A NOTION OF MINOR-BASED MATROID CONNECTIVITY

ZACHARY GERSHKOFF AND JAMES OXLEY

Abstract. For a matroid N, a matroid M is N-connected if every two elements of M are in an N-minor together. Thus a matroid is connected if and only if it is $U_{1,2}$-connected. This paper proves that $U_{1,2}$ is the only connected matroid N such that if M is N-connected with $|E(M)| > |E(N)|$, then $M\backslash e$ or M/e is N-connected for all elements e. Moreover, we show that $U_{1,2}$ and $M(\mathcal{W}_2)$ are the only matroids N such that, whenever a matroid has an N-minor using $\{e, f\}$ and an N-minor using $\{f, g\}$, it also has an N-minor using $\{e, g\}$. Finally, we show that M is $U_{0,1} \oplus U_{1,1}$-connected if and only if every clonal class of M is trivial.

1. Introduction

Our terminology follows Oxley [8]. We say that a matroid M *uses* an element e or a set Z of elements if $e \in E(M)$ or $Z \subseteq E(M)$. Let N be a matroid. A matroid M with $|E(M)| \geq 2$ is N-*connected* if, for every pair of distinct elements e, f of $E(M)$, there is a minor of M that is isomorphic to N and uses $\{e, f\}$.

We will assume, unless otherwise stated, that the matroids discussed here have at least two elements. Note that $U_{1,2}$-connectivity coincides with the usual notion of connectivity for matroids. Hence, relying on a well-known inductive property of matroid connectivity [13], we have that if M is $U_{1,2}$-connected, $e \in E(M)$, and $|E(M)| \geq 3$, then $M\backslash e$ or M/e is $U_{1,2}$-connected. Our first theorem shows that $U_{1,2}$ is the only connected matroid with this property.

Theorem 1.1. *Let N be a matroid. If, for every N-connected matroid M with $|E(M)| > |E(N)|$ and, for every e in $E(M)$, at least one of $M\backslash e$ or M/e is N-connected, then N is isomorphic to one of $U_{1,2}$, $U_{0,2}$, or $U_{2,2}$.*

One attractive property of matroid connectivity is that elements can be assigned to components. We say that a matroid N has the *transitivity property* if, for every matroid M and every triple $\{e, f, g\} \subseteq E(M)$, if e is in an N-minor with f, and f is in an N-minor with g, then e is in an N-minor with g. Let $M(\mathcal{W}_2)$ be the rank-2 wheel. In Section 6, we prove the following result.

Theorem 1.2. *The only matroids with the transitivity property are $U_{1,2}$ and $M(\mathcal{W}_2)$.*

On combining the last two theorems, we get the following result, which indicates how special the usual matroid connectivity is.

Corollary 1.3. *Let N be a matroid with the transitivity property such that whenever M is an N-connected matroid, $e \in E(M)$, and $|E(M)| > |E(N)|$, at least one of $M\backslash e$ and M/e is N-connected. Then $N \cong U_{1,2}$.*

Date: July 2, 2018.
1991 *Mathematics Subject Classification.* 05B35.
Key words and phrases. matroid connectivity, matroid roundedness.

The concept of N-connectivity can also convey interesting information when N is disconnected, as the next result indicates.

Theorem 1.4. *A matroid M is $U_{0,1} \oplus U_{1,1}$-connected if and only if every clonal class of M is trivial.*

The paper is structured as follows. In the next section, we recall Cunningham and Edmonds's decomposition theorem for connected matroids that are not 3-connected, which is a basic tool in our proofs. Sections 3, 4, and 5 treat the cases of N-connected matroids when N is 3-connected, connected, and disconnected, respectively. In particular, we prove Theorems 1.1, and 1.2 in Section 6 and Theorem 1.4 in Section 5. Finally, in Section 7, we consider what can be said when every set of three elements occurs in some minor. Moss [6] showed that 3-connected matroids can be characterized as those in which every set of four elements is contained in a minor isomorphic to a member of $\{\mathcal{W}^2, \mathcal{W}^3, \mathcal{W}^4, M(\mathcal{W}_3), M(\mathcal{W}_4), Q_6\}$.

2. Preliminaries

The concept of N-connectivity is closely related to roundedness, which is exemplified by Bixby's [1] result that if e is an element of a 2-connected non-binary matroid M, then M has a $U_{2,4}$-minor using e. Formally, let t be a positive integer and let $\mathcal{N}$ be a class of matroids. A matroid M *has an $\mathcal{N}$-minor* if M has a minor isomorphic to a member of $\mathcal{N}$. Seymour [11] defined $\mathcal{N}$ to be *t-rounded* if, for every $(t+1)$-connected matroid M with an $\mathcal{N}$-minor and every subset X of $E(M)$ with at most t elements, M has an $\mathcal{N}$-minor using X. Thus Bixby's result shows that $\{U_{2,4}\}$ is 1-rounded. Seymour [10] extended this result as follows.

Theorem 2.1. *Let M be a 3-connected matroid having a $U_{2,4}$-minor, and let e and f be elements of M. Then M has a $U_{2,4}$-minor using $\{e, f\}$.*

The connectivity function λ_M of a matroid M is defined for every subset X of $E(M)$ by $\lambda_M(X) = r(X) + r(E(M) - X) - r(M)$; equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. For disjoint subsets A, B of $E(M)$, define $\kappa_M(A, B) = \min\{\lambda_M(X) : A \subseteq X \subseteq E(M) - B\}$.

Lemma 2.2. *If N is a minor of M and A, B are disjoint subsets of $E(N)$, then $\kappa_N(A, B) \leq \kappa_M(A, B)$.*

Next we give a brief outline of Cunningham and Edmonds's decomposition [4] of matroids that are 2-connected but not 3-connected. More complete details can be found in [8, Section 8.3]. First recall that when (X, Y) is a 2-separation of a connected matroid M, we can write M as $M_X \oplus_2 M_Y$ where M_X and M_Y have ground sets $X \cup p$ and $Y \cup p$. A *matroid-labeled tree* is a tree T with vertex set $\{M_1, M_2, \dots, M_n\}$ such that each M_i is a matroid and, for distinct vertices M_j and M_k, the sets $E(M_j)$ and $E(M_k)$ are disjoint if M_j and M_k are non-adjacent, whereas if M_j and M_k are joined by an edge e, then $E(M_j) \cap E(M_k) = \{e\}$, and $\{e\}$ is not a separator in either M_j or M_k.

