ON TWO CLASSES OF NEARLY BINARY MATROIDS

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ABSTRACT. We give an excluded-minor characterization for the class of matroids M in which $M\backslash e$ or M/e is binary for all e in $E(M)$. This class is closely related to the class of matroids in which every member is binary or can be obtained from a binary matroid by relaxing a circuit-hyperplane. We also provide an excluded-minor characterization for the second class.

1. INTRODUCTION

The class of binary matroids is one of the most widely studied classes of matroids and its members have numerous attractive properties. This motivates the study of classes of matroids whose members are close to being binary. In this paper, we consider one very natural such minor-closed class $\mathcal{Z}$, which consists of those matroids M such that $M\backslash e$ or M/e is binary for all elements e of M. The main result of the paper is an excluded-minor characterization of $\mathcal{Z}$. This theorem can be restated in terms of matroid fragility, which has enjoyed a recent surge of research interest. Let N be a matroid. A matroid M is N-*fragile* if, for each element e of $E(M)$, at least one of $M\backslash e$ and M/e has no N-minor (see, for example, [5]). The class of N-fragile matroids is clearly minor-closed. The main result of this paper determines the set of excluded minors for the class of $U_{2,4}$-fragile matroids. Except where otherwise noted the notation and terminology follow [9].

It is well known that if H is a circuit and a hyperplane of a matroid M, then there is another matroid M' on $E(M)$ whose bases are the bases of M together with H. We say that M' is obtained from M by *relaxing* the *circuit-hyperplane* H and call M' a *relaxation* of M. A class of matroids that arises naturally in determining the excluded minors for $\mathcal{Z}$ is $\mathcal{R}$, those matroids M such that M is binary or M is a relaxation of a binary matroid.

The rank-three whirl is denoted by $\mathcal{W}^3$, while P_6 is the six-element rank-three matroid that has a single triangle as its only non-spanning circuit. Let Q_6 and R_6 be the six-element matroids of rank three for which geometric representations are given in Figure 1. Evidently $R_6 \cong U_{2,4} \oplus_2 U_{2,4}$. Let K be the seven-element rank-two matroid that is obtained by adding elements in parallel to three of the elements of $U_{2,4}$. The matroid K is depicted

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with its dual in Figure 2. In Section 2, we note that both $\mathcal{Z}$ and $\mathcal{R}$ are minor-closed and dual-closed classes of matroids and establish some excluded minors of each. We also introduce Cunningham and Edmonds's canonical tree decomposition of a 2-connected matroid, along with some preliminaries.

Let $\mathcal{D}$ denote the collection of all matroids that are obtained from connected binary matroids by relaxing two disjoint circuit-hyperplanes that partition the ground set. The collection $\mathcal{D}$ is in both our sets of excluded minors. Section 3 is devoted to proving the main result and another related result, both of which are stated next.
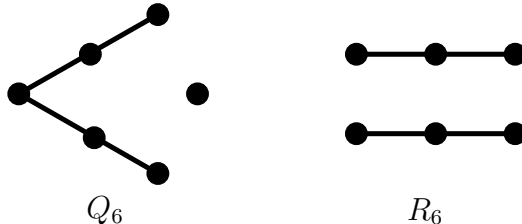

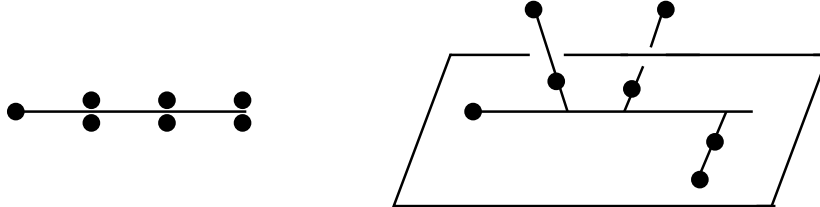FIGURE 1. Geometric representations of the six-element rank-three matroids Q_6 and R_6.

FIGURE 2. Representations of the matroid K and its (rank-5) dual K^*.

Theorem 1.1. *The set of excluded minors for the class of matroids $\mathcal{Z} = \{M : M \backslash e$ or M/e is binary for all e in $E(M)\}$ is $\{Q_6, P_6, U_{3,6}, R_6, U_{2,4} \oplus U_{1,1}, U_{2,4} \oplus U_{0,1}\}$ together with all the matroids in $\mathcal{D}$.*

Theorem 1.2. *The set of excluded minors for the class of matroids $\mathcal{R} = \{M : M$ is binary or can be obtained from a binary matroid by relaxing a circuit-hyperplane$\}$ is $\{U_{2,5}, U_{3,5}, K, K^*, R_6, U_{2,4} \oplus U_{1,1}, U_{2,4} \oplus U_{0,1}\}$ together with all the matroids in $\mathcal{D}$.*

For an even integer r exceeding two, let M_r be the rank-r tipless binary spike, that is, the vector matroid of the binary matrix $[I_r | J_r - I_r]$ where J_r is the matrix of all ones. Labeling the columns of this matrix $e_1, e_2, \ldots, e_{2r}$ in order, we see that $\{e_2, e_3, \ldots, e_r, e_{r+1}\}$ and its complement are both circuit-hyperplanes of M_r. By relaxing these circuit-hyperplanes,

we obtain a member of $\mathcal{D}$. Thus the sets of excluded minors in Theorems 1.1 and 1.2 are both infinite. However, these doubly relaxed spikes are not the only members of $\mathcal{D}$. In Section 4, we further discuss the complexity of $\mathcal{D}$.

