ON MATROID CONNECTIVITY

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[Received 5th March 1979; in revised form 22nd January 1980]

1. Introduction

THE concept of n-connection for matroids was introduced by W. T. Tutte **[12]** based upon the corresponding idea for graphs. The main result of Tutte's paper extends an earlier result of his **[11]** from 3-connected graphs to 3-connected matroids. Similarly, Murty **[7]** and Inukai and Weinberg **[6]** have given matroid generalizations of several results for n-connected graphs. This paper concentrates primarily on minimally 3-connected matroids. In Section 2 a result of Seymour **[9]** for minimally 2-connected matroids is extended when it is shown that in a minimally 3-connected matroid every circuit meets at least two distinct cocircuits of size 3. Sections 3, 4 and 5 generalize and extend work of Murty **[7]** and Halin **[4, 5]**. The greatest and least numbers of elements in a rank r minimally 3-connected matroid are determined and the matroids attaining these bounds are characterized.

The terminology used here for matroids and graphs will in general follow Welsh **[13]**. If S is a set, then $S = X_1 \dot\cup X_2 \dot\cup \cdots \dot\cup X_m$ indicates that S is the disjoint union of $X_1, X_2, \ldots, X_m$. The ground set and rank of the matroid M will be denoted by $E(M)$ and rk M respectively. If $T \subseteq E(M)$, its rank will be denoted by rk T. We shall sometimes denote the restriction of M to $E(M)-T$ by $M\backslash T$, or, if $T=\{x_1, x_2, \ldots, x_m\}$, by $M\backslash x_1, x_2, \ldots, x_m$. Likewise, the contraction of M to $E(M)-T$ will sometimes be written M/T or $M/x_1, x_2, \ldots, x_m$. Flats of M of ranks 1, 2 and 3 are called *points*, *lines* and *planes* respectively.

Familiarity will be assumed with the idea of n-connection for graphs (see, for example, **[1**, p. 42**]**). We now recall the definition of n-connection for matroids. If k is a positive integer, the matroid M is *k-separated* if there is a subset T of $E(M)$ such that $|T| \geq k$, $|E(M)-T| \geq k$ and

$$\text{rk}\, T + \text{rk}\,(E(M)-T) - \text{rk}\, M = k-1.$$

If there is a least positive integer j such that M is j-separated, it is called the *connectivity* $\lambda(M)$ of M. If there is no such integer we say that $\lambda(M) = \infty$.

The matroid M is said to be *n-connected* for any positive integer n such that $n \leq \lambda(M)$. It is routine to show **[12**, (12)**]** that

(1.1) *if M is n-connected, then M^* is n-connected.*

Quart. J. Math. Oxford (2), 32 (1981), 193–208

The familiar notion of connectivity for matroids is related to n-connection as follows.

(1.2) *A matroid is connected if and only if it is* 2*-connected.*

Let H be a matroid or a graph. Then we say that H is *minimally* n*-connected* if it is n-connected and, for every element x of $E(H)$, $H\backslash x$ is not n-connected. An element e of a 3-connected matroid M is *essential* if neither $M\backslash e$ nor M/e is 3-connected.

The notions of n-connection of a graph G and n-connection of the corresponding cycle matroid $M(G)$ do not, in general, coincide. However, [**13**, pp. 78–79]

(1.3) *if G has no loops and at least three vertices, then G is* 2*-connected if and only if $M(G)$ is* 2*-connected*;

and

(1.4) *if G is simple and has at least four vertices, then G is* 3*-connected if and only if $M(G)$ is* 3*-connected.*

Frequent use will be made of the following well-known result (see, for example, [**13**, Theorem 2.1.6]).

(1.5) LEMMA *A circuit and a cocircuit of a matroid cannot have exactly one common element.*

A circuit of size 3 of a matroid is called a *triangle*, and a cocircuit of size 3 a *triad*. It is easy to check that:

(1.6) LEMMA *If M is an n-connected matroid and $|E(M)|\geq 2n-1$, then $E(M)$ has no n-element subset which is both a circuit and a cocircuit. In particular, a* 3*-connected matroid with at least* 5 *elements has no triangle which is also a triad.*

2. Minimally 3-connected matroids

Dirac [**2**, Theorem 5] and Plummer [**8**, Corollary 2a] showed that every minimally 2-connected graph has a vertex of degree 2. The matroid analogue of this result was proved by Murty [**7**, Lemma 3.1] who showed that a minimally 2-connected matroid of rank at least two has a cocircuit of size two. This result was strengthened by Seymour [**9**, Lemma 2.2; **10**, (2.3)]. Halin [**3**] extended the graph result by showing that for $n\geq 1$, every minimally n-connected graph has a vertex of degree n. Recently, Wong [**14**, Theorem 3.4] has proved the analogue of Halin's result for minimally 3-connected matroids. In this section we give a strengthening of Wong's result that resembles Seymour's improvement of Murty's result. For $n>3$ it remains open to determine whether a minimally n-connected matroid has a cocircuit of size n. To deal with the $n=3$ case we shall use a number of results from [**12**] which we state without proof.

(**2.1**) Lemma [**12**, 7.1]. *If M is a 3-connected matroid and e is an essential element of M, then e is in a triangle or a triad.*

The next result follows easily by combining (1.1) and the generalization of [**12**, 6.7] to n-connected matroids.

(**2.2**) Lemma *If M is an n-connected matroid and $|E(M)|\geq 2(n-1)$, then every circuit and every cocircuit of M contains at least n elements.*

(**2.3**) Lemma [**12**, 7.2]. *Suppose M is a 3-connected matroid and $|E(M)|\geq 4$. Let $\{a, b, c\}$ be a triangle of M such that neither $M\backslash a$ nor $M\backslash b$ is 3-connected. Then M has a triad containing a and just one of b and c.*

(**2.4**) Proposition *If e is an essential element of a minimally 3-connected matroid M, then e is contained in a triad.*

Proof. By Lemma 2.1, e is contained in either a triangle or a triad. If e is contained in a triangle $\{e, f, h\}$, then, as M is minimally 3-connected, neither $M\backslash e$ nor $M\backslash f$ is 3-connected. Since M has an essential element, $|E(M)|\geq 5$ [**12**, 6.8], and so by Lemma 2.3, M has a triad containing e.

Halin [**5**, Satz 5] has proved that every circuit in a minimally 3-connected graph meets at least two vertices of degree 3. The next theorem is a matroid analogue of this. The proof extends Seymour's proof [**10**, (2.3)] of a similar result for 2-connected matroids.

