An upper bound for the circumference of a 3-connected binary matroid

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Submitted: Aug 18, 2022; Accepted: Feb 22, 2023; Published: TBD

Abstract

Jim Geelen and Peter Nelson proved that, for a loopless connected binary matroid M with an odd circuit, if a largest odd circuit of M has k elements, then a largest circuit of M has at most $2k-2$ elements. The goal of this note is to show that, when M is 3-connected, either M has a spanning circuit, or a largest circuit of M has at most $2k-4$ elements. Moreover, the latter holds when M is regular of rank at least four.

Mathematics Subject Classifications: 05B35

1 Introduction

We assume familiarity with matroid theory. Our notation and terminology will follow Oxley [9] except where otherwise indicated. For a positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. A circuit C in a matroid is *even* if $|C|$ is even; otherwise C is *odd*. A binary matroid is *affine* if all of its circuits are even. Let M be a matroid having at least one circuit. The *circumference*, $c(M)$, of M is the cardinality of a largest circuit of M. If M has an odd circuit, its *odd-circumference*, $c_{\text{odd}}(M)$, is the cardinality of a largest odd circuit of M. In a private communication to the second author, Jim Geelen and Peter Nelson proved the following result. The proof appears in [10].

Theorem 1. *Let M be a loopless connected binary matroid. If M is non-affine, then*

$$c(M) \leqslant 2c_{\text{odd}}(M) - 2.$$

The purpose of this note is to prove the next theorem, a refinement of the last result for 3-connected matroids.

Theorem 2. *Let M be a 3-connected binary matroid. If M is non-affine, then either M has a spanning circuit, or*

$$c(M) \leqslant 2c_{\text{odd}}(M) - 4.$$

As we shall show at the end of Section 4, the bound in the last theorem is sharp for infinitely many ranks. In the next section, we note some preliminary results that will be used in the proof of Theorem 2. The proof of the theorem will be given in Section 3. In Section 4, we show how Theorem 1 can be combined with results concerning matroids of small circumference due to Maia [7], Maia and Lemos [8], and Cordovil, Maia, and Lemos [2] to yield results of Oxley and Wetzler [11] and of Chun, Oxley, and Wetzler [1] up to some small-rank matroids. In Section 5, we conjecture a strengthening of Theorem 2 and we prove this conjecture when M is regular.

2 Preliminaries

Seymour [12] gave conditions under which a k-separation of a restriction of a matroid could be extended to a k-separation of the whole matroid. In particular, he proved the following result [12, (3.8)].

Theorem 3. *Let Z be a set in a matroid M and let (P_1, P_2) be a partition of Z. Then either M/Z has a circuit that is not a circuit of M/P_1 or of M/P_2, or $E(M)$ has a partition (X_1, X_2) such that $X_i \cap Z = P_i$ for each i in $\{1, 2\}$ and*

$$r(X_1) + r(X_2) - r(M) = r(P_1) + r(P_2) - r(Z).$$

For a matroid M, recall that $\mathcal{C}(M)$ denotes the set of circuits of M. A subset L of $E(M)$ is a *Tutte-line* of M if $(M|L)^*$ has rank two and has no loops [13]. As Tutte showed and is easily checked, a Tutte-line L has a partition into sets $P_1, P_2, \dots, P_n$ for some $n \geqslant 2$ such that $\mathcal{C}(M|L) = \{L - P_1, L - P_2, \dots, L - P_n\}$. A Tutte-line L is *connected* if $M|L$ is connected or, equivalently, if $n \geqslant 3$.

For a matroid M, a subset $\mathcal{S}$ of $\mathcal{C}(M)$ is a *linear subclass of circuits* of M provided that, for each Tutte-line L of M, either $|\mathcal{C}(M|L) \cap \mathcal{S}| \leqslant 1$, or $\mathcal{C}(M|L) \subseteq \mathcal{S}$. Tutte [13, (4.34)] proved the following result. We shall apply this result here by using the easily verified fact that, when M is binary, the set of all even circuits of M is a linear subclass of circuits of M.

