TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 300, Number 2, April 1987

ON NONBINARY 3-CONNECTED MATROIDS

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ABSTRACT. It is well known that a matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$, the 4-point line. Extending this result, Bixby proved that every element in a nonbinary connected matroid is in a $U_{2,4}$-minor. The result was further extended by Seymour who showed that every pair of elements in a nonbinary 3-connected matroid is in a $U_{2,4}$-minor. This paper extends Seymour's theorem by proving that if $\{x, y, z\}$ is contained in a nonbinary 3-connected matroid M, then either M has a $U_{2,4}$-minor using $\{x, y, z\}$, or M has a minor isomorphic to the rank-3 whirl that uses $\{x, y, z\}$ as its rim or its spokes.

1. Introduction. This paper proves a number of results on the structure of nonbinary 3-connected matroids. In [**8**], Seymour showed that if M is such a matroid and $\{x, y\} \subseteq E(M)$, then M has a $U_{2,4}$-minor that *uses* $\{x, y\}$, that is, M has a minor isomorphic to $U_{2,4}$ whose ground set contains $\{x, y\}$. Seymour also conjectured [**8**, **10**] that if M is nonbinary and 4-connected and $\{x, y, z\} \subseteq E(M)$, then M has a $U_{2,4}$-minor using $\{x, y, z\}$. In addition, he noted that this conjecture fails for certain 3-connected matroids such as the non-Fano matroid. This paper does not attack Seymour's conjecture directly but instead concentrates on characterizing precisely when the conjecture fails for nonbinary 3-connected matroids. The author has learned, since the original submission of this paper, that Seymour's conjecture has now been disproved by Kahn [**4**].

Most of the matroid terminology used here will follow Welsh [**12**]. The ground set, rank, and corank of the matroid M will be denoted by $E(M)$, $\operatorname{rk} M$, and $\operatorname{cork} M$ respectively. For an arbitrary subset T of $E(M)$, $\operatorname{rk} T$ and $\operatorname{cork} T$ will denote the rank and corank of T. The deletion of T from M will be denoted by $M \setminus T$ or $M|(E(M) - T)$, and the contraction of T from M by M/T or $M \cdot (E(M) - T)$. Flats of M of ranks one and two will be called *points* and *lines*. A 3-element circuit of M will be called a *triangle* and a 3-element cocircuit a *triad*.

If N and M are matroids on S and $S \cup e$ respectively and $e \notin S$, then M is an *extension* of N if $M \setminus e = N$, and M is a *lift* of N if M^* is an extension of N^*. We call M a *nontrivial extension* of N if e is neither a loop nor a coloop of M and e is not in a 2-element circuit of M. Likewise, M is a *nontrivial lift* of N if M^* is a nontrivial extension of N^*.

Received by the editors August 2, 1984. Presented at the Special Session on the Structure of Graphs and Matroids at the American Mathematical Society Summer Meeting in Eugene, Oregon, August, 1984.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05B35.
This research was partially supported by an LSU Faculty Summer Research Grant.

If k is a positive integer, the matroid M is *k-separated* if there is a subset T of $E(M)$ such that $|T| \geq k$, $|E(M) - T| \geq k$, and

$$\operatorname{rk} T + \operatorname{rk}(E(M) - T) - \operatorname{rk} M = k - 1.$$

When this occurs, $(T, E(M) - T)$ is an *exact k-separation* of M. We say that M is *n-connected* if M is not k-separated for any $k < n$. Thus a matroid is 2-connected precisely when it is connected [**12**, p. 69]. Moreover,

(1.1) *M is n-connected if and only if M^* is n-connected.*

We shall assume familiarity with the operations of series and parallel connection of matroids; a detailed discussion of these operations and their properties may be found in [**3**]. For matroids M_1 and M_2 on disjoint sets, if $p_1 \in E(M_1)$ and $p_2 \in E(M_2)$, then we shall denote the parallel connection of M_1 and M_2 with respect to the *basepoints* p_1 and p_2 by $P((M_1, p_1), (M_2, p_2))$ or just $P(M_1, M_2)$. If, for $i = 1, 2$, M_i has at least 3 elements and p_i is neither a loop nor a coloop of M_i, then $P((M_1, p_1), (M_2, p_2)) \setminus p$ will be called the 2-*sum* of M_1 and M_2 [**7**, p. 308]. In that case, M_1 and M_2 are the *parts* and p_1 and p_2 the *basepoints* of the 2-sum. The following fundamental link between 3-connection and 2-sums was proved by Seymour [**7**, (2.6)].

(1.2) THEOREM. *If (X_1, X_2) is an exact 2-separation of the matroid M, then there are matroids M_1 and M_2 on $X_1 \cup p_1$ and $X_2 \cup p_2$ respectively, where p_1 and p_2 are new elements, such that M is the 2-sum of M_1 and M_2 with respect to the basepoints p_1 and p_2. Conversely, if M is the 2-sum of M_1 and M_2, then $(E(M_1) - p_1, E(M_2) - p_2)$ is an exact 2-separation of M, and M_1 and M_2 are isomorphic to minors of M.*

We shall need to use the construction of a minor of M isomorphic to M_1. This proceeds as follows. Let C be a circuit of M meeting both $E(M_1) - p_1$ and $E(M_2) - p_2$. Choose an element z of C. Now delete $E(M_2) - p_2 - C$ and then contract $C - z - E(M_1)$ from M. The resulting minor of M is isomorphic to M_1.

The following two properties of 2-sums are straightforward to check.

(1.3) *The sum of M_1 and M_2 is connected if and only if both M_1 and M_2 are connected.*

(1.4) *If M is the 2-sum of matroids M_1 and M_2 and N is a 3-connected matroid which is a minor of M, then M_1 or M_2 has an N-minor.*

On combining Theorem 1.2 and (1.4) with the excluded-minor characterization of binary matroids, one easily gets that

(1.5) *The 2-sum of M_1 and M_2 is binary if and only if both M_1 and M_2 are binary.*

If $\{x, y\}$ is a circuit of the matroid M, we say that x and y are *in parallel* in M. If, instead, $\{x, y\}$ is a cocircuit, then x and y are *in series*. A *parallel class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct elements of A, then a and b are in parallel. *Series classes* are defined analogously. A series or parallel class is *nontrivial* if it contains more than one element. The matroid M' is a *series extension* of M if $M = M'/T$ and every element of T is in series with some element of M. *Parallel extensions* are defined analogously.

The following result of Bixby [**2**, Theorem 1] will be used repeatedly in the proof of the main theorem of this paper.

(1.6) THEOREM. *Let e be an element of the 3-connected matroid M. Then either $M \setminus e$ is a series extension of a 3-connected matroid or M/e is a parallel extension of a 3-connected matroid.*

Suppose that $r \geq 3$. The *wheel* $\mathcal{W}_r$ *of order* r is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex is joined by a single edge (a *spoke*) to each of the other vertices. The *whirl* $\mathcal{W}^r$ *of order* r is a matroid on $E(\mathcal{W}_r)$ having as its circuits all cycles of $\mathcal{W}_r$ other than the rim, as well as all sets of edges formed by adding a single spoke to the set of edges of the rim. The terms "rim" and "spoke" will be used in the obvious way in both $M(\mathcal{W}_r)$ and $\mathcal{W}^r$. It will also be convenient here to view the matroid $U_{2,4}$ as the whirl $\mathcal{W}^2$. Indeed, if one uses the construction of $\mathcal{W}^r$ from $M(\mathcal{W}_r)$ in the case when $r = 2$, it is easy to see that the resulting matroid is isomorphic to $U_{2,4}$.

The following result of Tutte [**11**, 8.3] indicates the fundamental role that whirls and the cycle matroids of wheels play in the class of 3-connected matroids.

(1.7) THEOREM. *Let M be a 3-connected matroid such that for all elements e neither $M \setminus e$ nor M/e is 3-connected. Then M has rank at least three and is a whirl or the cycle matroid of a wheel.*

An important tool in the proof of Theorem 1.7 and one that will also be needed in this paper is the following.

(1.8) LEMMA [**11**, 7.3]. *Let M be a 3-connected matroid having at least four elements. Suppose that $\{a, b, c\}$ is a triad of M such that neither M/a nor M/b is 3-connected. Then M has a triangle containing a and just one of b and c.*

The next theorem, the main result of this paper, will be proved in §4.

