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PROOF OF A CONJECTURE OF KAHN FOR NON-BINARY MATROIDS*

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This note proves a conjecture of Kahn by showing that if X is a 3-element independent set in a 3-connected non-binary matroid M, then M has a connected non-binary minor having X as a basis.

The matroid terminology used here will follow Welsh [8]. The matroid N *uses* the set T if T is a subset of the ground set $E(N)$ of N. The *rim* and set of *spokes* of a whirl are precisely the sets of elements which form the rim and set of spokes, respectively, of the corresponding wheel.

It is well-known that a matroid is non-binary if and only if it has a minor isomorphic to $U_{2,4}$, the 4-point line. For a non-binary matroid M, several authors have considered how the $U_{2,4}$-minors of M relate to specific sets of elements in M. In particular, Bixby [1] proved that if M is connected, then every element is in a $U_{2,4}$-minor, while Seymour [6] showed that if M is 3-connected, then every pair of elements is in a $U_{2,4}$-minor. In addition, Seymour [6, 7] conjectured that if M is 4-connected, then every triple of elements is in a $U_{2,4}$-minor. Kahn [3] disproved this conjecture and proposed the following weakening of it.

Conjecture. [3, Conjecture 1]. *If M is a 4-connected non-binary matroid and X is a 3-element subset of $E(M)$, then M has a non-binary minor having X as a basis.*

The truth of this conjecture for matroids of rank at least three follows easily from either of the next two theorems. In this note we prove these theorems and show that certain natural strengthenings of them do not hold. Notice that the conjecture fails for 4-connected non-binary matroids of rank less than three, the only two such matroids being $U_{2,4}$ and $U_{2,5}$ [2, 5].

Theorem 1. *Let M be a 3-connected non-binary matriod and X be a 3-element independent set in M. Then M has a connected non-binary minor having X as a basis.*

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Theorem 2. *Let M be a 3-connected non-binary matroid and X be a 3-element set that is both independent and coindependent in M. Then M has a connected non-binary minor in which X is both a basis and a cobasis.*

Both these theorems are fairly straightforward consequences of the next reult, the main theorem of [4].

Theorem 3. *Let X be a 3-element set in a non-binary 3-connected matroid M. Then either M has a $U_{2,4}$-minor using X or M has as a minor a rank-3 whirl that uses X as its rim or its set of spokes.*

Proof of Theorem 1. If the second alternative in the preceding theorem applies, then Theorem 1 is immediate. Thus we may assume that $E(M)$ has disjoint subsets T and U such that $M\setminus T/U$ uses X and is isomorphic to $U_{2,4}$. Now X is a circuit in $M\setminus T/U$ but is independent in M. Hence, for some element u of U, $X\cup\{u\}$ is contained in a circuit of M. Let $N=M\setminus T/(U-\{u\})$. Then N is a connected rank-3 non-binary minor of M having $X\cup\{u\}$ as a circuit and therefore having X as a basis. This completes the proof of Theorem 1. ▮

To prove Theorem 2, one extends the above argument by noting that if X is coindependent in M, then for some element t of T, $X\cup\{t\}$ is contained in a cocircuit of M. It follows that $M\setminus(T-\{t\})/(U-\{u\})$ is the required connected non-binary minor of M having X as both a basis and a cobasis.

It seems natural to ask whether the connected minors whose existence is guaranteed by Theorems 1 and 2 can also be required to be 3-connected. The following examples show that this is not the case. Let $X=\{x, y, z\}$ and M^* be isomorphic to the matroid M_1 for which a Euclidean representation is shown in Figure 1(a). Then M is 3-connected and non-binary and X is independent in M. However, M has no *3-connected* non-binary minor having X as a basis. Now suppose that M is obtained from the rank-4 matroid M_2 of Figure 1(b) by deleting the element e. Then M is non-binary and 3-connected and X is both independent and coindependent in M, yet M has no *3-connected* non-binary minor having X as both a basis and a cobasis.

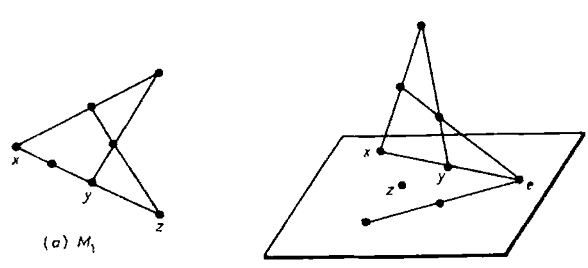

Fig. 1

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