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The Regular Matroids with No 5-Wheel Minor

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For r in $\{3, 4\}$, the class of binary matroids with no minor isomorphic to $M(\mathscr{W}_r)$, the rank-r wheel, has an easily described structure. This paper determines all graphs with no $\mathscr{W}_5$-minor and uses this to show that the class of regular matroids with no $M(\mathscr{W}_5)$-minor also has a relatively simple structure. © 1989 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to study the class of regular matroids with no minor isomorphic to $M(\mathscr{W}_5)$, the cycle matroid of the rank-5 wheel. This study is motivated by Tutte's wheels and whirls theorem [16], which implies that, for a 3-connected matroid M with at least four elements, there is a sequence $M_0, M_1, M_2, \ldots, M_n$ of 3-connected matroids, each a single-element deletion or contraction of its successor, such that $M_n = M$ and M_0 is a whirl of rank at least two or a wheel of rank at least three. Thus the wheels and whirls are the fundamental non-trivial building blocks for the class of 3-connected matroids. Indeed, since every matroid that is not 3-connected is a direct sum or a 2-sum of two matroids on fewer elements (Theorem 1.5), these building blocks are fundamental to the whole class of matroids. In view of this, it is natural to consider what matroids can arise when one excludes a small wheel or a small whirl as a minor. As the smallest whirl $\mathscr{W}^2$ is isomorphic to the 4-point line, the class of matroids with no $\mathscr{W}^2$-minor is precisely the class of binary matroids [15]. If one also excludes the smallest 3-connected wheel $M(\mathscr{W}_3)$, then the class of matroids one obtains is precisely the class of series-parallel networks [4, 7, 3]. In [11], the author determined all members of the class $EX(\mathscr{W}^2, M(\mathscr{W}_4))$ of matroids having no minor isomorphic to $\mathscr{W}^2$ or

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$M(\mathscr{W}_4)$. In this paper, as a step towards characterizing $EX(\mathscr{W}^2, M(\mathscr{W}_5))$, the class of binary matroids with no $M(\mathscr{W}_5)$-minor, we characterize the class of regular matroids with no $M(\mathscr{W}_5)$-minor.

The terminology used here for matroids and graphs will in general follow Welsh [18] and Bondy and Murty [1]. The ground set of a matroid M will be denoted by $E(M)$ and, if $T \subseteq E(M)$, we denote the rank of T by $\text{rk}\, T$. We shall write $\text{rk}\, M$ for $\text{rk}(E(M))$ and $\text{cork}\, M$ for the rank of the dual matroid M^* of M. If H is a matroid or a graph and $X \subseteq E(H)$, the deletion and contraction of X from H will be denoted by $H \backslash X$ and H/X, respectively.

Familiarity will be assumed with the concept of n-connection for graphs as defined, for example, in [1, p. 42]. For an integer n exceeding one, a matroid M is *n-connected* [16] if there is no positive integer $k < n$ so that $E(M)$ can be partitioned into subsets X and Y each having at least k elements such that $\text{rk}\, X + \text{rk}\, Y - \text{rk}\, M = k - 1$. Thus M is 2-connected if and only if it is connected. It is routine to verify that M is n-connected if and only if M^* is n-connected.

Let G be a graph without isolated vertices. The notions of n-connectedness of G and n-connectedness of its cycle matroid $M(G)$ do not, in general, coincide. However, by [16],

(1.1) *if G has at least three vertices, then G is 2-connected and loopless if and only if $M(G)$ is 2-connected*; and

(1.2) *if G has at least four vertices, then G is 3-connected and simple if and only if $M(G)$ is 3-connected.*

If G is a 3-connected simple graph, then a result of Whitney [19] establishes that, up to isomorphism, G has a unique planar dual G^*. We shall call G^* the *dual* graph of G.

