d$ and conditions (i)–(iii) hold. Then, by Proposition 3.4, since (ii) holds, $N$ is contraction-minimally 3-connected. Moreover, by Proposition 3.8 and Theorem 3.3(b), since (iii) holds, the set $D$ of deletable elements of $N$ is a subset of $E(N_0) - V$ of rank $r(N_0)$. Thus $N$ is a contraction-minimally 3-connected matroid in which the set of deletable elements has rank $d$.

To prove the converse, we now suppose that $M$ is a contraction-minimally 3-connected matroid in which the set $D$ of deletable elements has rank $d$ for some $d \geq 2$. We shall show that $M \in \mathcal{N}$ by constructing a root matroid $N_0$ and a family of wheels that are attached to this root in the prescibed manner.

Every element $e$ of $M$ that is not in $\mathrm{cl}_M(D)$ is in a fan. Since $M$ has no contractible elements, this fan must be of type-1 having both ends in $D$. As $e$ is not in $\mathrm{cl}_M(D)$, this fan has at least five elements. Moreover, by Theorem 2.1, this fan is unique unless $M$ has an $M(K_4)$-restriction having a line $L$ that lies in $\mathrm{cl}_M(D)$ and avoids $e$. Thus, if we take the set of fans of $M$ that contain an element of $E(M) - \mathrm{cl}_M(D)$ and, from each such fan, delete the ends, we obtain a collection of
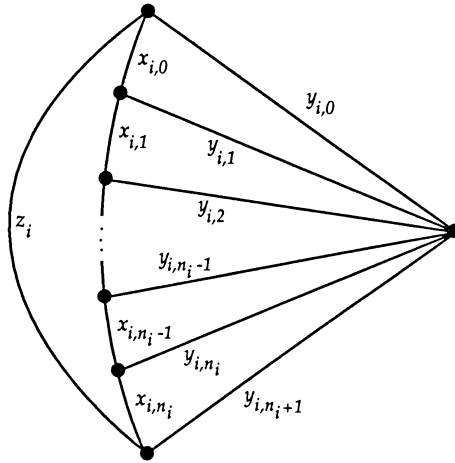
**Fig. 6.** $M(\mathcal{W}_{n_i+2})$

sets any two of which are either equal or disjoint. Look at the distinct sets in this collection and arbitrarily order these sets. For each such set, choose a type-1 fan from which this set arose. Let $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$ be this set of fans. Suppose that $\mathcal{F}_i$ is $\{y_{i,0}, x_{i,0}, y_{i,1}\}, \{x_{i,0}, y_{i,1}, x_{i,1}\}, \ldots, \{y_{i,n_i}, x_{i,n_i}, y_{i,n_i+1}\}$. Then, for all $j \neq i$, we have $E(\mathcal{F}_i) \cap E(\mathcal{F}_j) \subseteq \{y_{i,0}, y_{i,n_i+1}\}$.

Now construct the matroids $M_0, M_1, \ldots, M_m$ as follows: let $M_m = M$; and, for all $i$ with $1 \leq i \leq m$, let $M_{i-1}$ be the matroid $M_i \backslash Y_i / X_i$ with the element $x_{i,n_i}$ relabelled as $z_i$ where $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}\}$ and $X_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,n_i-1}\}$. For all $i$ in $\{1, 2, \ldots, m\}$, let $\Delta_i = \{y_{i,0}, z_i, y_{i,n_i+1}\}$ and let $N_i$ be the matroid $M(\mathcal{W}_{n_i+2})$ where the wheel is labelled as in Figure 6.

The next lemma contains the core of the argument that $M \in \mathcal{N}$. Its statement is somewhat awkward. This arises from the fact that, to construct the triangles of attachment, one must shrink fans down to a single triangle spanned by the ends of the fan. When two fans with the same ends are shrunk, parallel elements may result.

**3.9 Lemma.** *For all $i$ in $\{1, 2, \ldots, m\}$,*

(i)   $M_i = P_{\Delta_i}(N_i, M_{i-1}) \backslash z_i$;

(ii)  $\mathcal{F}_j$ *is a chain of $M_i$ for all $j \leq i$;*

(iii) *in $M_{i-1}$, every 2-circuit avoids $Y_j \cup X_j \cup x_{j,n_j}$ for all $j \leq i-1$; moreover, if $Z'_{i-1}$ is the subset of $\{z_i, z_{i+1}, \ldots, z_m\}$ consisting of those elements $z_j$ that are parallel in $M_{i-1}$ either to some element of $E(M)$ or to $z_p$ for some $p > j$, then $M_{i-1} \backslash Z'_{i-1}$ is 3-connected and is isomorphic to $\widetilde{M_{i-1}}$;*

(iv)  $\Delta_j$ *is a triangle of $M_{i-1}$ for all $j \geq i$; and*

(v)   $D$ *is a subset of $E(M_{i-1} \backslash Z'_{i-1})$; moreover, either $M_{i-1} \backslash Z'_{i-1} \cong U_{2,3}$ or $U_{2,4}$; or no element of $M_{i-1} \backslash Z'_{i-1}$ is contractible and the set of non-essential elements of this matroid is spanned by $D$.*

*Proof.* We shall prove all five parts simultaneously using an induction argument on $m - i$. For $m - i = 0$, part (ii) holds by assumption; parts (i), (iv), and the second part of (iii) are immediate consequences of Theorem 2.2. To establish the first part of (iii), we note that a 2-circuit of $M_{m-1}$ must contain $z_m$. If it meets $Y_j \cup X_j \cup x_{j,n_j}$ for some $j \le m - 1$, then $M_m$ has a circuit containing $x_{m,n_m}$, some subset of $X_m$, and some element of $Y_j \cup X_j \cup x_{j,n_j}$ for some $j \le m - 1$. This circuit meets some triad of the fan $\mathscr{F}_j$ in a single element; a contradiction. Hence (iii) holds when $m - i = 0$.

To prove (v) when $m - i = 0$, we note that, by Theorem 2.2, either (a) $M = P_{\Delta_m}(N_m, M_{m-1}) \backslash z_m$ and $M_{m-1}$ is 3-connected; or (b) $z_m$ is in a 2-circuit $\{z_m, h_m\}$ of $M_{m-1}$; the matroid $M_{m-1} \backslash z_m$ is 3-connected; and $M = P_{\Delta'_m}(N'_m, M_{m-1} \backslash z_m)$ where $N'_m$ is $N_m$ with $z_m$ relabelled as $h_m$, and $\Delta'_m = \{y_{m,0}, y_{m,n_m+1}, h_m\}$. By assumption, $M$ has no contractible elements and the set of deletable elements of $M$ is $D$, which contains $\{y_{m,0}, y_{m,n_m+1}\}$. Evidently $D \subseteq E(M_{i-1} \backslash Z'_{i-1})$. Assume that (a) holds and suppose first that $r(M_{m-1}) = 2$. Then either $M_{m-1} \cong U_{2,3}$ or $U_{2,4}$; or $|E(M_{m-1})| \ge 5$, $M_{m-1}$ has no contractible elements, and the set of deletable elements of $M_{m-1}$ is $E(M_{m-1})$ which is spanned by $\{y_{m,0}, y_{m,n_m+1}\}$, a subset of $D$. Thus (v) holds in case (a) when $r(M_{m-1}) = 2$. Suppose next that $r(M_{m-1}) > 2$. Then, by Proposition 2.4, since $M$ has no contractible elements, $M_{m-1}$ has no contractible elements except possibly for $z_m$. But this element is in a triangle of $M_{m-1}$ and $r(M_{m-1}) > 2$, so $z_m$ is also not contractible. Likewise, by Proposition 2.4, the set of deletable elements of $M_{m-1}$ is contained in $D \cup z_m$ and so is spanned by $D$. Thus (v) holds in case (a) when $r(M_{m-1}) > 2$.

