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Structure Theory and Connectivity for Matroids

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ABSTRACT. The concept of 3–connectedness for graphs was generalized to matroids by Tutte in 1966. Tutte identified wheels and whirls as being the only 3–connected matroids for which no single-element deletion or contraction remains 3–connected. This Wheels and Whirls Theorem has played a foundational role in the establishment of a coherent theory for 3–connected matroids. The Splitter Theorem, a powerful generalization of the Wheels and Whirls Theorem, was proved by Seymour in 1980. This survey of structure theory and connectivity results for matroids focuses particularly on how profoundly the Wheels and Whirls and Splitter Theorems have influenced the development of these areas.

1. Introduction

Throughout mathematics, there is widespread interest in breaking large objects into smaller, more easily understood, pieces. For matroids, the first such decomposition result was proved by Whitney [**106**] when he showed that every matroid can be uniquely written as the direct sum of its connected components. This transfers attention from arbitrary matroids to connected matroids. For such matroids, a decomposition theorem was established by Cunningham and Edmonds [**22**]. In particular, their result proved that every connected matroid can be built up from some of its 3–connected minors by a sequence of 2–sums. The attention is thus transferred from connected matroids to 3–connected matroids. These are the matroids that will be the focus of this paper. One important reason for stopping with 3–connected matroids and not looking at, say, 4–connected matroids is that, although there has been some work done for graphs in the latter case [**19, 73**], there is no known decomposition theorem for arbitrary 4–connected matroids.

2. Preliminaries

Any unexplained matroid terminology used here will follow Oxley [**62**]. For a matroid M, the simple matroid and the cosimple matroid associated with M will be denoted by $\widetilde{M}$ and $\underset{\sim}{M}$, respectively. We call these matroids the *simplification* and the *cosimplification* of M. The basic matroid property that a circuit and a cocircuit cannot have exactly one common element will be referred to as *orthogonality*.

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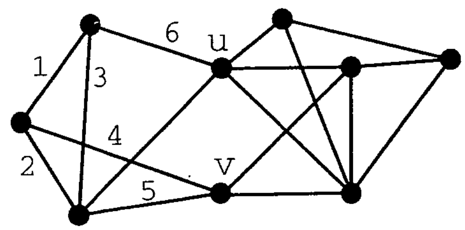

FIGURE 1. G.

The notion of a *connected* or *non-separable* matroid as one in which every pair of distinct elements is in a circuit was introduced by Whitney [**106**]. But it was Tutte [**99**] who originated the concept of higher connectivity for matroids. His motivation appears to derive from a desire to generalize the notion of n–connectedness for graphs and a wish to incorporate duality into the theory.

Consider the graph G shown in Figure 1. Evidently, G is not 3–connected. The 2–vertex cut $\{u, v\}$ of G induces a natural partition of $E(G)$. Letting $X = \{1, 2, 3, 4, 5, 6\}$, we have

$$r(X) + r(E(G) - X) - r(M(G)) = 1.$$

In general, for a positive integer k and a matroid M, a partition $\{X, Y\}$ of $E(M)$ is a *k–separation* if

$$min\{|X|, |Y|\} \geq k;$$

and

$$r(X) + r(Y) - r(M) \leq k - 1.$$

If equality holds in the last inequality, then the k–separation is *exact.* For $n \geq 2$, the matroid M is *n–connected* provided that, for all k in $\{1, 2, \ldots, n-1\}$, M has no k–separation. Hence M is 2–connected exactly when it is connected. Moreover, a routine rank argument establishes that a matroid is n–connected if and only if its dual is n–connected. The link between the graph and matroid concepts of n–connectedness is contained in the following result.

PROPOSITION 2.1. *For $n \geq 2$, let G be a graph without isolated vertices and suppose that $|V(G)| \geq n + 1$. Then $M(G)$ is n–connected if and only if G is n–connected and has no cycles with fewer than n edges.*

In particular, if $|V(G)| \geq 4$ and G has no isolated vertices, then $M(G)$ is 3–connected if and only if G is 3–connected and simple. Indeed, every 3–connected matroid with at least four elements is both simple and cosimple. It is straightforward to check that the only 3–connected matroids with fewer than four elements are $U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}$, and $U_{2,3}$. We shall frequently follow the common practice of restricting attention to 3–connected matroids with at least four elements.

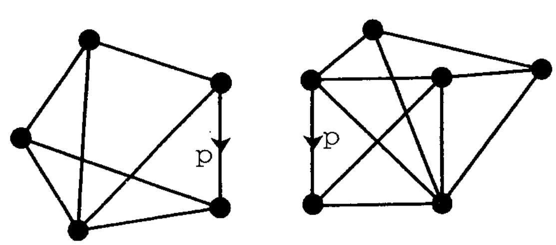

FIGURE 2. G is the 2–sum of G_1 and G_2 across the edge p.