When f is an edge of a matroid-labeled tree T joining vertices M_i and M_j, if we contract the edge f, we obtain a new matroid-labeled tree T/f by relabeling the composite vertex that results from this contraction as $M_i \oplus_2 M_j$, with every other vertex retaining its original label.

A *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree T such that if $V(T) = \{M_1, M_2, \dots, M_n\}$ and $E(T) = \{e_1, e_2, \dots, e_{n-1}\}$, then

(i) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)) - \{e_1, e_2, \ldots, e_{n-1}\}$;
(ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case, $n = 1$ and $M_1 = M$; and
(iii) the label of the single vertex of $T/\{e_1, e_2, \ldots, e_{n-1}\}$ is M.

We call the members of $\{e_1, e_2, \ldots, e_{n-1}\}$ *basepoints* since each member of this set is the basepoint of a 2-sum when we construct M. Cunningham and Edmonds (in [4]) proved the following (see also [8, Theorem 8.3.10]).

Theorem 2.3. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label that is not a circuit or a cocircuit is 3-connected, and there are no adjacent vertices that are both labeled by circuits or are both labeled by cocircuits. Moreover, T is unique up to relabeling of its edges.*

The tree decomposition T whose existence is guaranteed by the last theorem is called the *canonical tree decomposition* of M. Although circuits and cocircuits with at most three elements are 3-connected matroids, when we refer to a *3-connected vertex*, we shall mean one with at least four elements. Clearly, for each edge p of T, the graph $T\backslash p$ has two components. Thus p induces a partition of $V(T)$ and a corresponding partition (X_p, Y_p) of $E(M)$. The latter partition is a 2-separation of M; we say that it is *displayed* by the edge p. Moreover, $M = M_{X_p} \oplus_2 M_{Y_p}$ where M_{X_p} and M_{Y_p} have ground sets $X_p \cup p$ and $Y_p \cup p$, respectively. We shall refer to this 2-sum decomposition of M as having been *induced* by the edge p of T.

We shall frequently use the following well-known result, which appears, for example, as [9, Lemma 2.15].

Lemma 2.4. *Let M_1 and M_2 label distinct vertices in a tree decomposition T of a connected matroid M. Let P be the path in T joining M_1 and M_2, and let p_1 and p_2 be the edges of P meeting M_1 and M_2, respectively. Then M has a minor that uses $(E(M_1) \cup E(M_2)) \cap E(M)$ and is isomorphic to the 2-sum of M_1 and M_2, with respect to the basepoints p_1 and p_2.*

We will often use the next result, another consequence of Theorem 2.3.

Lemma 2.5. *Let (X, Y) be a 2-separation displayed by an edge p in a 2-connected matroid M. Suppose $y \in Y$. Then M has, as a minor, the matroid $M_X(y)$ that is obtained from M_X be relabeling p by y. In particular, let N be a 3-connected minor of M with $|E(N)| \geq 4$ and $|E(N) \cap Y| \leq 1$. If $|E(N) \cap Y| = 1$, let $y \in E(N) \cap Y$; otherwise let y be an arbitrary element of Y. Then $M_X(y)$ has N as a minor.*

Let T be the canonical tree decomposition of a 2-connected matroid M, and let M_0 label a vertex of T. Let $p_1, p_2, \ldots, p_d$ be the edges of T that meet M_0. For each p_i, let (X_i, Y_i) be the 2-separation of M displayed by p_i, where M_0 is on the X_i-side of the 2-separation. For each i, let $y_i \in Y_i$. Then, by repeated application of Lemma 2.5, we deduce that M has, as a minor, the matroid that is obtained from M_0 by relabeling p_i by y_i for all i in $\{1, 2, \ldots, d\}$. We denote this matroid by $M_0(y_1, y_2, \ldots, y_d)$ and call it a *specially relabeled* M_0-minor of M.

The following result, which is straightforward to prove by repeated application of Lemma 2.2, is well known.

Lemma 2.6. *Let N be a 3-connected matroid with $|E(N)| \geq 3$. Let M be a 2-connected matroid with canonical tree decomposition T. Then there is a unique vertex M' of T such that, for each edge p of T, the partition of $V(T)$ induced by p*

has the vertex M' on the same side as at least $|E(N)|-1$ elements of N. Moreover, there is a specially relabeled M'-minor of M that has N as a minor.

3. 3-connected matroids

Let $\mathcal{N}$ be a set of matroids. A matroid M is *$\mathcal{N}$-connected* if, for every two distinct elements e and f of M, there is an N-minor of M that uses $\{e, f\}$ for some N in $\mathcal{N}$. A consequence of [8, Proposition 4.3.6] is that a matroid with at least three elements is $\{U_{1,3}, U_{2,3}\}$-connected if and only if it is connected. The first result in this section characterizes $U_{2,3}$-connected matroids. One may hope for a characterization of 3-connectivity in terms of $\mathcal{N}$-connectivity, but no such characterization exists. To see this, note that if M is $\mathcal{N}$-connected, then so is $M \oplus_2 M$. A characterization of 3-connectivity in terms of minors containing 4-element sets, as opposed to the 2-element sets currently under consideration, is given in [6].

Proposition 3.1. *A matroid M is $U_{2,3}$-connected if and only if M is connected and simple.*

Proof. Suppose M is $U_{2,3}$-connected. Clearly M is connected and simple. Conversely, if M is connected and simple, and e and f are distinct elements of M, then M has a circuit C containing $\{e, f\}$ and $|C| \geq 3$. Hence M has a $U_{2,3}$-minor using $\{e, f\}$, so M is $U_{2,3}$-connected. □

Corollary 3.2. *A matroid M is $U_{1,3}$-connected if and only if M is connected and cosimple.*

We will describe N-connectivity for a 3-connected matroid N by first considering the case when N is $U_{2,4}$. We will refer to binary and non-binary matroids that label vertices of a canonical tree decomposition as *binary* and *non-binary vertices.*

Theorem 3.3. *A matroid M is $U_{2,4}$-connected if and only if M is connected and non-binary, and, in the canonical tree decomposition of M,*

(i) every binary vertex has at most one element that is not a basepoint; and
(ii) on every path between two binary vertices that each contain a unique element of $E(M)$, there is a non-binary vertex.