If M_1 and M_2 are matroids on the sets S and $S \cup e$, where $e \notin S$, then M_2 is an *extension* of M_1 if $M_2 \backslash e = M_1$, and M_2 is a *lift* of M_1 if M_2^* is an extension of M_1^*. We call M_2 a *non-trivial* extension of M_1 if e is neither a loop nor a coloop of M_2 and e is not in a 2-element circuit of M_2. Likewise, M_2 is a *non-trivial* lift of M_1 if M_2^* is a non-trivial extension of M_1^*. These terms will also be applied to graphs. Suppose v is a vertex of a loopless graph G such that $d(v) \geqslant 4$ and let G' be a graph constructed from G as follows: replace v by two new vertices v' and v'' that are joined by a new edge e; every edge of G that was incident with v is incident with exactly one of v' and v'' in G' so that both v' and v'' have degree at least three; the rest of G is left unchanged. Then G' is certainly a lift of G; we shall say that G' has been obtained from G by *splitting* v. The next two results, which were proved by Tutte [14], will be used frequently in the characterization of the class of simple 3-connected graphs having no $\mathscr{W}_5$-minor.

(1.3) LEMMA. *Let G be a simple 3-connected graph and suppose G' is a lift of G. Then the following are equivalent:*

(i) *G' is a non-trivial lift of G;*

(ii) *G' is simple and 3-connected;*

(iii) *G' is obtained from G by splitting a vertex of degree at least four.*

(1.4) LEMMA. *Let G be a simple 3-connected graph and suppose G' is an extension of G. Then G' is simple and 3-connected if and only if G' is a non-trivial extension of G.*

The last two lemmas have a common matroid generalization (see, for example, [9, Lemma 2.1]).

Suppose that the matroids M_1 and M_2 have disjoint ground sets and that $p_i \in E(M_i)$ for i in $\{1, 2\}$. Then the parallel connection [3] of M_1 and M_2 with respect to the basepoints p_1 and p_2 will be denoted by $P((M_1, p_1), (M_2, p_2))$ or just $P(M_1, M_2)$. Seymour [12] established the following basic link between 3-connection and parallel connection.

(1.5) THEOREM. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each of which has at least three elements and is isomorphic to a minor of M such that $M = P((M_1, p_1), (M_2, p_2))\backslash p$, where p is not a loop or a coloop of M_1 or M_2.*

When M decomposes as in this theorem, it is called the *2-sum of M_1 and M_2*. If $M_i \cong M(G_i)$ for i in $\{1, 2\}$, then the 2-sum of M_1 and M_2 is isomorphic to $M(G)$ where G is a 2-sum of the graphs G_1 and G_2 with respect to the edges p_1 and p_2. In general, for a positive integer m, if each of G_1 and G_2 has a distinguished K_m-subgraph, we form an *m-sum* of G_1 and G_2 as follows: define a bijection between the vertex-sets of the distinguished K_m-subgraphs; identify corresponding pairs of vertices; and, finally, delete the edges of both K_m-subgraphs.

Suppose that $r \geqslant 2$. The *wheel* $\mathscr{W}_r$ *of rank* r is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex (the *hub*) is joined by a single edge (a *spoke*) to each of the other vertices. The rank-r *whirl* $\mathscr{W}^r$ is a matroid on $E(\mathscr{W}_r)$ having as its circuits all cycles of $\mathscr{W}_r$ other than the rim, as well as all sets of edges formed by adding a single spoke to the edges of the rim. The smallest 3-connected whirl is $\mathscr{W}^2$; the smallest 3-connected wheel $M(\mathscr{W}_3)$.

A basic tool in the proof of the main result of this paper will be Seymour's splitter theorem [12]. Let **F** be a class of matroids that is closed under minors and under isomorphisms. A member N of **F** is a *splitter* for **F** if no 3-connected member of **F** has a proper minor isomorphic to N. A graph G will be called a splitter for a class of graphs if $M(G)$ is a splitter for the corresponding class of graphic matroids. We shall only need the following special case of the splitter theorem, its restriction to the class of graphic matroids. This result was also found by Negami [8].

(1.6) THEOREM. *Let G and H be 3-connected simple graphs such that H is a minor of G and if $H \cong \mathscr{W}_k$ for some $k \geqslant 3$, then G has no $\mathscr{W}_{k+1}$-minor. Then there is a sequence $G_0, G_1, G_2, ..., G_n$ of 3-connected simple graphs such that $G_0 \cong H$, $G_n = G$ and, for all i in $\{1, 2, ..., n\}$, G_i is an extension or lift of G_{i-1}.*

In Section 2 of this paper, we use the last result to determine all simple 3-connected graphs having no $\mathscr{W}_5$-minor. In Section 3, we combine this theorem with some results of Seymour to determine all 3-connected regular matroids with no $M(\mathscr{W}_5)$-minor. In Section 4, we prove a sharp upper bound on the number of elements in a rank-r simple regular matroid with no $M(\mathscr{W}_5)$-minor and solve the critical problem for this class of matroids.