(1.9) THEOREM. *Let M be a 3-connected nonbinary matroid and suppose that $\{x, y, z\} \subseteq E(M)$. Then either M has a $U_{2,4}$-minor using $\{x, y, z\}$ or M has a $\mathcal{W}^3$-minor in which $\{x, y, z\}$ is the rim or the set of spokes.*

A 3-connected matroid M is *internally* 4*-connected* if M has no exact 3-separation (X_1, X_2) with $|X_1|, |X_2| \geq 4$. In [**10**], Seymour characterized precisely when three elements in a 3-connected, internally 4-connected binary matroid are in a circuit. He also noted that the corresponding nonbinary problem is still open. As an immediate consequence of Theorem 1.9, we have the following result for the nonbinary problem.

(1.10) COROLLARY. *Let M be a 3-connected nonbinary matroid and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a circuit containing $\{x, y, z\}$ unless M has a $\mathcal{W}^3$-minor in which $\{x, y, z\}$ is the set of spokes.*

§§2 and 3 of this paper contain several results which are used in the proof of Theorem 1.9. Most of these results are also of interest in their own right. In particular, §3 proves the analogue of the main theorem for arbitrary 3-connected matroids.

2. A whirls theorem. In this section we shall use P. D. Seymour's theory of splitters [**7**] to prove that a nonbinary 3-connected matroid can be obtained from a whirl by a sequence of nontrivial lifts and extensions. The author's original proof of this result did not use splitters and he is indebted to Dr. Seymour for informing

him that the results of this section were already known and could be easily derived using splitters.

Let $\mathcal{F}$ be a class of matroids closed under minors and under isomorphism. A member N of $\mathcal{F}$ is a *splitter* for $\mathcal{F}$ if every 3-connected member of $\mathcal{F}$ having an N-minor is isomorphic to N.

The next result is Seymour's characterization of splitters [**7**, (7.3)].

(2.1) THEOREM. *Let $\mathcal{F}$ be a class of matroids closed under minors and under isomorphism. Suppose that N is a nonempty connected member of $\mathcal{F}$ for which the following conditions hold:*

(i) *both N and N^* are simple;*

(ii) *$\mathcal{F}$ contains no nontrivial extensions of N and no nontrivial lifts of N;*

(iii) *if $N \simeq M(\mathcal{W}_k)$ for some $k \geq 3$, then $M(\mathcal{W}_{k+1}) \notin \mathcal{F}$; and*

(iv) *if $N \simeq \mathcal{W}^k$ for some $k \geq 2$, then $\mathcal{W}^{k+1} \notin \mathcal{F}$.*

Then N is a splitter for $\mathcal{F}$.

An easy consequence of this is the following.

(2.2) THEOREM. *Let M and N be 3-connected matroids such that N is a minor of M, $|E(N)| \geq 4$, and if $N \simeq M(\mathcal{W}_k)$, M has no $M(\mathcal{W}_{k+1})$-minor, while if $N \simeq \mathcal{W}^k$, M has no $\mathcal{W}^{k+1}$-minor. Then there is a sequence $N_0, N_1, N_2, \ldots, N_n$ of 3-connected matroids such that $N_0 \simeq N$, $N_n = M$, and, for all i in $\{0, 1, 2, \ldots, n-1\}$, N_i is a single-element deletion or contraction of N_{i+1}.*

PROOF. Let $\mathcal{F}$ be the class of matroids M_1 that are isomorphic to some minor of M. Then $\mathcal{F}$ is closed under minors and under isomorphism. Let M' be a 3-connected member of $\mathcal{F}$ which is maximal with the property that there is a sequence $M'_0, M'_1, \ldots, M'_m$ of 3-connected matroids such that $M'_0 \simeq N$, $M'_m = M'$, and, for all i in $\{0, 1, 2, \ldots, m-1\}$, M'_i is a single-element deletion or contraction of M'_{i+1}. Then, as $|E(N)| \geq 4$, Theorem 3.1 implies that M' is a splitter for $\mathcal{F}$. Hence $M' \simeq M$.

The following result for whirls, an immediate consequence of the last theorem, will be used in the proof of Theorem 1.9.

(2.3) COROLLARY. *Let M be a nonbinary 3-connected matroid. Then M can be obtained from a whirl by a sequence of nontrivial lifts and nontrivial extensions.*

3. Some structural results. This section contains a number of results which will be needed in the proof of the main theorem. The first of these is a natural extension of Theorem 2.5 of [**6**] for nonbinary matroids. Euclidean representations for the 6-element matroids P_6 and Q_6 are shown in Figure 1.

(3.1) THEOREM. *Let M be a 3-connected nonbinary matroid having rank and corank at least three. Then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$.*

The proof of this theorem will use the following two lemmas.

(3.2) LEMMA. *Let N_1 be a 3-connected matroid having an element e such that $N_1/e \simeq U_{2,k}$ for some $k \geq 5$. Then N_1 has a minor isomorphic to one of $U_{3,6}$, P_6, or Q_6.*

PROOF. We argue by induction on k. If $k = 5$, then N_1 is a nontrivial lift of $U_{2,5}$, and it is not difficult to check that N_1 is isomorphic to one of $U_{3,6}$, P_6, or Q_6. Thus

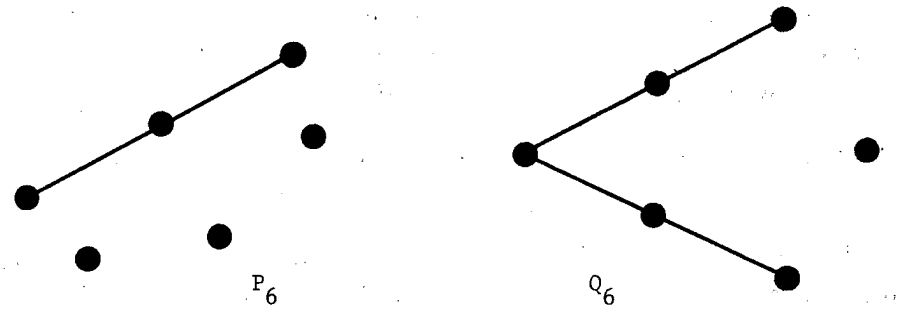

FIGURE 1

the proposition is true for $k = 5$. Now assume it true for $k = n$ where $n \geq 5$ and let $k = n+1$. Suppose that f is an element of $E(N_1/e)$. Then $N_1 \setminus f/e \simeq U_{2,n}$ and the induction assumption implies that, provided $N_1 \setminus f$ is 3-connected, $N_1 \setminus f$ and hence N_1 has a minor isomorphic to one of $U_{3,6}$, P_6, or Q_6. Thus we may assume that, for all elements f of $E(N_1) - \{e\}$, $N_1 \setminus f$ is not 3-connected. But, both N_1 and $N_1 \setminus f/e$ are 3-connected and so, by [**5**, Lemma 2.6], N_1 has a triad containing $\{e, f\}$. Therefore $N_1 \setminus e$ is not 3-connected and so N_1 is minimally 3-connected. Since $\operatorname{rk} N_1 = 3$, it follows by [**5**, Theorem 4.7] that $|E(N_1)| \leq 6$, a contradiction. This completes the proof of the lemma.

(3.3) LEMMA. *Let M be a 3-connected matroid having rank at least three and suppose that M has a $U_{2,5}$-minor. Then M has a 3-connected minor N_1 which has a single-element contraction that is isomorphic to $U_{2,k}$ for some $k \geq 5$.*

PROOF. Let k be the greatest integer m for which M has a $U_{2,m}$-minor. Then $k \geq 5$. Now let N be a $U_{2,k}$-minor of M and apply Theorem 2.2. The choice of N guarantees that no 3-connected extension of it is a minor of M. It follows that M has as a minor a 3-connected lift N_1 of N.