As $\mathcal{D}$ shows, the class of matroids that can be obtained from binary matroids by relaxing at most two circuit-hyperplanes does contain an infinite antichain. Geelen, Gerards, and Whittle announced in 2009 that the class of binary matroids itself contains no infinite antichains. These observations raise the interesting question, which was asked by a referee of this paper, as to whether or not the class $\mathcal{R}$ of matroids that can be obtained from binary matroids by relaxing at most one circuit-hyperplane contains an infinite antichain. It is not difficult to check using, for example, [3, Lemma 2.6], that $\mathcal{Z}$ contains an infinite antichain if and only if $\mathcal{R}$ does.

2. Preliminaries

This section first notes that both $\mathcal{Z}$ and $\mathcal{R}$ are minor- and dual-closed, and then determines some excluded minors for each class.

Lemma 2.1. *The classes $\mathcal{Z}$ and $\mathcal{R}$ are both closed under duality and the taking of minors.*

This lemma is immediate for $\mathcal{Z}$ and is a straightforward consequence of the following result of Kahn [4] for $\mathcal{R}$ (see also [9, p. 115]).

Lemma 2.2. *Let X be a circuit-hyperplane of a matroid M and let M' be the matroid obtained from M by relaxing X. Then $(M')^*$ is obtained from M^* by relaxing the circuit-hyperplane $E(M) - X$ of the latter. Moreover, when $e \in E(M) - X$, M/e and M'/e are equal and, unless M has e as a coloop, $M'\backslash e$ is obtained from $M\backslash e$ by relaxing the circuit-hyperplane X of the latter, and the dual situation holds when $e \in X$.*

It is not difficult to deduce from the above result that the class $\mathcal{R}$ is contained in the class $\mathcal{Z}$. We say a matroid N is a *series extension* of a matroid M if $M = N/T$ and every element of T is in series with some element of M. We call N a *parallel extension* of M if N^* is a series extension of M^*. Note that this differs from the terminology used in [9]. The following result from [8] will be used extensively throughout the paper.

Theorem 2.3. *The following two statements are equivalent.*
   (i) *A matroid M is non-binary and, for every element e, at least one of $M\backslash e$ and M/e is binary.*
  (ii) (a) *Both $r(M)$ and $r^*(M)$ exceed two for a matroid M and M can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or*
       (b) *M is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geq 5$; or*

(c) M is isomorphic to a series extension of $U_{n-2,n}$ for some
   $n \geq 5$; or
(d) M can be obtained from $U_{2,4}$ by series extension of a subset S of
   $E(U_{2,4})$ and parallel extension of a disjoint subset T of $E(U_{2,4})$
   where S or T may be empty.

Let $EX(\mathcal{M})$ denote the class of excluded minors for a class of matroids
$\mathcal{M}$. Some excluded minors for $\mathcal{Z}$ and $\mathcal{R}$ are easy to identify. We omit the
routine argument that establishes the following.

Lemma 2.4. *The matroids $U_{2,4} \oplus U_{1,1}$, $U_{2,4} \oplus U_{0,1}$, and R_6 are in both
$EX(\mathcal{Z})$ and $EX(\mathcal{R})$.*

The following three results will also be useful, the first is from [4]; the
second is elementary; the third is from [10].

Lemma 2.5. *Let M' be obtained from M by relaxing a circuit-hyperplane.*
   (i) *If M is connected, then M' is non-binary; and*
   (ii) *if M is n-connected, then so is M'.*

Lemma 2.6. *The only disconnected matroids having a circuit-hyperplane
are $U_{n-1,n} \oplus U_{1,k}$, for integers n, $k \geq 1$.*

Lemma 2.7. *Let M be a binary matroid, H be a circuit-hyperplane of M,
and M' be obtained from M by relaxing H. Then M' is binary if and only
if M is $U_{n-1,n} \oplus U_{1,k}$, for integers n, $k \geq 1$.*

Note that, in Lemmas 2.6 and 2.7, the disconnected matroids are graphic
and carry the name *enlarged 1-wheels* in [10].

Recall, $\mathcal{D}$ is the collection of all matroids that are obtained from connected
binary matroids by relaxing two disjoint circuit-hyperplanes that partition
the ground set.

Lemma 2.8. *All matroids in $\mathcal{D}$ are in both $EX(\mathcal{Z})$ and $EX(\mathcal{R})$.*

Proof. Take a matroid M_2 in $\mathcal{D}$. Let X and Y be the disjoint circuit-
hyperplanes of the connected binary matroid M that are relaxed to obtain
M_2. Let M_X and M_Y denote the matroids obtained from M by relaxing X
and Y, respectively, and take e in $E(M_2)$. Note that the case with $e \in X$
is symmetric to the case with $e \in Y$; both $\mathcal{Z}$ and $\mathcal{R}$ are dual-closed classes,
and if X and Y are complementary circuit-hyperplanes of M, they are so
for M^* as well.