(**2.5**) Theorem *If C is a circuit of a minimally 3-connected matroid M and $|E(M)|\geq 4$, then M has at least two distinct triads intersecting C.*

To prove this theorem we require the following additional result which generalizes a result of Murty [**7**, p. 54] for 2-connected matroids.

(**2.6**) Lemma *Suppose that x and y are distinct elements of an n-connected matroid M where $n\geq 2$ and $|E(M)|\geq 2(n-1)$. Assume that $M\backslash x/y$ is n-connected but that $M\backslash x$ is not n-connected. Then M has a cocircuit of size n containing x and y.*

Proof. It is straightforward to show that $M\backslash x$ is $(n-1)$-separated. That is, $E(M\backslash x)=X\,\dot\cup\, Y$ where $|X|, |Y|\geq n-1$. and

(**2.7**) $\operatorname{rk} X+\operatorname{rk} Y-\operatorname{rk}(M\backslash x)=n-2.$

Suppose, without loss of generality, that $y\in X$. Then, denoting by rk′ the rank function of $M\backslash x/y$, we have, by Lemma 2.2, that $\operatorname{rk}'(M\backslash x/y)=\operatorname{rk}(M\backslash x)-1$. Hence

$$\operatorname{rk}'(X-y)+\operatorname{rk}'(Y)-\operatorname{rk}'(M\backslash x/y)=\operatorname{rk} X+\operatorname{rk}(Y\cup y)-\operatorname{rk}(M\backslash x)-1.$$

Therefore, by (2.7), as $M\backslash x/y$ is n-connected, $|X-y|<n-1$. But $|X|\geq n-1$, hence $|X|=n-1$. Thus, as M is n-connected, it follows from Lemma 2.2 that X is independent and so

$$\text{rk}\, X=|X|=n-1.$$

It follows from (2.7) and Lemma 2.2 that $\text{rk}\, M=\text{rk}\,(M\backslash x)=\text{rk}\, Y+1$. Thus Y is contained in a hyperplane of M. But $E(M)-Y=X\cup x$ and $|X\cup x|=n$. Therefore, since M has no cocircuits of size less than n, we conclude the $X\cup x$ is a cocircuit of M containing both x and y.

Proof of Theorem 2.5. The argument is by induction on $|C|$. As $|E(M)|\geq 4$, we have, by Lemma 2.2, that $|C|\geq 3$. Suppose $|C|=3$ and let $C=\{a, b, c\}$. Then neither $M\backslash a$ nor $M\backslash b$ is 3-connected. Thus, by Lemma 2.3, M has a triad T_1 containing a and just one of b and c, say b. Similarly, as neither $M\backslash c$ nor $M\backslash b$ is 3-connected, there is a triad T_2 containing c and just one of a and b. Clearly T_1 and T_2 are distinct and so the result is established for $|C|=3$.

Assume the proposition holds for all circuits having fewer than m elements and let $|C|=m\geq 4$. Then, as M is 3-connected, $E(M)\neq C$ and so $|E(M)|\geq 5$.

If C does not contain a non-essential element, then by Proposition 2.4, every element of C is in a triad. Since $|C|\geq 4$, the required result follows. We may therefore suppose that C contains an element z_1 such that M/z_1 is 3-connected. Now, if M/z_1 is minimally 3-connected, then, as $C\backslash z_1$ is a circuit of M/z_1, the required result follows by induction. Thus we may assume that there is an element x of M/z_1 such that $M/z_1\backslash x$ is 3-connected. Then, as $M\backslash x$ is not 3-connected, it follows by Lemma 2.6 that M has a triad C_1^* containing z_1 and x.

Since $|C|\geq 4$, $C-C_1^*$ is non-empty and so contains an element z_2. Now either (I) M/z_2 is 3-connected; of (II) M/z_2 is not 3-connected. In case I, arguing as above; we get that either M/z_2 is minimally 3-connected, in which case the required result follows by induction; or z_2 is contained in a triad C_2^* of M. Since $C_1^*\neq C_2^*$, the proposition holds in case I.

In case II, z_2 is an essential element of M and so, by Proposition 2.4, z_2 is contained in a triad of M, and the required result holds.

(2.8) COROLLARY *If M is a minimally 3-connected matroid having at least 4 elements, then M has a cobase of elements each of which is in a triad.*

The next result is a matroid analogue of another graph-theoretic result of Halin [**5**, Satz 4].

(2.9) THEOREM *Let M be a minimally 3-connected matroid having at least four elements. Then for all elements e of M such that e is not in a triad, M/e is minimally 3-connected.*

Proof. If $e \in E(M)$ and e is not in a triad, then by Proposition 2.4, e is not essential. As M is minimally 3-connected, $M \backslash e$ is not 3-connected and so M/e is 3-connected. If M/e is not minimally 3-connected, then there is an element x of $E(M/e)$ such that $M/e \backslash x$ is 3-connected. Now $M \backslash x$ is not 3-connected and so, by Lemma 2.6, M has a triad containing e and x. This contradiction implies that M/e is minimally 3-connected, as required.

(2.10) COROLLARY *Let M be a minimally 3-connected matroid having at least four elements and let U be the set of elements of M which are not contained in a triad. Then, if $V \subseteq U$, M/V is minimally 3-connected.*

3. Minimally n-connected matroids with the least number of elements

Murty's main result in [**7**, Corollary 3.3, Theorem 3.4] is the following.

(3.1) THEOREM *If $r \geq 3$, a minimally 2-connected matroid M of rank r has at most $2r-2$ elements, the upper bound being attained if and only if $M \cong M(K_{2,r-1})$.*

In the next section we obtain the corresponding result for minimally 3-connected matroids. In this section we solve the easier problem of determining the least number of elements in a minimally n-connected matroid of rank r, again characterizing when the bound is attained.

(3.2) THEOREM *Let M be a minimally n-connected matroid of rank r where $r, n \geq 2$. If $n \leq r$, then $|E(M)| \geq r+n-1$ with equality being attained if and only if $M \cong U_{r,r+n-1}$. If $n > r$, then $|E(M)| \geq 2r-1$ with equality being attained if and only if $M \cong U_{r,2r-1}$.*

The proof of this theorem uses the following two lemmas.