Theorem 4. *Let $\mathcal{S}$ be a linear subclass of circuits of a connected matroid M. If C and D are circuits of M such that $D \notin \mathcal{S}$, then there is a sequence $X_0, X_1, \dots, X_m$ of distinct circuits of M with $X_0 = C$ and $X_m = D$ such that $\{X_1, X_2, \dots, X_m\} \cap \mathcal{S} = \emptyset$ and, for each i in $[m]$, the set $X_{i-1} \cup X_i$ is a connected Tutte-line of M.*

3 Proof of the Main Theorem

Throughout this section, we assume that M is a connected non-affine binary matroid such that

$$c_{\text{odd}}(M) = 2k + 1,$$

for some integer $k \geqslant 1$. By Theorem 1, $c(M) \leqslant 4k$.

Lemma 5. *Let C be a circuit of M such that $|C| = 4k$. If C_1 and C_2 are odd circuits of M such that $C = C_1 \triangle C_2$, then $|C_1| = |C_2| = 2k+1$ and $|C_1 \cap C_2| = 1$.*

Proof. As C_1 and C_2 are not properly contained in C, it follows that $|C_1 \cap C_2| \geqslant 1$. Therefore

$$4k = |C| = |C_1| + |C_2| - 2|C_1 \cap C_2| \leqslant 2(2k+1) - 2 = 4k, \tag{1}$$

so equality holds throughout (1). Thus $|C_1| = |C_2| = 2k+1$ and $|C_1 \cap C_2| = 1$. □

Let C, C_1, C_2 be as in Lemma 5. If $e \in C_1 \cap C_2$, then we say that e is a *good chord* of C having C_1 and C_2 as its *associated circuits*.

Lemma 6. *If C is a circuit of M such that $|C| = 4k$, then there are circuits C_1 and C_2 of M with $C = C_1 \triangle C_2$ such that $|C_1 \cap C_2| = 1$ and $|C_1| = |C_2| = 2k+1$.*

Proof. As noted above, the set $\mathcal{S}$ of even circuits of M is a linear subclass of circuits of M. Choose $D \notin \mathcal{S}$. By Theorem 4, M has a sequence $X_1, X_2, \dots, X_m$ of distinct odd circuits with $X_m = D$ such that $X_{i-1} \cup X_i$ is a connected Tutte-line of M for all i in $[m]$, where $X_0 = C$. Take $C_1 = X_1$ and $C_2 = C_1 \triangle C$. Then $C = C_1 \triangle C_2$. As $|C|$ is even and $|C_1|$ is odd, $|C_2|$ is odd. The result follows from Lemma 5. □

The last lemma can also be proved by applying Lemma 3.2 of [10]. We have presented the proof above to recognize Tutte's contribution to this area.

For a subset F of $E(M)$, an *F-arc* [12, Section 3] is a circuit of M/F that is not a circuit of M. Let C be a circuit of M such that $|C| = 4k$. By Lemma 6, there is a good chord e for C. Note that $\{e\}$ is a C-arc of M. Thus $\{e\}$ is the only C-arc of M containing e. Let A be a C-arc of M. Then there are circuits C_A and D_A of M such that $C_A \cap D_A = A$ and $C_A \triangle D_A = C$. We say that A *crosses* e provided $M|(C \cup A \cup e)$ is obtained from $M(K_4)$ by a sequence of series extensions. In particular, if A crosses e, then $A \neq \{e\}$.

Lemma 7. *For a circuit C of M with $|C| = 4k$, let e be a good chord of C having C_1 and C_2 as its associated circuits. Let A be a C-arc of M and let C_A and D_A be circuits of M such that $C_A \cap D_A = A$ and $C_A \triangle D_A = C$. Then A crosses e if and only if*

$$\emptyset \notin \{C_A \cap C_1, C_A \cap C_2, D_A \cap C_1, D_A \cap C_2\}. \tag{2}$$

Moreover, when A crosses e, the circuit space of $M|(C \cup A \cup e)$, which is spanned by $\{C, C_1, C_A\}$, contains seven non-zero members each of which is the support of a circuit of $M|(C \cup A \cup e)$.