In case (b), $M_{m-1} \backslash Z'_{m-1} = M_{m-1} \backslash z_m$. By Proposition 2.3, every deletable element of $M_{m-1} \backslash Z'_{m-1}$ is in $D$. Moreover $M_{m-1} \backslash Z'_{m-1} \cong U_{2,3}$ or $U_{2,4}$; or $M_{m-1} \backslash Z'_{m-1}$ has no contractible elements. Thus (v) holds when $m - i = 0$.

Now assume that (i)–(v) hold for $m - i \le p - 1$ and let $m - i = p \le m - 1$. Thus $i = m - p$. We show first that (ii) holds in this case. By definition, $M_i$ is $M_{i+1} \backslash Y_{i+1} / X_{i+1}$ with $x_{i+1,n_{i+1}}$ relabelled as $z_{i+1}$. Hence, for all $j \le i$, all the elements of $\mathscr{F}_j$ are elements of $M_i$. If $T$ is a link of $\mathscr{F}_j$ that is a triangle of $M_{i+1}$, then $T$ is a triangle of $M_i$ unless $M_{i+1}$ has a circuit of the form $T' \cup X'_{i+1}$ where $T'$ is a proper non-empty subset of $T$, and $X'_{i+1}$ is a non-empty subset of $X_{i+1}$. But, in the exceptional case, by orthogonality, $T' \cup X'_{i+1}$ must contain $X_{i+1} \cup x_{i+1,n_{i+1}}$; a contradiction since $x_{i+1,n_{i+1}} \notin T$. Hence each link of $\mathscr{F}_j$ that is a triangle of $M_{i+1}$ is also a triangle of $M_i$. Now let $T$ be a link of $\mathscr{F}_j$ that is a triad of $M_{i+1}$. Then $T$ is a triad of $M_i$ unless $M_{i+1}$ has a cocircuit of the form $T' \cup Y'_{i+1}$ where $T'$ is a proper non-empty subset of $T$, and $Y'_{i+1}$ is a non-empty subset of $Y_{i+1}$. In the exceptional case, by orthogonality, $T' \cup Y'_{i+1}$ must contain $Y_{i+1} \cup \{y_{i+1,0}, y_{i+1,n_{i+1}+1}\}$; a contradiction since no triad of $\mathscr{F}_j$ contains $y_{i+1,0}$. We conclude that every link of $\mathscr{F}_j$ that is a triad of $M_{i+1}$ is also a triad of $M_i$. Hence $\mathscr{F}_j$ is a chain of $M_i$ for all $j \le i$; that is, (ii) holds when $m - i = p$.

We shall show next that $M_{i-1}$ is loopless. Assume, to the contrary, that $M_{i-1}$ has a loop. Then $M_i \backslash Y_i / X_i$ has a loop, $e$ say. By (iii) of the induction assumption, $M_i$ is loopless. Hence $M_i$ has $e \cup X'_i$ as a circuit for some non-empty subset $X'_i$ of $X_i$. We showed above that $\mathscr{F}_i$ is a chain of $M_i$. Since $e \notin Y_i$, it follows, by orthogonality, that $X'_i = X_i$ and $e = x_{i,n_i}$. By a sequence of circuit eliminations begin-

ning with the circuit $e \cup X_i$, using each of the triangles of $\mathscr{F}_i$ in order, and exploiting orthogonality after each elimination, we deduce that $M_i$ has a circuit that is contained in $\{y_{i,0}, y_{i,n_i+1}\}$. This contradiction to (iii) of the induction assumption completes the proof that $M_{i-1}$ is loopless.

By the induction assumption, $\widetilde{M}_i$ is isomorphic to $M_i \backslash Z'_i$ and is 3-connected. Moreover, $Z'_i \subseteq \{z_{i+1}, z_{i+2}, \dots, z_m\}$. Thus $M_i \backslash Z'_i$ has $\mathscr{F}_j$ as a chain for all $j \leq i$. Since $M_i \backslash Z'_i$ is 3-connected, Theorem 2.2 implies that

$$(1) \qquad\qquad M_i \backslash Z'_i = P_{\Delta_i}(N_i, M'_{i-1}) \backslash z_i$$

where $M'_{i-1}$ is $(M_i \backslash Z'_i) \backslash Y_i / X_i$ with $x_{i,n_i}$ relabelled as $z_i$, and $N_i$ is $M(\mathscr{W}_{n_i+2})$ where the wheel is labelled as in Figure 6. Thus $\Delta_i$ is a triangle of $M'_{i-1}$. Moreover, either

(a) $M'_{i-1}$ is 3-connected; or
(b) $M'_{i-1}$ has a 2-circuit $C$ containing $z_i$, and $M'_{i-1} \backslash z_i$ is 3-connected.

In the latter case, by the same argument that we used to prove the first part of (iii) when $m - i = 0$, we deduce that $C$ is disjoint from $Y_j \cup X_j \cup x_{j,n_j}$ for all $j \leq i - 1$.

By definition, $M_{i-1}$ is $M_i \backslash Y_i / X_i$ with $x_{i,n_i}$ relabelled as $z_i$. Thus $M'_{i-1} = M_{i-1} \backslash Z'_i$, and so $\Delta_i$ is a triangle of $M_{i-1}$. We showed above that $M_{i-1}$ is loopless. Therefore, if $\pi$ is a parallel class of $M_i$ and $\pi \subseteq \{z_{i+1}, z_{i+2}, \dots, z_m\}$, then the elements of $\pi$ are parallel in $M_{i-1}$. Moreover, if $\pi' \subseteq \{z_{i+1}, z_{i+2}, \dots, z_m\}$ and the elements of $\pi'$ are parallel in $M_i$ to some element $e$ of $E(M)$, then, by the induction assumption, $\{e\}$ is disjoint from $Y_j \cup X_j \cup x_{j,n_j}$ for all $j \leq i$. Thus $e \cup \pi' \subseteq E(M_{i-1})$ and the elements of $e \cup \pi'$ are parallel in $M_{i-1}$. Thus every non-trivial parallel class of $M_i$ is contained in a parallel class of $M_{i-1}$. Hence $Z'_i \subseteq Z'_{i-1}$. In case (a), $M_{i-1} \backslash Z'_i$ is 3-connected and is isomorphic to $\widetilde{M}_{i-1}$; hence $Z'_{i-1} = Z'_i$. In case (b), $M_{i-1} \backslash (Z'_i \cup z_i)$ is 3-connected and is isomorphic to $\widetilde{M}_{i-1}$; hence $Z'_{i-1} = Z'_i \cup z_i$. The observation concerning the circuit $C$ and the induction assumption imply that, in both cases, every 2-circuit of $M_{i-1}$ avoids $Y_j \cup X_j \cup x_{j,n_j}$ for all $j \leq i - 1$. We conclude that (iii) holds when $m - i = p$. Furthermore, since, in $M_{i-1}$, each element of $Z'_i$ is parallel to an element of $M_{i-1}$ to which it is parallel in $M_i$, we deduce that $M_i = P_{\Delta_i}(N_i, M_{i-1}) \backslash z_i$; that is, (i) holds for $i = m - p$.