2. THE GRAPHIC CASE

In this section we shall determine all 3-connected simple graphs having no $\mathscr{W}_5$-minor. First we consider some examples of such graphs. For each $k \geqslant 3$, consider the graph $K_{3,k}$, labelling its vertex classes V_1 and V_2 where $|V_1| = 3$. Now add edges x, y, and z to this graph so that all pairs of vertices in V_1 are joined. We shall call the resulting graph A_k. It is straightforward to check that, for all $k \geqslant 3$, all of A_k, $A_k \backslash x$, $A_k \backslash x, y$, and $K_{3,k}$ are simple, 3-connected, and have no $\mathscr{W}_5$-minor. Similarly, one can check that the graphs H_6, Q_3, $K_{2,2,2}$, and H_7 shown in Fig. 1 also have these properties. The first of these is the graph obtained from K_5 by splitting a vertex, while the second and third are the graphs of the cube and the octahedron and hence are duals of each other. The graph H_7 is isomorphic to its own dual. Notice also that H_7 can be obtained by taking a 3-sum of $K_{2,2,2}$ and K_4.

(2.1) THEOREM. *Let G be a graph. Then G is simple and 3-connected having no $\mathscr{W}_5$-minor if and only if*

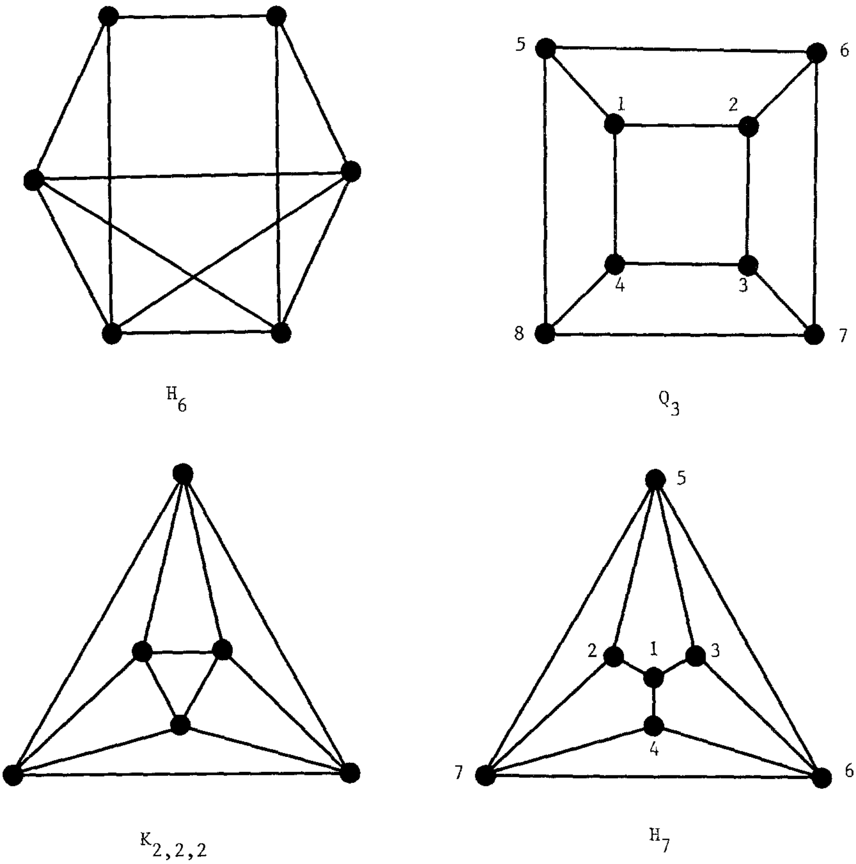


FIGURE 1

(i) $G \cong \mathcal{W}_3$ *or* $\mathcal{W}_4$;

(ii) *G is isomorphic to a simple 3-connected minor of* H_6, Q_3, $K_{2,2,2}$, *or* H_7; *or*