Proof. We may assume that $e \notin A$ otherwise the result follows easily. Let $N = M|(C \cup A \cup e)$ and $\mathcal{P} = \{C_A \cap C_1, C_A \cap C_2, D_A \cap C_1, D_A \cap C_2\}$. As $\{e\}$ and A are circuits of N/C, we see that $N/C = [(N/C)|A] \oplus [(N/C)|\{e\}]$ and $r^*(N/C) = 2$. Thus $r^*(N) = 3$. Moreover, A and $\{e\}$ are series classes of N because C is a circuit of N, and A and $\{e\}$

are series classes of N/C. Now the circuit space of N has dimension 3 and is spanned by $\{C_1, C_A, C\}$. Thus N^* is represented over $GF(2)$ by the matrix whose columns are labelled by the elements of $E(N)$ and whose rows are the incidence vectors of C_1, C_A, and C. One can now check that the parallel classes of N^* are A, $\{e\}$, and the non-empty members of $\mathcal{P}$. We deduce that N^* is isomorphic to a parallel extension of $M(K_4)$ if and only if (2) holds. $\square$

Lemma 8. *For a circuit C of M satisfying $|C| = 4k$, let A be a C-arc of M and let e be a good chord of C. Suppose that C_A and D_A are circuits of M such that $C_A \cap D_A = A$ and $C_A \triangle D_A = C$. If A crosses e, then $|A| = 1$ and $|C_A| = |D_A| = 2k + 1$.*

Proof. As $C_A \triangle D_A = C$, the parities of $|C_A|$ and $|D_A|$ are the same. We may assume that C_A and D_A are even otherwise the result follows by Lemma 5. Let C_1 and D_1 be the associated circuits of e with respect to C. We may assume that

$$|D_1 \cap C_A| \leqslant |C_1 \cap C_A|.$$

By the last part of Lemma 7, $C_A \triangle C \triangle C_1$ is a circuit D of M. Then $D = C_A \triangle D_1$. As C_A is even and D_1 is odd, D is odd. Observe that

$$\begin{aligned}
|D| &= |A| + |\{e\}| + |C_1 \cap C_A| + |(D_1 \cap C) - C_A| \\
&= |A| + 1 + |C_1 \cap C_A| + (|D_1 \cap C| - |D_1 \cap C_A|) \\
&= |A| + 1 + |D_1 \cap C| + (|C_1 \cap C_A| - |D_1 \cap C_A|) \\
&\geqslant |A| + 1 + 2k
\end{aligned}$$

where the last step follows because $|C_1 \cap C_A| \geqslant |D_1 \cap C_A|$. As A is non-empty and D is odd, we have a contradiction to the assumption that $c_{\text{odd}}(M) = 2k + 1$. $\square$

Lemma 9. *Let M be 3-connected. For a circuit C of M having $4k$ elements, let $e_1, e_2, \dots, e_n$ be the good chords of C, and let $N = M|(C \cup \{e_1, e_2, \dots, e_n\})$. For each i in $[n]$, let C_i and D_i be the associated circuits of e_i with respect to C.*

(i) If S be a series class of N contained in C, then, for each i in $[n]$, there is an X_i in $\{C_i, D_i\}$ such that $S = C \cap X_1 \cap X_2 \cap \cdots \cap X_n$.

(ii) Every series class of N is trivial.

Proof. By Lemma 6, $n \geqslant 1$. Clearly the circuit space of N has $\{C, C_1, C_2, \dots, C_n\}$ as a basis. We shall first show *(i)* and use this to deduce *(ii)*.

Let S be a series class of N contained in C. Choose a in S. For each i in $[n]$, there is an X_i in $\{C_i, D_i\}$ such that $a \in X_i$. Hence $S \subseteq X_i$ and so $S \subseteq C \cap X_1 \cap X_2 \cap \cdots \cap X_n$. Let Z be the matrix whose columns are labelled by the elements of $E(N)$ and whose rows are the incidence vectors of $C, X_1, X_2, \dots, X_n$. Then Z represents N^* over $GF(2)$. As $C \cap X_1 \cap X_2 \cap \cdots \cap X_n$ is non-empty, it is a parallel class of N^*. Hence $S = C \cap X_1 \cap X_2 \cap \cdots \cap X_n$, so *(i)* holds.

Now assume that *(ii)* fails, and let S be a series class of N such that $|S| \geqslant 2$. By *(i)*, for each i in $[n]$, there is an X_i in $\{C_i, D_i\}$ such that

$$S = C \cap X_1 \cap X_2 \cap \cdots \cap X_n. \tag{3}$$

Note that $\{S, E(N)-S\}$ is a 2-separation for N. As M is 3-connected, Theorem 3 implies that there is a circuit A of $M/E(N)$ that is not a circuit of M/S or of $M/(E(N)-S)$. As $e_1, e_2, \ldots, e_n$ are loops of M/C, it follows that A is a circuit of M/C. Then there are circuits C_A and D_A of M such that $C_A \cap D_A = A$ and $C_A \triangle D_A = C$. Choose C_A such that $|C_A \cap C| \leqslant 2k$. Now $C_A - E(N) = A$ and $C_A \cap E(N) \neq \emptyset$. As A is not a circuit of M/S, it follows that