Proof. Suppose M is non-binary and connected, and the canonical tree decomposition T of M satisfies the above conditions. Suppose e and f are distinct elements of M. If e and f are in the same 3-connected vertex M_0 of T, then, by (i), M_0 is non-binary. Thus, by Theorem 2.1, M has a $U_{2,4}$-minor using $\{e, f\}$.

Next suppose e belongs to a binary vertex M_1 of T, and f belongs to a non-binary vertex M_0 of degree d. By Lemma 2.5, M contains a specially labeled M_0-minor $M_0(e, y_2, y_3, \dots, y_d)$ using $\{e, f\}$. Similarly, let e and f belong to binary vertices M_1 and M_2, and let M_0 be a non-binary vertex on the path between them in T. Then M contains a specially labeled M_0-minor $M_0(e, f, y_3, y_4, \dots, y_d)$. Thus, by Theorem 2.1, M has a $U_{2,4}$-minor using $\{e, f\}$.

Suppose now that M is $U_{2,4}$-connected. Clearly M is non-binary and connected. If a binary vertex M_1 in T contains two non-basepoints e and f, then, by Lemma 2.6, a $U_{2,4}$-minor of M using $\{e, f\}$ must be a minor of M_1; a contradiction.

Now suppose e and f are the unique non-basepoints of binary vertices M_1 and M_2, respectively, in T, and let N be a $U_{2,4}$-minor of M using $\{e, f\}$. By Lemma 2.6, T has a nonbinary vertex M_0 such that, for every edge p of T, the partition of $V(T)$

induced by p has M_0 on the same side as at least $|E(N)| - 1$ elements of N. Let p_1 be the edge incident with M_0 such that M_1 and M_0 are on opposite sides of the induced partition of $V(T)$. Then M_2 must be on the same side of this partition as M_0. Hence M_0 lies on the path in T between M_1 and M_2. □

The last theorem can be generalized as follows.

Theorem 3.4. *Let N be a 3-connected matroid with at least four elements. A matroid M is N-connected if and only if M is connected, has N as a minor, and, in the canonical tree decomposition of M,*

(i) every vertex that is not N-connected has at most one element that is not a basepoint; and
(ii) on every path between two vertices that are not N-connected and that each have unique non-basepoints, there is an N-connected vertex.

4. Connected matroids

In this section, we consider N-connected matroids when N is connected but not 3-connected.

Theorem 4.1. *A matroid M is $M(\mathcal{W}_2)$-connected if and only M is connected and non-uniform.*

Proof. If M is $M(\mathcal{W}_2)$-connected, then it is clearly both connected and non-uniform. To prove the converse, suppose M is connected and non-uniform. We argue by induction that M is $M(\mathcal{W}_2)$-connected. This is immediate if $|E(M)| = 4$, since $M(\mathcal{W}_2)$ is the unique 4-element connected, non-uniform matroid. Assume it holds for $|E(M)| < n$ and let $|E(M)| = n > 4$. Distinguish two elements x and y of $E(M)$.

Suppose there is an element e of $E(M) - \{x, y\}$ such that M/e is disconnected. Then M is the parallel connection, with basepoint e, of two matroids M_1 and M_2. Now $M\backslash e$ is connected. We may assume that it is uniform; otherwise, by the induction assumption, $M\backslash e$ and hence M has an $M(\mathcal{W}_2)$-minor using $\{x, y\}$. Now $r(E(M_1) - e) + r(E(M_2) - e) - r(M\backslash e) = 1$. Suppose each of $|E(M_1) - e|$ and $|E(M_2) - e|$ has at least two elements. Then $M\backslash e$ has a 2-separation. Since $M\backslash e$ is uniform, it follows that $M\backslash e$ is a circuit or a cocircuit. In the latter case, M is also a cocircuit; a contradiction. If $M\backslash e$ is a circuit, then M is the parallel connection of two circuits, and M is easily seen to have an $M(\mathcal{W}_2)$-minor using $\{x, y\}$.

Now suppose that $|E(M_1) - e| = 1$. Thus M has a circuit, $\{e, f\}$ say, containing e. As $M\backslash e$ is uniform but M is not, $r(M) \geq 2$, so $M\backslash e$ has a circuit containing $\{f, x, y\}$. It follows that M has an $M(\mathcal{W}_2)$-minor with ground set $\{e, f, x, y\}$.

We may now assume that M/e is connected for all e in $E(M) - \{x, y\}$. Moreover, by replacing M with M^* in the argument above, we may also assume that $M\backslash e$ is connected for all such e. If $M\backslash e$ or M/e is non-uniform, then, by the induction assumption, M has an $M(\mathcal{W}_2)$-minor using $\{x, y\}$. Thus both $M\backslash e$ and M/e are uniform. Let $r(M\backslash e) = r$. Then every circuit of $M\backslash e$ has $r+1$ elements. Since M is not uniform, it has a circuit containing e that has at most r elements. Contracting e from M produces a rank-$(r - 1)$ matroid having a circuit with at most $r - 1$ elements. Since M/e is uniform, this is a contradiction. □

We omit the straightforward proof of the next result.

Lemma 4.2. *If M, N, and N' are matroids such that M is N-connected and N is N'-connected, then M is N'-connected.*

If we wish to describe the class of N-connected matroids for a 3-connected matroid N, it suffices to describe the N-connected matroids that are 3-connected and then apply Theorem 3.4. If N is not 3-connected, the task of describing N-connected matroids becomes harder, and we omit any attempt to provide a general theorem for N-connectivity in this case. We will instead give characterizations for two specific matroids that are not 3-connected, namely $U_{1,4}$ and its dual $U_{3.4}$. We will use the following theorem of Oxley [7].