PROOF OF THEOREM 3.1. Since M is nonbinary, it has a $U_{2,4}$-minor N. Now clearly $U_{2,5}$ is the only nontrivial extension of $U_{2,4}$ and, by duality, $U_{3,5}$ is the only nontrivial lift. Thus, on applying Theorem 2.2 and recalling that $U_{2,4} \simeq \mathcal{W}^2$, we obtain that M has a minor isomorphic to one of $\mathcal{W}^3$, $U_{2,5}$, or $U_{3,5}$. In the first case, the theorem is immediate, while in the second and third cases, it follows on applying the combination of Lemmas 3.2 and 3.3 to M and M^* respectively.

Theorem 3.1 can be extended by using the following result.

(3.4) LEMMA. *Let M be a 3-connected matroid and $\{x, y\}$ be a subset of $E(M)$. Suppose that M has a minor isomorphic to a member of $\{U_{3,6}, P_6, Q_6, \mathcal{W}^3\}$. Then M has such a minor using $\{x, y\}$.*

PROOF. We use the main result of Seymour's paper [**9**]. As each of $U_{3,6}$, P_6, Q_6, and $\mathcal{W}^3$ is self-dual, we need only check that if M is a 3-connected matroid such that $M \setminus e$ is isomorphic to one of the four specified matroids and $f \in E(M \setminus e)$, then M has a deletion using $\{e, f\}$ that is isomorphic to one of the four specified matroids. We omit the straightforward checking of cases that is needed to complete the proof.

On combining this lemma with Theorem 3.1, we immediately obtain the following.

(3.5) COROLLARY. *Let M be a 3-connected nonbinary matroid having rank and corank at least three and suppose that $\{x, y\} \subseteq E(M)$. Then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$ that uses $\{x, y\}$.*

The next result corresponds to the main theorem in the case that one does not restrict to nonbinary matroids.

(3.6) THEOREM. *Let M be a 3-connected matroid having rank and corank at least three and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, $\mathcal{W}^3$, or $M(K_4)$ that uses $\{x, y, z\}$.*

PROOF. As M is 3-connected having rank and corank at least three, $|E(M)| \geq 6$ and, by Theorem 2.5 of [**6**], M has a minor isomorphic to one of the five specified matroids. The result therefore follows immediately if $|E(M)| = 6$. We shall argue by induction on $|E(M)|$ so suppose that $|E(M)| > 6$ and that the result is true for all matroids with fewer elements than M. Let $\operatorname{rk} M = 3$. If M has no element e such that $M \setminus e$ is 3-connected, then, by [**5**, Theorem 4.7], $|E(M)| \leq 6$; a contradiction. Hence, M does have such an element e. If $e \notin \{x, y, z\}$, then the result follows by the induction assumption. Thus we may suppose, without loss of generality, that $e = x$. By [**6**, Theorem 4.2], $M \setminus x$ has a restriction isomorphic to one of $U_{3,6}$, P_6, Q_6, $\mathcal{W}^3$, or $M(K_4)$ that uses $\{y, z\}$. It is now a straightforward matter to check that in all five of these cases the required result holds. We note that the case checking required here is very similar to that required in the proof of Lemma 3.4. We conclude that the theorem holds if M has rank 3 and, by duality, it also holds if M has corank 3. We shall now assume that both the rank and corank of M exceed 3.

By Theorem 1.6, for all elements e of M, either $M \setminus e$ is a series extension of a 3-connected matroid or M/e is a parallel extension of a 3-connected matroid. Choose e in $E(M) - \{x, y, z\}$ and suppose that $M \setminus e$ is a series extension of the 3-connected matroid N and that x, y, and z are in different series classes of N. Then we may assume that $\{x, y, z\} \subseteq E(N)$. Now either

(i) N has rank at least 3; or,

(ii) N has rank 2.

In both cases, since $\operatorname{cork} M > 3$, $\operatorname{cork} N \geq 3$. The result therefore follows in the first case by applying the induction assumption to N. In the second case, $N \simeq U_{2,k}$ for some k and, since $\operatorname{cork} N \geq 3$, $k \geq 5$. Now as $\operatorname{rk} M \geq 4$ and $\operatorname{rk} N = 2$, $M \setminus e$ has at least one nontrivial series class. Choose x_1 and x_2 in this class taking x_1 equal to x, y, or z if one of these elements is in the class. Contract all the elements from this class except x_1 and x_2 and then contract all but one element from each of the other nontrivial series classes of $M \setminus e$ ensuring that x, y, and z are kept. The resulting contraction of M has rank 3 and has $\{e, x_1, x_2\}$ as a cocircuit. Now delete all but three elements, a, b, and c, from the line which is complementary to $\{e, x_1, x_2\}$ again ensuring that x, y, and z are kept.

In the resulting matroid M', the closure of $\{x_1, x_2\}$ does not contain e otherwise e is in the closure in M of the series class S_1 of $M \setminus e$ containing $\{x_1, x_2\}$. In that case, $\operatorname{rk}(S_1 \cup e) = \operatorname{rk} S_1$ and if $S_2 = E(M \setminus e) - S_1$, then $\operatorname{rk} M = \operatorname{rk}(M \setminus e) = \operatorname{rk} S_2 + |S_1| - 1$ and $\operatorname{rk} S_1 = |S_1|$. Thus $\operatorname{rk} S_1 + \operatorname{rk} S_2 = \operatorname{rk} M + 1$, so $\operatorname{rk}(S_1 \cup e) + \operatorname{rk} S_2 = \operatorname{rk} M + 1$, a contradiction to the fact that M is 3-connected. We conclude that, in M', $\{e, x_1, x_2\}$

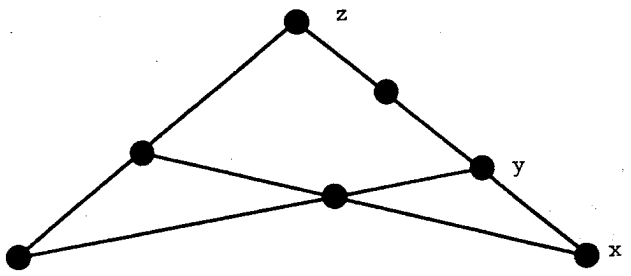

FIGURE 2

is a basis. It follows without difficulty that M' is 3-connected, hence the required result holds in case (ii).

We may now assume that, for all elements e of $E(M) - \{x, y, z\}$, either at least two of x, y, and z are in series in $M \setminus e$, or at least two of x, y, and z are in parallel in M/e. If x and y are in series in $M \setminus f_1$ and x and z are in parallel in M/f_2 where $f_2 \neq f_1$, then $\{x, y, f_1\}$ is a cocircuit of M and $\{x, z, f_2\}$ is a circuit of M. But $|\{x, y, f_1\} \cap \{x, z, f_2\}| = 1$; a contradiction.

Suppose next that x and y are in series in both $M \setminus f_1$ and $M \setminus f_2$. Then $\{x, y, f_1\}$ and $\{x, y, f_2\}$ are cocircuits of M and hence, by exchange, $\{x, f_1, f_2\}$ and $\{y, f_1, f_2\}$ are cocircuits of M. It follows then that, for f_3 in $E(M) - \{x, y, z, f_1, f_2\}$, no pair of x, y, and z can be in a parallel class in M/f_3, otherwise we will again get a circuit and a cocircuit of M having exactly one common element.

As $|E(M) - \{x, y, z\}| \geq 5$, it follows that either, for all e in $E(M) - \{x, y, z\}$, at least two of x, y, and z are in series in $M \setminus e$, or, for all such e, at least two of x, y, and z are in parallel in M/e. In the first case, $\{x, y, z\}$ is spanning in M^*; in the second case, it is spanning in M. Thus we obtain the contradiction that M has rank or corank at most 3, and this completes the proof of the theorem.

The next result comes from applying the last theorem to binary matroids.

(3.7) COROLLARY. *Let M be a 3-connected binary matroid having rank and corank at least three and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a minor isomorphic to $M(K_4)$ that uses $\{x, y, z\}$.*

In view of this result it seems tempting to conjecture the following extension of Corollary 3.5: If M is a 3-connected *nonbinary* matroid having rank and corank at least three and $\{x, y, z\} \subseteq E(M)$, then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$ using $\{x, y, z\}$. However, the matroid shown in Figure 2 satisfies the hypotheses of this conjecture but not its conclusion. Theorem 1.9 arises fairly naturally when one attempts to modify this conjecture in light of the above example.