$$(C_A \cap C) - S \neq \emptyset.$$

Likewise, as A is not a circuit of $M/(E(N)-S)$,

$$C_A \cap S \neq \emptyset.$$

Take a in $(C_A \cap C) - S$ and s in $S \cap C_A$. Then $a \notin X_j$ for some j in $[n]$. Let $Y_j = X_j \triangle C$. Then $a \in Y_j \cap C_A$ and $s \in X_j \cap C_A$. Hence

$$\emptyset \notin \{X_j \cap C_A, Y_j \cap C_A\}. \tag{4}$$

If $S \subseteq C_A$, then D_A is a circuit of M such that $A \subseteq D_A \subseteq C \cup A$ and $D_A \cap S = \emptyset$. Thus A is a circuit of $M/(E(N)-S)$, a contradiction. Therefore

$$S \nsubseteq C_A. \tag{5}$$

As $|Y_j \cap C| = 2k \geqslant |C_A \cap C|$ and $s \in C_A - Y_j$, it follows that $Y_j \cap D_A = (Y_j \cap C) - C_A \neq \emptyset$. We also have that $X_j \cap D_A = (C - Y_j) - C_A \neq \emptyset$ because $S \subseteq C - Y_j$ and $S \nsubseteq C_A$, by (5). We deduce that

$$\emptyset \notin \{X_j \cap D_A, Y_j \cap D_A\}. \tag{6}$$

By (4), (6), and Lemma 7, A crosses $\{e_j\}$. Thus, by Lemma 8, $|A| = 1$ and $|C_A| = |D_A| = 2k+1$. If $A = \{f\}$, then f is a good chord of C. This is a contradiction as $f \notin \{e_1, e_2, \ldots, e_n\}$. Hence *(ii)* holds. □

Next we prove the main result.

Proof of Theorem 2. We continue to assume that M is a non-affine binary matroid for which $c_{\mathrm{odd}}(M) = 2k+1$. In addition, we assume that M is 3-connected and that C is a circuit of M with $4k$ elements. We shall prove that C spans M. Let $e_1, e_2, \ldots, e_n$ be the good chords of C. By Lemma 6, $n \geqslant 1$. For i in $[n]$, let C_i and D_i be the associated circuits of e_i with respect to C. Let $N = M|(C \cup \{e_1, e_2, \ldots, e_n\})$.

We may assume that $E(M) - E(N) \neq \emptyset$ otherwise the result holds. Let A be a circuit of $M/E(N)$ that is not a circuit of M. Note that A is a circuit of M/C because C spans N. Let C_A and D_A be circuits of M such that $A = C_A \cap D_A$ and $C = C_A \triangle D_A$. Choose C_A such that $|C_A \cap C| \leqslant 2k$.

Suppose that $|C_A \cap C| < |A|$. Then $|D_A| = [|C| - |C_A \cap C|] + |A| > |C| = 4k$, a contradiction. We conclude that

$$|C_A \cap C| \geqslant |A|.$$

Therefore $|C_A \cap C| \geqslant 2$, otherwise $|C_A \cap C| = 1$, so $|A| = 1$ and $|C_A| = 2$, a contradiction.

Let a and b distinct elements of $C_A \cap C$. By Lemma 9(ii), $\{a\}$ is a series class of N. By Lemma 9(i), for each i in $[n]$, there is an X_i in $\{C_i, D_i\}$ such that $\{a\} = C \cap X_1 \cap X_2 \cap \cdots \cap X_n$. Thus $b \notin X_j$ for some j in $[n]$. Take $Y_j = C \triangle X_j$. Thus $C_A \cap X_j \neq \emptyset$ and $C_A \cap Y_j \neq \emptyset$. As $|C_A \cap C| \leqslant 2k = |X_j \cap C| = |Y_j \cap C|$, it follows that $X_j \cap D_A = (X_j \cap C) - C_A \neq \emptyset$ and $Y_j \cap D_A \neq \emptyset$. By Lemma 7, A crosses $\{e_j\}$. By Lemma 8, $|A| = 1$, say $A = \{f\}$, and f is a good chord of C in M, a contradiction. Thus C spans M and the theorem is proved. □

4 Consequences

In this section, we note some implications of Theorem 1. We begin with a quick proof of this theorem based on the following 2007 result of Lemos [4, Corollary 1]. For an element e of a connected matroid M other than $U_{1,1}$, let $c_e(M)$ be the size of a largest circuit of M containing e.