Theorem 4.3. *Let M be a 3-connected matroid having rank and corank at least three, and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a minor isomorphic to one of $U_{3,6}, P_6, Q_6, \mathcal{W}^3$, or $M(K_4)$ that uses $\{x, y, z\}$.*

Proposition 4.4. *A 3-connected matroid M is $U_{1,4}$-connected if and only if either $M \cong U_{2,n}$ for some $n \geq 5$, or M has rank and corank at least three.*

Proof. Clearly if $n \geq 5$, then $U_{2,n}$ is $U_{1,4}$-connected. Now assume that $r(M) \geq 3$ and $r^*(M) \geq 3$. Suppose $\{x, y\} \subseteq E(M)$. Then, by Theorem 4.3, M has an $\mathcal{N}$-minor using $\{x, y\}$ where $\mathcal{N}$ is $\{U_{3,6}, P_6, Q_6, \mathcal{W}^3, M(K_4)\}$. One easily checks that each member of $\mathcal{N}$ is $U_{1,4}$-connected. Hence, by Lemma 4.2, M is $U_{1,4}$-connected.

To prove the converse, assume that M is $U_{1,4}$-connected. Since $r^*(U_{1,4}) = 3$, it follows that $r^*(M) \geq 3$. The required result holds if $r(M) \geq 3$. But, since M is 3-connected and $U_{1,4}$-connected, $r(M) \geq 2$. Moreover, if $r(M) = 2$, then $M \cong U_{2,n}$ for some $n \geq 5$. □

Duality gives a corresponding result for $U_{3,4}$-connectivity.

Corollary 4.5. *A 3-connected matroid M is $U_{3,4}$-connected if and only if either $M \cong U_{n-2,n}$ where $n \geq 5$, or M has rank and corank at least 3.*

Observe that this fails to fully characterize $U_{3,4}$-connectivity for if we let $M = M(K_{2,3})$, then M is $U_{3,4}$-connected but none of the matroids in its canonical tree decomposition is $U_{3,4}$-connected. We can instead describe $U_{3,4}$-connectivity in terms of forbidden configurations of matroids in the canonical tree decomposition.

Proposition 4.6. *Suppose M is not 3-connected. Then M is $U_{3,4}$-connected if and only if M is connected and simple, and, in the canonical tree decomposition T of M, there is no vertex of degree at most two that is labeled by some $U_{2,n}$ such that its only neighbors in T are cocircuits that use elements of $E(M)$.*

Proof. Let T be the canonical tree decomposition of M. Assume M is $U_{3,4}$-connected. Then, by Lemma 4.2, M is $U_{2,3}$-connected, so M is connected and simple. Suppose that T has a vertex M_0 whose degree d is at most two such that M_0 is labeled by some $U_{2,n}$ and has its only neighbors $M_1, \dots, M_d$ labeled by cocircuits that use elements of $E(M)$. For each i in $\{1, \dots, d\}$, suppose $f_i \in E(M_i) \cap E(M)$. Then M can be obtained from a copy of $U_{2,n}$ using $\{f_1, \dots, f_d\}$ by, for each i, adjoining some matroid via parallel connection across the basepoint f_i. If $d = 1$, let f_2 be an element of M_0 other than f_1. Clearly M has no circuit using $\{f_1, f_2\}$ that has more than three elements.

Now assume that M is connected and simple and that T satisfies the specified conditions. Let $\{e, f\}$ be a subset of $E(M)$ that is not contained in a $U_{3,4}$-minor.

Assume first that e and f belong to the same vertex M_1 of T. As M is simple, M_1 is not a cocircuit. Now M has a specially relabeled M_1-minor using $\{e, f\}$. Thus, by Corollary 4.5, $M_1 \cong U_{2,n}$ for some $n \geq 3$. Let p be an edge of T that meets M_1. Consider the 2-sum $N_1 \oplus_2 N_2$ induced by p where $\{e, f\} \subseteq E(N_1)$. Certainly N_1 has a circuit containing $\{e, f, p\}$, and N_2 has a circuit of size at least three containing p. Thus M has a $U_{3,4}$-minor containing $\{e, f\}$; a contradiction.

We now know that e and f belong to distinct vertices M_1 and M_2 of T. Each edge p of the path P in T joining M_1 and M_2 induces a 2-sum decomposition of M into two matroids, N_{1p} and N_{2p}. Moreover, an element x_i of $E(N_{ip})$ is in a circuit of N_{ip} of size at least three containing p unless x_i is parallel to p in N_{ip}. Thus e or f is parallel to p in N_{1p} or N_{2p}, respectively. Let the edges of P, in order, be $p_1, p_2, \ldots, p_k$ where p_1 meets M_1. We may assume that e is parallel to p_1 in N_{1p_1}. Then the vertex M_1 of T containing e is a cocircuit.

Suppose $k \geq 3$. As no two adjacent vertices of T are cocircuits, neither e nor f is parallel to p_2 in N_{1p_2} or N_{2p_2}. Hence M has a $U_{3,4}$-minor using $\{e, f\}$. This contradiction implies that $k \in \{1, 2\}$. Suppose $k = 2$. Then f is parallel to p_2 in N_{2p_2}. Thus M_2 is a cocircuit. Since M has no $U_{3,4}$-minor using $\{e, f\}$, the vertex M_3 of T that is adjacent to both M_1 and M_2 is isomorphic to some $U_{2,n}$. By assumption, M_3 must have another neighbor in T to which it is joined by the edge q, say. Then, for the 2-sum decomposition $Q_1 \oplus_2 Q_2$ of M induced by q, there is a circuit of Q_1 containing $\{e, f, q\}$ and a circuit of Q_2 of size at least three containing q. Thus M has a $U_{3,4}$-minor using $\{e, f\}$. This contradiction implies that $k = 1$. Then $M = N_{1p_1} \oplus_2 N_{2p_1}$. Thus the specially relabeled minor $N_{2p_1}(e)$ uses $\{e, f\}$. Now the canonical tree decomposition T' of $N_{2p_1}(e)$ can be obtained from the component of $T \backslash p_1$ using N_{2p_1} by replacing M_2 by $M_2(e)$. As e and f are contained in the same vertex of T', we deduce from the second paragraph that $N_{2p_1}(e)$, and hence M, has a $U_{3,4}$-minor using $\{e, f\}$; a contradiction. □

5. Disconnected matroids

We now turn our attention to N-connectivity where N is disconnected. The following is essentially immediate.