The next result characterizes those connected nonbinary matorids having an element which is used by every $U_{2,4}$-minor.

(3.8) THEOREM. *Let M be a nonbinary connected matroid such that for some element e of M both $M \setminus e$ and M/e are binary. Then M is obtained from a 4-point line having ground set $\{e, e_1, e_2, e_3\}$ by a sequence of at most three 2-sums where the basepoints of these 2-sums are e_1, e_2, and e_3, the other part of each 2-sum is connected and binary, and each of e_1, e_2, and e_3 is the basepoint of at most one of these 2-sums.*

PROOF. Evidently M has rank and corank at least two. We shall suppose first that M is 3-connected. If M has rank two, then $M \simeq U_{2,k}$ for some $k \geq 4$. If

$k > 4$, then M does not have the required property. Thus if M has rank 2, then the required result holds. By duality, if M has corank two, then $M^* \simeq U_{2,4}$ and so $M \simeq U_{2,4}$.

We may now assume that if M is 3-connected, then $\operatorname{rk} M$, $\operatorname{cork} M \geq 3$. In that case, by Corollary 3.5, M has a minor N isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$ that uses e. It is straightforward to check that, in each case, one of $N \setminus e$ and N/e is nonbinary. Hence we obtain the contradiction that one of $M \setminus e$ and M/e is nonbinary.

We may now assume that M is not 3-connected. Then

$$M = P((M_1, p_1), (M_2, p_2)) \setminus p$$

for some matroids M_1 and M_2. If M_i is nonbinary for $i = 1$ and 2, then as M_i is connected, by [**1**, Theorem 3.7], it has a $U_{2,4}$-minor using p_i. Thus M has a minor isomorphic to the 2-sum of two copies of $U_{2,4}$ and so one of $M \setminus e$ and M/e is nonbinary; a contradiction. It follows that we may assume that M_1 is nonbinary and M_2 is binary and that $|E(M_2)|$ is maximum subject to these conditions. Clearly $e \in E(M_1)$. If M_1 is 3-connected, then $M_1 \simeq U_{2,4}$. If M_1 is not 3-connected, it is the 2-sum of a nonbinary and a binary matroid. Again choose the binary part of M_1 to have cardinality as large as possible. If the nonbinary part of M_1 is not 3-connected, then decompose it. We may repeat such decompositions, in each case choosing the binary part to have maximum cardinality, until we obtain $U_{2,4}$ as one of the parts of the 2-sum. The other part of this 2-sum must be a connected binary matroid. One of the three elements of $U_{2,4}$ different from e is the basepoint of this 2-sum. In building back up to M, it follows, by the fact the cardinality of the binary part of each 2-sum was chosen to be maximum, that M can indeed be obtained from $U_{2,4}$ as described.

(3.9) COROLLARY. *Let M be a 3-connected nonbinary matroid having an element e such that both $M \setminus e$ and M/e are binary. Then $M \simeq U_{2,4}$.*

4. The main theorem. In this section we shall prove Theorem 1.9, the main result of this paper. This theorem extends the following result of Seymour [**8**, (3.1)].

(4.1) THEOREM. *Let M be a nonbinary 3-connected matroid and suppose that $\{x, y\} \subseteq E(M)$. Then M has a $U_{2,4}$-minor using $\{x, y\}$.*

The proof of Theorem 1.9 is a long induction argument and will be presented as a sequence of lemmas. The basic idea of the proof is to find, in a minimal counterexample M, elements f that are not in $\{x, y, z\}$, such that both $M \setminus f$ and M/f are nonbinary. Given such an element, we apply Theorem 1.6 to get that either $M \setminus f$ is a nonbinary series extension of a 3-connected matroid or M/f is a nonbinary parallel extension of a 3-connected matroid. But then the induction assumption can be applied to one of $M \setminus f$ and $M^* \setminus f$ unless x, y, and z do not all lie in different series classes. In the exceptional case, if one has sufficiently many such elements f, then one can deduce enough about the structure of M to get the theorem. In order to find elements f with the required property, we shall rely on the following lemma.

(4.2) LEMMA. *Suppose that M is the 2-sum of M_1 and M_2 where M_2 is nonbinary. If f is an element of M_1 that is neither in series nor in parallel with p_1, then both $M \setminus f$ and M/f are nonbinary.*

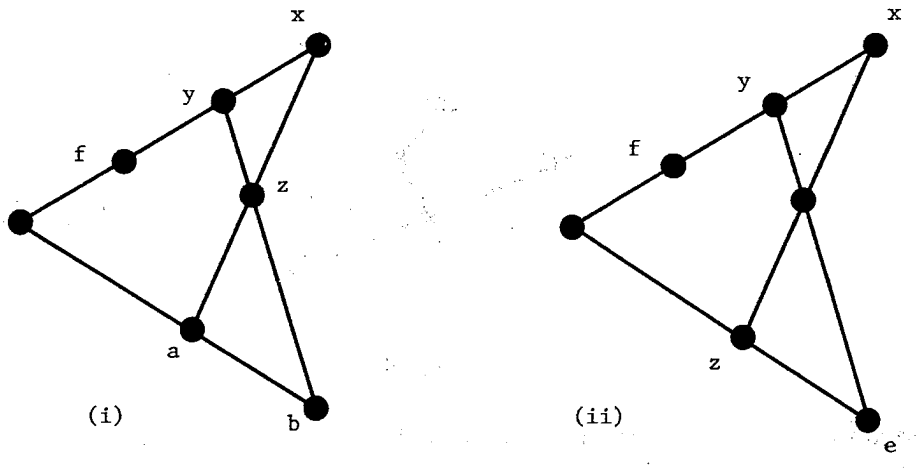

FIGURE 3

PROOF. The result is immediate if f and p_1 are in different components of M_1. Thus assume that f and p_1 are in the same component of M_1. As f and p_1 are not in series, M_1 has a circuit containing p_1 and avoiding f. Thus $M \setminus f$ has an M_2-minor and so is nonbinary. On applying the same argument to M^*, we get that $M^* \setminus f$ and hence M/f is nonbinary.

The next lemma proves the theorem in the case that two of x, y, and z lie on a line in M having at least four points.

(4.3) LEMMA. *If M satisfies the hypotheses of Theorem* 1.9 *and, in addition, has a restriction isomorphic to $U_{2,4}$ that uses two of x, y, and z, then M satisfies the conclusion of Theorem* 1.9.

PROOF. We shall suppose that M has a $U_{2,4}$-restriction using $\{x, y\}$. If M has rank 2, then the lemma is immediate. Since M has $U_{2,4}$ as a restriction, M cannot have corank 2 unless $M \simeq U_{2,4}$. It follows that we may assume that both the rank and corank of M exceed two. Then as M is 3-connected, by Theorem 3.6, M has a minor N isomorphic to one of $M(K_4)$, $U_{3,6}$, $\mathcal{W}^3$, P_6 or Q_6 that uses $\{x, y, z\}$. In the last four cases, it is routine to check that the lemma holds. We note, for future reference, that in these cases the existence of a $U_{2,4}$-restriction of M using two of x, y, and z is not needed to obtain the conclusion. In the remaining case, $N \simeq M(K_4)$.

Let L be the line of M containing $\{x, y\}$. Then, in forming N from M, no element of L can be contracted. The lemma is immediate if $z \in L$, so we assume that this is not so. If x and y are on one of the 3-point lines of N, then choose an element f of $L - E(N)$ and let N' be the extension of N by f. Without loss of generality, we may assume that x, y, and z are as shown in Figure 3(i) or (ii). In the first case, on labelling the elements a and b as shown, we get that $N'|\{x, y, z, a, b, f\} \simeq \mathcal{W}^3$ and has $\{x, y, z\}$ as its set of spokes. In the second case, it is clear that N'/e has a $U_{2,4}$-restriction using $\{x, y, z\}$.

If x and y are not on one of the 3-point lines of N, then we extend N by adding two elements of $L - E(N)$. This can be done in several different ways but it is routine to show that, in each case, M has a $U_{2,4}$-minor using $\{x, y, z\}$.