Theorem 10. *Let e be an element of a connected matroid M such that $r(M) \geqslant 3$. If M/e is connected, then*

$$c_e(M) \geqslant \left\lceil \frac{c(M)}{2} \right\rceil + 2.$$

Proof of Theorem 1. As M is non-affine, connected, and loopless, $r(M) \geqslant 2$. Let $[I_r|Z]$ be a binary representation of M. Adjoin a new column to $[I_r|Z]$ labelled by e and a new row consisting entirely of ones. Every entry in the column labelled by e is zero except for the entry in the new row, which is one. Let N be the binary matroid represented by this new matrix. Clearly N is affine. Moreover, since N/e is connected, N is connected otherwise N has e as a loop or a coloop. But e is clearly not a loop of N, and e is not a coloop of N because N/e has an odd circuit K, and $K \cup e$ is a circuit of the affine matroid N. Now

$$\begin{aligned} \{C \in \mathcal{C}(N) : e \in C\} &= \{D \cup e : D \in \mathcal{C}(M) \text{ and } D \text{ is odd}\}, \text{ and} \\ \{C \in \mathcal{C}(N) : e \notin C\} &\supseteq \{D : D \in \mathcal{C}(M) \text{ and } D \text{ is even}\}. \end{aligned}$$

By Theorem 10,

$$c_{\text{odd}}(M) + 1 = c_e(N) \geqslant \lceil \tfrac{c(N)}{2} \rceil + 2 \geqslant \lceil \tfrac{c(M)}{2} \rceil + 2.$$

Therefore $c_{\text{odd}}(M) - 1 \geqslant \frac{c(M)}{2}$ and the theorem follows. □

Theorem 1 was motivated in part by results of Oxley and Wetzler [11] and Chun, Oxley, and Wetzler [1] determining the simple connected binary matroids with odd-circumference three and the 3-connected binary matroids with odd-circumference five. As Theorem 1 bounds the circumference of a matroid in terms of its odd-circumference, the first of these results can be derived from theorems that determine all simple connected matroids of small circumference. Such matroids having circumference in $\{3,4,5\}$ were found by Maia [7] and Maia and Lemos [8]. We begin with the 3-connected case [8, Theorem 1.2]. The matroid Z_5 is the rank-5 binary spike with tip t.

Theorem 11. *Let M be a 3-connected matroid that is not isomorphic to $U_{1,1}$, F_7^*, $AG(3,2)$, $Z_5\backslash\{e,t\}$, or $Z_5\backslash t$, where e is an element of Z_5 other than t. If $r(M) \leqslant 5$, then $c(M) = r(M) + 1$.*

In [5, Theorem 1.5], we proved the following.

Theorem 12. *For a 3-connected matroid M of rank at least six, $c(M) \geqslant 6$.*

A matroid M is an *e-book* if, for some $n \geqslant 1$, there are 3-connected rank-2 matroids $M_1, M_2, \dots, M_n$ and an element e such that $E(M_i) \cap E(M_j) = \{e\}$, when $i \neq j$, and M is $P_e(M_1, M_2, \dots, M_n)$, the parallel connection, with basepoint e, of $M_1, M_2, \dots, M_n$. The next result combines Propositions 1, 3, and 4 and Theorem 8 of Maia [7].

Theorem 13. *Let M be a simple connected matroid that is not 3-connected. If $c(M) \leqslant 5$, then there are matroids N, B_1, and B_2 such that N is 3-connected, B_1 is an e-book, B_2 is an f-book, $E(B_1)$ and $E(B_2)$ are disjoint, $E(B_1) \cap E(N) = \{e\}$, and $E(B_2) \cap E(N) = \{f\}$. Moreover,*

(i) $r(N) \in \{2,3\}$ *and* $M = P_e(N, B_1)\backslash X$, *for some* $X \subseteq \{e\}$; *or*

(ii) N *is isomorphic to* F_7^* *or to* $AG(3,2)$ *and* $M = P_e(N, B_1)\backslash X$, *for some* $X \subseteq \{e\}$; *or*

(iii) $r(N) = 2$ *and* $M = P_f(P_e(N, B_1), B_2)\backslash X$, *for some* $X \subseteq \{e, f\}$.