Proposition 5.1. *Let n be an integer exceeding one. A matroid M is $U_{n,n}$-connected if and only if M is simple with rank at least n.*

Recall that elements x and y of a matroid M are *clones* if the bijection on $E(M)$ that interchanges x and y but fixes every other element yields the same matroid. Next we prove Theorem 1.4, showing that a matroid is $U_{0,1} \oplus U_{1,1}$-connected if and only if no element has a clone. The proof will use the well-known fact (see, for example, [2]) that two elements in a matroid are clones if and only if they are in precisely the same cyclic flats.

Proof of Theorem 1.4. Suppose every clonal class of M is trivial and let x and y be distinct elements of M. Then M has a cyclic flat F that contains exactly one of x and y, say x. In $M/(F - x)$, the element x is a loop but y is not. Thus M has a $U_{0,1} \oplus U_{1,1}$-minor using $\{x, y\}$, so M is $U_{0,1} \oplus U_{1,1}$-connected.

Conversely, assume M is $U_{0,1} \oplus U_{1,1}$-connected, but M has elements x and y that are in the same cyclic flats. Suppose that $M/C\backslash D \cong U_{0,1} \oplus U_{1,1}$ and $E(M/C\backslash D) = \{x, y\}$. Let x be the loop of $M/C\backslash D$. Then $x \in \mathrm{cl}_M(C)$. Thus $y \in \mathrm{cl}_M(C)$, so y is a loop in $M/C\backslash D$; a contradiction. □

Recall, for the next result, that an element is *free* in a matroid if it is not a coloop and every circuit that contains it is spanning.

Theorem 5.2. *A matroid M is $U_{1,2} \oplus U_{1,1}$-connected if and only if M is loopless, has at most one coloop, and has at most one free element.*

Proof. Clearly if M is $U_{1,2} \oplus U_{1,1}$-connected, then it obeys the specified conditions. Conversely, suppose M is loopless, has at most one coloop, and has at most one free element. Let e and f be elements of M. Suppose first that M is disconnected. If e and f are in the same component, then they are in a $U_{1,2}$-minor of that component, so M has a $U_{1,2} \oplus U_{1.1}$-minor using $\{e, f\}$. If e and f are in different components, then one of these components is not a coloop. That component has a $U_{1,2}$-minor using e or f. It follows that M has a $U_{1,2} \oplus U_{1,1}$-minor using $\{e, f\}$.

Now suppose M is connected. Suppose that e is free in M. Then f is in some non-spanning circuit, C_f. Choose g in $C_f - f$. Contracting $C_f - \{f, g\}$ and deleting every other element of M yields a $U_{1,2} \oplus U_{1,1}$-minor of M using $\{e, f\}$.

Suppose neither e nor f is free in M. If there is a non-spanning circuit C containing $\{e, f\}$, we can find a $U_{1,2} \oplus U_{1,1}$-minor by contracting every element of C except e and f, and deleting every other element except for one. Now suppose every circuit containing $\{e, f\}$ is spanning. Since e is not free, there is a non-spanning circuit C containing e. Clearly $f \notin \mathrm{cl}(C)$ otherwise $M|\mathrm{cl}(C)$ is a connected matroid of rank less than $r(M)$ so it contains a circuit containing $\{e, f\}$; a contradiction. Therefore, after we contract all of C except for e and one other element, we see that f will not be a loop. Thus we can find a $U_{1,2} \oplus U_{1,1}$-minor using $\{e, f\}$. □

Corollary 5.3. *A matroid M is $U_{1,2} \oplus U_{0,1}$-connected if and only if M is coloopless and has at most one element that is in every dependent flat.*

6. N-connectivity as compared to connectivity

Before proving Theorem 1.1, we state and prove its converse.

Proposition 6.1. *If $N \in \{U_{1,2}, U_{0,2}, U_{2,2}\}$, then, for every N-connected matroid M with $|E(M)| \geq 3$ and for every e in $E(M)$, at least one of $M \backslash e$ or M/e is N-connected.*

Proof. The result is immediate if $N \cong U_{1,2}$. By duality, it suffices to deal with the case when $N \cong U_{2,2}$. Suppose M is $U_{2,2}$-connected, and $|E(M)| \geq 3$. By Proposition 5.1, M is simple with rank at least two. Therefore if M is $U_{2,2}$-connected and $r(M) > 2$, we can delete any element e of M and still have an N-connected matroid. Observe that if $r(M) = 2$, then M must be connected since it is simple. Therefore M has no coloops, so $r(M\backslash e) = 2$ for all e of $E(M)$. Thus $M \backslash e$ is $U_{2,2}$-connected. □

Proof of Theorem 1.1. First we consider the case when N is connected. Then N is $U_{1,2}$-connected. Thus, by Lemma 4.2, every N-connected matroid is $U_{1,2}$-connected and so is connected. Suppose M is an N-connected matroid with $|E(M)| > |E(N)|$.

Assume N is simple. Then, by Proposition 3.1 and Lemma 4.2, N, and hence M, is $U_{2,3}$-connected. Let M_1 and M_2 be isomorphic copies of M with disjoint ground sets. Pick arbitrary elements g_1 and g_2 in M_1 and M_2, and let M_3 be the parallel connection of M_1 and M_2 with respect to the basepoints g_1 and g_2, which we relabel as g in M_3. Then one easily sees that M_3 is N-connected. Let $e, f \in E(M_1) - g$.

By assumption, we can remove all the elements of $E(M_1) - \{e, f, g\}$ from M_3 via deletion or contraction to obtain a matroid M_4 that is still N-connected. Since M_4 is $U_{2,3}$-connnected, it follows that $\{e, f, g\}$ is a triangle in M_4. Moreover, $\{e, f\}$ is a series pair in M_4. However, neither $M_4\backslash e$ nor M_4/e is $U_{2,3}$-connected since $M_4\backslash e$ is disconnected, and M_4/e has f and g in parallel. We deduce that N is not simple. Dually, N is not cosimple. The only uniform matroid that is neither simple nor cosimple is $U_{1,2}$, so either $N \cong U_{1,2}$, or N is non-uniform.