(3.3) LEMMA [**6**, p. 312]. *The connectivity of the uniform matroid $U_{r,k}$ is as follows:*

$$\lambda(U_{r,k}) = \begin{cases} r+1, & \text{if } \ k \geq 2r+2, \\ \infty, & \text{if } \ 2r-1 \leq k \leq 2r+1, \\ k-r+1, & \text{if } \ r \geq 1 \ \text{ and } \ k \leq 2r-2. \end{cases}$$

(3.4) LEMMA *Let M be an n-connected matroid of rank r. If M is not uniform, then $|E(M)| \geq r+n$.*

Proof. As M is not uniform, there is a circuit C and a cocircuit C^* having empty intersection. Let $k = \min\{|C|, |C^*|\}$ and $|D| = k$ where $D = C$ or C^*. Then $|E(M) - D| \geqslant k$ and $\mathrm{rk}\, D + \mathrm{rk}\,(E(M) - D) - \mathrm{rk}\, M \leqslant k-1$. Hence $k \geqslant n$ and so $|C|$, $|C^*| \geqslant n$. Now $\mathrm{rk}\,(E(M) - C^*) = r-1$, and $E(M) - C^*$ contains C and is therefore dependent. Thus $|E(M) - C^*| \geqslant r$ and so $|E(M)| \geqslant n + r$.

Proof of Theorem 3.2. As any restriction of a uniform matroid is uniform, it is straightforward to deduce the required result by combining Lemmas 3.3 and 3.4.

4. The greatest number of elements in a minimally 3-connected matroid

To determine the greatest number of elements in a minimally 3-connected matroid of rank r we shall require several preliminary results including the next theorem, the main result of [**12**, 8.3].

Suppose that $r \geqslant 3$. The *wheel* $\mathcal{W}_r$ *of order* r is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex (the *hub*) is joined by a single edge (a *spoke*) to each of the other vertices. The *whirl* $\mathcal{W}^r$ *of order* r is a matroid on $E(\mathcal{W}_r)$ having as its circuits all cycles of $\mathcal{W}_r$ other than the rim, as well as all sets of edges formed by adding a single spoke to the edges of the rim. The terms "rim" and "spokes" will be applied in the obvious way in both $M(\mathcal{W}_r)$ and $\mathcal{W}^r$. Each of $M(\mathcal{W}_r)$ and $\mathcal{W}^r$ has rank r and is isomorphic to its dual [**12**, 4.7].

(**4.1**) Theorem *A* 3-*connected matroid has all of its elements essential if and only if it is a whirl or the cycle matroid of a wheel.*

(**4.2**) Lemma *Let M be an n-connected matroid having at least* $2(n-1)$ *elements and suppose $x_1, x_2, \ldots, x_m \in E(M)$ where $m < n$. If $M \backslash x_1, x_2, \ldots, x_m$ is n-connected, then so is $M \backslash x_1, x_2, \ldots, x_k$ for $k \leqslant m$.*

Proof. We shall prove this result by induction on $m-k$, it being trivially true for $m-k=0$. Assume it true for $m-k<t$ and let $m-k=t$. Then $M \backslash x_1, x_2, \ldots, x_{m-t+1}$ is n-connected. If $M \backslash x_1, x_2, \ldots, x_{m-t}$ is not n-connected, then for some $j \leqslant n-1$, $M \backslash x_1, x_2, \ldots, x_{m-t}$ is j-separated. That is, $E(M \backslash x_1, x_2, \ldots, x_{m-t}) = X \dot{\cup} Y$ where $|X|, |Y| \geqslant j$ and

(**4.3**) $\mathrm{rk}\, X + \mathrm{rk}\, Y - \mathrm{rk}\,(M \backslash x_1, x_2, \ldots, x_{m-t}) = j-1$.

Now, by Lemma 2.2, M has no cocircuit of size less than n, hence $\mathrm{rk}\,(M \backslash x_1, x_2, \ldots, x_{m-t}) = \mathrm{rk}\, M = \mathrm{rk}\,(M \backslash x_1, x_2, \ldots, x_{m-t+1})$, and so, from (4.3), we get

(**4.4**) $\mathrm{rk}\, X + \mathrm{rk}\, Y - \mathrm{rk}\,(M \backslash x_1, x_2, \ldots, x_{m-t+1}) = j-1$.

Suppose, without loss of generality, that $x_{m-t+1} \in X$. If $|X|>j$, then $|X-x_{m-t+1}| \geq j$ and, by (4.4), $\text{rk}\,(X-x_{m+t-1})+\text{rk}\,Y-\text{rk}\,(M\backslash x_1, x_2, \ldots, x_{m-t+1}) \leq j-1$, contrary to the fact that $M\backslash x_1, x_2, \ldots, x_{m-t+1}$ is n-connected. We may therefore assume that $|X|=j$. Now, since $j \leq n-1$, M has no circuits of size less than $j+1$, hence $\text{rk}\,(X-x_{m-t+1})=\text{rk}\,X-1$. Therefore, from (4.4), we have,

$$\text{rk}\,(X-x_{m-t+1})+\text{rk}\,Y-\text{rk}\,(M\backslash x_1, x_2, \ldots, x_{m-t+1})=j-2.$$

This contradiction to the fact that $M\backslash x_1, x_2, \ldots, x_{m-t+1}$ is n-connected completes the proof of the lemma.

The next result contains the core of the argument which determines the greatest number of elements in a minimally 3-connected matroid of rank r. It is essentially the matroid analogue of a result of Halin [**4**, Satz 7.5].

(**4.5**) LEMMA *Let M be a minimally 3-connected matroid and suppose that $|E(M)| \geq 6$. Then either*

(i) *M is isomorphic to a whirl or the cycle matroid of a wheel; or*

(ii) *M/e is minimally 3-connected for some element e of M; or*

(iii) *every element of M is in a triad, M has a non-essential element and, for all non-essential elements f, either*

   (a) *$M/f\backslash x$ is minimally 3-connected where $\{f, x, y\}$ is a triad of M, it being the unique triad containing x and the unique triad containing both f and y; or*

   (b) *$M/f\backslash x, y$ is minimally 3-connected where $\{f, x, y\}$ is the unique triad intersecting $\{f, x, y\}$.*

Proof. If $|E(M)|=6$, then it is straightforward to check that either M is isomorphic to $U_{4,6}$, in which case (ii) holds, or M is isomorphic to $M(\mathcal{W}_3)$ or $\mathcal{W}^3$, in which case (i) holds. We may therefore assume that $|E(M)| \geq 7$. Suppose, in addition, that neither (i) nor (ii) holds. Then, by Theorem 2.9, every element of M is in a triad. Moreover, by Theorem 4.1, M certainly has a non-essential element. Let f be any such element. Then, as M/f is not minimally 3-connected, there is an element x of M/f such that $M/f\backslash x$ is 3-connected. Since $M\backslash x$ is not 3-connected, it follows that for some element y of $M/f\backslash x$, the set $\{f, x, y\}$ is a triad. Now since $|E(M/f\backslash x)| \geq 5$, every triad of M containing x also contains f, as otherwise $M/f\backslash x$ has a cocircuit of size 2; a contradiction. It follows from this, using circuit exchange, that $\{f, x, y\}$ is the unique triad containing both f and x and hence $\{f, x, y\}$ is the unique triad of M containing x. Applying circuit exchange again gives that $\{f, x, y\}$ is the unique triad containing both f and y.