These results can now be used to prove the following result of Oxley and Wetzler's [11] that determines all connected binary matroids with odd-circumference three. We denote by $K'_{2,n}$ the graph obtained from $K_{2,n}$ by adding an edge joining the vertices in the 2-vertex class.

Theorem 14. *A connected simple binary matroid M has no odd circuits other than triangles if and only if*

(i) M *is affine; or*

(ii) M *is isomorphic to* $M(K_4)$ *or* F_7; *or*

(iii) M *is isomorphic to* $M(K'_{2,n})$ *for some* $n \geqslant 1$.

Proof. If $c_{\text{odd}}(M) = 3$, then, by Theorem 1, $c(M) \leqslant 4$. The theorem now follows from Theorems 11, 12, and 13. $\square$

For $n \geqslant 2$, a binary matroid M is a *book* having *pages* $M_1, M_2, \dots, M_n$ and *spine* T provided that

(i) $M_1, M_2, \dots, M_n$ are binary matroids; and

(ii) $T = E(M_1) \cap E(M_2) \cap \cdots \cap E(M_n)$; and

(iii) $E(M_1) - T, E(M_2) - T, \dots, E(M_n) - T$ are pairwise disjoint sets; and

(iv) T is a triangle of each M_i; and

(v) M is $P_T(M_1, M_2, \dots, M_n)$, the generalized parallel connection across the triangle T of $M_1, M_2, \dots, M_n$.

Theorem 15 (Cordovil, Maia, and Lemos [2])**.** *Let M be a 3-connected binary matroid such that $r(M) \geqslant 8$. Then $c(M) = 6$ if and only if, for $n = r(M) - 2$, there is a book M' with pages $M_1, M_2, \dots, M_n$ and spine T such that each M_i is isomorphic to $M(K_4)$ or F_7, and $M = M' \backslash S$ for some $S \subseteq T$.*

This theorem can be combined with Theorem 2 to prove the following result of Chun, Oxley, and Wetzler [1] for matroids of rank at least eight. Note that all of the matroids described in (iii) of this theorem attain the bound in Theorem 2.

Theorem 16. *A 3-connected binary matroid M has no odd circuits of size exceeding five if and only if*

(i) M is affine; or

(ii) $r(M) \leqslant 5$; or

(iii) M is obtained from an n-page book, whose pages are isomorphic to $M(K_4)$ or F_7, for some $n \geqslant 4$ by deleting up to two elements of its spine; or

(iv) M has rank six and is one of nine non-regular matroids.

Proof for $r(M) \geqslant 8$. By Theorem 14, we may assume that $c_{\text{odd}}(M) = 5$. Then, by Theorem 2, either $c(M) = r(M) + 1$, or $c(M) \leqslant 6$. If $c(M) = r(M) + 1$, then, by Theorem 1, $r(M) \leqslant 7$, a contradiction. If $c(M) \leqslant 6$, then, by Theorem 12, $c(M) = 6$ and the theorem follows by Theorem 15. $\square$

One may hope to be able to bound the circumference of a binary matroid in terms of the maximum size of an even circuit. But this is not possible. For positive integers c and d such that $c < d$ and d is odd, Lemos, Reid, and Wu [6, Theorem 1.3] describe all connected binary matroids such that

$$\{c, d\} = \{|C| : C \in \mathcal{C}(M)\}.$$

Their results show that the number c must be even. Moreover, it is possible to construct these matroids keeping c fixed and taking d as large as one desires.

5 Extensions

We believe that the following extension of Theorem 2 holds.

Conjecture 17. Let M be a 3-connected binary matroid. If M is non-affine, then either M is isomorphic to $U_{0,1}$, $U_{2,3}$, $M(K_4)$, or F_7, or

$$c(M) \leqslant 2c_{\text{odd}}(M) - 4.$$

We show next that the conjecture holds when M is regular.