We have already noted that $\Delta_i$ is a triangle of $M_{i-1}$. To complete the proof that (iv) holds when $i = m - p$, we need to show that $\Delta_j$ is a triangle of $M_{i-1}$ for all $j \geq i + 1$. By the induction assumption, for all such $j$, the set $\Delta_j$ is a triangle of $M_i$. Since $E(\mathscr{F}_j) \cap E(\mathscr{F}_i) \subseteq \{y_{i,0}, y_{i,n_i+1}\}$, every such triangle $\Delta_j$ avoids $Y_i \cup X_i \cup x_{i,n_i}$ and so is a union of circuits of $M_{i-1}$. But, since we have shown that (iii) holds for $i = m - p$, it follows that $M_{i-1}$ is loopless with all its 2-circuits meeting $\{z_i, z_{i+1}, \dots, z_m\}$. Hence $\Delta_j$ is a triangle of $M_{i-1}$ for all $j \geq i - 1$. Therefore (iv) holds when $m - i = p$.

Finally, to show that (v) holds when $m - i = p$, we note that, by (1), $M_i \backslash Z'_i = P_{\Delta_i}(N_i, M'_{i-1}) \backslash z_i$. Since $i \geq 1$, it follows by (iii) and (v) of the induction assumption that $M_i \backslash Z'_i$ is 3-connected having no contractible elements and having its set of non-essential elements spanned by $D$. As noted above, either

(a) $M'_{i-1}$ is 3-connected, in which case, $Z'_{i-1} = Z'_i$; or
(b) $M'_{i-1} \backslash z_i$ is 3-connected, in which case, $Z'_{i-1} = Z'_i \cup z_i$.

Now, precisely the same argument that was used to establish (v) when $m - i = 0$ completes the proof of (v) when $m - i = p$. □

**3.10 Corollary.** *For all $i$ in $\{0, 1, \ldots, m\}$,*

(i) $r(M) = r(M_i) + \sum_{j=i+1}^{m} n_j$; *and*
(ii) $r_{M_i}(D) = r_M(D)$.

*Moreover, $D$ spans $M_0$.*

*Proof.* Let $B_D$ be a basis for $D$ in $M$ and consider $B_D \cup (\bigcup_{j=1}^{m} X_j)$. Clearly this set spans $M$. Suppose that it contains a circuit $C$. If $C$ meets $X_j$ for some $j$ in $\{1, 2, \ldots, m\}$, then, by orthogonality using the triads of the chain $\mathscr{F}_j$, we deduce that $C$ contains $X_j \cup x_{j, n_j}$. But $x_{j, n_j} \notin B_D$ so $C$ avoids $X_j$. Hence $C \subseteq B_D$; a contradiction. We conclude that $B_D \cup (\bigcup_{j=1}^{m} X_j)$ is a basis for $M$. Thus $r(M) = r(M_m) = r(D) + \sum_{j=1}^{m} |X_j| = r(D) + \sum_{j=1}^{m} n_j$. But, for all $j$ in $\{1, 2, \ldots, m\}$, we have $r(M_j) = r(M_{j-1}) + |X_j|$ by Lemma 3.9(i). It follows easily that, for all $i$ in $\{0, 1, \ldots, m\}$,

$$r(M) = r(M_m) = r(M_i) + \sum_{j=i+1}^{m} |X_j| = r(M_i) + \sum_{j=i+1}^{m} n_j.$$

Thus $r(D) = r(M_0)$, so $D$ spans $M_0$. Moreover,

$$r_{M_i}(D) = r_M \left( D \cup \left( \bigcup_{j=i+1}^{m} X_j \right) \right) - r_M \left( \bigcup_{j=i+1}^{m} X_j \right)$$

$$= r_M \left( B_D \cup \left( \bigcup_{j=i+1}^{m} X_j \right) \right) - r_M \left( \bigcup_{j=i+1}^{m} X_j \right)$$

$$= r_M(B_D) = r_M(D). \quad \square$$

We now complete the proof of Theorem 3.1. First note that, by repeatedly applying Lemma 3.9(i), we deduce that

$$M_m = P_{\Delta_m}(N_m, P_{\Delta_{m-1}}(N_{m-1}, \ldots, P_{\Delta_1}(N_1, M_0) \ldots)) \backslash \{z_1, z_2, \ldots, z_m\}.$$

By Lemma 3.9, the matroid $M_0 \backslash Z_0'$ is 3-connected and $\Delta_i$ is a triangle of $M_0$ for all $i$ in $\{1, 2, \ldots, m\}$. For all such $i$, let $\Delta_i'$ be the three elements of $M_0 \backslash Z_0'$ that are parallel in $M_0$ to some element of $\Delta_i$. Relabel the elements of the triangle $\Delta_i$ of $N_i$ with the corresponding elements of $\Delta_i'$. Then, for $j \neq i$, we have $E(N_j) \cap E(N_i) \subseteq \Delta_j' \cap \Delta_i'$. Construct the sequence of matroids $A_0, A_1, \ldots, A_m$ by letting $A_0 = M_0 \backslash Z_0' = N_0$ and, for all $i$ in $\{1, 2, \ldots, m\}$, letting $A_i = P_{\Delta_i'}(N_i, A_{i-1})$.

Let $V = E(M_0 \backslash Z_0') \cap \{z_1, z_2, \ldots, z_m\}$. Clearly $M = A_m \backslash V$, and $V$ satisfies (vi). Moreover, by Corollary 3.10, $r_M(D) = r(M_0)$. Since $r(M_0 \backslash Z_0') = r(M_0)$, we deduce that $r_M(D) = r(N_0)$. We conclude that $M \in \mathscr{N}$ and its set of deletable elements has rank $r(N_0)$. Since, by assumption, $M$ is contraction-minimally 3-connected, it follows, by Proposition 3.4, that every element $e$ of $N_0$ obeys one of (ii)(a)–(c) of Theorem 3.1.

Finally, we need to show that $N_0$ satisfies (iii) of Theorem 3.1. Let $C^*$ be a $t$-cocircuit of $N_0$ for some $t$ in $\{2, 3\}$ such that $C^*$ meets at most $3 - |C^* - V|$ of the triangles of attachment.

Suppose first that $t = 2$. Then $N_0 \cong U_{2,3}$ and $|C^* - V|$ is 2 or 1. In the former case, $C^*$ meets at most one of the triangles of attachment, so $k \leq 1$ and $M$ is $U_{2,3}$ or a wheel; a contradiction. In the latter case, as $|C^* \cap V| = 1$, the cocircuit $C^*$ must meet one or two of the triangles of attachment. Indeed, by (VI)(c), $k = 2$ and $M$ is obtained by taking the generalized parallel connection of two wheels across a triangle, and then deleting an element of this triangle that is a rim element of both wheels. Thus $M$ is a wheel; a contradiction.

We may now assume that $t = 3$, that is, $C^*$ is a triad of $N_0$. Since each element of $V$ is in a triangle of attachment, it is a straightforward consequence of orthogonality that $V$ cannot contain $C^*$. Hence we need only consider the cases when $|V \cap C^*|$ is 0, 1, or 2. In the first case, $C^*$ avoids $\Delta_i'$ for all $i$. Thus $C^*$ avoids $\text{cl}_{N_0}(\Delta_i')$ for all $i$, and so $C^*$ is a triad of $A_m$. Hence $C^*$ is a union of cocircuits of $A_m \backslash V$. The last matroid, which equals $M$, is 3-connected. Hence $C^*$ is a triad of $M$. Therefore, for all $e$ in $C^*$, the matroid $M \backslash e$ is not 3-connected. Thus the set of deletable elements of $M$ is contained in $E(N_0) - C^*$ and so has rank less than $r(N_0)$; a contradiction. Thus $|V \cap C^*| \neq 0$.