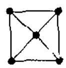
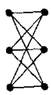
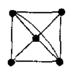
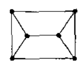
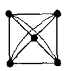
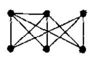
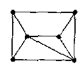
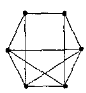
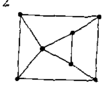
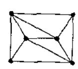
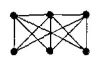
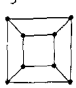
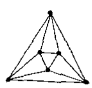
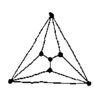
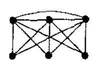
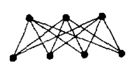
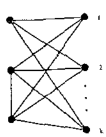
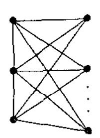
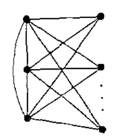
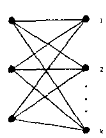
(iii) *for some* $k \geqslant 3$, *G is isomorphic to one of* A_k, $A_k \backslash x$, $A_k \backslash x, y$, *or* $K_{3,k}$.

A complete list of the 3-connected simple graphs having no $\mathcal{W}_5$-minor is given in Table I.

Proof of Theorem 2.1. It was noted above that each of the graphs listed in (i)–(iii) is 3-connected, simple, and has no $\mathcal{W}_5$-minor. Now suppose that G is simple, 3-connected, and has no $\mathcal{W}_5$-minor. We shall use the following result of Dirac [6, Theorem 1], a short proof of which was given in [2]. We denote by $K_5 - e$ the graph obtained from K_5 by deleting a single edge.

TABLE I

The 3-Connected Simple Graphs with No $\mathscr{W}_5$-Minor

Number of edges	Graphs				
6	W_3 				
8	W_4 				
9	$K_{3,3}$ 	K_5-e 	$(K_5-e)^*$ 		
10	K_5 	$A_3\backslash x,y$ 	J_1 		
11	H_6 	J_2 	J_3 	$A_3\backslash x$ 	
12	Q_3 	$K_{2,2,2}$ 	H_7 	A_3 	$K_{3,4}$ 
$3k+i$ $k\geqslant 4$ $1\leqslant i\leqslant 3$	$i=1$ $A_k\backslash x,y$ 	$i=2$ $A_k\backslash x$ 	$i=3$ A_k 		
$3k$ $k\geqslant 5$	$K_{3,k}$ 				

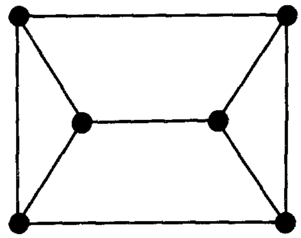


FIG. 2. $(K_5-e)^*$.

(2.2) THEOREM. *Let H be a simple 3-connected graph. Then exactly one of the following holds*:

(i) *H has two vertex-disjoint cycles*;

(ii) *H is isomorphic to $\mathcal{W}_r$ for some $r \geqslant 3$*;

(iii) *H is isomorphic to K_5 or K_5-e*;

(iv) *for some $k \geqslant 3$, H is isomorphic to one of A_k, $A_k\backslash x$, $A_k\backslash x, y$, or $K_{3,k}$*.

On applying this result to G, we obtain that either G is listed under (2.1) (i)–(iii), or G has two vertex-disjoint cycles. Thus we may assume that the latter occurs. It follows easily by Menger's theorem that G has a minor isomorphic to $(K_5-e)^*$, the dual of K_5-e (see Fig. 2). Therefore, by Theorem 1.6, there is a sequence $G_0, G_1, G_2, ..., G_n$ of simple 3-connected graphs such that $G_0 \cong (K_5-e)^*$, $G_n = G$, and, for all i in $\{1, 2, ..., n\}$, G_{i-1} is a single-element deletion or contraction of G_i. The rest of the proof of Theorem 2.1 will concentrate on this sequence. We shall repeatedly use the fact that none of the graphs in the sequence has a $\mathcal{W}_5$-minor. Since every vertex of $(K_5-e)^*$ has degree 3, G_1 cannot be a lift of G_0. It follows, by the symmetry of $(K_5-e)^*$, that G_1 is isomorphic to the graph J_1 shown in Fig. 3.