If $M/f\backslash x$ is not minimally 3-connected, then there is an element z of $M/f\backslash x$ such that $M/f\backslash x, z$ is 3-connected. But M/f and $M/f\backslash x, z$ are 3-connected and $|E(M/f)| \geq 6$, hence by Lemma 4.2, $M/f\backslash z$ is 3-connected. It follows by Lemma 2.6 that for some element w, $\{f, z, w\}$ is a triad of M. Moreover, as above, every triad containing z also contains f.

Now suppose $z \neq y$. Then $\{f, x, y\}$ and $\{f, z, w\}$ are distinct cocircuits of M, so by exchange, there is a cocircuit C^* contained in $\{x, y, z, w\}$. As $|C^*| \geq 3$, $C^* - \{x, z\}$ contains a cocircuit D^* of $M/f \backslash x, z$ and $|D^*| \leq 2$. As $M/f \backslash x, z$ is 3-connected, this contradicts Lemma 2.2. We conclude that $z = y$ and hence that $M/f \backslash x, y$ is 3-connected and $\{f, x, y\}$ is the unique triad intersecting $\{x, y\}$. Another application of the argument just used gives that $\{f, x, y\}$ is the unique triad of M intersecting $\{f, x, y\}$. Since $M/f \backslash x, y = M \backslash f, x, y$ and every element of M is in a triad, every element of $M/f \backslash x, y$ is in a triad of $M/f \backslash x, y$, and so $M/f \backslash x, y$ is minimally 3-connected.

(4.6) LEMMA *Let M be a minimally* 3*-connected matroid of rank r for which none of* (4.5) (i), (4.5) (ii) *or* (4.5) (iii) (a) *holds. If $|E(M)| \geq 2r$ and $r \geq 4$, then $r \geq 6$. Moreover, if $r = 6$, then $|E(M)| = 2r$.*

Proof. By Lemma 4.5 every element of M is in a triad. Moreover, M has a non-essential element f such that M/f is 3-connected and $M/f \backslash x, y$ is minimally 3-connected, where $\{f, x, y\}$ is the unique triad of M intersecting $\{f, x, y\}$. Now $M/f \backslash x, y = M \backslash f, x, y$ and $|E(M \backslash f, x, y)| \geq 2r - 3 \geq 5$, so $M \backslash f, x, y$ has no cocircuits of size less than 3. Therefore if $a_1 \in E(M \backslash f, x, y)$, then there is a triad of M containing a_1 and this triad must avoid $\{f, x, y\}$. Hence the complementary hyperplane H_1 of this triad contains $\{f, x, y\}$ and $|E(M \backslash f, x, y) \cap H_1| \geq 2r - 6 \geq 2$. We may therefore choose an element a_2 from $H_1 \cap E(M \backslash f, x, y)$. Now M has a triad containing a_2 and this triad avoids $\{f, x, y\}$. It follows that the corresponding hyperplane H_2 contains $\{f, x, y\}$ and $|E(M \backslash f, x, y) \cap H_1 \cap H_2| \geq 2r - 9$.

If $r = 4$, then $H_1 \cap H_2$ has rank 2 and so $\{f, x, y\}$ is a circuit of M. But $\{f, x, y\}$ is also a cocircuit of M and this is a contradiction to Lemma 1.6.

If $r = 5$, then we can choose an element a_3 from $E(M \backslash f, x, y) \cap H_1 \cap H_2$. Now M has a triad containing a_3 and avoiding $\{f, x, y\}$. Thus the corresponding hyperplane H_3 contains $\{f, x, y\}$, and so $H_1 \cap H_2 \cap H_3 \supseteq \{f, x, y\}$. Therefore $\{f, x, y\}$ is both a triangle and a triad of M; a contradiction.

If $r = 6$ and $|E(M)| > 2r$, then $|E(M \backslash f, x, y) \cap H_1 \cap H_2 \cap H_3| \geq 1$ and we can extend the above argument to again obtain the contradiction that $\{f, x, y\}$ is both a triangle and a triad of M.

(4.7) THEOREM *Let M be a minimally* 3*-connected matroid of rank r where $r \geq 3$. Then*

$$|E(M)| \leq \begin{cases} 2r, & \text{if } r \leq 6; \\ 3r - 6, & \text{if } r \geq 7. \end{cases}$$

Proof. If r is a positive integer and $r \geq 2$, then let $g(r) = \max\{|E(M)|: M$ is a minimally 3-connected matroid of rank $r\}$. It is easy to check that $g(2) = 3$, with the unique minimally 3-connected matroid of rank 2 on 3 elements being $U_{2,3}$. Since $M(\mathcal{W}_3)$ is minimally 3-connected of rank 3, $g(3) \geq 6$. But $g(3) \leq 6$ as, by Lemma 4.5,

(4.8) $g(r+1) \leq \max\{2(r+1), g(r)+3\}$.

Thus $g(3) = 6$.

We now show by induction on r that

(4.9) $g(r) = 2r$ for $r = 3, 4, 5, 6$.

Suppose that $3 \leq r \leq 5$ and $g(r) = 2r$. As $M(\mathcal{W}_{r+1})$ is minimally 3-connected, $g(r+1) \geq 2(r+1)$. Now let M be a minimally 3-connected matroid of rank $r+1$ having $g(r+1)$ elements. If (4.5) (i) holds, then $|E(M)| = 2(r+1)$. If (4.5) (ii) holds, then $g(r) \geq g(r+1) - 1 \geq 2(r+1) - 1 > 2r$; a contradiction. If (4.5) (iii) (a) occurs, then $|E(M)| \leq 2r+2$. It follows that we may assume that none of (4.5) (i), (ii) or (iii) (a) occurs. Thus by Lemma 4.6, $|E(M)| \leq 2(r+1)$ and the proof of (4.9) is complete.

If $r \geq 7$, then, by (4.8) and (4.9), $g(r) \leq 12 + 3(r-6)$. That is, $g(r) \leq 3r - 6$. Now as $r \geq 7$, $K_{3,r-2}$ is a 3-connected graph. Thus, by (1.4), $M(K_{3,r-2})$ is a 3-connected matroid and hence, as every element of $M(K_{3,r-2})$ is in a triad, $M(K_{3,r-2})$ is minimally 3-connected. But $|E(M(K_{3,r-2}))| = 3r - 6$, hence for $r \geq 7$, $g(r) = 3r - 6$, as required.