PROOF OF THEOREM 1.9. Let M be a nonbinary 3-connected matroid which is a minor-minimal counterexample to the theorem. Then M^* is also a minimal

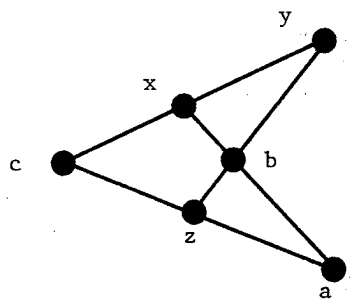

FIGURE 4

counterexample to the theorem. If $\operatorname{rk} M = 2$, then $M \simeq U_{2,k}$ for some $k \geq 4$ and so the theorem holds for M; a contradiction. By duality, if $\operatorname{cork} M = 2$, then the theorem also holds for M. Thus $\operatorname{rk} M$, $\operatorname{cork} M \geq 3$. Suppose that M has rank 3. Then, by Theorem 3.6, M has a restriction isomorphic to one of $U_{3,6}$, $\mathcal{W}^3$, P_6, Q_6, or $M(K_4)$ that uses $\{x, y, z\}$. It was noted in the preceding proof that, in the first four cases, the theorem holds. Thus M has a restriction N isomorphic to $M(K_4)$ that uses $\{x, y, z\}$. Since M is nonbinary, it is not isomorphic to $M(K_4)$ or the Fano matroid. It therefore has an element e such that the extension N' of N by e is nonbinary. Evidently N' is 3-connected. Therefore, as $N' \setminus e$ is binary, Corollary 3.9 implies that N'/e is nonbinary. But N'/e has rank two and therefore has a $U_{2,4}$-restriction using $\{x, y, z\}$ unless e is collinear with two of x, y, and z in N'. By Lemma 4.3, N' has no 4-point line using two of x, y, and z. It follows that, without loss of generality, we may assume that x, y, and z are as shown in Figure 4, and that e, x, and z are collinear. Now e, a, and y must also be collinear, otherwise N'/a has a $U_{2,4}$-restriction using $\{x, y, z\}$ and the theorem holds for M; a contradiction. Furthermore, b, c, and e are not collinear, otherwise N' is isomorphic to the Fano matroid and hence is binary. Thus $N'|\{x, y, z, b, c, e\} \simeq \mathcal{W}^3$ and has $\{x, y, z\}$ as its set of spokes.

We conclude that if M has rank 3 and, by duality, if M has corank 3, then the theorem holds for M; a contradiction. Thus both the rank and corank of M exceed 3.

As M is nonbinary and 3-connected, by Corollary 2.3, M is obtained from a whirl by a sequence of nontrivial lifts and nontrivial extensions. Now M itself is not a whirl, otherwise, as is easily checked, it is not a counterexample to the theorem. Thus M has an element e such that either $M \setminus e$ or M/e is obtained from a whirl in the manner described. We may assume that the first of these occurs, otherwise we replace M by M^* in the argument that follows. If $e \notin \{x, y, z\}$, then we get a contradiction by applying the induction assumption to $M \setminus e$. Thus, we can assume that $e = x$. Notice that this assumption distinguishes x from y and z.

The rest of the proof of Theorem 1.9 will consist of a sequence of eight lemmas. For some of these, both the lemma and its dual will be used in the proof of the theorem.

(4.4) LEMMA. *Suppose that a is an element of $E(M) - \{x, y, z\}$ for which $M \setminus a$ is a nonbinary series extension of a 3-connected matroid. Then a is the unique such element. Moreover, y and z are in the same series class of $M \setminus a$, this class does not contain x, and $\{a, y, z\}$ is a triad of M.*

PROOF. Let $M\backslash a$ be a series extension of the 3-connected matroid N. If x, y, and z are in distinct series classes of $M \setminus a$, then we may assume that $\{x, y, z\} \subseteq E(N)$. Since the theorem holds for N, it follows that it also holds for M; a contradiction. If x, y, and z are all in the same series class of $M \setminus a$, then every 3-element subset of $\{a, x, y, z\}$ is a triad of M. Thus $M \cdot \{a, x, y, z\} \simeq U_{2,4}$; a contradiction. This leaves the possibility that two of x, y, and z are in the same series class of $M \setminus a$, while the third is in a different series class. As $M \setminus x$ is 3-connected, x is not in a triad of M and therefore x is not in a 2-cocircuit of $M \setminus a$. Thus $\{y, z\}$ is a cocircuit of $M \setminus a$ and hence $\{a, y, z\}$ is a cocircuit of M. To establish the uniqueness of a, suppose that $M \setminus b$ is a nonbinary series extension of a 3-connected matroid where $b \in E(M) - \{x, y, z, a\}$. Then, applying the above argument with b replacing a, we get that $\{b, y, z\}$ is a cocircuit of M and so $M^*|\{a, b, y, z\} \simeq U_{2,4}$. It now follows by applying Lemma 4.3 to M^* that the conclusion of the theorem holds for M^* and hence for M; a contradiction.

(4.5) LEMMA. *Suppose that $a \in E(M) - \{x, y, z\}$ and $M \setminus a$ is a nonbinary series extension of a 3-connected matroid. Then M has no element b different from a, x, y, and z such that both $M \setminus b$ and M/b are nonbinary.*

PROOF. Assume that M has such an element b. By the preceding lemma, $M \setminus b$ is not a series extension of a 3-connected matroid. Thus, by Theorem 1.6, M/b is a parallel extension of a 3-connected matroid. By dualizing the argument given in the preceding lemma, it follows that we may suppose that two of x, y, and z lie in the same parallel class of M/b while the third lies in a different parallel class. Thus $\{b, g, h\}$ is a triangle of M for some subset $\{g, h\}$ of $\{x, y, z\}$. As $\{a, y, z\}$ is a triad of M, it follows that $\{g, h\} = \{y, z\}$, otherwise we have a triangle and a triad meeting in just a single element. Thus $\{b, y, z\}$ is a triangle of M.

We show next that $M \setminus b$ is a series extension of a 3-connected matroid. If not, then by [**2**, Lemma 2], $M \setminus b$ has an exact 2-separation (Y, Z) in which both $|Y|$ and $|Z|$ exceed two. Now $\{y, z\} \not\subseteq Y$, otherwise $(Y \cup b, Z)$ is an exact 2-separation of the 3-connected matroid M; a contradiction. Likewise, $\{y, z\} \not\subseteq Z$. Thus we may assume that $y \in Y$ and $z \in Z$. We may also suppose that $a \in Y$. Then, as $\{a, y, z\}$ is a triad of M, z is a coloop of $M|Z$. Hence $\mathrm{rk}(Z - z) = \mathrm{rk}\, Z - 1$. In addition, $\mathrm{rk}(Y \cup \{z, b\}) = \mathrm{rk}(Y \cup z)$ and therefore

$$\mathrm{rk}(Y \cup \{z, b\}) + \mathrm{rk}(Z - z) = \mathrm{rk}(Y \cup z) + \mathrm{rk}(Z - z) \leq \mathrm{rk}\, Y + \mathrm{rk}\, Z = \mathrm{rk}\, M + 1.$$

Since M is 3-connected, it follows that $|Z - z| \leq 1$, that is, $|Z| \leq 2$; a contradiction. We conclude that $M \setminus b$ is indeed a series extension of a 3-connected matroid. But $M \setminus b$ is also nonbinary and so we have a contradiction to Lemma 4.4.