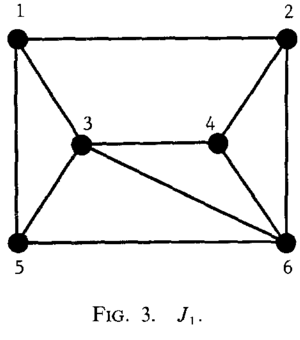


FIG. 3. J_1.

A consequence of the next lemma is that if G_2 is a non-trivial lift of J_1, then $G_2 \cong J_2$, where J_2 is the graph shown in Fig. 4. The graphs $J_1 + 45$ and $J_1 + 14$ appearing in the lemma are obtained from J_1 by adding the edges 45 and 14, respectively.

(2.3) LEMMA. *Let J_1' be one of J_1, $J_1 + 45$, and $J_1 + 14$ and let N_1 be a non-trivial lift of J_1' having no $\mathscr{W}_5$-minor. Then $J_1' = J_1$ and $N_1 \cong J_2$, or $J_1' = J_1 + 14$ and $N_1 \cong H_7$.*

Proof. By the symmetry of J_1', we can assume that 6 is split, say into 6′ and 6″, when N_1 is formed. Now, in N_1, either a member of $\{6', 6''\}$ is adjacent to both 2 and 5, or not. In the first case, N_1 is isomorphic to J_2, $J_2 + ab$, or H_7 according to whether J_1' is isomorphic to J_1, $J_1 + 45$, or $J_1 + 14$. Since $(J_2 + ab)/ab$ has a $\mathscr{W}_5$-minor, it follows that $N_1 \cong J_2$ or H_7. In the second case, we may assume that 2 is adjacent to 6′, and 5 is adjacent to 6″. Moreover, one of 3 and 4 is adjacent to 6′ and the other to 6″. It follows that $N_1/34$ has a $\mathscr{W}_5$-minor. ∎

Next we consider when G_2 is a non-trivial extension of J_1. The graph J_3 appearing in the next lemma is $J_1 + 14$.

(2.4) LEMMA. *If G_2 is a non-trivial extension of J_1, then G_2 is isomorphic to H_6 or J_3.*

Proof. By the symmetry of J_1, G_2 is isomorphic to $J_1 + 14$, $J_1 + 45$, or $J_1 + 32$. The second of these is isomorphic to H_6 and the third has a $\mathscr{W}_5$-minor with rim 124651 and hub 3. ∎

It follows from the above that we may assume that G_2 is isomorphic to J_2, J_3, or H_6.

(2.5) LEMMA. *H_6 is a splitter for the class of graphs with no $\mathscr{W}_5$-minor.*

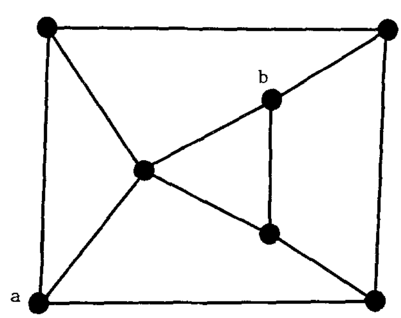


FIG. 4. J_2.

Proof. We shall identify H_6 with J_1+45 to which it is isomorphic. By Lemma 2.3, every non-trivial lift of J_1+45 has a $\mathscr{W}_5$-minor. If H_6' is a non-trivial extension of J_1+45, then, without loss of generality, $H_6' \cong J_1+45+32$. But this matroid has J_1+32 as a minor and so, by the preceding proof, has a $\mathscr{W}_5$-minor. ∎

Next we suppose that G_2 is isomorphic to J_3 where we recall that $J_3=J_1+14$. By Lemma 2.3, if G_3 is a non-trivial lift of J_3, then $G_3 \cong H_7$.