5. The minimally 3-connected matroids with the greatest number of elements

The characterization of those minimially 3-connected matroids attaining equality in Theorem 4.7 will be broken up into the cases $3 \leq r \leq 5$ and $r \geq 6$. The first of these is quite long.

(5.1) Lemma *Let M be a minimally 3-connected matroid of rank $r+1$ on a set of $2(r+1)$ elements where $r \geq 4$. Suppose that M satisfies (4.5)(iii)(a). Then $M/f \backslash x$ is not isomorphic to $M(\mathcal{W}_r)$ or $\mathcal{W}^r$.*

Proof. By (4.5)(iii)(a), every element of M is in a triad and $M/f\backslash x$ is minimally 3-connected where $\{f, x, y\}$ is a triad of M. We shall assume that $M/f\backslash x$ is isomorphic to $M(\mathcal{W}_r)$ or $\mathcal{W}^r$ and obtain a contradiction. Labelling both $E(\mathcal{W}_r)$ and $E(\mathcal{W}^r)$ as shown we may suppose, without loss of generality, that either (I) $y = 1$, or (II) $y = r+1$.

Case I. If $y = 1$, then $\{f, x, 1\}$ is a triad of M and $\{1, r+1, 2r\}$, $\{2, r+1, r+2\}$, $\{3, r+2, r+3\}, \ldots, \{r, 2r-1, 2r\}$ are the triads of $M/f\backslash x$. Now, in $M\backslash x$, the set $\{f, 1\}$ is a cocircuit, hence $M/f\backslash x \cong M/1\backslash x$ and the

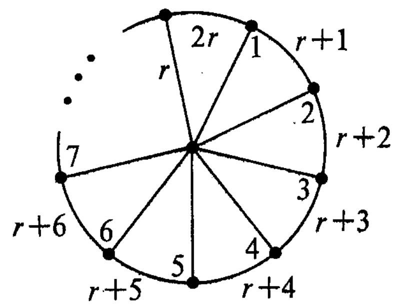

FIG. 1.

triads of $M/1\backslash x$ are $\{f, r+1, 2r\}$, $\{2, r+1, r+2\}$, $\{3, r+2, r+3\}, \dots, \{r, 2r-1, 2r\}$. By (4.5)(iii) (a), $\{f, x, 1\}$ is the unique triad of M containing x, and the unique triad containing both f and 1.

Now every element of M is in a triad and the only triads of $M/f\backslash x$ containing $2, 3, \dots, r$ respectively are $\{2, r+1, r+2\}, \{3, r+2, r+3\}, \dots,$ and $\{r, 2r-1, 2r\}$. These are also the only triads of $M/1\backslash x$ containing $2, 3, \dots, r$ respectively, hence each of these $r-1$ sets is a triad of M. The corresponding hyperplanes all have rank r and their intersection has rank 2 and contains $\{f, x, 1\}$. But this again gives the contradiction that M has a subset which is both a triangle and a triad.

Case II. If $y=r+1$, then from (4.5) (iii) (a), $\{f, x, r+1\}$ is the unique triad containing x, and the unique triad containing both f and $r+1$. The triads of $M/f\backslash x$ are

$$\{1, r+1, 2r\}, \{2, r+1, r+2\}, \{3, r+2, r+3\}, \dots, \{r, 2r-1, 2r\},$$

and of $M/r+1\backslash x$ are

$$\{1, f, 2r\},\ \{2, f, r+2\}, \{3, r+2, r+3\}, \dots, \{r, 2r-1, 2r\}.$$

It follows that

$$\{3, r+2, r+3\}, \{4, r+3, r+4\}, \dots, \{r, 2r-1, 2r\}$$

are triads of M, they being the only triads of $M/f\backslash x$ and of $M/r+1\backslash x$ which contain $3, 4, \dots, r$ respectively.

M has a triad containing 1 and so at least one of $\{1, f, 2r\}$ and $\{1, r+1, 2r\}$ is a triad of M. Note that if both $\{1, r+1, 2r\}$ and $\{1, f, 2r\}$ are triads, then $\{f, 1, r+1\}$ is a triad of M; a contradiction. Similarly exactly one of $\{2, r+1, r+2\}$ and $\{2, f, r+2\}$ is a triad of M.

Now taking the intersection of the hyperplanes of M corresponding to the triads $\{3, r+2, r+3\}$, $\{4, r+3, r+4\}, \dots, \{r, 2r-1, 2r\}$, we get that $\{f, x, 1, 2, r+1\}$ is a plane of M.

We now distinguish 3 cases.

(A) $\{1, r+1, 2r\}$ is a triad of M;
(B) $\{2, r+1, r+2\}$ is a triad of M; and
(C) $\{1, f, 2r\}$ and $\{2, f, r+2\}$ are triads of M.

(A) The hyperplane corresponding to $\{1, r+1, 2r\}$ is

$$\{f, x, 2, 3, 4, \ldots, r, r+2, r+3, \ldots, 2r-1\}.$$

Intersecting this with the plane $\{f, x, 1, 2, r+1\}$ gives that $\{f, x, 2\}$ is a line of M. Thus $\{2, x\}$ is a circuit of the 3-connected matroid M/f; a contradiction.

(B) A similar argument to that given in (A) again leads to a contradiction.

(C) In this case, $\{x, 2, 3, \ldots, 2r-1\}$ and $\{x, 1, 3, 4, \ldots, r, r+1, r+3, r+4, \ldots, 2r\}$ are hyperplanes of M. Intersecting each of these hyperplanes with the plane $\{f, x, 1, 2, r+1\}$ gives that $\{x, 2, r+1\}$ and $\{x, 1, r+1\}$ are both lines of M; a contradiction.

(5.2) THEOREM *Let M be a minimally 3-connected matroid of rank r having precisely $2r$ elements. If $3 \leq r \leq 5$, then M is isomorphic to $M(\mathcal{W}_r)$ or $\mathcal{W}^r$.*

Proof. As noted earlier it is straightforward to check that the only rank 3 minimally 3-connected matroids having 6 elements are $M(\mathcal{W}_3)$ and $\mathcal{W}^3$.