(4.6) LEMMA. *M does not have three distinct elements a, b, and c that are not in $\{x, y, z\}$ such that both $M \setminus e$ and M/e are nonbinary for all e in $\{a, b, c\}$.*

PROOF. Assume that M does have three such elements, a, b, and c. Then, by Lemma 4.5, none of $M \setminus a$, $M \setminus b$, or $M \setminus c$ is a series extension of a 3-connected matroid. Therefore, by Theorem 1.6, each of M/a, M/b and M/c is a parallel extension of a 3-connected matroid. We obtain the contradiction that the theorem holds for M unless in each of M/a, M/b, and M/c, two of x, y, and z lie in the same parallel class while the third lies in a different parallel class. If $\{j, k\} \subseteq \{x, y, z\}$ and $\{j, k\}$ lies in a parallel class in two of M/a, M/b, and M/c, then, by Lemma

4.3, the theorem holds for M. Thus, we may assume that x and y are parallel in M/a, that y and z are parallel in M/b, and that x and z are parallel in M/c. Then $\{a,x,y\}$, $\{b,y,z\}$, and $\{c,x,z\}$ are triangles of M. The theorem follows unless $M|\{a,b,c,x,y,z\} \simeq M(K_4)$ and $\{a,b,c\}$ is also a triangle of M. In that case, as none of $M \setminus a$, $M \setminus b$, or $M \setminus c$ is 3-connected, the dual of Lemma 1.8 implies that M has a triad T_a containing a and just one of b and c. If $c \in T_a$, then, from the intersection of T_a with the triangles $\{a,x,y\}$ and $\{c,x,z\}$, we get the contradiction that $x \in T_a$. Thus $b \in T_a$ and, from considering intersections with the triangles $\{a,x,y\}$ and $\{b,y,z\}$, we get that $T_a = \{a,b,y\}$. By a similar argument, we have that M has a triad T_c containing c and just one of a and b, and hence that $T_c = \{c,b,z\}$. Now

$$\mathrm{rk}(T_a \cup T_c) + \mathrm{rk}(E(M) - (T_a \cup T_c)) - \mathrm{rk}\, M \leq 3 + \mathrm{rk}\, M - 2 - \mathrm{rk}\, M = 1.$$

But M is 3-connected, hence $|E(M) - (T_a \cup T_c)| \leq 1$. It follows that $|E(M)| = 6$ and $M \simeq M(K_4)$; a contradiction to the fact that M is nonbinary.

(4.7) LEMMA. *Suppose that M has an element e different from x, y, and z such that $M \setminus e$ is nonbinary and is not a series extension of a 3-connected matroid. Then M has two distinct elements f_1 and f_2 that are not in $\{x,y,z\}$ such that all of $M \setminus f_1$, M/f_1, $M \setminus f_2$, and M/f_2 are nonbinary.*

The next two rather straightforward lemmas will be used in the proof of Lemma 4.7.

(4.8) LEMMA. *Let N be a 3-connected matroid. Suppose that $e \in E(N)$ and $N \setminus e$ is the 2-sum of N_1 and N_2 where N_2 is nonbinary. If N_1 has an element q_1 that is parallel to the basepoint p_1, then N/e is nonbinary.*

PROOF. Let $E(N_1) - p_1 = E_1$ and $E(N_2) - p_2 = E_2$. By (1.3), N_2 is connected. Hence the closure of E_2 in N contains q_1. But N is 3-connected, so e is not in the closure of $E_2 \cup \{q_1\}$ in N. Therefore, in $N \setminus (E_1 - \{q_1\})$, the element e is a coloop. Thus

$$N/e \setminus (E_1 - \{q_1\}) \simeq N \setminus e \setminus (E_1 - \{q_1\}) \simeq N_2.$$

Hence N/e is nonbinary and the lemma is proved.

(4.9) LEMMA. *Let e be an element of M different from x, y, and z such that $M \setminus e$ is the 2-sum of two nonbinary matroids M_1 and M_2. Then $E(M) - \{x,y,z,e\}$ contains three distinct elements a, b, and c such that both $M \setminus f$ and M/f are nonbinary for all f in $\{a,b,c\}$.*

PROOF. For i in $\{1,2\}$, let A_i be the subset of $E(M_i) - p_i$ consisting of those elements that are neither in series nor in parallel with p_i in M_i. Then, since M_i is nonbinary, $|A_i| \geq 3$. Thus $|(A_1 \cup A_2) - \{x,y,z\}| \geq 3$. The lemma follows immediately since if $f \in A_1 \cup A_2$, then, by Lemma 4.2, both $M \setminus f$ and M/f are nonbinary.

PROOF OF LEMMA 4.7. As $M \setminus e$ is not a series extension of a 3-connected matroid,

$$M \setminus e = P((M_1,p_1),(M_2,p_2)) \setminus p$$

where M_1 has at least four elements and is not a single circuit [**7**, (5.1)(ii)] and M_2 is nonbinary. Moreover, it follows on combining Lemmas 4.6 and 4.9 that M_1

is binary. We shall prove Lemma 4.7 first in the case when p_1 is parallel with another element q_1 of M_1. As $M \setminus e$ has no 2-circuits, $\{p_1, q_1\}$ is a parallel class of M_1. Now $M \setminus q_1$ is nonbinary since it has M_2 as a minor. Moreover, $M \setminus e/q_1$ has rank at least 3 and is not connected, so M/q_1 cannot be a parallel extension of a 3-connected matroid. Thus, by Theorem 1.6, $M \setminus q_1$ is a series extension of a 3-connected matroid. As $M \setminus q_1$ is nonbinary, it follows by Lemma 4.4 that either

(I) $q_1 \notin \{x, y, z\}$ and $\{q_1, y, z\}$ is a triad of M, or

(II) $q_1 \in \{x, y, z\}$.

In case I, $\{q_1, y, z\}$ is also a triad of $M \setminus e$, otherwise $M^*|\{e, q_1, y, z\} \simeq U_{2,4}$ and we obtain a contradiction by applying Lemma 4.3 to M^*. As $\{p_1, q_1\}$ is a circuit of M_1, $\{q_1, y, z\}$ is not a cocircuit of M_1. Thus $\{q_1, p_1\}$, $\{q_1, y, p_1\}$, or $\{q_1, z, p_1\}$ is a cocircuit of M_1. But, as M_1 is connected and $|E(M_1)| \geq 4$, it cannot have $\{p_1, q_1\}$ as both a circuit and a cocircuit. Hence, without loss of generality, we may assume that $\{q_1, y, p_1\}$ is a cocircuit of M_1. Then $\{p_2, z\}$ is a cocircuit of M_2 and so $z \in E(M_2)$. If $E(M_1) = \{x, y, p_1, q_1\}$, then x is in a 2-cocircuit of M_1 avoiding p_1 and we get the contradiction that x is in a triad of M. Therefore, $E(M_1) - \{x, y, p_1, q_1\}$ is nonempty and, by Lemma 4.2, if f is in this set, then both $M \setminus f$ and M/f are nonbinary. But $M \setminus q_1$ is a nonbinary series extension of a 3-connected matroid. Hence, by Lemma 4.5, we obtain a contradiction. Thus case I cannot occur.

Consider case II. If we can show that $E(M_1) - \{x, y, z, p_1\}$ contains at least two elements, then, by Lemma 4.2, we may take f_1 and f_2 to be two elements of this set. We note that we could take $f_2 = e$ for $M \setminus e$ is certainly nonbinary and, by Lemma 4.8, M/e is also nonbinary. Thus to prove this lemma, we need only find a single element in $E(M_1) - \{x, y, z, p_1\}$. However, for use in the proof of Lemma 4.11, we shall determine when we may need to take one of f_1 and f_2 equal to e. If $E(M_1) = \{x, y, z, p_1\}$, then M_1 and hence M has $\{x, y, z\}$ as a triangle and the theorem follows easily by Theorem 4.1; a contradiction. If $E(M_1) = \{x, y, z, p_1, f\}$ for some element f, then, as noted above, $\{x, y, z\}$ cannot be a triangle of M_1. It follows, since x is not in a triad of M, that M_1 is obtained from a 4-element circuit on $\{x, y, z, f\}$ by adding p_1 in parallel with x. Thus $M^*|\{e, f, y, z\} \simeq U_{2,4}$ and we get a contradiction by applying Lemma 4.3 to M^*. If $E(M_1) = \{p_1, f, q_1, s\}$ where $\{q_1, s\} \subseteq \{x, y, z\}$, then M_1 is obtained from a triangle on $\{q_1, f, s\}$ by adding p_1 in parallel with q_1. Since x is not in a triad of M, $s \neq x$. Now, by Lemma 4.2, both $M \setminus f$ and M/f are nonbinary. In this case, we take $f_1 = f$ and $f_2 = e$.

