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A circuit covering result for matroids

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The purpose of this note is to prove the following result.

THEOREM 1. *Let M be a connected matroid. Suppose that C is a circuit of M and p and q are elements of M. Then M has circuits C_p and C_q such that $p \in C_p$, $q \in C_q$, and $C \subseteq C_p \cup C_q$.*

Note that p and q are not required to be distinct in this theorem. Indeed on letting $p = q$, we obtain the following corollary which may also be easily deduced from the proof of Lehman's result ((1), p. 721) that, if e is an element of a connected matroid M, then M is uniquely determined by the collection of circuits containing e.

COROLLARY 1. *Let M be a connected matroid. Suppose that C is a circuit of M and p is an element of M. Then M has circuits C_1 and C_2 such that $p \in C_1 \cap C_2$ and $C \subseteq C_1 \cup C_2$.*

The matroid terminology used here will in general follow Welsh (3). The ground set of the matroid M will be denoted by $E(M)$ and, if $x \in E(M)$, then we shall sometimes write $M\backslash x$ and M/x for the restriction and contraction respectively of M to $E(M)\backslash x$. A flat of rank one in a matroid will be called a *point*; a flat of rank two a *line*.

The proof of Theorem 1 will use the following well-known result.

LEMMA (Tutte (2), 6·5). *If M is a connected matroid and $e \in E(M)$, then either $M\backslash e$ or M/e is connected.*

Proof of Theorem 1. If $p, q \in C$, then let $C_p = C_q = C$. We may therefore suppose, without loss of generality, that $p \notin C$. If $q \in C$, then let $q' = p$. If the required result can be established for p and q', then it also holds for p and q. Thus assume that $p, q \notin C$.

We argue by induction on $|E(M)|$. Clearly $|C| \geqslant 2$, hence $|E(M)| \geqslant 3$. If $|E(M)| = 3$, then $M \cong U_{1,3}$ and, since $q \notin C$, $p = q$. The required result follows immediately. Now assume that the theorem holds for all connected matroids having fewer than n elements and let M be a connected matroid having exactly n elements. As M is connected, there is a circuit containing p and intersecting C. Choose such a circuit P so that $|P\backslash C|$ is minimal. Similarly, choose a circuit Q containing q and intersecting C so that $|Q\backslash C|$ is minimal. Evidently $E(M) = P \cup Q \cup C$ for otherwise the result follows by the induction assumption.

We now distinguish two cases:

(i) M has an element x such that $x \notin C \cup p \cup q$; and

(ii) $E(M) = C \cup p \cup q$.

Case (i). By the lemma, either $M\backslash x$ or M/x is connected. But, if $M\backslash x$ is connected, then the required result follows by the induction assumption. On the other hand, if M/x is connected, then the result again follows by the induction assumption provided that C is a circuit of M/x.

If C is not a circuit of M/x, then there is a proper subset D of C such that $D \cup x$ is a circuit of M. Now $x \in (P \cup Q)\backslash(C \cup p \cup q)$; hence suppose that $x \in P$. Then since $x \in P \cap (D \cup x)$ and $p \in P\backslash(D \cup x)$ it follows, by circuit exchange, that there is a circuit P' of M such that $p \in P' \subseteq (P \cup D \cup x)\backslash x \subseteq (P \cup C)\backslash x$. But $|P'\backslash C| < |P\backslash C|$. Moreover, $P' \cap C \neq \varnothing$ as otherwise $P' \subsetneqq P$. Thus the choice of P is contradicted. Similarly, if $x \in Q$, then the choice of Q is contradicted. Hence the proof of (i) is complete.

Case (ii). In this case we distinguish the following three possibilities:

(*a*) $p = q$;

(*b*) $p \neq q$ and C is not spanning in M; and

(*c*) $p \neq q$ and C is spanning in M.

(*a*) If $p = q$, then M is a connected single-element extension of a circuit. It follows that $\{p\}$ is a hyperplane in M^* and hence M^* has rank 2. But M^* is connected so there exist disjoint hyperplanes H_1 and H_2 of M^* neither of which contains p. If we let $C_p = S\backslash H_1$ and $C_q = S\backslash H_2$, then the required result follows.

(*b*) If $p \neq q$ and C is not spanning in M, then C is a hyperplane of M. Thus $\{p, q\}$ is both a circuit and a hyperplane of M^*. A similar argument to that given in (*a*) completes the proof of this case.

(*c*) In this case assume that the required circuits C_p and C_q do not exist. Consider M^* and observe that, as $\{p, q\} = L_{p,q}$ is a hyperplane of M^* and C is spanning in M, the matroid M^* has rank 3. Moreover, it follows by the induction assumption that we may suppose that M^* is simple. Since we have assumed that C_p and C_q do not exist, it follows that, if L_1 and L_2 are lines of M^* such that $p \notin L_1$ and $q \notin L_2$, then $L_1 \cap L_2 \neq \varnothing$. Thus if L and L' are lines of M containing p and not q, then L and L' contain the same number, say $m+1$, of points. Likewise all lines through q other than $L_{p,q}$ contain the same number, $n+1$, of points. Thus p is incident with precisely n lines other than $L_{p,q}$, and q is incident with precisely m lines other than $L_{p,q}$. If X is a line of M^* avoiding both p and q, then X meets each of the lines through p other than $L_{p,q}$. Hence X contains exactly n points. Similarly X meets each of the lines through q other than $L_{p,q}$, and so X contains exactly m points. Thus $m = n$. It is now easy to check that M^* has precisely n^2+2 points and that each point of M^* meets exactly $n+1$ lines.

As M^* is connected, $n \geqslant 2$ and we may choose two distinct points x and y of M^* such that $L_{x,y} \cap L_{p,q} = \varnothing$, where $L_{x,y}$ is the line of M^* through x and y. Now every point not on $L_{x,y}$ is uniquely determined as the intersection of two lines, one through x and the other through y. Moreover, every such pair of lines of M^* determines a point, otherwise we may take the corresponding circuits of M to be C_p and C_q; a contradiction. As the number N of points of M^* is the sum of the number of points on the line through x and y and the number of points not on this line, $N = n + ((n+1)-1)((n+1)-1) = n^2+n$. But we showed earlier that $N = n^2+2$; therefore $n = 2$.

It follows easily that $M^* \cong M(K_4)$. Hence $M \cong M(K_4)$, C is a four-element circuit of M and p and q are the two elements of M which are not in C. It is straightforward to check in this case that the required circuits C_p and C_q exist. This contradiction finishes the proof of (c) and thereby completes the proof of the theorem.

The following result generalizes Theorem 1.

COROLLARY 2. *Let M be a connected matroid of rank r and let F be a flat of M of rank k, where $0 \leqslant k \leqslant r-1$. If $p, q \in E(M)$, then M has rank k flats F_p and F_q such that $p \notin F_p$, $q \notin F_q$ and $F_p \cap F_q \subseteq F$.*

Proof. Let M be truncated $(r-k-1)$ times to obtain $T^{r-k-1}(M)$. Clearly this matroid is connected and has F as a hyperplane.

The result now follows by applying Theorem 1 to $(T^{r-k-1}(M))^*$.

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