Theorem 18. *Let M be a 3-connected regular matroid. If M has an odd circuit, then either M is isomorphic to $U_{0,1}$, $U_{2,3}$, or $M(K_4)$, or*

$$c(M) \leqslant 2c_{\text{odd}}(M) - 4.$$

Proof. As M is regular and 3-connected having an odd circuit, we may assume that $c_{\text{odd}}(M) \geqslant 5$ otherwise, by Theorem 14, M is isomorphic to $U_{0,1}$, $U_{2,3}$, or $M(K_4)$. We may also assume that $c(M) > 2c_{\text{odd}}(M) - 4$ and, by Theorem 2, that M has a spanning circuit. By Theorem 1, $c(M) \leqslant 2c_{\text{odd}}(M) - 2$. Moreover, equality holds here since $c_{\text{odd}}(M) \geqslant 5$. Thus, for some $k \geqslant 2$, the matroid M has a spanning circuit C having $4k$ elements, and $c_{\text{odd}}(M) = 2k + 1$.

Let $e_1, e_2, \dots, e_n$ be the good chords of C. Then, by Lemma 6, $n \geqslant 1$. Let $N = M|(C \cup \{e_1, e_2, \dots, e_n\})$. Then N has a binary representation of the form $[I_{4k-1}|\mathbf{1}|D]$ where the columns of I_{4k-1} are labelled by the elements of $C - z$ for some z in C, the column $\mathbf{1}$ of all ones is labelled by z, and the columns of D are labelled by $\{e_1, e_2, \dots, e_n\}$. Then each column of D has exactly $2k$ ones. Moreover, since by Lemma 9, all series classes of N are trivial, all of the rows of D are distinct and non-zero. Since $k \geqslant 2$, it follows that D has at least seven rows, so D has at least three columns.

For columns a, b, and c of D, let $D[a, b, c]$ be the submatrix of D whose columns are labelled by a, b, and c. Evidently, $D[a, b, c]$ has exactly eight possible different rows. Let x_i be the number of rows equal to (t_2, t_1, t_0) where $i = t_2 2^2 + t_1 2 + t_0$. Hence, by permuting rows, the matrix that is obtained from $D[a, b, c]$ by adjoining the column sums $a + b + c$ and $a + b + c + z$ is as shown in Figure 1 where the labels on the rows indicate the multiplicities of the rows and may be zero.

Because each of a, b, and c has exactly $2k$ ones, we have

$$\begin{aligned} x_4 + x_5 + x_6 + x_7 &= 2k, \\ x_2 + x_3 + x_6 + x_7 &= 2k, \\ x_1 + x_3 + x_5 + x_7 &= 2k. \end{aligned}$$

Thus

$$x_1 + x_2 + x_4 + x_7 + 2(x_3 + x_5 + x_6) = 6k - 2x_7,$$

so

$$x_1 + x_2 + x_4 + x_7 \equiv 0 \mod 2.$$

As $\sum_{i=0}^{7} x_i = 4k-1$, we deduce that

$$x_0 + x_3 + x_5 + x_6 \equiv 1 \mod 2.$$

We will be interested in the sets $C'\{a,b,c\}$ and $C'\{a,b,c,z\}$ where, for example, the first of these is the union of $\{a,b,c\}$ and all of the elements of $C-z$ in which the corresponding row of $a+b+c$ is one. Clearly

$$|C'\{a,b,c\}| = x_1 + x_2 + x_4 + x_7 + 3,$$

and

$$|C'\{a,b,c,z\}| = x_0 + x_3 + x_5 + x_6 + 4.$$

Because the sum of the columns in each of the sets $C'\{a,b,c\}$ and $C'\{a,b,c,z\}$ is the zero vector, each set is a disjoint union of circuits of M.

18.1. At most one of $C'\{a,b,c\}$ and $C'\{a,b,c,z\}$ is a circuit of M

Assume that both sets are circuits of M. As each is odd, each has at most $2k+1$ elements. Thus

$$(4k-1)+7 = \sum_{i=0}^{7} x_i + 7 = |C'\{a,b,c\}| + |C'\{a,b,c,z\}| \leqslant (2k+1)+(2k+1),$$

a contradiction. Thus 18.1 holds.

18.2. For every two columns a and b of D, each of $(1,1)$, $(1,0)$, and $(0,1)$ must occur as a row of $D[a,b]$. Moreover, the number s of rows equal to $(1,0)$ equals the number of rows equal to $(0,1)$. The number t of rows equal to $(1,1)$ is in $[2k-1]$, and number of rows equal to $(0,0)$ is $4k-1-2s-t$.

This follows because each column of D has $2k$ ones and $2k-1$ zeros.