This finishes the proof of the lemma in the case that p_1 is parallel to some other element of M_1. From now on, we shall assume that this does not occur. We complete the proof of the lemma by first establishing the existence of the element f_1. We then extend the argument to obtain f_2. Now f_1 certainly exists unless every element of $E(M_1) - \{p_1, x, y, z\}$ is in series with p_1. In the exceptional case, we consider M_1^*. It has a parallel class P containing p_1 and has at least two and at most three other elements. Since M_1 has no 2-circuits, M_1^* has no 2-cocircuits. Moreover, M_1^* is connected. Thus M_1^* has rank 2 and $\{x, y, z\} = E(M_1^*) - P$. Hence $\{x, y, z\}$ is a triad of M_1^* and so $\{x, y, z\}$ is a triangle of M_1 and hence of M. It follows, by Theorem 4.1, that the theorem holds for M; a contradiction. This establishes the existence of an element f_1 with the properties claimed.

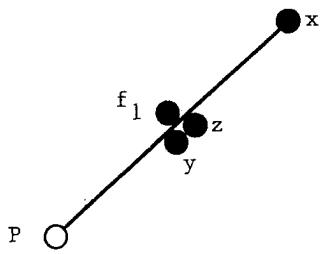

FIGURE 5

To establish the existence of the element f_2, we shall begin with the assumption that this element does not exist. Thus consider $M \setminus e$ as before and assume that f_1 is the only element of $E(M_1) - \{p_1, x, y, z\}$ that is not in series with p_1. As in the case when p_1 was parallel to one of x, y, and z, we shall try to choose f_2 distinct from e and we shall note when this may not be possible.

As before we consider M_1^*. It contains the element f_1, some subset of $\{x, y, z\}$, and a parallel class P containing p_1. We know that M_1^* is connected and has no 2-cocircuits. We look first at the case when all of x, y, and z are in $E(M_1^*) - P$. Then $\operatorname{rk} M_1^* \leq 3$. Suppose that $\operatorname{rk} M_1^* = 3$. As $\{x, y, z\}$ is not a triad of M_1^*, the complement of the line of M_1^* containing P and f_1 is a cocircuit of M_1^* properly contained in $\{x, y, z\}$; a contradiction. Thus $\operatorname{rk} M_1^* = 2$. As M_1^* has no 2-circuit containing x and no $U_{2,4}$-minor, M_1^* is as shown in Figure 5. But then $M^*|\{e, f_1, y, z\} \simeq U_{2,4}$ and we get a contradiction by Lemma 4.3.

We conclude that $E(M_1^*) - P \not\supseteq \{x, y, z\}$. If $E(M_1^*) - P$ contains only one element of $\{x, y, z\}$, then f_2 exists or else we get the contradiction that M_1^* has a 2-cocircuit. Thus $E(M_1^*) - P$ contains f_1 together with exactly two elements, say g and h, of $\{x, y, z\}$. Moreover, M_1^* has rank 2 and is binary. Hence M_1^* is a line having three points, P, $\{a, b\}$, and $\{c\}$, where $\{a, b, c\} = \{f_1, g, h\}$. Therefore $\{e, a, b\}$ is a triad of M and so $x \notin \{a, b\}$. Now look at $M \setminus f_1$. If it is a series extension of a 3-connected matroid, then, by Lemma 4.4, $\{f_1, y, z\}$ is a triad of M. But $\{e, a, b\}$ is one of $\{e, f_1, y\}$, $\{e, f_1, z\}$, or $\{e, y, z\}$. Thus $M^*|\{e, f_1, y, z\} \simeq U_{2,4}$ and we get a contradiction by Lemma 4.3. Hence $M \setminus f_1$ is not a series extension of a 3-connected matroid. Now apply the first part of the proof of this lemma using $M \setminus f_1$ in place of $M \setminus e$ to obtain an element f_2 not in $\{f_1, x, y, z\}$ for which both $M \setminus f_2$ and M/f_2 are nonbinary. We observe that possibly $f_2 = e$ but, in that case, the structure of M_1^* is as indicated above. This completes the proof of Lemma 4.7.

(4.10) LEMMA. *M has two distinct elements f_1 and f_2 that are not in $\{x, y, z\}$ such that all of $M \setminus f_1$, M/f_1, $M \setminus f_2$, and M/f_2 are nonbinary.*

PROOF. By Theorem 4.1, M^* has a $U_{2,4}$-minor using $\{y, z\}$. Therefore, as $\operatorname{rk} M^* \geq 4$, M^* has an element e not in $\{x, y, z\}$ such that M^*/e and hence $M \setminus e$ is nonbinary. If $M \setminus e$ is not a series extension of a 3-connected matroid, then this lemma is immediate from Lemma 4.7. Thus we may assume that $M \setminus e$ is a series extension of a 3-connected matroid. In that case, by Lemma 4.4, $\{e, y, z\}$ is a triad of M and so $\{e, y, z\}$ is a triangle of M^*. By Theorem 4.1, it follows that M^* has a $U_{2,4}$-minor using $\{e, y, z\}$, hence, as $\operatorname{rk} M^* \geq 4$, M^* has an element f not in $\{e, x, y, z\}$ such that M^*/f is nonbinary. Thus $M \setminus f$ is nonbinary. If $M \setminus f$ is not a series extension of a 3-connected matroid, the lemma is immediate from Lemma 4.7. If $M \setminus f$ is a series extension of a 3-connected matroid, then $\{f, y, z\}$ is a triad

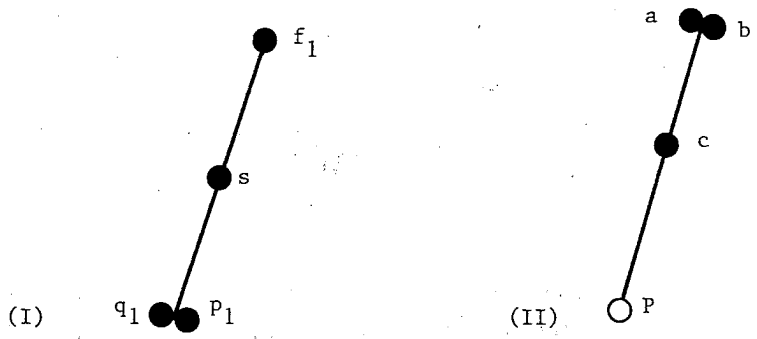

FIGURE 6

of M and so $M^*|\{e,f,y,z\} \simeq U_{2,4}$. We now obtain a contradiction by applying Lemma 4.3 to M^*.

On combining the next result with the preceding lemma, Theorem 1.9 will follow.

(4.11) LEMMA. *$E(M) - \{x,y,z\}$ does not contain two distinct elements f_1 and f_2 such that all of $M \setminus f_1$, M/f_1, $M \setminus f_2$, and M/f_2 are nonbinary.*

PROOF. We assume that the elements f_1 and f_2 do exist. By Lemma 4.5 and Theorem 1.6, both M/f_1 and M/f_2 are parallel extensions of 3-connected matroids, and neither $M \setminus f_1$ nor $M \setminus f_2$ is a series extension of a 3-connected matroid. Moreover, by Lemma 4.3 and the dual of the argument in Lemma 4.4, two of x, y, and z form a triangle with f_1, and a different pair forms a triangle with f_2. Now x has been distinguished in $\{x,y,z\}$ by the assumption that $M \setminus x$ is nonbinary and 3-connected. It follows then that, without loss of generality, we may assume that $\{f_1,x,y\}$ is a triangle of M and that either (i) $\{f_2,x,z\}$ or (ii) $\{f_2,y,z\}$ is also a triangle of M. In both cases, if M has an element f_3 that is not in $\{f_1,f_2,x,y,z\}$ such that both $M \setminus f_3$ and M/f_3 are nonbinary, then we obtain a contradiction to Lemma 4.6. A contradiction is also obtained if $\{f_1,x,y,z\}$ contains a triangle other than $\{f_1,x,y\}$. Now consider $M \setminus f_2$. It is nonbinary and is not a series extension of a 3-connected matroid. Thus $M \setminus f_2 = P((M_1,p_1),(M_2,p_2)) \setminus p$ where M_2 is nonbinary, M_1 is not a single circuit, and $|E(M_1)| \geq 4$. We choose M_1 here so that $|E(M_1)|$ is maximum subject to these conditions. Now $f_1 \in E(M_1)$, otherwise the argument of Lemma 4.7 establishes the existence of the required element f_3. From the proof of that lemma, it follows that the element f_3 exists unless either M_1 is as shown in Figure 6(I) where $\{q_1,s\} \subseteq \{x,y,z\}$ and $s \neq x$; or $E(M_1^*) - P$ contains f_1 and exactly two elements of $\{x,y,z\}$ and M_1^* is as shown in Figure 6(II) where $\{a,b,c\} = \{f_1,g,h\}$, $\{g,h\} \subseteq \{x,y,z\}$, and $x \notin \{a,b\}$. In the former case, $\{f_1,q_1,s\}$ is a triangle of M contained in $\{f_1,x,y,z\}$. Thus $\{q_1,s\} = \{x,y\}$. But $s \neq x$, so $q_1 = x$ and $s = y$. Therefore, in case (I), $\{f_1,f_2,y\}$ is a triad of M. This is also true in case (II) for, in that case, $\{a,b,c\}$ is a triad of M_1^* and hence is a triangle of M. Since it is contained in $\{f_1,x,y,z\}$, it must be $\{f_1,x,y\}$. Now $x \notin \{a,b\}$, so $\{a,b\} = \{f_1,y\}$. Hence we do indeed have that in both cases (I) and (II), $\{f_1,y,f_2\}$ is a triad of M. Therefore if $\{f_2,x,z\}$ is a triangle of M it meets this triad in a single element. This contradiction eliminates case (i) above. Hence we need only consider case (ii). It follows that in both cases (I) and (II) we may assume that M_1 is obtained from the cycle matroid of the graph shown in Figure 7 by adding a (possibly empty) set of edges in series with p_1.