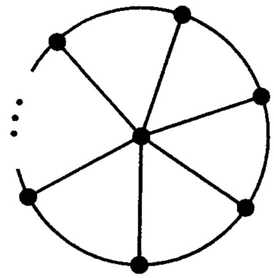

FIGURE 3. A wheel.

The graph G in Figure 1 is the 2–sum across the edge p of the graphs G_1 and G_2 in Figure 2, that is, G is obtained from G_1 and G_2 by identifying the (directed) edges labeled p, respecting their directions, and then deleting this composite edge. This graph operation has a natural matroid generalization because the cycles in G can be specified in terms of those of G_1 and G_2. Let M_1 and M_2 be 2–connected matroids each having at least three elements such that $E(M_1) \cap E(M_2) = \{p\}$. The *2–sum* of M_1 and M_2 (with respect to p) is the matroid $M_1 \oplus_2 M_2$ with ground set $[E(M_1) \cup E(M_2)] - \{p\}$ for which the circuits are the following: all circuits of M_1 avoiding p; all circuits of M_2 avoiding p; and all sets of the form $(C_1 \cup C_2) - \{p\}$, where C_i is a circuit of M_i containing p.

The following basic link between 2–sum and 3–connectedness was proved by several authors [**8, 20, 82**].

THEOREM 2.2. *A 2–connected matroid M is not 3–connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2 each of which is isomorphic to a proper minor of M.*

A complete decomposition of a 2–connected matroid into 3–connected pieces was determined by Cunningham and Edmonds [**22**]. The details of this decomposition together with an example may be found on pp. 290–291 of [**62**].

We now know that a matroid that is not 3–connected can be built up by direct sums and 2–sums from 3–connected matroids, each of which is isomorphic to a minor of the original matroid. It is straightforward to show that many basic matroid properties are preserved under both direct sum and 2–sum. Hence it is natural to focus on 3–connected matroids. Two particularly important families of 3–connected matroids are the wheels and the whirls. For $r \geq 2$, the *wheel $\mathcal{W}_r$ of rank r* is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex is joined by a single edge (a *spoke*) to each of the other vertices (see Figure 3). The *rank-r whirl $\mathcal{W}^r$* is the matroid on $E(\mathcal{W}_r)$ that has as its circuits all cycles of $\mathcal{W}_r$ other than the rim as well as all sets of edges formed by adding a single spoke to the edges of the rim. The terms "rim" and "spoke" will be applied in the obvious way in $M(\mathcal{W}_r)$ with the following warning. The case $r = 3$ differs from all other cases in that one cannot distinguish rim elements from spokes by looking just at the matroid. In that case, we arbitrarily designate one of the 3–element circuits of $M(\mathcal{W}_3)$ to be the rim. We shall usually refer to the cycle matroid of a wheel as just a wheel. The smallest 3–connected whirl is $\mathcal{W}^2$, which is isomorphic to $U_{2,4}$; the smallest 3–connected wheel is $M(\mathcal{W}_3)$, which is isomorphic to $M(K_4)$ (see Figure 4).

Every element of a wheel or whirl is in both a *triangle*, a 3–element circuit, and a *triad*, a 3–element cocircuit. Thus if M is a wheel or whirl with at least six elements,