Next we show that N cannot be non-uniform. Suppose, instead, that N is non-uniform. Then, as N is connected, by Theorem 4.1, N is $M(\mathcal{W}_2)$-connected.

Recall that M is N-connected with $|E(M)| > |E(N)|$. Let $n = |E(N)| + 1$ and distinguish elements e, f of $E(M)$. Let each of $M_1, M_2, \ldots, M_n$ be a copy of M and let e_i and f_i be the elements of M_i corresponding to e and f. Let M' be the parallel connection of $M_1, M_2, \ldots, M_n$ with respect to the basepoints $e_1, e_2, \ldots, e_n$ where these elements are relabeled as e in M'. By assumption, for each M_i, we can remove $E(M_i) - \{e, f_i\}$ from M' in such a way that the resulting matroid M'' is N-connected. Since M'' is connected, it must be isomorphic to $U_{1,n+1}$, which is clearly not $M(\mathcal{W}_2)$-connected; a contradiction. We conclude that N cannot be non-uniform, and hence the theorem holds when N is connected.

Next we consider the case when N is disconnected, first showing the following.

6.2.1. *If each element of N is a loop or a coloop, then $N \cong U_{0,2}$ or $U_{2,2}$.*

Suppose $n \geq 3$ and let $N \cong U_{n,n}$. Let $M = U_{2,3} \oplus U_{n-2,n-2}$. Then M is N-connected, but if e is a coloop of M, then neither $M\backslash e$ nor M/e has a $U_{n,n}$-minor. Therefore $N \ncong U_{n,n}$; dually, $N \ncong U_{0,n}$.

If $N = U_{0,1} \oplus U_{1,1}$, then let $M = M(K_4)$. By Theorem 1.4, M is N-connected, but, for every e of $E(M)$, both $M\backslash e$ and M/e have nontrivial clonal classes and are therefore not N-connected. Now assume $N \cong U_{0,n} \oplus U_{m,m}$ for some $n \geq 2$ and $m \geq 1$. Then $U_{0,n+1} \oplus U_{m,m}$ is an N-connected matroid, say M. But if e is a coloop, then neither $M\backslash e$ nor M/e has an N-minor. On combining this contradiction with duality, we conclude that 6.2.1 holds.

Now assume that N has $k + s$ components $N_1, N_2, \ldots, N_{k+s}$ where those with at least two elements are $N_1, N_2, \ldots, N_k$. Then $k \geq 1$. For each i in $\{1, 2, \ldots, k\}$, choose an element e_i of N_i and relabel it as p. Let M' be the parallel connection of $N_1, N_2, \ldots, N_k$ with respect to the basepoint p where we take $M' = N_1$ if $k = 1$. Let N' be a copy of N whose ground set is disjoint from $E(N)$, and let n'_i be the component of N' corresponding to N_i. Let $M_1 = N' \oplus M'$. We show next that

6.2.2. *M_1 is N-connected.*

Suppose $\{e, f\} \subseteq E(M_1)$. Certainly M_1 has an N-minor using $\{e, f\}$ if $\{e, f\} \subseteq E(N')$. Next suppose that $e \in E(M')$. Then, since M' is a connected parallel connection, we see that, for each i in $\{1, 2, \ldots, k + s\}$, there is an N_i-minor of M' using e. Thus, if $f \in E(N')$, say $f \in E(N'_j)$, then we can choose $i \neq j$ and get an N-minor of M_1 using $\{e, f\}$ unless $k = 1 = j$. In the exceptional case, M' has an N_2-minor with ground set $\{e\}$ and again we get an N-minor of M_1 using $\{e, f\}$. We may now assume that $f \in E(M')$, say $f \in E(N_j)$. Then M' has an N_j-minor using $\{e, f\}$, so M_1 has an N-minor using $\{e, f\}$. Thus 6.2.2 holds.

Since M_1 is N-connected, by assumption, we may delete or contract elements of M_1 until we obtain an N-connected matroid M_2 with $|E(M_2)| = |E(N)| + 1$. In particular, we may remove elements from M' in M_1 until a single element g remains.

Now choose e in $E(N'_1)$. Then $M_2\backslash e$ or M_2/e is isomorphic to N. But both $M_2\backslash e$ and M_2/e have more one-element components than N'; a contradiction. □

Recall that we say that a matroid N has the transitivity property if, for every matroid M and every triple $\{e, f, g\} \subseteq E(M)$, if e is in an N-minor with f, and f is in an N-minor with g, then e is in an N-minor with g. Clearly N has the transitivity property if and only if N^* has the transitivity property.

Lemma 6.3. *Suppose N is a matroid having the transitivity property. Let N' be obtained from N by adding an element f in parallel to a non-loop element e of N. Then there is an element g of $E(N')$ such that $N'\backslash g$ is isomorphic to N and has $\{e, f\}$ as a 2-circuit. Moreover, g is in a 2-circuit in N.*

Proof. The transitivity property implies that $\{e, f\}$ is in an N-minor of N'. Since $r^*(N') > r^*(N)$, there must be an element g of $E(N') - \{e, f\}$ such that $N'\backslash g \cong N$. Since we have introduced a new 2-circuit in constructing N', when we delete g, we must destroy a 2-circuit. □

By the last lemma and duality, we obtain the following result.

Corollary 6.4. *If N is a matroid having the transitivity property, then N has a component with more than one element.*

The following elementary observation and its dual will be used repeatedly in the proof of Theorem 1.2.

Lemma 6.5. *Suppose N is a matroid with the transitivity property. Let N_0 be a component of N with the largest number of elements. Suppose f is added in parallel to an element e of N_0. Let N'_0 and N' be the resulting extensions of N_0 and N, respectively. Suppose $g \in E(N')$ such that $N'\backslash g \cong N$. Then $g \in E(N'_0)$.*

Recall that a set S of elements of a matroid M is a *fan* if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \dots, s_n)$ of the elements of S such that, for all i in $\{1, 2, \dots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad; and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad; and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

Note that the above extends the definition given in [8] by eliminating the requirement that M be simple and cosimple. We shall follow the familiar practice here of blurring the distinction between a fan and a fan ordering.