Suppose $e \in X$. By Lemma 2.2, M_2/e is obtained from M_Y/e by relax-
ing the circuit-hyperplane $X - e$ of the latter and $M_Y/e = M/e$. If M/e is
connected, then M_2/e is non-binary by Lemma 2.5. Now assume M/e is
disconnected. Then $M/e = U_{n-1,n} \oplus U_{1,k}$ for some n, $k \geq 1$, by Lemma 2.6.
But Y is a spanning circuit in M/e, which is a contradiction since M/e has
no spanning circuits. We conclude that M_2/e is non-binary. By symmetry

and duality the same argument holds for $M_2 \backslash e$, and for both M_2/f and $M_2 \backslash f$ when $f \in Y$.

Any deletion $M_2 \backslash z$ equals $M_Y \backslash z$ or $M_X \backslash z$. By symmetry we only need to consider the case with $z \in X$. The matroid $M_Y \backslash z$ can be obtained by relaxing a circuit-hyperplane in a binary matroid. By duality, the same holds for M_2/z. Therefore any minor of M_2 is in $\mathcal{R}$ and so is in $\mathcal{Z}$. Thus M_2 is in $EX(\mathcal{R})$ and in $EX(\mathcal{Z})$. $\qquad \square$

The next two lemmas list matroids that are excluded minors for exactly one of $\mathcal{R}$ and $\mathcal{Z}$. Their routine proofs are omitted.

Lemma 2.9. *The matroids $U_{2,5}$, $U_{3,5}$, K, and K^* are excluded minors for the class $\mathcal{R}$.*

Lemma 2.10. *The matroids Q_6, P_6, and $U_{3,6}$ are excluded minors for the class $\mathcal{Z}$.*

A class $\mathcal{N}$ of matroids is 1-*rounded* [12] if every member of $\mathcal{N}$ is connected and, whenever e is an element of a connected matroid M having an $\mathcal{N}$-minor, M has an $\mathcal{N}$-minor using e. The following three results will be useful in our proofs, they come from [1], [12], and [6], respectively.

Lemma 2.11. *The set $\{U_{2,4}\}$ is 1-rounded.*

Lemma 2.12. *The set $\{M(K_4), U_{2,4}\}$ is 1-rounded.*

Lemma 2.13. *The set $\{\mathcal{W}^3, P_6, Q_6, U_{3,6}\}$ is 1-rounded.*

Next we introduce Cunningham and Edmonds's tree decomposition for connected matroids [2]. Our treatment of this material follows [9, pp. 307–310]. A *matroid-labeled tree* is a tree T with vertex set $\{M_1, M_2, \ldots, M_k\}$ for some positive integer k such that

    (i) each M_i is a matroid;
    (ii) if M_{j_1} and M_{j_2} are joined by an edge e_i of T, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and $\{e_i\}$ is not a separator of M_{j_1} or M_{j_2}; and
    (iii) if M_{j_1} and M_{j_2} are non-adjacent, then $E(M_{j_1}) \cap E(M_{j_2})$ is empty.

Let e be an edge of a matroid-labeled tree T and suppose e joins vertices labeled by M_1 and M_2. Suppose that we contract e and relabel by $M_1 \oplus_2 M_2$ the composite vertex that results by identifying the endpoints of e. Then we retain a matroid-labeled tree and we denote this tree by T/e. This process can be repeated and since the operation of 2-sum is associative, for every subset $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\}$ of $E(T)$, the matroid-labeled tree $T/e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ is well-defined.

A *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree T such that if $V(T) = \{M_1, M_2, \ldots, M_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$, then

    (i) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k)) - \{e_1, e_2, \ldots, e_{k-1}\}$;

(ii) $E(M_i) \geq 3$ for all i unless $|E(M)| < 3$, in which case $k=1$ and $M_1=M$; and

(iii) M labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

In general, a tree decomposition of a matroid is not unique. However, Cunningham and Edmonds were able to guarantee uniqueness of the *canonical tree decomposition* described in the following theorem from [2].

Theorem 2.14. *Each 2-connected matroid M has a tree decomposition T in which every vertex is labeled by a 3-connected matroid, $U_{m-1,m}$ for some $m \geq 3$, or $U_{1,n}$ for some $n \geq 3$. Moreover, there are no two adjacent vertices that are both labeled by uniform matroids of rank one or are both labeled by uniform matroids of corank one, and T is unique to within a relabeling of its edges.*

The canonical tree decomposition provides a unique way to break up a 2-connected matroid M into 3-connected pieces, uniform matroids of rank one, and uniform matroids of corank one. Moreover, we can reconstruct M from these pieces using the 2-sum operation with the common elements between matroids as basepoints. A basic property of the 2-sum operation is that M_1 and M_2 are minors of $M_1 \oplus_2 M_2$. The following result is well known; its routine proof is omitted.

Lemma 2.15. *Let M_1 and M_2 label vertices in a tree decomposition T of a connected matroid M. Let P be the path in T joining M_1 and M_2, and let p_1 and p_2 be the edges of P meeting M_1 and M_2 respectively. In other words, p_1 and p_2 are basepoints for 2-sums in the reconstruction of M. Then M has a minor isomorphic to the 2-sum of M_1 and M_2, where $p_1=p_2$ is the basepoint of the 2-sum.*

The following two results will also be needed. The first is basic and its proof is omitted. The second result comes from [7].