Now let M be a minimally 3-connected matroid of rank 4 on a set of 8 elements. By Lemma 4.6, one of (4.5) (i), (ii) or (iii) (a) must hold. If M is not isomorphic to $M(\mathcal{W}_4)$ or $\mathcal{W}^4$, then (4.5) (i) does not hold. Moreover, since $g(3)=6$, (4.5) (ii) does not hold. Therefore we may assume that (4.5) (iii) (a) holds. Hence, every element of M is in a triad and $M/f\backslash x$ is a minimally 3-connected matroid where $\{f, x, y\}$ is a triad of M. Since $M/f\backslash x$ has rank 3 and 6 elements it follows that $M/f\backslash x$ is isomorphic to $M(\mathcal{W}_3)$ or $\mathcal{W}^3$. Thus $M\backslash f, x, y$ is isomorphic to one of the matroids N_1 or N_2 in Figure 2.

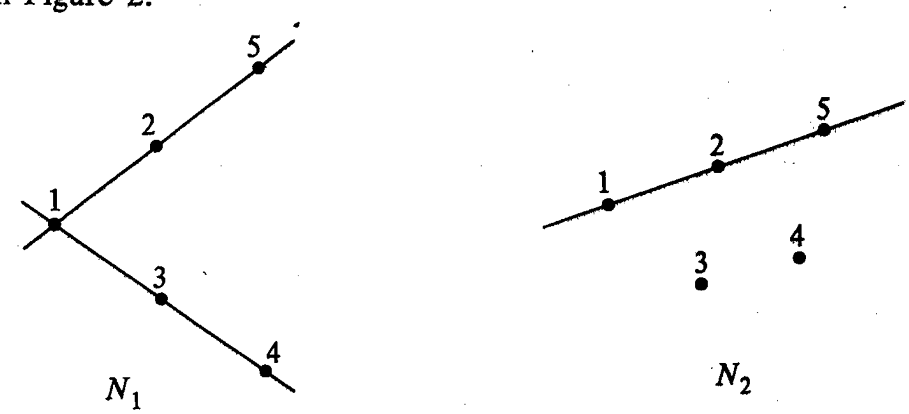

FIG. 2.

Since every element of M is in a triad, corresponding to every element e of M, there is a 5-point plane avoiding e. With N_1 and N_2 labelled as shown, let $F_0=\{1,2,3,4,5\}$ and F_1 be a 5-point plane of M avoiding 1. Then, since $|E(M)-F_0|=3$, $|F_0\cap F_1|\geq 2$. Moreover, since $F_0\cap F_1$ has rank 2 and all the 3-point lines of F_0 contain 1, $|F_0\cap F_1|\leq 2$. Thus $|F_0\cap F_1|=2$ and so $F_1\supseteq\{f,x,y\}$. Now, if $F_1\cap\{1,2,5\}=\emptyset$, then $\{1,2,5\}$ is both a triangle and a triad of M. Thus, without loss of generality, suppose that $F_0\cap F_1=\{2,3\}$. Let F_2 be a 5-point plane of M avoiding 2. Then $|F_0\cap F_2|\geq 2$. Furthermore, in N_2, $|F_0\cap F_2|\leq 2$, since the only 3-point line of F_0 in N_2 contains 2. Thus in N_2, $|F_0\cap F_2|=2$, and so $F_1\cap F_2\supseteq\{f,x,y\}$. This implies that $\{f,x,y\}$ is both a triangle and a triad; a contradiction. If $|F_0\cap F_2|=2$ in N_1 we obtain the same contradiction. However, in N_1, we may have $|F_0\cap F_2|=3$, in which case $F_0\cap F_2=\{1,3,4\}$. Then $F_1\cap F_2\cap F_0=\{3\}$ and $|F_1\cap F_2\cap(E(M)\setminus F_0|=2$. Now M/f is 3-connected and is therefore simple, hence $f\notin F_1\cap F_2$. Therefore $F_1\cap F_2=\{x,y,3\}$. Let F_3 be a 5-point plane of M avoiding the element 3. If $|F_3\cap F_0|=2$, then $F_1\cap F_3\supseteq\{f,x,y\}$, and M has $\{f,x,y\}$ as both a circuit and a cocircuit; a contradiction. If $|F_3\cap F_0|=3$, then $F_3\cap F_0=\{1,2,5\}$. Now, arguing as for $F_1\cap F_2$, we get that $F_1\cap F_3=\{x,y,2\}$. But this means that $\{x,y,2\}$ and $\{x,y,3\}$ are both lines of the simple matroid M. This contradiction completes the proof of the case $r=4$. The rest of the proof uses the result for this case together with Lemmas 4.6 and 5.1.

(5.3) LEMMA *Let M be a minimally 3-connected matroid of rank r on a set of $3r-6$ elements and suppose $r\geq 6$. If $r=6$ and (4.5) (iii) (b) occurs, or $r\geq 7$, then $E(M)$ is the disjoint union of $r-2$ triads. Moreover, no circuit of M has fewer than 4 elements and M has precisely $r-2$ distinct triads.*

Proof. If $r\geq 7$, then since no minimally 3-connected matroid of rank $r-1$ has more than $3r-9$ elements, it follows that M satisfies (4.5) (iii) (b). That is, every element of M is in a triad and $M\setminus f,x,y$ is minimally 3-connected where $\{f,x,y\}$ is the unique triad of M intersecting $\{f,x,y\}$.

Now $|E(M\setminus f,x,y)|=3r-9\geq 9$. Choose a_1 from $E(M\setminus f,x,y)$. There is a triad of M containing a_1 and avoiding $\{f,x,y\}$. Let H_1 be the hyperplane which is complementary to this triad. Then $H_1\supseteq\{f,x,y\}$ and $|E(M\setminus f,x,y)\cap H_1|\geq 3r-12$. Assume that for some i in $\{1,2,\ldots,r-4\}$, the elements, $a_1,a_2,\ldots,a_i$ and the hyperplanes $H_1,H_2,\ldots,H_i$ have been chosen such that

$$a_j\in(E(M\setminus f,x,y)\cap H_1\cap H_2\cdots\cap H_{j-1})-H_j,\quad |H_j|=|E(M)|-3$$

and $H_j\supseteq\{f,x,y\}$ for all $1\leq j\leq i$. Then

$$|E(M\setminus f,x,y)\cap H_1\cap H_2\cdots\cap H_i|\geq 3r-6-3(i+1)\geq 3.$$

Now choose a_{i+1} from $E(M\backslash f, x, y)\cap H_1\cap H_2\cap\cdots\cap H_i$. Then M has a triad containing a_{i+1} and avoiding $\{f, x, y\}$. Let H_{i+1} be the hyperplane complementary to this triad. Then $H_{i+1}\supseteq\{f, x, y\}$ and

$$\textbf{(5.4)}\quad |E(M\backslash f, x, y)\cap H_1\cap H_2\cap\cdots\cap H_i\cap H_{i+1}| \geqslant |E(M\backslash f, x, y)\cap H_1\cap H_2\cap\cdots\cap H_i|-3.$$

It follows that

$$\textbf{(5.5)}\quad |E(M\backslash f, x, y)\cap H_1\cap H_2\cap\cdots\cap H_{r-3}|\geqslant 3r-6-3(r-2)\geqslant 0.$$

If the inequality (5.4) is strict for some i, then the inequality (5.5) is strict and so we may choose an element a_{r-2} from $E(M\backslash f, x, y)\cap H_1\cap H_2\cap\cdots\cap H_{r-3}$ and a hyperplane H_{r-2} of M such that $|H_{r-2}|=|E(M)|-3$, $a_{r-2}\notin H_{r-2}$ and $\{f, x, y\}\subseteq H_{r-2}$. Then $H_1\cap H_2\cap\cdots\cap H_{r-2}$ is a line of M containing the triad $\{f, x, y\}$. This contradiction to Lemma 1.6 means that equality holds in (5.4) for all i and hence that M is a disjoint union of $r-2$ triads.