Suppose that $|V \cap C^*| = 1$. Then $C^*$ meets some $\Delta_i'$ in an element $v_i$ of $V$ and in some other element $u_i$. Moreover, $C^*$ meets no other triangle of attachment. Now $\Delta_i'$ is a triangle of the wheel $N_i$ and this wheel has a triad $T_i^*$ containing $\{u_i, v_i\}$. It follows from properties of the generalized parallel connection that $T_i^* \cup C^*$ is a cocircuit of $A_m$. Therefore the 3-element set $(T_i^* \cup C^*) - v_i$ is a union of cocircuits of $M$. Hence this set is a triad of $M$ containing $u_i$. But $u_i$ is a fan end in $M$ and so is deletable in $M$, a contradiction. Thus $|V \cap C^*| \neq 1$.

It remains to consider the case when $|V \cap C^*| = 2$. Then $C^*$ meets some $\Delta_i'$ and $\Delta_j'$ in distinct elements $v_i$ and $v_j$, respectively, of $V$. Moreover, these are the only triangles of attachment meeting $C^*$. If $r_{N_0}(\Delta_i' \cup \Delta_j') = 2$, then the line of $N_0$ that contains $\Delta_i' \cup \Delta_j'$ has at least four points and so contains $C^*$. Thus $N_0 \cong U_{2,4}$ and it follows that $k = 2$ and that $M$ is a whirl; a contradiction. We may now assume that $r_{N_0}(\Delta_i' \cup \Delta_j') = 3$. Then $C^* = \{v_i, v_j, x\}$ where $x$ is the unique element in $\Delta_i' \cap \Delta_j'$. Now the wheels $N_i$ and $N_j$ have triads $T_i^*$ and $T_j^*$ that meet $\Delta_i'$ and $\Delta_j'$ in $\{x, v_i\}$ and $\{x, v_j\}$, respectively. Thus $T_i^* \cup T_j^* \cup C^*$ is a cocircuit of $A_m$. Therefore $(T_i^* \cup T_j^* \cup C^*) - \{v_i, v_j\}$ is a triad of $M$ containing $x$. But $x$ is a fan end in $M$ and so is deletable in $M$; a contradiction. This completes the proof that $N_0$ satisfies (iii) and thereby finishes the proof of Theorem 3.1.                                      □

Next we shall show how Theorem 1.3 can be deduced from Theorem 3.1.

*Proof of Theorem 1.3.* Suppose $M$ is a 3-connected matroid having exactly two non-essential elements, $x$ and $y$, each of which is deletable. Then $M$ is contraction-minimally 3-connected whose set of deletable elements has rank 2. Thus, by Theorem 3.1, $M$ is a member of $\mathcal{N}$ in which the root $N_0$ has rank 2 and contains $\{x, y\}$. Suppose $|E(N_0) \cap E(M)| \geq 4$. Since, for each element $e$ of $(E(N_0) \cap E(M)) - \{x, y\}$, the matroid $M \backslash e$ is not 3-connected, Tutte's Triangle Lemma [15] implies that $M$ has a triad contained in $E(N_0) \cap E(M)$ which implies that $M \cong$

$U_{2,4}$; a contradiction. Thus $|E(N_0) \cap E(M)| \leq 3$. If $|E(N_0) \cap E(M)| = 2$, then $E(N_0) \cap E(M) = \{x, y\}$. In that case, we may view $x$ and $y$ as spokes of each $N_i$. Moreover, $k \geq 3$ otherwise $M$ is a wheel or whirl. Hence, if $|E(N_0) \cap E(M)| = 2$, then $M$ can be constructed by following steps (i)–(vi) of Theorem 1.3. It remains to consider the case when $|E(N_0) \cap E(M)| = 3$. If this occurs, then, by (VI)(c) in the constructive description of $M$ as a member of $\mathcal{N}$, it follows that $k \geq 2$. Thus, if $z$ is the element of $(E(N_0) \cap E(M)) - \{x, y\}$, then the simplification of $M/z$ is not 3-connected. Hence the cosimplification of $M \backslash z$ is 3-connected. But $M \backslash z$ is not 3-connected since $z \notin \{x, y\}$, so $M$ has a triad containing $z$. This triad must also contain $x$ or $y$, contradicting the fact that each of these elements is deletable.

Conversely, suppose that $M$ is constructed as described in (i)–(vi) of Theorem 1.3. Then $M$ is a member of $\mathcal{N}$ in which the root $N_0$ has rank 2 and is spanned by $\{x, y\}$, and so (ii)(a) of Theorem 3.1 holds for all elements $e$ of $N_0$. Moreover, as $k \geq 3$, it follows that (iii) of Theorem 3.1 holds. Hence, by Theorem 3.1, $M$ is contraction-minimally 3-connected and its set of deletable elements has rank 2 and is contained in $E(N_0) \cap E(M)$. Thus $x$ and $y$ are the only deletable elements of $M$ and the theorem holds.                                                                    □

## 4. One Deletable and One Contractible Element

In this section, we shall prove the characterization of type-3 matroids given in Theorem 1.4. First, recall that a type-3 matroid is a 3-connected matroid having exactly two non-essential elements, one of which is deletable and one of which is contractible.

A big step in proving Theorem 1.4 is to identify the type-3 matroids in which all the fans are trivial, that is, have exactly four elements. For all $k \geq 2$, let $\mathcal{N}_k$ be the class of such matroids that have exactly $k$ fans.

**4.1 Proposition.** *For all $k \geq 2$, the class $\mathcal{N}_k$ coincides precisely with the class of matroids $M$, other than the 3-wheel and the 3-whirl, that can be obtained from a $(k + 1)$-spike by deleting an element other than the tip.*

We remark that the exclusion of the 3-wheel and the 3-whirl in the last result only takes effect in the case when $k = 2$.

*Proof.* First, suppose that $M$ is neither a 3-wheel nor a 3-whirl and that $M$ can be obtained from a $(k + 1)$-spike $S_{k+1}$ by deleting an element $z$ that is different from the tip, $y$. It is straightforward to check that $M$ is 3-connected. Let $\{x, y, z\}$ be the triangle of $S_{k+1}$ that contains $\{y, z\}$, and let $\{y, a_1, b_1\}, \{y, a_2, b_2\}, \ldots, \{y, a_k, b_k\}$ be the other $k$ triangles through the tip. Then these $k$ triangles are the only triangles of $M$ unless $k = 2$. But, even in the exceptional case, since $M$ is not a 3-wheel or a 3-whirl, it has no other triangles. The only triads of $M$ are the $k$ sets of the form $\{x, a_i, b_i\}$. It now follows easily that $M$ has exactly $k$ fans, each set of the form $\{x, y, a_i, b_i\}$ being the ground set of such a fan. Thus $M \in \mathcal{N}_k$.

Now suppose that $N$ is a member of $\mathcal{N}_k$. Let the $k$ fans in $N$ be $\mathcal{F}_1, \mathcal{F}_2, \ldots,$ $\mathcal{F}_k$ and suppose that the links of the fan $\mathcal{F}_i$ are $\{y, a_i, b_i\}$ and $\{x, a_i, b_i\}$, the first

of which is a triangle and the second a triad. Evidently $E(N) = \bigcup_{i=1}^{k} E(\mathscr{F}_i)$, so $|E(M)| = 2k + 2$. Deleting the $k$ triads $\{x, a_1, b_1\}, \{x, a_2, b_2\}, \ldots, \{x, a_k, b_k\}$ from $M$ reduces the rank by at least $k$ and leaves just the element $y$. Thus $r(M) \geq k + 1$. Dually, $r^*(M) \geq k + 1$. Since $|E(M)| = 2k + 2$, it follows that both $M$ and its dual have rank $k + 1$. Now $M \backslash x$ has $\{a_i, b_i\}$ as a cocircuit for all $i$. Thus $M \backslash x$ is the parallel connection of $k$ three-point lines, $L_1, L_2, \ldots, L_k$, with common basepoint $y$ where $L_i = \{y, a_i, b_i\}$ for all $i$.