Lemma 2.16. *The class of binary matroids is closed under the operation of 2-sum.*

Lemma 2.17. *The following statements are equivalent for a 3-connected matroid M having rank and corank at least three:*

(i) *M has a $U_{2,5}$-minor;*
(ii) *M has a $U_{3,5}$-minor;*
(iii) *M has a minor isomorphic to one of P_6, Q_6, or $U_{3,6}$.*

3. Main Result

In this section we prove the main results of the paper, Theorems 1.1 and 1.2. We begin by finding all the disconnected excluded minors of each class. Due to the similarity of the proofs for each class, we combine the arguments where possible.

Lemma 3.1. *Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. The only disconnected members of $EX(\mathcal{U})$ are $U_{2,4} \oplus U_{1,1}$ and $U_{2,4} \oplus U_{0,1}$.*

Proof. By Lemma 2.4, both matroids are in $EX(\mathcal{U})$. Now let M be an arbitrary disconnected member of $EX(\mathcal{U})$. As M is non-binary and disconnected, it has distinct components M_1 and M_2 where M_1 is non-binary. Since M_1 has a $U_{2,4}$-minor and M_2 has a $U_{0,1}$- or $U_{1,1}$-minor, the lemma follows. $\square$

The following result from [7] will be useful in our proofs.

Theorem 3.2. *Let M be a 3-connected matroid having rank and corank exceeding two.*
  (i) *If M is binary, then M has an $M(K_4)$-minor.*
  (ii) *If M is non-binary, then M has one of $\mathcal{W}^3$, Q_6, P_6, and $U_{3,6}$ as a minor.*

Before finding the complete list of 2-connected excluded minors, we need the following lemmas. The first lemma comes from [9, Section 1.5, Exercise 14]; its proof is omitted.

Lemma 3.3. *The following statements are equivalent for a matroid M:*
  (a) *M is a relaxation of some matroid,*
  (b) *M has a basis B such that $B \cup e$ is a circuit of M for every e in $E(M) - B$ and neither B nor $E(M) - B$ is empty.*

Lemma 3.4. *Let M be a matroid that can be obtained from a binary matroid N by relaxing a circuit-hyperplane X of the latter. If M contains a $\mathcal{W}^k$-minor for some $k \geq 3$, then, in every $\mathcal{W}^k$-minor of M, the rim elements are contained in X and no element of X is a spoke.*

Proof. Let M_1 be a $\mathcal{W}^k$-minor of M. If e is in the rim of M_1, then M_1/e is non-binary. But, for all f in $E(M) - X$, by Lemma 2.2, M/f is binary. Therefore $e \in X$. The assertion about spokes follows by duality. $\square$

Lemma 3.5. *Let M be a connected non-binary matroid. Either M has an R_6-, $U_{2,4} \oplus U_{0,1}$-, or $U_{2,4} \oplus U_{1,1}$-minor, or M is obtained from a 3-connected non-binary matroid M_0 by parallel and series extension of disjoint subsets T and S of $E(M_0)$, where both S and T are possibly empty.*

Proof. Consider the canonical tree decomposition T of M. As M is non-binary, by Lemma 2.16 there must be a non-binary matroid M_0 in T. Assume there is another vertex labeled by a non-binary matroid M_1. Then, by Lemma 2.15, we see that M has an $M_0 \oplus_2 M_1$-minor. Let p_1 be the basepoint of this 2-sum. Each of M_0 and M_1 is connected and non-binary, so by Lemma 2.11 each of M_0 and M_1 has a $U_{2,4}$-minor that uses p_1. Thus M has an R_6-minor, and the lemma holds when M_1 exists.

We may now assume that M_0 is the unique non-binary matroid labeling a vertex of T. Suppose there is a vertex labeled by a 3-connected binary matroid M_2 with at least four elements. Then M has an $M_0 \oplus_2 M_2$-minor. Now M_0 has a $U_{2,4}$-minor and, as M_2 is 3-connected and binary, Theorem 3.2 tells us that M_2 has an $M(K_4)$-minor. Let p_2 be the basepoint of $M_0 \oplus_2 M_2$. As above, M_0 has a $U_{2,4}$-minor using p_2. By Lemma 2.12, M_2 has an $M(K_4)$-minor using p_2. Thus M has a $U_{2,4} \oplus_2 M(K_4)$-minor and therefore has a $U_{2,4} \oplus U_{1,1}$-minor. Hence the lemma holds when M_2 exists.

We may now assume all matroids other than M_0 labeling vertices in T are $U_{1,n}$ or $U_{m-1,m}$ for varying $n, m \geq 3$. If we have a path in T beginning at M_0 that has the form M_0—$U_{m-1,m}$—$U_{1,n}$, then M has a $U_{2,4} \oplus U_{0,1}$-minor. By duality, we may not have a path of the form M_0—$U_{1,n}$—$U_{m-1,m}$. Therefore we may assume the only non-trivial paths beginning at M_0 in T are of the form M_0—$U_{m-1,m}$, or M_0—$U_{1,n}$. In other words, M is obtained from M_0 by parallel and series extension of disjoint subsets of $E(M_0)$. $\square$

Recall that the matroid K is the matroid obtained from $U_{2,4}$ by adding elements in parallel to three of its elements.