Lemma 6.6. *Let $(s_1, s_2, \dots, s_n)$ be a fan X in a matroid M such that each of $\{s_1, s_2\}$ and $\{s_{n-1}, s_n\}$ is a circuit or a cocircuit. Then X is a component of M.*

Proof. By switching to the dual if necessary, we may assume that $\{s_1, s_2, s_3\}$ is a triangle of M. Thus $\{s_1, s_2\}$ is a cocircuit. Observe that $\{s_i : i \text{ is odd}\}$ spans X. If n is odd, this is immediate, and if n is even, it follows from the fact that $\{s_{n-1}, s_n\}$ is a circuit in this case. By duality, $\{s_i : i \text{ is even}\}$ spans X in M^*. Hence $r(X) + r^*(X) \leq |X|$; that is, $\lambda(X) \leq 0$, so X is a component of M. □

We define a *special fan* to be a fan $(s_1, s_2, \dots s_k)$ such that $\{s_1, s_2\}$ is a cocircuit of M. We will now show that $U_{1,2}$ and $M(\mathcal{W}_2)$ are the only connected matroids with the transitivity property.

Proof of Theorem 1.2. It is clear that $U_{1,2}$ has the transitivity property. By Theorem 4.1, two elements of M are in an $M(\mathcal{W}_2)$-minor together if and only if they are in a connected, non-uniform component together. It follows that $M(\mathcal{W}_2)$ has the transitivity property.

Suppose that N has the transitivity property. Assume that N is not isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$. Next we show the following.

6.7.1. *Let N_0 be a largest component of N. Then N_0 is isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$.*

Assume that this assertion fails. Then, by Corollary 6.4, N_0 has at least two, and hence at least three, elements. Take an element e of N_0 and add an element f in series with it. Let the resulting coextensions of N_0 and N be N_0' and N', respectively. Then, by the transitivity property, $N'/a \cong N$ for some element a of $E(N') - \{e, f\}$. Furthermore, by the dual of Lemma 6.5, $a \in E(N_0)$. We deduce that N_0 has a 2-cocircuit, say $\{a, b\}$. In N_0, add an element c in parallel to a to get N_1. Then, by transitivity and Lemma 6.5, there is an element s_1 of $E(N_1) - \{a, c\}$ such that $N_1 \backslash s_1 \cong N_0$. Since $N_1 \backslash b$ has $\{a, c\}$ as a component, the component sizes of $N_1 \backslash b$ and N_0 do not match, so $s_1 \neq b$. Thus $s_1 \in E(N) - \{a, b, c\}$, so $N_1 \backslash s_1$ has $\{c, a, b\}$ as a cocircuit. Next add an element d to $N_1 \backslash s_1$, putting it in series with c. Let the resulting matroid be N_2. By the dual of Lemma 6.5, there is an element s_2 of $E(N_2) - \{c, d\}$ such that $N_2/d \cong N_0$. Moreover, s_2 must be in a 2-cocircuit of N_2, and s_2 is in a triangle in N_2 as N_2/s_2 must have a 2-circuit that is not present in N_2 since adding d destroyed the 2-circuit $\{a, c\}$. Now $s_2 \neq a$ since N_2/a has $\{c, d\}$ as a component.

Suppose $s_2 = b$. Then b is in a 2-cocircuit $\{b, e\}$ in N_2. Moreover, N_2 has a triangle T containing b. By orthogonality, $T = \{b, e, a\}$. Then (d, c, a, e) is a fan X in N_2/b having $\{c, d\}$ as a cocircuit and $\{a, e\}$ as a circuit. By Lemma 6.6, $X = E(N_2/b)$, so $N_0 \cong N_2/b \cong M(\mathcal{W}_2)$; a contradiction.

We now know that $s_2 \neq b$, so $s_2 \notin \{a, b, c, d\}$. Thus N_0 has (d, c, a, b) as a special fan. Among all the special fans of N_0 and N_0^*, take one, $(a_1, a_2, \ldots, a_k)$, with the maximum number of elements. Then $k \geq 4$. First assume $\{a_{k-2}, a_{k-1}, a_k\}$ is a triad. Suppose $\{a_{k-1}, a_k\}$ is a 2-circuit of N_0. Then, by Lemma 6.6, the special fan is the whole component N_0. As $N_0 \not\cong M(\mathcal{W}_2)$, we see that $k \geq 6$. Add an element f in parallel to a_3 to form a new matroid N_0'. Then $\{a_1, a_3\}$ is in an N_0-minor of N_0', and so is $\{a_1, f\}$. By the transitivity property, N_0' has $\{a_3, f\}$ in an N_0-minor. Since N_0' has $\{a_3, f\}$ and $\{a_{k-1}, a_k\}$ as its only 2-circuits, while N_0 has a single 2-circuit, we deduce that $N_0' \backslash a_k \cong N_0$. But every element of N_0 is in a cocircuit of size at most three, yet f is in no such cocircuit of $N_0' \backslash a_k$; a contradiction.

It remains to deal with the cases when, in N_0, either $\{a_{k-2}, a_{k-1}, a_k\}$ is a triad and $\{a_{k-1}, a_k\}$ is not a circuit, or $\{a_{k-2}, a_{k-1}, a_k\}$ is a triangle. In these cases, add a_0 in parallel with a_1 to produce N_3. To obtain an N_0-minor of N_3 using $\{a_0, a_1\}$, we must delete an element z of N_3 that belongs to a 2-circuit. Now z is not in $\{a_2, a_3, \ldots, a_k\}$ as none of these elements is in a 2-circuit, so $N_3 \backslash z$ is isomorphic to N_0 and has $(a_0, a_1, \ldots, a_k)$ as a special fan. This contradicts our assumption that a special fan in N_0 or N_0^* has at most k elements. We conclude that 6.7.1 holds.

6.7.2. *N has no single-element component.*

To see this, let N_0 be a largest component of N. By 6.7.1, N_0 is isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$. Assume that N has a single-element component N_1 with $E(N_1) = \{a\}$.