To show that $M$ can be obtained from a spike by deleting a non-tip element, we construct a single-element extension of $M$ that is a spike. Let $M^+$ be obtained from $M$ by freely adding the element $z$ to the line $\{x, y\}$ of $M$. Let $L_0 = \{x, y, z\}$. Then, in $M^+$, the $k + 1$ lines $L_0, L_1, \ldots, L_k$ all have three points and all pass through $y$. Moreover, their union is $E(M^+)$ and so has rank $k + 1$. Now take a set $\mathscr{X}$ of $m$ of the lines $L_0, L_1, \ldots, L_k$ where $m \leq k$ and choose $j$ such that $L_j \notin \mathscr{X}$. If $L_0 \notin \mathscr{X}$, the union of the $m$ lines in $\mathscr{X}$ has rank $m + 1$. If $L_0 \in \mathscr{X}$, then the union of the $m - 1$ lines in $\mathscr{X} - \{L_0\}$ has rank $m$. Since this union avoids the cocircuit $\{x\} \cup (L_j - \{y\})$, the union of all of the lines in $\mathscr{X}$ has rank $m + 1$. Hence $M^+$ is indeed a spike. $\qquad\square$

In order to describe all type-3 matroids, we shall want to shrink a non-trivial fan down to a trivial fan. The next result describes how this can be done.

**4.2 Lemma.** *Suppose $k \geq 2$. Let $M$ be a type-3 matroid with exactly $k$ fans and $x$ and $y$ be the two non-essential elements where $x$ is contractible and $y$ is deletable. Every fan of $M$ has $x$ and $y$ as its ends. Let*

$$\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \ldots, \{x_n, y_{n+1}, x_{n+1}\}$$

*be a fan with at least six elements, where $y = y_0$ and $x = x_{n+1}$. Then*

$$M = P_{\Delta_1}(M(\mathscr{W}_{n+2}), M_1) \backslash z$$

*where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathscr{W}_{n+2}$ is labelled as in Figure 4; and $M_1$ is 3-connected and is obtained from the matroid $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n$ by relabelling $x_n$ as $z$. Moreover,*

(i) *if $M$ is binary, then $M_1$ is also binary;*
(ii) *if $k \geq 3$, then $M_1$ is a type-3 matroid with exactly $k$ fans;*
(iii) *if $k = 2$, then $M_1$ is a wheel, or a whirl, or a type-3 matroid with exactly two fans; and*
(iv) *if $M_1$ is a type-3 matroid, then $\Delta_1$ is the unique triangle of a trivial fan of $M_1$.*

*Proof.* Consider the given fan and omit the last link $\{x_n, y_{n+1}, x_{n+1}\}$ from this fan. This leaves a chain to which Theorem 2.2 can be applied. Hence

$$M = P_{\Delta_1}(M(\mathscr{W}_{n+2}), M_1) \backslash z$$

where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathscr{W}_{n+2}$ is labelled as in Figure 4; and $M_1$ is the matroid $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n$ with $x_n$ relabelled as $z$. Moreover, $M_1$ is 3-connected, or $M_1 \backslash z$ is 3-connected. If the latter occurs, then we deduce that

$M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n, x_n$ is 3-connected. But $M$ has $\{x_n, y_{n+1}, x_{n+1}\}$ as a cocircuit, so $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n, x_n$ has a cocircuit contained in $\{y_{n+1}, x_{n+1}\}$. This matroid has at least four elements since $M$ has at least two fans. Thus $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n, x_n$ cannot be 3-connected. We conclude that $M_1 \backslash z$ is not 3-connected. Hence $M_1$ is 3-connected.

If $M$ is binary, then $M_1$ is also binary since it is isomorphic to a minor of $M$. Next we prove that $M_1$ has at most two non-essential elements. By Proposition 2.4, if $e \notin \{x, y, z\}$, then, as $e$ is essential in $M$, it is also essential in $M_1$. Moreover, it is easy to see that $\{z, x_{n+1}, y_{n+1}\}$ is a triad in $M_1$. Therefore, $z$ is in both a triangle and a triad in $M_1$, and thus it is essential. We conclude that $M_1$ has at most two non-essential elements. Thus $M_1$ is not uniform and, since it is 3-connected, $M_1$ has at least six elements.

Now we show that (ii)–(iv) hold. The element $y$ is in at least $k$ triangles of $M_1$. Thus either (a) $k = 2$ and $M_1$ is a wheel or a whirl, or (b) $M_1$ is not a wheel or a whirl. Hence we may assume that the latter holds. Therefore, by Theorem 2.1, $M_1$ has exactly two non-essential elements, namely, $x$ and $y$. Since $x$ is in a triad and $y$ is in a triangle, we conclude that $x$ is contractible and $y$ is deletable in $M_1$. Thus $M_1$ is a type-3 matroid. Since $x = x_{n+1}$ and $y = y_0$, we conclude that $\{y_0, z, y_{n+1}\}$, $\{z, y_{n+1}, x_{n+1}\}$ is a maximal chain and therefore a type-3 fan of $M_1$. Therefore $\Delta_1$ is the unique triangle of a trivial fan of $M_1$. As $M_1$ is the matroid $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n$ with $x_n$ relabelled as $z$, it is straightforward to check, by using orthogonality, that $M_1$, like $M$, has exactly $k$ fans. We conclude that (ii)–(iv) hold. $\qquad\square$

Next, for all $k \geq 2$, we construct a class $\mathcal{R}_k$ of matroids as follows. Start with a matroid $N_0$ that can be obtained from a $(k + 1)$-spike by deleting an element $z$ other than the tip $y$. We shall call $N_0$ the *root matroid*. $N_0$ has exactly $k$ triangles, $\Delta_1, \Delta_2, \ldots, \Delta_k$, that contain $y$. Suppose $t \leq k$. Let $N_1, N_2, \ldots, N_t$ be a collection of wheels, each having rank at least three such that $E(N_0), E(N_1), \ldots, E(N_t)$ are all disjoint. Let $\Delta_1', \Delta_2', \ldots, \Delta_t'$ be triangles of $N_1, N_2, \ldots, N_t$, respectively. For all $i$ in $\{1, 2, \ldots, t\}$, take a bijection from $\Delta_i$ to $\Delta_i'$ that maps $y$ to a spoke of $N_i$, and relabel the elements of each $\Delta_i'$ by the corresponding elements of $\Delta_i$ so that $z_i$ labels the element of $\Delta_i'$ that is a rim element of $N_i$. Construct a sequence of matroids as follows: let $R_0 = N_0$, and, for each $i$ in $\{1, 2, \ldots, t\}$, let $R_i = P_{\Delta_i}(N_i, R_{i-1})$. Finally, let $M = R_t \backslash z_1, z_2, \ldots, z_t$. The class $\mathcal{R}_k$ consists of all matroids $M$ that can be constructed in this way.

By a similar argument to that given in Lemma 3.2, it is not difficult to show that $R_t$ is independent of the order in which the wheels $N_1, N_2, \ldots, N_t$ are attached to the root matroid $N_0$. Thus $R_t$ is well-defined, although it should not be over-looked that it does depend on the bijections that map each $\Delta_i$ onto the corresponding $\Delta_i'$.

The next lemma shows that, for all $k \geq 2$, every member of $\mathcal{R}_k$ that is not a wheel or whirl is a type-3 matroid.