Lemma 3.6. *The matroid R_6 is the only connected, but not 3-connected, member of $EX(\mathcal{Z})$. The connected, but not 3-connected, members of $EX(\mathcal{R})$ are R_6, K, and K^*.*

Proof. Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. By Lemmas 2.4 and 2.9, R_6 is in $EX(\mathcal{U})$ and K and K^* are in $EX(\mathcal{R})$. Let M be a 2-connected member of $EX(\mathcal{U})$ that is not 3-connected and is not R_6, K, or K^*. By Lemma 3.5, M is obtained from a 3-connected non-binary matroid M_0 by parallel and series extension of disjoint subsets T and S of $E(M_0)$ where $S \cup T \neq \emptyset$.

Let $M_0 \cong U_{2,4}$. If $\mathcal{U}=\mathcal{Z}$, then M is in $\mathcal{U}$, as it satisfies (ii)(d) in Theorem 2.3, which is a contradiction, so let $\mathcal{U}=\mathcal{R}$. As M has neither K nor K^* as a minor, both S and T have size less than three. By duality, we may assume that $0 \leq |T| \leq |S| \leq 2$. In each case, M can be realized as a relaxation of a binary matroid. For example, when $|S| = |T| = 2$, assume the non-trivial series classes have sizes s_1 and s_2, and the non-trivial parallel classes have sizes p_1 and p_2. We can obtain M by relaxing the circuit-hyperplane in $M(G)$ where G is a graph on three vertices $\{a, b, c\}$ with p_1 parallel edges between a and c, p_2 parallel edges between b and c, and two internal vertex disjoint paths with sizes s_1 and s_2 between a and b. The other cases can be checked similarly. We deduce contradictorily that $M \in \mathcal{U}$.

We may now assume $|E(M_0)| \geq 5$ and consider $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. By switching to the dual if necessary, we may also assume that M has at least one non-trivial parallel class and let $\{x, y\}$ be in that class.

3.6.1. *The matroid $M \backslash x$ can be obtained from a binary matroid by relaxing a circuit-hyperplane.*

Proof. We know $M\backslash x$ is in $\mathcal{U}$. Thus it satisfies one of (ii)(a)-(ii)(d) in Theorem 2.3. If $M\backslash x$ satisfies (ii)(a), then the result follows. Assume $M\backslash x$ satisfies (ii)(b). Then $M\backslash x$ is a parallel extension of $U_{2,n}$, for some $n \geq 5$. Hence M is also a parallel extension of this matroid and $M \in \mathcal{Z}$, and M has a $U_{2,5}$-minor, which contradicts Lemma 2.9 if $\mathcal{U}{=}\mathcal{R}$. Next assume that $M\backslash x$ satisfies (ii)(c). Then $M\backslash x$ is a series extension of $U_{n-2,n}$ for some $n \geq 5$, and M is a parallel extension of this series extension. Then $M\backslash x$, and hence M, contains the excluded minor $U_{2,4} \oplus U_{0,1}$. Lastly, assume $M\backslash x$ satisfies (ii)(d). Let $U = E(U_{2,4}) - S - T$, where S and T are as defined in Theorem 2.3. Recall that $\{x, y\}$ is a circuit of M. If y is in a non-trivial series class of $M\backslash x$, then M contains the excluded minor $U_{2,4} \oplus U_{0,1}$, so $y \in T \cup U$. Therefore M satisfies (ii)(d), so $M \in \mathcal{Z}$ and we assume $\mathcal{U}{=}\mathcal{R}$. As M has neither K nor K^* as a minor, $|S| < 3$ and $|T \cup y| < 3$. As noted above, in these cases M can be realized as a relaxation of a binary matroid. $\qquad\square$

If $r(M_0) = 2$, then M_0 has a $U_{2,5}$-minor, so assume $\mathcal{U}{=}\mathcal{Z}$. It is not hard to check that we get a contradiction in this case by establishing that either $M \in \mathcal{Z}$, or M contains a $U_{2,4} \oplus U_{1,1}$-minor. Thus we may assume $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$, $r(M_0) \geq 3$ and, by duality, $r^*(M_0) \geq 3$. As M_0 is non-binary and 3-connected, and all of P_6, Q_6, and $U_{3,6}$ are either in $EX(\mathcal{U})$ or contain members of $EX(\mathcal{U})$, Theorem 3.2 implies that M_0 contains a $\mathcal{W}^3$-minor. By 3.6.1, $M\backslash x$ is a relaxation of a binary matroid N, so let B be the circuit-hyperplane relaxed in N to produce $M\backslash x$. Assume $y \notin B$ and let N_1 be obtained from N by adding x back in parallel to y. Then B is a circuit-hyperplane of N_1 whose relaxation is M, a contradiction.

We may now assume $y \in B$. Since M_0 has a $\mathcal{W}^3$-minor and no P_6-, Q_6-, or $U_{3,6}$-minor, by Lemma 2.13 M_0 has a $\mathcal{W}^3$-minor M_y using y. By Lemma 3.4, we know that y is a rim element of M_y. This implies that M has a $\mathcal{W}^3$-minor in which one of the rim elements is replaced by the parallel class containing $\{x, y\}$. This is a contradiction since it implies M has a $U_{2,4} \oplus U_{0,1}$-minor. $\qquad\square$

In finding the complete list of 3-connected excluded minors for each class, we use the following lemma.