The fact that every circuit of M has at least 4 elements follows from Lemmas 1.5 and 1.6. To see that M has precisely $r-2$ triads, first recall that $\{f, x, y\}$ is the unique triad intersecting $\{f, x, y\}$. If $e\in E(M\backslash f, x, y)$ and e is in two distinct triads C_1^* and C_2^*, then let $e=a_1$ and $H_1=E(M)-C_1^*$. Now $C_2^*\cap H_1\neq\emptyset$, so choose a_2 from this set. Then $H_2=E(M)-C_2^*$ is a hyperplane avoiding a_2 and containing $\{f, x, y\}$. But $|C_1^*\cup C_2^*|\leqslant 5$, hence $|E(M\backslash f, x, y)\cap H_1\cap H_2|\geqslant E(M\backslash f, x, y)\cap H_1|-2$, contrary to the fact that equality holds in (5.4). This completes the proof of the lemma.

(5.6) Theorem *Let M be a minimally 3-connected matroid of rank r having precisely $3r-6$ elements and suppose that $r\geqslant 6$. Then either*
(i) *$r=6$ and M is isomorphic to $M(\mathcal{W}_6)$ or $\mathcal{W}^6$; or*
(ii) *no circuit of M has size less than 4 and M is a disjoint union of triads.*

Proof. If $r=6$ and (i) above does not hold, then (4.5) (i) does not hold. As $g(5)=10$, (4.5) (ii) does not hold, and, by Lemma 5.1 and Theorem 5.2, (4.5) (iii) (a) does not hold. Thus (4.5) (iii) (b) holds. To complete the proof one needs only to apply Lemma 5.3.

We show next that every rank r matroid having $3r-6$ elements and satisfying (5.6) (ii) is minimally 3-connected.

(5.7) Lemma *Let M be a rank r matroid where $r\geqslant 5$ and suppose that M has no circuits of size less then 4. If $E(M)=T_1\dot\cup T_2\dot\cup\cdots\dot\cup T_{r-2}$ where $T_1, T_2, \ldots, T_{r-2}$ are triads of M, then M is minimally 3-connected.*

Proof. Let $T_i=\{a_i, b_i, c_i\}$ for $1\leqslant i\leqslant r-2$. As M has no circuits of size less than 4, $B=T_1\cup\{d_2, d_3, \ldots, d_{r-2}\}$ is a basis of M where $d_i\in T_i$ for all i such that $2\leqslant i\leqslant r-2$. Now if $e_i\in T_i-d_i$, then the fundamental circuit of

e_i with respect to B meets T_1 in at least two elements. Thus, by (1.2), M is 2-connected. Now to establish that M is 3-connected we need to show that for all pairs of subsets X and Y of $E(M)$ such that $X \dot\cup Y = E(M)$ and $|X|, |Y| \geq 2$,

(5.8) $\text{rk}\, X + \text{rk}\, Y \geq r+2$.

If $X \supseteq T_i$ for some i, then either $Y \supseteq T_j$ for some $j \neq i$, or not. In the first case, on letting $t = |\{k\colon X \cap T_k \neq \emptyset\}|$, we have $\text{rk}\, X \geq t+2$ and $\text{rk}\, Y \geq (r-2-t)+2$, hence (5.8) holds. In the second case, $\text{rk}\, X = r$ and, as $|Y| \geq 2$, $\text{rk}\, Y \geq 2$, thus again (5.8) holds. We may now assume that for all i such that $1 \leq i \leq r-2$, $X \cap T_i \neq \emptyset \neq Y \cap T_i$. Then $\text{rk}\, X \geq r-2$ and $\text{rk}\, Y \geq r-2$. But either $|X \cap T_1| = 2$ or $|Y \cap T_1| = 2$. Thus $\text{rk}\, X \geq r-1$ or $\text{rk}\, Y \geq r-1$, and hence $\text{rk}\, X + \text{rk}\, Y \geq (r-1)+(r-2) \geq r+2$, where the second inequality holds since $r-2 \geq 3$. We conclude that M is 3-connected. As every element is in a triad, it follows that M is minimally 3-connected.

If M satisfies the hypotheses of Lemma 5.7, it is not difficult to see that M is uniquely determined by its set $\mathcal{D}$ of circuits of size 4. Now, by Lemma 1.5, circuit exchange, and the fact that M has no circuits of size less than 4, we get that

(5.9) (i) *if* $D \in \mathcal{D}$, *then* $|D \cap T_i|$ *is even for all* i;
(ii) *if* $D_1, D_2 \in \mathcal{D}$, *then* $|D_1 \cap D_2| \leq 2$; *and*
(iii) *if* $D_1, D_2 \in \mathcal{D}$ *and* $|D_1 \cap D_2 \cap T_i| = 2$ *for some* i, *then* $D_1 \triangle D_2 \in \mathcal{D}$.

(5.10) THEOREM *Let* S *be a set of* $3r-6$ *elements where* $r \geq 6$ *and suppose that* S *is the disjoint union of* $r-2$ *sets* $T_1, T_2, \ldots, T_{r-2}$ *of size* 3. *Let* $\mathcal{D}$ *be a collection of* 4-*element subsets of* S *satisfying* (5.9) (i)–(iii) *and*

$$\mathcal{B} = \Big\{A \cup F\colon |A| = 4, A \subseteq T_i \cup T_j \quad \text{where} \quad \{i, j\} \subseteq \{1, 2, \ldots, r-2\}, \\ A \notin \mathcal{D} \quad \text{and} \quad F \text{ is a transversal of } \{T_k\}_{k=1}^{r-2}\Big\}.$$

Then $\mathcal{B}$ *is the set of bases and* $\{T_1, T_2, \ldots, T_{r-2}\}$ *the set of triads of a minimally* 3-*connected matroid on* S.