(2.6) LEMMA. *If G_3 is a non-trivial extension of J_3, then $G_3 \cong K_{2,2,2}$. Moreover, every non-trivial extension of $K_{2,2,2}$ has a $\mathscr{W}_5$-minor.*

Proof. By the symmetry of J_3, G_3 is isomorphic to J_3+45 or J_3+25. The second of these is isomorphic to $K_{2,2,2}$; the first is isomorphic to a non-trivial extension of H_6 and so. by Lemma 2.5, has a $\mathscr{W}_5$-minor. Since a non-trivial extension of J_3+25 has a minor isomorphic to J_3+45, it also follows that every non-trivial extension of $K_{2,2,2}$ has a $\mathscr{W}_5$-minor. ∎

The graphs J_2 and J_3 are duals of each other. Moreover, from above, if $G_2 \cong J_3$, then $G_3 \cong K_{2,2,2}$ or H_7. Therefore, if $G_2 \cong J_2 \cong J_3^*$ and G_3 is planar, then $G_3 \cong K_{2,2,2}^*$ or H_7^*. The first of these is isomorphic to Q_3 and the second to H_7. If $G_2 \cong J_2$ and G_3 is non-planar, it is straightforward to check that G_3 has a $\mathscr{W}_5$-minor.

We may now assume that G_3 is isomorphic to Q_3, $K_{2,2,2}$, or H_7. The next three lemmas complete the proof of Theorem 2.1 by showing that each of these graphs is a splitter for the class of graphs having no $\mathscr{W}_5$-minor.

(2.7) LEMMA. *Q_3 is a splitter for the class of graphs with no $\mathscr{W}_5$-minor.*

Proof. Every vertex of Q_3 has degree 3, so Q_3 has no non-trivial lifts. By the symmetry of Q_3, a non-trivial extension of it is isomorphic to one of Q_1+13 and Q_1+17 where the vertices of Q_3 are labelled as in Fig. 1. In the first case, on contracting 37 and 56 and deleting 15, we obtain a graph isomorphic to $\mathscr{W}_5$. In the second case, on contracting 17, we obtain a graph isomorphic to $\mathscr{W}_6$. ∎

(2.8) LEMMA. *$K_{2,2,2}$ is a splitter for the class of graphs with no $\mathscr{W}_5$-minor.*

Proof. By Lemma 2.6, $K_{2,2,2}$ has no non-trivial extension without a $\mathscr{W}_5$-minor. Now let N_2 be a non-trivial lift of $K_{2,2,2}$ having no $\mathscr{W}_5$-minor. If N_2 is planar, then N_2^* is a non-trivial extension of $K_{2,2,2}^*$. But, as the last graph is isomorphic to Q_3, it follows by the preceding lemma that N_2^* has a $\mathscr{W}_5$-minor. Hence N_2 has a $\mathscr{W}_5$-minor, a contradiction. Thus we may suppose that N_2 is non-planar. By symmetry, N_2 is uniquely determined up to isomorphism and one easily checks that N_2 has a $\mathscr{W}_5$-minor. ∎

(2.9) LEMMA. *H_7 is a splitter for the class of graphs with no $\mathscr{W}_5$-minor.*

Proof. Let N_3 be a non-trivial lift of H_7. Then N_3 is obtained by splitting one of the degree-4 vertices of H_7. With the vertices of H_7 labelled as in Fig. 1, we may assume, by symmetry, that it is the vertex 5 which is split, say into vertices $5'$ and $5''$. In N_3, both $5'$ and $5''$ have degree three. If each of them is joined to one vertex in $\{2, 3\}$ and one vertex in $\{6, 7\}$, then, on contracting 12 and 13 and deleting 67 from N_3, we obtain a graph isomorphic to $\mathscr{W}_5$. Thus we may assume that $5'$ is adjacent to both 2 and 3, and $5''$ is adjacent to both 6 and 7. But then, $N_3 \backslash 67 \cong Q_3$ and so, by Lemma 2.7, N_3 has a $\mathscr{W}_5$-minor.

Since H_7 is isomorphic to its dual, it follows from the above argument that every planar non-trivial extension of H_7 has a $\mathscr{W}_5$-minor. Moreover, by symmetry, H_7 has a unique non-planar non-trivial extension and this extension is easily shown to have a $\mathscr{W}_5$-minor. ∎

A referee has observed that an alternative proof of Theorem 2.1 may be derived from results of Truemper [13, Theorems 4.5 and 4.7].

To conclude this section, we note the following result that is a straightforward combination of Theorems 2.1 and 2.2.