Lemma 3.7. *If a matroid N' is obtained from a non-binary matroid N by relaxing a circuit-hyperplane X, then*

    (i) *N' has a $U_{2,5}$- or $U_{3,5}$-minor; or*
    (ii) *N' has a matroid in the class $\mathcal{D}$ as a minor.*

Proof. As N is non-binary and has a circuit-hyperplane, $|E(N)| \geq 5$. If $|E(N)| = 5$, then either N is $U_{2,4} \oplus_2 U_{1,3}$, in which case N' is $U_{2,5}$, or N is $U_{2,4} \oplus_2 U_{2,3}$, in which case N' is $U_{3,5}$. Thus the result holds if $|E(N)| = 5$. Now assume that the result holds for $|E(N)| < k$, and consider the case where $|E(N)| = k \geq 6$. If $r(N) = 2$, then N' has a $U_{2,5}$-minor and the result holds. Dually, the result holds if $r^*(N) = 2$, so assume $r(N), r^*(N) \geq 3$.

Take $e \in X$ and consider N/e. By Lemma 2.2, N'/e is obtained from N/e by relaxing $X - e$. If N/e is non-binary, then we invoke the induction hypothesis to see that the result holds. Hence N/e is binary for all $e \in X$. By duality, $N \backslash e$ is binary for all $e \notin X$. Thus, for every $e \in E(N)$, at least one of $N \backslash e$ and N/e is binary. By Theorem 2.3, we deduce that one of (ii)(a)-(ii)(d) holds for N.

As $r(N), r^*(N) \geq 3$, we know N cannot satisfy (ii)(b) or (ii)(c). Assume N satisfies (ii)(d). If $|S| = 0$ or $|T| = 0$, we contradict our rank or corank assumptions, so $|S|, |T| \geq 1$. It is straightforward to check that N cannot have a circuit-hyperplane, which is a contradiction.

Finally, assume N satisfies (ii)(a). Then N can be obtained from some connected binary matroid M by relaxing a circuit-hyperplane Y in M. Assume $X \cap Y \neq \emptyset$ and take $e \in X \cap Y$. Then N/e is binary and is obtained from the binary matroid M/e via relaxation. By Lemma 2.7, $M/e \cong U_{n-1,n} \oplus U_{1,k}$ for some $n, k \geq 1$, and $N/e \cong U_{n,n+1} \oplus_2 U_{1,k+1}$. However, this implies N/e has no circuit-hyperplane unless $n = 2$ and $k = 2$, so we assume these values for n and k. But this means $N'/e \cong U_{2,4}$, which is a contradiction since $r(N'), r^*(N') \geq 3$. Thus $X \cap Y = \emptyset$ and, by duality, $(E(M) - X) \cap (E(N) - Y) = \emptyset$. As both $(X, E(N) - X)$ and $(Y, E(N) - Y)$ partition the ground set, $X = E(N) - Y$ and $E(N) - X = Y$. Hence N' is obtained from the connected binary matroid M by relaxing the two disjoint circuit-hyperplanes X and Y, so N' is in $\mathcal{D}$ and the result holds. $\square$

Lemma 3.8. *The complete list of* 3*-connected members of* $EX(\mathcal{Z})$ *is* Q_6, P_6, $U_{3,6}$, *and the matroids in* $\mathcal{D}$. *The complete list of* 3*-connected members of* $EX(\mathcal{R})$ *is* $U_{2,5}$, $U_{3,5}$, *and the matroids in* $\mathcal{D}$

Proof. Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. Let M be a 3-connected excluded minor of $\mathcal{U}$ that is not Q_6, P_6, $U_{3,6}$, $U_{2,5}$, $U_{3,5}$, or any of the matroids in $\mathcal{D}$. Clearly $r(M) \geq 3$ and $r^*(M) \geq 3$. Either (a) M is a relaxation of a non-binary matroid; or (b) M is not a relaxation of any matroid at all. Consider case (a). By Lemma 3.7, since M has no matroid in $\mathcal{D}$ as a minor, M has a $U_{2,5}$- or $U_{3,5}$-minor. As M has rank and corank at least three, by Lemma 2.17, M has a Q_6-, P_6-, or $U_{3,6}$-minor. This is a contradiction to Lemma 2.9 or 2.10.

Now consider case (b). By Theorem 3.2, M must contain one of $\mathcal{W}^3$, Q_6, P_6, and $U_{3,6}$. As all of these except $\mathcal{W}^3$ contain excluded minors of $\mathcal{U}$, we know that M has a $\mathcal{W}^3$-minor. Let $\mathcal{W}^k$ be the largest whirl-minor of M. We use Seymour's Splitter Theorem [11] to grow M from $\mathcal{W}^k$. Let x be the element added with the last move. By duality, we may assume that x is added via extension. Thus $M \backslash x$ is a non-binary 3-connected member of $\mathcal{U}$. If $\mathcal{U} = \mathcal{R}$, then $M \backslash x$ is a relaxation of a binary matroid. If $\mathcal{U} = \mathcal{Z}$, then $M \backslash x$ satisfies one of (ii)(a)-(ii)(d) in Theorem 2.3. As $M \backslash x$ is 3-connected, it cannot satisfy (ii)(b)-(ii)(d). Hence, in both cases, $M \backslash x$ is a relaxation of a binary matroid N_1.