The proof of this theorem, although long, is routine and will not be given here.

(5.11) COROLLARY *Let* M *be a minimally* 3-*connected matroid of rank* r *and suppose* $|E(M)| = 3r-6$. *Then for* $r \geq 6$, *either* M *is isomorphic to* $M(\mathcal{W}_6)$ *or* $\mathcal{W}^6$, *or* M *is as in Theorem* 5.10.

Recall, from the proof of Theorem 4.7, that $g(r)$ denotes the greatest number of elements in a minimally 3-connected matroid of rank r.

(**5.12**) THEOREM *Let M be a minimally 3-connected matroid of rank r having $g(r)$ elements where $r \geq 2$. If M is binary, then*

$$\begin{array}{llll} M \cong U_{2,3} & & \text{for} & r=2; \\ M \cong M(\mathcal{W}_r) & & \text{for} & 3 \leq r \leq 5; \\ M \cong M(\mathcal{W}_6) & \text{or} \quad M(K_{3,4}) & \text{for} & r=6; \quad \text{and} \\ M \cong M(K_{3,r-2}) & & \text{for} & r \geq 7. \end{array}$$

Proof. Since $\mathcal{W}^r$ is not binary, if $3 \leq r \leq 5$, the result follows by Theorem 5.2. If $r \geq 6$, we shall use (5.6), (5.7), (5.9) and (5.10) to get the result. In particular, we show that for each $r \geq 6$ there is a unique binary matroid of the type specified in Theorem 5.10. Suppose that M is a binary minimally 3-connected matroid of rank r where $r \geq 6$. Assume, moreover, that $E(M) = T_1 \dot\cup T_2 \dot\cup \cdots \dot\cup T_{r-2}$ where $T_1, T_2, \ldots, T_{r-2}$ are triads of M and, for all i, $T_i = \{a_i, b_i, c_i\}$. Consider $M \mid (T_i \cup T_j)$ where $1 \leq i < j \leq r-2$. This is a binary matroid on a set of 6 elements. Since the $r-4$ disjoint cocircuits $T_1, T_2, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{j-1}, T_{j+1}, \ldots, T_{r-2}$ have been deleted from M, $\text{rk}(M \mid (T_i \cup T_j)) \leq 4$. But, by (5.6) (ii) and (5.9) (i), $\{a_i, b_i, c_i, a_j\}$ is independent in $M \mid (T_i \cup T_j)$, and so $\text{rk}(M \mid (T_i \cup T_j)) = 4$. By (5.9) (i) again, $T_i \cup \{a_j\}$, $T_i \cup \{b_j\}$, $T_i \cup \{c_j\}$ and $T_j \cup \{a_i\}$, $T_j \cup \{b_i\}$, $T_j \cup \{c_i\}$ are bases of $M \mid (T_i \cup T_j)$. Consider $(M \mid (T_i \cup T_j))^*$. This is a rank 2 binary matroid on a set of 6 elements. Moreover, if $|B^*| = 2$ and $B^* \subseteq T_i$ or $B^* \subseteq T_j$, then B^* is a base of $(M \mid (T_i \cup T_j))^*$. Since M is binary, it follows that $(M \mid (T_i \cup T_j))^*$ is isomorphic to a 3-point line where each point consists of a pair of parallel elements. If we relabel the points of $T_2, T_3, \ldots, T_{r-2}$ as necessary so that in $(M \mid (T_1 \cup T_i))^*$, $\{a_1, a_i\}$, $\{b_1, b_i\}$, and $\{c_1, c_i\}$ are circuits, then $\{a_1, a_i, b_1, b_i\}$, $\{a_1, a_i, c_1, c_i\}$ and $\{b_1, b_i, c_1, c_i\}$ are circuits of M, the only 4-element circuits contained in $T_1 \cup T_i$. We want to check that in $M \mid (T_i \cup T_j)$, $\{a_i, a_j, b_i, b_j\}$, $\{a_i, a_j, c_i, c_j\}$ and $\{b_i, b_j, c_i, c_j\}$ are circuits. Suppose then that $\{d_j, e_j, f_j\} = \{a_j, b_j, c_j\}$ and that $\{a_i, d_j, b_i, e_j\}$, $\{a_i, d_j, c_i, f_j\}$ and $\{b_i, e_j, c_i, f_j\}$ are circuits of $M \mid (T_i \cup T_j)$. Then, as M is binary, taking the

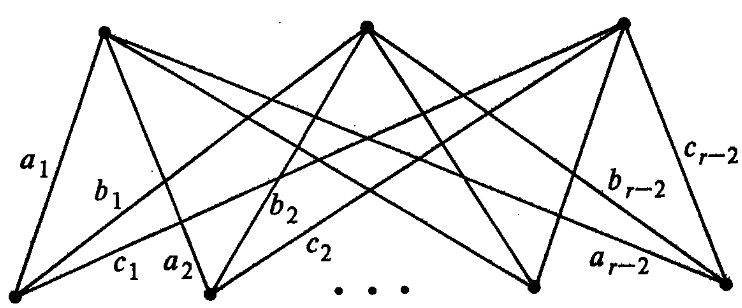

FIG. 3.

symmetric difference of these circuits with $\{a_1, a_i, b_1, b_i\}$, $\{a_1, a_i, c_1, c_i\}$ and $\{b_1, b_i, c_1, c_i\}$ respectively gives that each of $\{a_1, d_j, b_1, e_j\}$, $\{a_1, d_j, c_1, f_j\}$ and $\{b_1, e_j, c_1, f_j\}$ must be a circuit of M. Thus $d_j = a_j$, $e_j = b_j$ and $f_j = c_j$. We conclude that the collection of circuits of size 4 of M is

$$\{\{x_i, x_j, y_i, y_j\}: \{x, y\} \subseteq \{a, b, c\} \quad \text{and} \quad 1 \leqslant i < j \leqslant r-2\}.$$

Labelling $K_{3,r-2}$ as shown we see that $M \cong M(K_{3,r-2})$.

(**5.13** COROLLARY [**4**, Satz 7.6]. *Let G be a minimally 3-connected graph. If $|V(G)| \geqslant 7$, then $|E(G)| \leqslant 3|V(G)| - 9$. Equality is attained here for $|V(G)| \geqslant 8$ if and only if $G \cong K_{3,|V(G)|-3}$, and for $|V(G)| = 7$ if and only if $G \cong K_{3,4}$ or $\mathcal{W}_6$.*

Acknowledgement.

The author thanks the referee for several helpful suggestions.

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