(2.10) COROLLARY. *Let G be a 3-connected simple graph that is not isomorphic to a wheel and has at least thirteen edges. Then G has two vertex-disjoint cycles if and only if G has a $\mathscr{W}_5$-minor.*

3. THE REGULAR CASE

In the preceding section, we determined all 3-connected simple graphs having no $\mathscr{W}_5$-minor. By (1.2), the cycle matroids of these graphs are precisely the 3-connected graphic matroids with four or more elements having no $M(\mathscr{W}_5)$-minor. Evidently the five 3-connected matroids, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$, with fewer than four elements are all graphic and have no $M(\mathscr{W}_5)$-minor. The cocycle matroids of the graphs in Theorem 2.1 are precisely the 3-connected cographic matroids with four or more elements having no $M(\mathscr{W}_5)$-minor. In this section, we shall determine all 3-connected regular matroids with no $M(\mathscr{W}_5)$-minor. We begin by proving the following result.

(3.1) THEOREM. *Let M be a 3-connected regular matroid having no $M(\mathscr{W}_5)$-minor. Then $M \cong R_{10}$, or M is graphic or cographic.*

The proof of this is based on the following two theorems of Seymour [12, (14.2) and (7.4)]. The matroids R_{10} and R_{12} are the linear dependence matroids of the following matrices over $GF(2)$:

$$R_{10} \qquad \left[\begin{array}{c|ccccc} & 1&1&0&0&1\\ & 1&1&1&0&0\\ I_5 & 0&1&1&1&0\\ & 0&0&1&1&1\\ & 1&0&0&1&1 \end{array}\right]$$

$$R_{12} \qquad \begin{array}{c} \begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array} \\ \left[\begin{array}{c|cccccc} & 1&1&1&0&0&0\\ & 1&1&0&1&0&0\\ I_6 & 1&0&0&0&1&0\\ & 0&1&0&0&0&1\\ & 0&0&1&0&1&1\\ & 0&0&0&1&1&1 \end{array}\right] \end{array}$$

(3.2) THEOREM. *Let M be a 3-connected regular matroid. Then either M is graphic or cographic, or M has a minor isomorphic to one of R_{10} and R_{12}.*

(3.3) THEOREM. *If M is a 3-connected regular matroid having an R_{10}-minor, then $M \cong R_{10}$.*

Proof of Theorem 3.1. The result follows immediately from the last two theorems provided that we can show that R_{12} has an $M(\mathcal{W}_5)$-minor. But it is routine to check that $R_{12}/3\backslash 10 \cong M(\mathcal{W}_5)$, with the elements of the latter being labelled as in Fig. 5.

Evidently R_{10} has no $M(\mathcal{W}_5)$-minor. Using this, together with Theorems 2.1 and 3.1 and the remarks at the start of this section, we get the following result.

(3.4) THEOREM. *Let M be a regular matroid. Then M is 3-connected and has no $M(\mathcal{W}_5)$-minor if and only if*

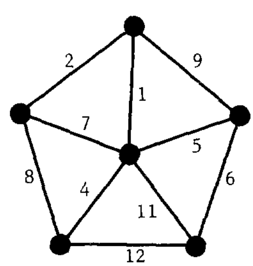


FIGURE 5.

(i) *for some* $k \geqslant 3$, M *is isomorphic to one of* $M(A_k)$, $M(A_k \backslash x)$, $M(A_k \backslash x, y)$, $M(K_{3,k})$, *or their duals*; *or*

(ii) M *is isomorphic to a* 3-*connected minor of* R_{10}, $M(Q_3)$, $M(K_{2,2,2})$, $M(H_7)$, $M(H_6)$, *or* $M^*(H_6)$.

It follows from this theorem that the 3-connected regular matroids with no $M(\mathscr{W}_5)$-minor are the cycle and cocycle matroids of the graphs in Table I, R_{10}, and the five 3-connected matroids with fewer than four elements, namely, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$. Hence, apart from these five trivial examples, the only regular matroid with no $M(\mathscr{W}_5)$-minor that is 4-connected is R_{10}.

A consequence of Theorem 1.5 is that one can construct all regular matroids having no $M(\mathscr{W}_5)$-minor by beginning with the 3-connected such matroids and repeatedly using the operations of direct sum and 2-sum.

4. Some Consequences of the Characterization

In this section we use Theorem 3.4 to determine some properties of the class of regular matroids with no $M(\mathscr{W}_5)$-minor.