**4.3 Lemma.** *Suppose $k \geq 2$ and $t \in \{0, 1, \ldots, k\}$. Let $M$ be a member $R_t \backslash z_1, z_2, \ldots, z_t$ of $\mathcal{R}_k$. Then $N_0$ is a minor of $M$ and either*

(i) $k = 2$ *and M is a wheel or whirl; or*
(ii) *M is a type-3 matroid having exactly k fans, the ground sets of which are* $(E(N_1) \cup \{x\}) - \{z_1\}$, $(E(N_2) \cup \{x\}) - \{z_2\}, \ldots, (E(N_t) \cup \{x\}) - \{z_t\}$, $\varDelta_{t+1} \cup \{x\}, \varDelta_{t+2} \cup \{x\}, \ldots, \varDelta_k \cup \{x\}$.

*Proof.* If $t = 0$, then $R_t = N_0$ and the result follows by Proposition 4.1. Now suppose the lemma is true for $t < n$. Then, for $t = n \geq 1$, by elementary properties of the generalized parallel connection (see, for example, [9, Proposition [12.4.14]]),

$$M = R_n \backslash z_1, z_2, \ldots, z_n$$
$$= P_{\varDelta_n}(N_n, R_{n-1}) \backslash z_1, z_2, \ldots, z_n$$
$$= P_{\varDelta_n}(N_n, R_{n-1} \backslash z_1, z_2, \ldots, z_{n-1}) \backslash z_n.$$

Now write $M_1$ for $R_{n-1} \backslash z_1, z_2, \ldots, z_{n-1}$. By the induction assumption, $N_0$ is a minor of $M_1$ and either

(i) $k = 2$ and $M_1$ is a wheel or a whirl; or
(ii) $M_1$ is a type-3 matroid having exactly $k$ fans, the ground sets of which are $(E(N_1) \cup \{x\}) - \{z_1\}$, $(E(N_2) \cup \{x\}) - \{z_2\}, \ldots, (E(N_{n-1}) \cup \{x\}) - \{z_{n-1}\}$, $\varDelta_n \cup \{x\}, \varDelta_{n+1} \cup \{x\}, \ldots, \varDelta_k \cup \{x\}$.

It follows easily that $N_0$ is a minor of $M$. Moreover, in case (i), it is not difficult to see that $M$ is also a wheel or a whirl. Thus we may assume that (ii) holds. By Lemma 2.5, since both $N_n$ and $M_1$ are 3-connected having at least four elements, $P_{\varDelta_n}(N_n, M_1)$ is 3-connected. As the simplification of $P_{\varDelta_n}(N_n, M_1)/z_n$ is not 3-connected, it follows that the cosimplification of $P_{\varDelta_n}(N_n, M_1) \backslash z_n$ is 3-connected. But $P_{\varDelta_n}(N_n, M_1) \backslash z_n$ is cosimple unless there is a triad in $P_{\varDelta_n}(N_n, M_1)$ containing $z_n$. Such a triad contains exactly two elements of $\varDelta_n$. Thus its intersection with one of $E(N_n)$ and $E(M_1)$ has size two, so $N_n$ or $M_1$ is not cosimple; a contradiction. We conclude that $P_{\varDelta_n}(N_n, M_1) \backslash z_n$ is indeed cosimple. Hence $M$ is 3-connected.

Next we consider the collections of triangles and triads of $M$. Evidently the triangles of $M$ consist of the triangles of $M_1$ that avoid $z_n$ along with the triangles of $N_n$ that avoid $z_n$. In particular, every element of $E(M) - \{x\}$ is in a triangle of $M$. A set that is a triad of $N_n$ or of $M_1$ and that avoids $\varDelta_n$ is a triad of $M$. In particular, every element of $E(M) - \varDelta_n$ is in a triad of $M$. Moreover, $M_1$ has a triad $\{x_n, z_n, x\}$ contained in $\varDelta_n \cup \{x\}$, and $N_n$ has a triad $\{e, x_n, z_n\}$ containing $\{x_n, z_n\}$. Thus $P_{\varDelta_n}(N_n, M_1)$ has $\{e, x_n, z_n, x\}$ as a cocircuit, so $M$ has $\{e, x_n, x\}$ as a triad. We conclude that every element of $E(M) - \{y\}$ is in a triad of $M$.

We now know that every element of $E(M) - \{x, y\}$ is essential in $M$, that $y$ is in a triangle, and that $x$ is in a triad. Indeed, $y$ is in at least $k$ triangles. Thus, if $k \geq 3$, then $M$ is certainly not a wheel or a whirl, so $M$ is a type-3 matroid in which $y$ is deletable and $x$ is contractible. Moreover, the ground sets of the fans of $M$ are as specified in (ii) of the theorem. If $k = 2$ and $M$ is not a wheel or a whirl, then $y$ is deletable and $x$ is contractible in $M$; and $M$ has exactly two fans whose ground sets are $(E(N_1) \cup \{x\}) - \{z_1\}$ and $\varDelta_2 \cup \{x\}$, or $(E(N_1) \cup \{x\}) - \{z_1\}$ and $(E(N_2) \cup \{x\}) - \{z_2\}$. $\qquad \square$

**4.4 Lemma.** *Let $M$ be a type-3 matroid having exactly $k$ fans for some $k \geq 2$. Then $M \in \mathscr{R}_k$.*

*Proof.* Let $M$ be a counterexample to the lemma having the minimum number of elements. Let $x$ be the unique contractible element of $M$ and $y$ be the unique deletable element. By Theorem 2.1, the set of essential elements of $M$ can be partitioned into $k$ classes such that two elements are in the same class if and only if they are in a common fan. Let $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$ be the fans of $M$. Then $E(M) = \bigcup_{i=1}^{k} E(\mathscr{F}_i)$. If, for all $i$ in $\{1, 2, \ldots, k\}$, the fan $\mathscr{F}_i$ has exactly four elements, then $M \in \mathscr{N}_k$ and so $M \in \mathscr{R}_k$; a contradiction. Thus we may assume that there is a fan, say $\mathscr{F}_1$, having at least six elements. Suppose the links of $\mathscr{F}_1$ are $\{y_0, x_0, y_1\}$, $\{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \ldots, \{x_n, y_{n+1}, x_{n+1}\}$ where $y_0 = y$. By Lemma 4.2,

$$M = P_\Delta(M(\mathscr{W}_{n+2}), M_1) \backslash z$$

where $\Delta = \{y_0, y_{n+1}, z\}$; $\mathscr{W}_{n+2}$ is labelled as in Figure 4; and $M_1$ is obtained from the matroid $M/x_0, x_1, \ldots, x_{n-1} \backslash y_1, y_2, \ldots, y_n$ by relabelling $x_n$ as $z$. Moreover, either

(i) $k = 2$ and $M_1$ is a wheel or a whirl; or
(ii) $M_1$ is a type-3 matroid having exactly $k$ fans.

In case (i), $M$ is a wheel or a whirl; a contradiction. Thus (ii) holds. Since $|E(M_1)| < |E(M)|$, we have that $M_1 \in R_k$. Hence we may assume that $M_1 = R_t \backslash z_1, z_2, \ldots, z_t$ for some $R_t$ constructed as above and some $t \leq k$. Now, by (iv) of Lemma 4.2, $\Delta$ is the unique triangle of a trivial fan of $M_1$. Thus, by Lemma 4.3, $\Delta$ is a triangle of the root $N_0$. Therefore, since

$$M = P_\Delta(M(\mathscr{W}_{n+2}), R_t) \backslash z_1, z_2, \ldots, z_t, z,$$

we obtain the contradiction that $M \in \mathscr{R}_k$. □

To prove Theorem 1.4, we need only to combine the earlier lemmas.