Let B be the special basis in $M\backslash x$ that is a circuit-hyperplane in N_1. For all $e \in E(M\backslash x) - B$, the set $B \cup e$ is a circuit in $M\backslash x$. Now B is also a basis of M and $B \cup e$ is a circuit of M for all $e \in E(M) - (B \cup x)$. If $B \cup x$ is a circuit of M, then M can be realized as a relaxation of some matroid by Lemma 3.3, which is a contradiction. Thus there is some $y \in B$ such that y is not in the circuit contained in $B \cup x$. Now, by Lemma 2.2, $M\backslash x/y$ can be obtained by relaxing the circuit-hyperplane $B - y$ in N_1/y.

Assume that $M\backslash x/y$ is binary. It follows from Lemma 2.7 that $M\backslash x/y$ can be obtained from a circuit C by adding some, possibly empty, set of elements in parallel with some element z of C where $C - z = B - y$. As $M\backslash x$ is 3-connected, it has no non-trivial series classes. Hence $|C - z| = 1$, so $r(M\backslash x/y) = 1$, which contradicts the fact that $r(M) \geq 3$. Therefore $M\backslash x/y$ must be non-binary, and so M/y is also non-binary.

3.8.1. *The matroid M/y can be obtained from a binary matroid via relaxation.*

Proof. This is certainly true if $\mathcal{U}=\mathcal{R}$, so assume $\mathcal{U}=\mathcal{Z}$. Then M/y satisfies one of (ii)(a)-(ii)(d) in Theorem 2.3. First note that M/y cannot satisfy (ii)(c), because a connected single-element coextension of a series extension of $U_{n-2,n}$ has corank two, and so has no $\mathcal{W}^3$-minor. Assume M/y satisfies (ii)(b). Then $r(M) = 3$. As M has a $\mathcal{W}^3$-minor, it is not hard to check that we must coextend M/y by y in a way that creates a matroid having a $U_{2,4} \oplus U_{1,1}$-, Q_6-, or P_6-minor. Now assume M/y satisfies (ii)(d). As M is 3-connected, M/y cannot have any non-trivial series classes. A routine check shows that either M contains an excluded minor or M can be realized as a relaxation of a binary matroid, both of which are contradictions. Thus (ii)(a) holds, and so does the result. $\square$

We now revert to working in generality, where $\mathcal{U} \in \{\mathcal{Z},\mathcal{R}\}$. We know that $M\backslash x$ is obtained by relaxing a circuit-hyperplane B in a binary matroid N_1, and M/y is obtained by relaxing a circuit-hyperplane B' in a binary matroid N_2. We show next that

3.8.2. $B'=B - y$.

Proof. By Lemma 2.2, $M\backslash x/y$ is obtained by relaxing the circuit-hyperplane $B - y$ in N_1/y. Consider $(M\backslash x/y)\backslash e$ for $e \notin B - y$, and assume $(M\backslash x/y)\backslash e$ is binary. Then, as $(M\backslash x/y)\backslash e$ is a relaxation of a binary matroid, we know $(M\backslash x/y)\backslash e \cong U_{t-1,t} \oplus_2 U_{1,v}$ for some $t, v \geq 1$. Now $E(M\backslash x/y\backslash e)$ has a partition (S, P) where S is the relaxed set $B - y$ and P is its complement. Then $S \cup y$ is the relaxed set of $M\backslash x$ and $P \cup e$ is the relaxed set of $(M\backslash x)^*$. As $r_{M\backslash x}(P \cup y) = 2$, we know $|P| \leq 2$, else the matroid N_1 would be non-binary, and, by duality, $|S| \leq 2$. However, $r(M\backslash x) = |S \cup y|$ and $r^*(M\backslash x) = |P \cup e|$. Thus, as $M\backslash x$ has a $\mathcal{W}^k$-minor for some $k \geq 3$, we know $|S \cup y| = |P \cup e| = 3$. Therefore, $M\backslash x \cong \mathcal{W}^3$. The only 3-connected single-element extension of $\mathcal{W}^3$ that does not contain an excluded minor is

F_7^-, depicted in Figure 3. But F_7^- is a relaxation of the Fano plane, which is binary, giving us a contradiction. Therefore we may assume $(M \backslash x / y) \backslash e$ is non-binary for all $e \notin B - y$. Then, for all such e, the matroid $M / y \backslash e$ is also non-binary. By Lemma 2.2, for every $e \in B'$ the matroid $M / y \backslash e$ is binary. Thus if e is not in $B - y$, then it is not in B'. Therefore, $B' \subseteq B - y$. As $B - y$ and B' are both bases for M / y, the result holds. $\square$