By replacing N by its dual if necessary, we may assume that a is a coloop of N. Let c be an element that is in a 2-cocircuit of N_0. Now let N' be obtained from N by adding an element b so that N' has $\{a, b, c\}$ as a triangle and $\{a, b\}$ as a cocircuit. Then, by the transitivity property, $N' \backslash g \cong N$ for some element g not in $\{a, b\}$. By the choice of N_0, we deduce that g must be in the same component N_0' of N' as $\{a, b, c\}$. Moreover, g must be in a 2-cocircuit of N_0'. But N_0' contains no such element. Hence 6.7.2 holds.

6.7.3. *N has a single component of maximum size.*

Assume that this fails, and let N_0 and N_1 be components of N of maximum size. Let $\{a_i, b_i\}$ be a 2-circuit of N_i. Let N_i' be obtained from N_i by adding c_i in series with b_i. Now take a copy of $U_{2,3}$ with ground set $\{c_0, z, c_1\}$ and adjoin N_0' and N_1' via parallel connection across c_0 and c_1, respectively. Truncate the resulting matroid to get N_{01}. Then $r(N_{01}) = r(N_0) + r(N_1) + 1$. Let N' be obtained from N by replacing $N_0 \oplus N_1$ by N_{01}. Now N_{01}/c_0 and N_{01}/c_1 have $(N_0 \oplus N_1)$-minors using $\{z, c_1\}$ and $\{z, c_0\}$, respectively. Hence N'/c_0 and N'/c_1 have N-minors using $\{z, c_1\}$ and $\{z, c_0\}$. Thus, by transitivity, N' has an N-minor $\tilde{N}$ using $\{c_0, c_1\}$. As $r(N') = r(N) + 1$, there are elements e, f, and g of $E(N') - \{c_0, c_1\}$ such that $\tilde{N} = N'/e\backslash f, g$. Now N'/e must have two disjoint 2-circuits that are not in N'. Thus $e \in E(N_{01})$. As $e \notin \{c_0, c_1\}$, it follows that $N_0 \cong M(\mathcal{W}_2) \cong N_1$ and, by symmetry, we may assume that $e = a_0$. But N_{01}/a_0 does not have an $(M(\mathcal{W}_2) \oplus M(\mathcal{W}_2))$-minor. Thus 6.7.3 holds.

By 6.7.1 and 6.7.3, N has a single largest component N_0 and it is isomorphic to $M(\mathcal{W}_2)$. As N is disconnected, we may assume by duality that N has a component N_1 that is isomorphic to $U_{1,k}$ for some k in $\{2, 3\}$. Now take a copy of $U_{2,3}$ with ground set $\{c_0, z, c_1\}$ and adjoin copies of $U_{2,k+1}$ via parallel connection across c_0 and c_1, letting the resulting matroid be N_{01}. Replacing $N_0 \oplus N_1$ by N_{01} in N to give N', we see that $r(N') = r(N) + 1$. Moreover, N'/c_0 and N'/c_1 have N-minors using $\{c_1, z\}$ and $\{c_0, z\}$, respectively. But c_0 and c_1 are the only elements e of N' such that N'/e has two disjoint 2-circuits that are not in N'. Thus N' has no N-minor using $\{c_0, c_1\}$. This contradiction completes the proof of the theorem. □

We conclude this section by proving Corollary 1.3, which demonstrates how two of the basic properties of matroid connectivity are enough to characterize it.

Proof of Corollary 1.3. Assume that $N \ncong U_{1,2}$. Then, by Theorem 1.1 and duality, we may assume that $N \cong U_{2,2}$. But $U_{2,2}$ does not have the transitivity property as the matroid $U_{1,2} \oplus U_{1,1}$ shows. □

7. Three-element sets

The notion of N-connectivity defined here relies on sets of two elements. Sets of size three have already been an object of some study. Seymour asked whether every 3-element set in a 4-connected non-binary matroid belongs to a $U_{2,4}$-minor but Kahn [5] and Coullard [3] answered this question negatively. Seymour [12] characterized the internally 4-connected binary matroids that are $U_{2,3}$-connected, but the problem of completely characterizing when every triple of elements in an internally 4-connected matroid is in a $U_{2,3}$-minor remains open [8, Problem 15.9.7].

For a 3-connected binary matroid M having rank and corank at least three, Theorem 4.3 shows that every triple of elements of M is in an $M(K_4)$-minor. The next result extends this theorem to connected binary matroids. As the proof, which is based on Lemma 2.6, is so similar to those appearing earlier, we omit the details.

Proposition 7.1. *Let M be a connected binary matroid. For every triple $\{x, y, z\} \subseteq E(M)$, there is an $M(K_4)$-minor using $\{x, y, z\}$ if and only if every matroid in the canonical tree decomposition of M has rank and corank at least* 3*.*

References

[1] R. E. Bixby, *l*-matrices and a characterization of non-binary matroids, *Discrete Math.* **8** (1974), 139–145.
[2] J. E. Bonin and W. R. Schmitt, Splicing matroids, *European J. Combin.* **32** (2011), 722–744.
[3] C. R. Coullard, Counterexamples to conjectures on 4-connected matroids, *Combinatorica* **6** (1986), 315–320.
[4] W. H. Cunningham, *A combinatorial decomposition theory*, Ph.D. thesis, University of Waterloo, 1973.
[5] J. Kahn, A problem of P. Seymour on nonbinary matroids, *Combinatorica* **5** (1985), 319–323.
[6] T. Moss, A minor-based characterization of matroid 3-connectivity, *Adv. in Appl. Math.* **50** (2013), 132–141.
[7] J. G. Oxley, On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.
[8] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.
[9] J. Oxley and J. Taylor, On two classes of nearly binary matroids, *European J. Combin.* **36** (2014), 251–260.
[10] P. D. Seymour, On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.
[11] P. D. Seymour, Minors of 3-connected matroids, *European. J. Combin.* **6** (1985), 375–382.
[12] P. D. Seymour, Triples in matroid circuits, *European J. Combin.* **7** (1986), 177–185.
[13] W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: `zgersh2@math.lsu.edu`

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: `oxley@math.lsu.edu`