*Proof of Theorem 1.4.* Every 3-connected matroid $M$ with just two non-essential elements, one deletable and one contractible, is a type-3 matroid with $k$ fans for some $k \geq 2$. Thus, by Lemma 4.4, $M \in \mathscr{R}_k$ and so $M$ can be constructed as described in the theorem. Conversely, if $M \in \mathscr{R}_k$, then, by Lemma 4.3, $M$ is a wheel or whirl, or a type-3 matroid with exactly $k$ fans. □

## 5. Consequences

In this section, we look briefly at some of the consequences of the main results of the earlier sections. In particular, we shall prove Theorem 1.2, thereby completing the description of all 3-connected graphic matroids with exactly two non-essential elements. Moreover, we shall determine all 3-connected binary matroids in which the set of non-essential elements has rank two. In the binary case, one can be more explicit than in the general case in describing exactly which matroids arise.

Let $M$ be a binary matroid in which the set $X$ of non-essential elements has rank two. Then, as $M$ is binary, $|X|$ is 2 or 3. In the latter case, either (i) $M \cong U_{2,3}$, or (ii) no element of $X$ is contractible. In case (ii), every element of $X$ is deletable, so $M$ is contraction-minimally 3-connected. If $|X| = 2$, then $M$ is a matroid of type-1, type-2, or type-3. But, if $M$ is of type-1, then it is again contraction-minimally 3-connected. Thus, to find all the possibilities for $M$, it will suffice to determine (a) all binary type-3 matroids and (b) all contraction-minimally 3-connected binary matroids in which the set of deletable elements has rank two and hence has size two or three. The first result of this section notes that all the matroids of type (b) are graphic. In order to state this result, we introduce another class of graphs. For all $k \geq 2$, *a triangle-sum of $k$ wheels* is the graph that can be obtained from $k$ disjoint wheels and a single triangle with edge set $\{x, y, z\}$ by identifying $\{x, y, z\}$ with a triangle in each of the $k$ wheels.

**5.1 Theorem.** *Let $M$ be a binary matroid. Then $M$ is a contraction-minimally 3-connected matroid in which the set of deletable elements has rank two if and only if $M$ is isomorphic to the cycle matroid of an m-dimensional wheel for some $m \geq 3$ or a triangle-sum of n wheels for some $n \geq 2$.*

The proofs of the results in this section will be delayed until all the results have been stated. The results in Section 4 imply that every binary type-3 matroid can be constructed from a matroid that is obtained by deleting a non-tip element from a binary spike. We noted earlier that, for all $k \geq 2$, the unique binary $(k + 1)$-spike is the vector matroid of the matrix $[I_{k+1}|J_{k+1} - I_{k+1}|\mathbf{1}]$ where the column $\mathbf{1}$ of all ones corresponds to the tip of the spike. Let $\Lambda_k$ be the vector matroid of the matrix $[I_{k+1}|A]$ that is obtained from $[I_{k+1}|J_{k+1} - I_{k+1}|\mathbf{1}]$ by deleting the second last column.

**5.2 Lemma.** *Let $M$ be a binary matroid and suppose $k \geq 3$. Then $M$ is a type-3 matroid having exactly $k$ fans each of which is trivial if and only if $M \cong \Lambda_k$.*

Next we describe how to construct all binary type-3 matroids starting from $\Lambda_k$. This construction is a special case of the construction described in Section 4, so we shall be somewhat less formal here. We label the elements of $\Lambda_k$ as follows. If $k$ is odd, label the columns of $[I_{k+1}|A]$, in order, by $b_1, b_2, \ldots, b_k, x, a_1, a_2, \ldots, a_k, y$; if $k$ is even, then interchange $a_k$ and $b_k$ in the above labelling. It is easy to check that, in both cases, (i) $\{a_1, a_2, \ldots, a_k, x\}$ is a circuit; and (ii) $\{y, a_i, b_i\}$ is a triangle and $\{x, a_i, b_i\}$ is a triad for all $i$ in $\{1, 2, \ldots, k\}$. Let $\Delta_i = \{y, a_i, b_i\}$ for all $i$, and let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 \geq n_2 \geq \cdots \geq n_k$. Let $t = \max\{j : n_j > 0\}$. For all $i \leq t$, attach $M(\mathscr{W}_{n_i+2})$ to $\Lambda_k$ across the triangle $\Delta_i$ such that $a_i$ labels a rim element of the attached wheel. From the resulting matroid, delete all the elements $a_i$ for which $i \leq t$ to produce the matroid $M_1(n_1, n_2, \ldots, n_k)$. When $t = k$, the matroid $M_2(n_1, n_2, \ldots, n_k)$ is obtained by modifying the above construction so that $a_1, a_2, \ldots, a_{k-1}$ and $b_k$ are identified with rim elements of the attached wheels, and these $k$ elements are deleted at the last step of the construction. If $n_k = 0$, we define $M_2(n_1, n_2, \ldots, n_k)$ to be equal to $M_1(n_1, n_2, \ldots, n_k)$.

The next theorem characterizes all binary type-3 matroids identifying, in particular, exactly when two such matroids are isomorphic.

**5.3 Theorem.** *Let $M$ be a binary matroid. Then $M$ is of type-3 if and only if $M$ is isomorphic to the cycle matroid of a twisted wheel, or to $M_1(n_1, n_2, \ldots, n_k)$ or $M_2(n_1, n_2, \ldots, n_k)$ for some $k \geq 3$. Moreover, the matroids $M_1(n_1, n_2, \ldots, n_k)$ and $M_2(n_1, n_2, \ldots, n_k)$ are isomorphic if and only if $M$ has at least one trivial fan.*

On combining Theorem 5.1 and Theorem 5.3 and using duality, we immediately obtain the following result that explicitly identifies all 3-connected binary matroids whose set of non-essential elements has rank at most two.

**5.4 Corollary.** *Let $M$ be a 3-connected binary matroid other than a wheel. Then the set of non-essential elements of $M$ has rank one if and only if $M \cong U_{1,3}$, and has rank two if and only if $M$ is isomorphic to one of the following:*

(i)   $U_{2,3}$;
(ii)  *the cycle matroid of a twisted wheel;*
(iii) *the cycle matroid of a triangle-sum of $n$ wheels for some $n \geq 2$;*
(iv)  *the cycle or cocycle matroid of an $m$-dimensional wheel for some $m \geq 3$; or*
(v)   $M_1(n_1, n_2, \ldots, n_k)$ *or* $M_2(n_1, n_2, \ldots, n_k)$ *for some $k \geq 3$.*

The remainder of this section will be devoted to proving the results stated above. We begin by proving Theorem 5.1 thereby specifying all binary matroids of type-1.

*Proof of Theorem 5.1.* If $M$ is isomorphic to the cycle matroid of an $m$-dimensional wheel for some $m \geq 3$ or a triangle-sum of $n$ wheels for some $n \geq 2$, then it is straightforward to check that $M$ is contraction-minimally 3-connected and its set of deletable elements has rank two.

Now suppose that $M$ is contraction-minimally 3-connected and that its set of deletable elements has rank two. Then, by Theorem 3.1, $M$ is a member of $\mathcal{N}$ in which the root $N_0$ has rank two. Moreover, by Theorem 3.3, $N_0$ is isomorphic to a minor of $M$. Hence, as $M$ is binary, $|E(N_0)| \leq 3$. Moreover, since $N_0$ is 3-connected, equality must hold here. From the construction for members of $\mathcal{N}$ and condition (iii) of Theorem 3.1, it follows easily that $M$ is indeed isomorphic to the cycle matroid of an $m$-dimensional wheel for some $m \geq 3$ or a triangle-sum of $n$ wheels for some $n \geq 2$.                                                                          $\square$

We now turn to binary matroids of type-3 first specifying all such matroids in which all fans are trivial.