We know $B \cup e$ is a circuit of M for all $e \in E(M) - (B \cup x)$, and that $(B - y) \cup e$ is a circuit of M / y for all $e \in E(M) - B$. Thus, since $B \cup x$ is not a circuit of M, we see $(B - y) \cup x$ is a circuit of M. As $M \backslash x$ has a $\mathcal{W}^3$-minor, but no Q_6-, P_6-, or $U_{3,6}$-minor, Lemma 2.13 implies that $M \backslash x$ has a $\mathcal{W}^3$-minor using y. By Lemma 3.4, as y is in B, it follows that this $\mathcal{W}^3$-minor has y as a rim element. Hence by adding x back, M has a single-element extension of $\mathcal{W}^3$ as a minor. Let $\{b_1, b_2, y\}$ be the set of rim elements in $\mathcal{W}^3$. There are only two single-element extensions of $\mathcal{W}^3$ that do not contain excluded minors, and they are F_7^- and a parallel extension of a spoke element of $\mathcal{W}^3$ (see Figure 3). But, in each of them, the set $\{b_1, b_2, x\}$ should be a circuit because $(B - y) \cup x$ is a circuit of M and the only elements of $M \backslash x$ that can be contracted to produce the $\mathcal{W}^3$-minor must belong to B. This contradiction completes the proof. $\square$
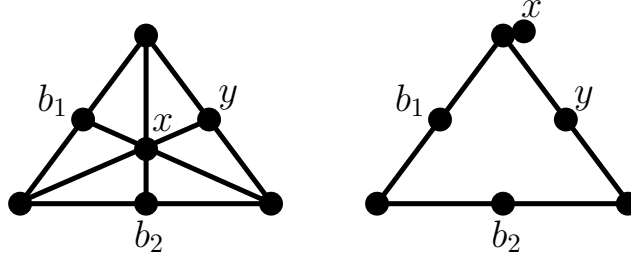

FIGURE 3. Geometric representations of F_7^-, and a parallel extension of a spoke element of $\mathcal{W}^3$.

Proofs of Theorems 1.1 and 1.2. These follow immediately by combining Lemmas 3.1, 3.6, and 3.8. $\square$

4. THE COMPLEXITY OF $\mathcal{D}$

Jim Geelen asked (private communication) whether members of $\mathcal{D}$ could contain arbitrarily large projective geometries. In this section, we observe that they can. Note that all sums in this section are modulo two. Let A be a $k \times (2^k - 1)$ matrix representing the rank-k binary projective geometry $PG(k - 1, 2)$, where k is odd. Let $n = 2^k + k + 1$, let $t = 2^k + k - 1$, and consider the rank-n binary matrix Z in Figure 4. The entries α_i and β_j are defined the next paragraph.
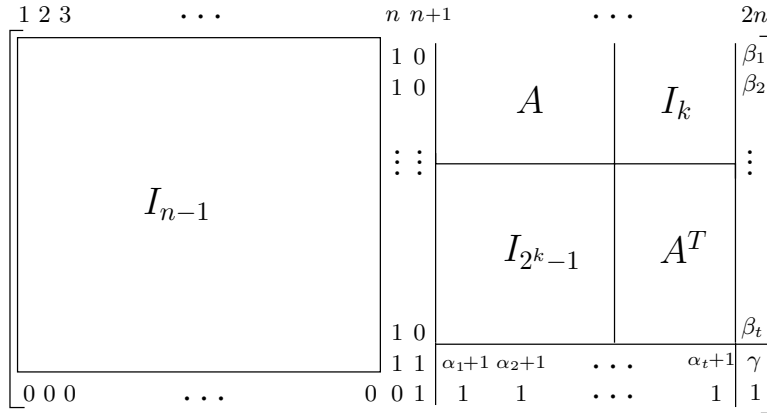
FIGURE 4. The matrix Z.

Let z_{sc} denote the entry in row s and column c of Z. Let $\alpha_i = \Sigma_{s=1}^{n-2} z_{s(n+1+i)}$, for $1 \le i \le t$. Let $\beta_j = \Sigma_{c=n+2}^{2n-1} z_{jc}$, for $1 \le j \le t$, and let $\gamma = 1 + \Sigma_{j=1}^{t} \beta_j$. Let Z' be the submatrix of Z whose columns are labeled by $n+1$, $n+2$, ..., $2n$. Then each column in Z' is contained in the hyperplane of $PG(n-1, 2)$ consisting of those vectors whose coordinates sum to zero. Moreover, no other column of Z is in this hyperplane. The definitions ensure that all the rows of Z', except possibly row $n-1$, sum to zero. To see that row $n-1$ also sums to zero, note that $\Sigma_{j=1}^{t} \beta_j = \Sigma_{i=1}^{t} \alpha_i$ since both of these sums count the number of non-zero entries in the same submatrix. We know that $\Sigma_{c=n+2}^{2n-1} z_{(n-1)c} = 1 + \Sigma_{i=1}^{t} (\alpha_i + 1) = 1 + t + \Sigma_{i=1}^{t} \alpha_i$. As t is even, $1 + t + \Sigma_{i=1}^{t} \alpha_i = 1 + \Sigma_{i=1}^{t} \alpha_i = 1 + \Sigma_{j=1}^{t} \beta_j = \gamma$. Thus $\{n+1, n+2, \ldots, 2n\}$ is a circuit-hyperplane of $M[Z]$ and it is easy to see that its complement is as well. By relaxing both these circuit-hyperplanes, we get a member of $\mathcal{D}$ that contains a $PG(k-1, 2)$-minor.

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