Let M be a simple rank-r binary matroid. It follows from results of Dirac [4, 5] that if M has no $M(\mathscr{W}_3)$-minor, then $|E(M)| \leqslant 2r-1$. Furthermore, it was shown in [11] that if M has no $M(\mathscr{W}_4)$-minor, then

$$|E(M)| \leqslant \begin{cases} 3r-2, & \text{if } r \text{ is odd,} \\ 3r-3, & \text{if } r \text{ is even.} \end{cases}$$

Based on these results, one may hope that if M has no $M(\mathscr{W}_5)$-minor, then $|E(M)| \leqslant 4r$. To see that this fails, one can take the parallel connection of k copies of $PG(3, 2)$ for $k \geqslant 2$. From this example, one deduces that the best possible upper bound on $|E(M)|/r$ that could be valid for all r is $\frac{14}{3}$. No such result has yet been proved. However, in this section, we note that if M is simple and regular having no $M(\mathscr{W}_5)$-minor, then $|E(M)|/r$ is less than 3. In particular, the following result holds. The proof of this is similar to the proof of [10, Theorem 5.1] so the details are omitted.

(4.1) Theorem. *Let* M *be a rank-r member of the class* $\mathscr{Q}$ *of simple regular matroids having no* $M(\mathscr{W}_5)$-*minor. Then*

$$|E(M)| \leqslant \begin{cases} 3r-2, & \textit{if } \; r \equiv 1 \pmod 3, \\ 3r-3, & \textit{otherwise.} \end{cases}$$

Moreover, M attains this bound if and only if either

(i) *$M = P(M_1, M_2)$, where each of M_1 and M_2 is a member of $\mathscr{D}$ attaining the bound and having rank at least two, and* rk M_1 *or* rk M_2 *is congruent to* 1 (mod 3); *or*

(ii) *$M \cong M(K_n)$ for some n in* $\{2, 3, 4, 5\}$, *$M \cong M(K_{2,2,2})$, or $M \cong M(A_k)$ for some $k \geq 3$ such that $k \not\equiv 2$* (mod 3).

In particular, the bound is attained for all ranks.

It follows from this theorem that, for a simple n-vertex graph G having no $\mathscr{W}_5$-minor,

$$|E(G)| \leq \begin{cases} 3n-5, & \text{if } n \equiv 2 \pmod 3, \\ 3n-6, & \text{otherwise.} \end{cases}$$

Furthermore, this bound is attained for all $n \geq 2$ and one can determine from the theorem precisely which graphs attain the bound.

The second consequence of Theorem 3.4 that we shall look at involves the chromatic numbers of loopless regular matroids with no $M(\mathscr{W}_5)$-minor. If M is a loopless regular matroid having chromatic polynomial $P(M; \lambda)$ (see, for example, [18, p. 262]), its *chromatic number* $\chi(M)$ is $\min\{j \in \mathbb{Z}^+ : P(M; j) > 0\}$.

(4.2) THEOREM. *Let M be a loopless regular matroid having no $M(\mathscr{W}_5)$-minor. Then $\chi(M) \leq 5$. Moreover, if M is 3-connected, equality holds here if and only if $M \cong M(K_5)$.*

Proof. If M is $M_1 \oplus M_2$ or $P(M_1, M_2) \backslash p$, where M_1 and M_2 are loopless, then $\chi(M) \leq \max\{\chi(M_1), \chi(M_2)\}$ [17]. Thus it suffices to prove that if M is 3-connected and regular having no $M(\mathscr{W}_5)$-minor, then $\chi(M) \leq 5$ with equality if and only if $M \cong M(K_5)$. Using Table I, it is straightforward to check this if M is graphic or cographic. By Theorem 3.4, if M is neither graphic nor cographic, then $M \cong R_{10}$ and this matroid, being a disjoint union of cocircuits, has chromatic number 2 (see, for example, [17, p. 7]). ∎

It follows from this theorem that, for a loopless regular matroid M having no $M(\mathscr{W}_5)$-minor, the critical exponent [18, p. 273] $c(M; 2)$ is at most 3. It is an open problem to determine whether the weaker bound $c(N; 2) \leq 4$ holds for all loopless *binary* matroids N with no $M(\mathscr{W}_5)$-minor. If this bound holds it is best possible since, for example, $c(PG(3, 2); 2) = 4$.

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