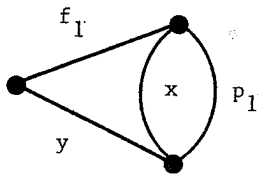

FIGURE 7

If the series class of M_1 containing p_1 contains elements u_1 and v_1 distinct from p_1, then $\{f_2, u_1, v_1\}$ is a triad of M. As $z \in E(M_2)$, this triad meets the triangle $\{f_2, y, z\}$ in a single element; a contradiction. Thus either

(1) M_1 is as shown in Figure 7; or

(2) M_1 has one extra element u_1 in series with p_1.

In case (2), $\{f_1, x, u_1\}$ and $\{x, y, u_1\}$ are triads of M_1 and hence of $M \setminus f_2$. Since x is not in a triad of M, neither $\{f_1, x, u_1\}$ nor $\{x, y, u_1\}$ is a cocircuit of M and therefore both $\{f_2, f_1, x, u_1\}$ and $\{f_2, x, y, u_1\}$ are cocircuits of M. Thus $M \setminus u_1$ has $\{f_1, f_2, x\}$ and $\{f_2, x, y\}$ as cocircuits. Therefore $(M \setminus u_1)^* | \{x, y, f_1, f_2\}$ is either a 4-point line or a line having three points, $\{x\}$, $\{y, f_1\}$, $\{f_2\}$. In the first case, $(M \setminus u_1)^*$ is nonbinary, so $M \setminus u_1$ is also nonbinary. Since M/u_1 has an M_2-minor, it too is nonbinary. Thus we can take $f_3 = u_1$ and the required result follows. In the second case, $\{y, f_1\}$ is a cocircuit of $M \setminus u_1$. Thus $\{u_1, y, f_1\}$ is a triad of M and hence of M_1. But $\{x, p_1, u_1\}$ is a triangle of M_1 meeting this triad in a single element; a contradiction. Thus case (2) cannot occur.

Now consider case (1). Because x is parallel to p_1 in M_1, the matroid $M \setminus f_1, f_2, y$ is obtained from M_2 simply by relabelling the element p_2 by x. For the rest of the proof we shall identify M_2 with $M \setminus f_1, f_2, y$.

We shall show next that M_2 has a $U_{2,4}$-minor using $\{x, z\}$. If M_2 is 3-connected this is immediate from Theorem 4.1. If M_2 is not 3-connected, then, by Theorem 1.2, M_2 is the 2-sum of matroids $M_{2,1}$ and $M_{2,2}$ with respect to the basepoints s_1 and s_2 where $x \in E(M_{2,2})$ and $|E(M_{2,1})|$ is maximum subject to these conditions. If $M_{2,2}$ is not 3-connected, then it is the 2-sum of matroids $M_{2,3}$ and $M_{2,4}$ with respect to the basepoints s_3 and s_4 where $x \in E(M_{2,4})$ and $|E(M_{2,3})|$ is maximum. By the choice of $M_{2,1}$, we must have that $s_2 \in E(M_{2,4})$. If $M_{2,4}$ is not 3-connected, then it is the 2-sum of matroids $M_{2,5}$ and $M_{2,6}$ with respect to the basepoints s_5 and s_6 where $x \in E(M_{2,6})$ and $|E(M_{2,5})|$ is maximum. Again the choice of $M_{2,1}$ and $M_{2,3}$ guarantees that s_2 and s_4 are in $E(M_{2,6})$. Repeating this process, we eventually obtain a 3-connected matroid $M_{2,2k}$ containing $x, s_2, s_4, \ldots, s_{2k}$ such that M_2 is formed from $M_{2,2k}$ by taking the 2-sum of $M_{2,2k}$ and $M_{2,1}$ with respect to s_2 and s_1; then taking the 2-sum of the result and $M_{2,3}$ with respect to s_4 and s_3 and so on. Evidently, for all j in $\{1, 2, \ldots, k\}$, M_2 is the 2-sum of $M_{2,2j-1}$ and another matroid, say $M'_{2,2j-1}$, which contains x. As M_2 is nonbinary, at least one of the parts of this 2-sum is nonbinary. If $M_{2,2j-1}$ is nonbinary, then $M \setminus f_2$ is the 2-sum of $M_{2,2j-1}$ and another matroid M'_1, which is the 2-sum of M_1 and $M'_{2,2j-1}$. But M'_1 has more elements than M_1; a contradiction to the choice of M_1. It follows that $M_{2,2j-1}$ is binary. Since this is true for all j in $\{1, 2, \ldots, k\}$, we must have that $M_{2,2k}$ is nonbinary. It follows, by Theorem 4.1 (or see [8, (4.2)]), that, whether it is 3-connected or not, M_2 has a $U_{2,4}$-minor using $\{x, z\}$. Suppose that this minor is $M_2/A \setminus B$. then $M \setminus f_2/A \setminus B$ is the matroid shown in Figure 8.

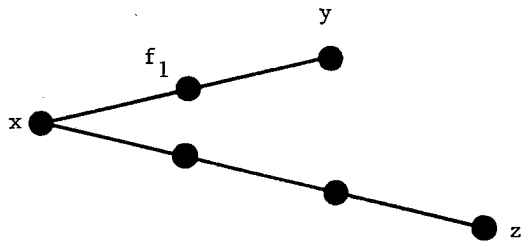

FIGURE 8

Now, we may assume that A is independent in M_2. As $A \cap \{y, f_1, f_2\}$ is empty and $\{y, f_1, f_2\}$ is a cocircuit of M, both $A \cup f_2$ and $A \cup y$ are independent in M. We recall that $\{f_2, y, z\}$ is a circuit of M. Thus either $\{f_2, y, z\}$ is a circuit of M/A or $\{f_2, y, z\}$ properly contains a circuit of M/A. In the first case, we are forced to get a $\mathcal{W}^3$-minor of M having $\{x, y, z\}$ as its set of spokes; a contradiction. In the second case, since every circuit of M containing f_2 must meet both $E(M_1)$ and $E(M_2)$, and $\{y, z\}$ is independent in M/A, we must have that $\{f_2, y\}$ is a circuit of M/A. Thus $A \cup \{f_2, y\}$ contains a circuit C of M. Moreover, $\{f_2, y\} \subseteq C$ since both $A \cup f_2$ and $A \cup y$ are independent. But $\{f_2, y, z\}$ is also a circuit of M and so, by exchange, M has a circuit containing z and contained in $A \cup \{f_2, z\}$. The element f_2 is not in this circuit, otherwise the exact 2-separation we have of $M \setminus f_2$ extends to an exact 2-separation of the 3-connected matroid M; a contradiction. But now in M/A, the element z is a loop; a contradiction. This completes the proof of the lemma, thereby finishing the proof of the theorem.

ACKNOWLEDGMENT. The author thanks the referee for several suggested improvements to this paper.

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