*Proof of Lemma 5.2.* By Proposition 4.1, it suffices to show that every single-element deletion of the unique binary $(k+1)$-spike $M$ is isomorphic to $\Lambda_k$. Label the columns in the representation $[I_{k+1}|J_{k+1} - I_{k+1}|\mathbf{1}]$ for $M$ by $x_1, x_2, \ldots, x_{k+1}$, $y_1, y_2, \ldots, y_{k+1}, p$. Then the set of non-spanning circuits of $M$ consists of

(i)  all sets of the form $\{p, x_i, y_i\}$ for $1 \leq i \leq k+1$;
(ii) all sets of the form $\{x_i, y_i, x_j, y_j\}$ for $1 \leq i < j \leq k+1$;

(iii)  all sets of the form $\{d_1, d_2, \ldots, d_{k+1}\}$ such that $d_i \in \{x_i, y_i\}$ for all $i$ and there
       are an odd number of elements in $\{d_1, d_2, \ldots, d_{k+1}\} \cap \{y_1, y_2, \ldots, y_{k+1}\}$.

Evidently $M \backslash y_i \cong \Lambda_k$ for all $i$, and $M \backslash x_i \cong M \backslash x_j$ for all $i$ and $j$. Moreover, from
the list of circuits of $M$, it is easy to see that $M$ has an automorphism that inter-
changes $x_1$ with $y_1$, interchanges $x_2$ with $y_2$, and fixes every other element. Thus
$M \backslash y_1 \cong M \backslash x_1 \cong \Lambda_k$ and the lemma follows easily.                              □

We are now ready to prove the characterization of all binary type-3 matroids.

*Proof of Theorem 5.3.* The construction for a twisted wheel described in Section 1
shows that the cycle matroid of this graph is a member of $\mathcal{R}_2$ with the root being a
3-wheel. Thus, by Theorem 1.4, this cycle matroid is of type-3. The same theorem
guarantees that, for all $k \geq 3$, each of $M_1(n_1, n_2, \ldots, n_k)$ and $M_2(n_1, n_2, \ldots, n_k)$ is
of type-3. Moreover, each is certainly binary since the generalized parallel con-
nection of two binary matroids is always binary [2, Corollary 6.13].
    Conversely, suppose that the binary matroid $M$ is of type-3. Let $M$ have
exactly $k$ fans. Then, by Lemma 4.4, $M \in \mathcal{R}_k$. Since the root $N_0$ is a minor of $M$,
Proposition 4.1 and Lemma 5.2 imply that either $k = 2$ and $N_0$ is isomorphic to a
3-wheel, or $k \geq 3$ and $N_0 \cong \Lambda_k$. In the first case, since the generalized parallel
connection of two graphic matroids is graphic [2, Theorem 6.17], it is straightfor-
ward to check that $M$ is the cycle matroid of a twisted wheel. We may now assume
that $k \geq 3$. Let the $k$ fans of $M$ have $2n_1 + 4, 2n_2 + 4, \ldots, 2n_k + 4$ elements, re-
spectively, where $n_1 \geq n_2 \geq \cdots \geq n_k$ and let $t = \max\{j : n_j > 0\}$. Then, by the
symmetry of $\Lambda_k$, we may assume that, in the construction of $M$, the wheel $\mathcal{W}_{n_i+2}$ is
attached across the triangle $\Delta_i$ for all $i$ in $\{1, 2, \ldots, t\}$. To specify $M$ exactly, we
need to indicate, for each $i$, whether $a_i$ or $b_i$ is identified with a rim element of
$\mathcal{W}_{n_i+2}$. Now the circuits of $\Lambda_k$ include all the sets $\Delta_i$ and all the sets $(\Delta_i \cap \Delta_j) - \{y\}$
for $i \neq j$. From the list of circuits of $M[I_{k+1}|J_{k+1} - I_{k+1}|\mathbf{1}]$ given in the proof of
Lemma 5.2 and from the remarks made following the statement of that lemma, we
deduce that the only other non-spanning circuits of $\Lambda_k$ are the sets of the form
$\{d_1, d_2, \ldots, d_k, x\}$ where $d_i \in \{a_i, b_i\}$ for all $i$ and $\{d_1, d_2, \ldots, d_k\}$ contains an even
number of members of $\{b_1, b_2, \ldots, b_k\}$. We deduce that $\Lambda_k$ has an automorphism
that interchanges an even number of $a_i$'s with the corresponding $b_i$'s. Then, by
applying such an automorphism, we may assume that each $\mathcal{W}_{n_i+2}$ is attached to $\Delta_i$
so that $a_i$ is a rim element of the wheel for all $i < k$. Thus, when $t < k$, we deduce
that $M \cong M_1(n_1, n_2, \ldots, n_k)$. When $t = k$, a choice arises as to whether $a_k$ or $b_k$ is
identified with a rim element of $\mathcal{W}_{n_k+2}$, so $M$ is isomorphic to $M_1(n_1, n_2, \ldots, n_k)$ or
$M_2(n_1, n_2, \ldots, n_k)$.
    To complete the proof of the theorem, it remains only to decide when the
matroids $M_1(n_1, n_2, \ldots, n_k)$ and $M_2(n_1, n_2, \ldots, n_k)$ are isomorphic. By definition
they certainly are if $n_k = 0$. We shall now show that when $n_k > 0$, these matroids
are not isomorphic. Abbreviate these two matroids as $M_1$ and $M_2$. Assume that $\varphi$
is an isomorphism between $M_1$ and $M_2$. By Theorem 2.1, every essential element
of $M_1$ and $M_2$ is in a unique type-3 fan. Thus every essential element is in at most
two triangles and two triads. In $M_1$ and $M_2$, the element $y$ is the only element that
is in more than two triangles and $x$ is the only element in more than two triads.

Thus the isomorphism must fix each of these elements. Moreover, because every fan in each matroid has at least six elements, there is a unique element in each fan for which the only triad of the matroid containing this element also contains $x$. In $M_1$ and $M_2$, the collections $U_1$ and $U_2$ of such elements equal $\{b_1, b_2, \ldots, b_k\}$ and $\{b_1, b_2, \ldots, b_{k-1}, a_k\}$, respectively. Under the map $\varphi$, a fan in $M_1$ must map to a fan in $M_2$. Moreover, $\varphi(U_1) = U_2$. Thus $\varphi(\{x\} \cup U_1) = \{x\} \cup U_2$. Now if $k$ is even, then $\{x\} \cup U_1$ is a circuit of $M_1$, but $\{x\} \cup U_2$ is not a circuit of $M_2$; and if $k$ is odd, then $\{x\} \cup U_1$ is not a circuit of $M_1$, but $\{x\} \cup U_2$ is a circuit of $M_2$. In either case, we get a contradiction.                                                                 $\square$

Finally, Theorem 1.2 is quite straightforward to deduce from Corollary 5.4. Alternatively, it is not difficult to prove this theorem directly using Theorem 2.1. We conclude by presenting the first of these proofs.

*Proof of Theorem 1.2.* From the list given in Corollary 5.4 of binary 3-connected matroids whose set of non-essential elements has rank two, we eliminate the matroids under (i) and (iii) as having more than two non-essential elements. If $m \geq 3$, then an $m$-dimensional wheel has $K_{3,3}$ as a minor and so the cocycle matroid of this graph is not graphic. For $k \geq 3$, the matroids $M_1(n_1, n_2, \ldots, n_k)$ and $M_2(n_1, n_2, \ldots n_k)$ have $\Lambda_k$ as a minor and so have $\Lambda_3$ as a minor. But $\Lambda_3$ has the dual of the Fano matroid as a restriction and so is not graphic. The only matroids that remain in the list in Corollary 5.4 are the cycle matroids of twisted wheels and $m$-dimensional wheels for $m \geq 3$, so the theorem is proved.                                     $\square$

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