18.3. $D[a,b,c]$ does not have as a submatrix either the matrix F in Figure 2 or any row or column pemutation of F.

$$\begin{array}{c} \\ x_0\\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & a+b+c & a+b+c+z \end{array}\\
\begin{bmatrix}
0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 1 & 0\\
0 & 1 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 1\\
1 & 0 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & 1\\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}$$

Figure 1: $D[a,b,c]$ with $a+b+c$ and $a+b+c+z$ adjoined.

Assume that $D[a,b,c]$ has F as a submatrix. Observe that a circuit of M contained in $C'\{a,b,c\}$ and containing c must also contain a and b. As $C-z$ is independent, it follows that $C'\{a,b,c\}$ is a circuit of M. Moreover, a circuit of M contained in $C'\{a,b,c,z\}$ and containing z must also contain a and b, and hence c. Thus $C'\{a,b,c,z\}$ is a circuit of M, so we have a contradiction to 18.1. Hence 18.3 holds.

18.4. *$D[a,b]$ does not have exactly one row in which both entries are one.*

Assume that $D[a,b]$ has exactly one row, say row $2k$, equal to $(1,1)$. Then we may assume that the first $2k-1$ rows of $D[a,b]$ equal $(1,0)$ and the last $2k-1$ rows equal $(0,1)$. Suppose that D has a column c other than a or b such that its entry in row $2k$ is equal to one. Then, because $D[b,c]$ has $(0,1)$ as a row, there must be a one in column c among the first $2k-1$ rows. By symmetry, as $D[a,c]$ has $(0,1)$ as a row, there is a one in column c among the last $2k-1$ rows. Because c has exactly $2k$ ones, it must also have a zero among each of its first $2k-1$ and last $2k-1$ rows. It follows that, after a possible row permutation, $D[a,b,c]$ has F as submatrix, a contradiction to 18.3.

We may now assume that, for every column c other than a and b, the entry in row $2k$ is 0. As c has rows among its first $2k-1$ and last $2k-1$ having entries equal to one, it follows that $D[a,b,c]$ has as a submatrix the 3×3 matrix whose rows are $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$. Since z is a column of all ones, it follows that N has the Fano matroid as a minor, a contradiction. We conclude that 18.4 holds.

We may now assume that $D[a,b]$ has at least two rows equal to $(1,1)$. As the corresponding rows of D are distinct, there is a column c in which the entries in these two rows are distinct. Hence $D[a,b,c]$ has $(1,1,1)$ and $(1,1,0)$ as rows. Since $D[a,c]$ has $(0,1)$ as a row, $D[a,b,c]$ has $(0,1,1)$ or $(0,0,1)$ as a row. Consider the first case. As $D[b,c]$ has $(0,1)$ as a row, $D[a,b,c]$ has $(1,0,1)$ or $(0,0,1)$ as a row. The first of these options yields F_7^* as a minor of M. Thus $D[a,b,c]$ has $(0,0,1)$ as a row. As $D[a,b]$ has $(1,0)$ as a row, $D[a,b,c]$ has $(1,0,1)$ or $(1,0,0)$ as a row. Again, the first of these options yields F_7^* as a minor of M. Thus $D[a,b,c]$ has $(1,0,0)$ as a row. Hence $D[a,b,c]$ has as a submatrix a row and column permutation of F, a contradiction to 18.3.

It remains to consider the case when $D[a,b,c]$ has $(1,1,1)$, $(1,1,0)$, and $(0,0,1)$ as rows. We showed above that $D[a,b,c]$ does not have $(0,1,1)$ as a row. By symmetry between the first two columns, $D[a,b,c]$ does not have $(1,0,1)$ as a row. As $D[a,b]$ has $(1,0)$ as a row, $D[a,b,c]$ has $(1,0,0)$ as a row. Also, as $D[a,b]$ has $(0,1)$ as a row, $D[a,b,c]$ has $(0,1,0)$ as a row since it does not have $(0,1,1)$ as a row. It follows that, after adjoining

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 2: F does not occur as a submatrix of $D[a,b,c]$.

column z to $D[a, b, c]$, we obtain a matrix that has, as a submatrix, a row permutation of the matrix

$$D' = \begin{array}{c} \begin{array}{cccc} z & a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Then $[I_4|D']$ represents a minor of M. This minor is the non-regular matroid S_8, so we have a contradiction that completes the proof. □

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