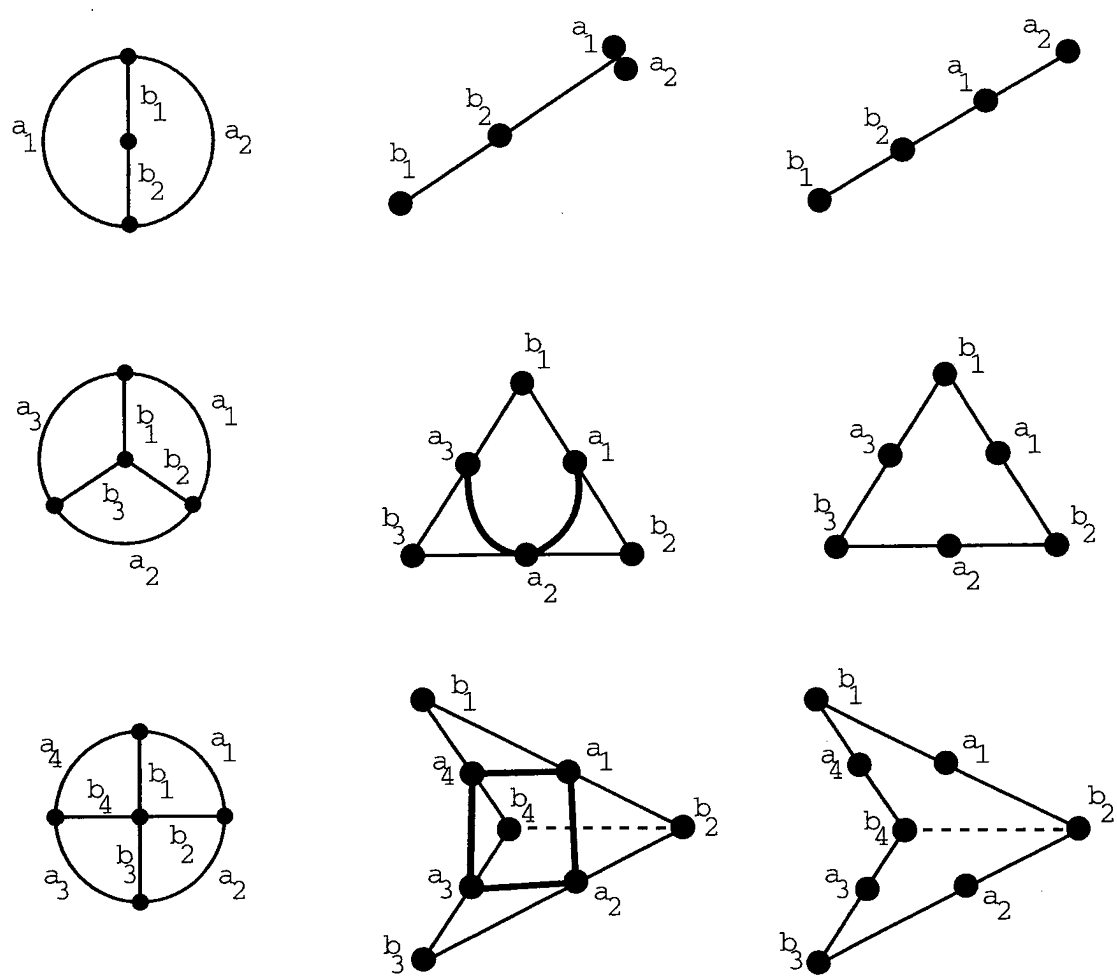

FIGURE 4. Three wheel graphs with geometric representations of their cycle matroids and of the corresponding whirls.

then M has no single-element deletion or contraction that is 3–connected. Tutte's Wheels and Whirls Theorem [**99**], which we state next, asserts that the wheels and whirls are the only matroids with this property. The usefulness of such a result in induction arguments is clear and, indeed, this theorem has had a profound influence on the development of results for 3–connected matroids.

THEOREM 2.3 (The Wheels and Whirls Theorem). *The following statements are equivalent for a 3-connected matroid M having at least one element.*

(i) *For every element e of M, neither $M\backslash e$ nor M/e is 3-connected.*
(ii) *M has rank at least three and is isomorphic to a wheel or a whirl.*

The fact that no corresponding result is known for arbitrary n–connected matroids when $n \geq 4$ goes a long way towards explaining the relative lack of development of results for such matroids. Evidently Theorem 2.3 distinguishes a prominent role for wheels and whirls within the class of 3–connected matroids. Indeed, using the elementary observation that the only 4–element 3–connected matroid is $U_{2,4}$, which is isomorphic to the rank–2 whirl, we deduce the following consequence of the last theorem.

COROLLARY 2.4. *Let M be a 3–connected matroid having at least four elements. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 = M$; M_n is a wheel of rank at least three, or a whirl of rank at least two; and, for all i in $\{1, 2, \ldots, n\}$, the matroid M_i is a single-element deletion or a single-element contraction of M_{i-1}.*

The Wheels and Whirls Theorem and the corollary just noted have served as the foundation upon which the theory of 3–connected matroids has been built. The importance of this foundation will be apparent throughout this paper. The next section gives some examples of well-known structural results for graphs. Section 4 presents a powerful extension of Corollary 2.4 known as the Splitter Theorem. This theorem is then used to derive some structural results for matroids that are similar to the graph results in Section 3. In particular, these results describe the structure of certain excluded-minor classes of matroids with much of the attention focusing on cases where a small wheel or a small whirl is among the excluded minors.

The Splitter Theorem was originally used to prove an important decomposition theorem for the class of regular matroids, and this theorem is discussed in Section 5. Section 6 considers the problem of characterizing the 3–connected matroids for which every sequence of the type discussed in Corollary 2.4 ends in an isomorphic copy of the same matroid. In Section 7, some examples are given of other structural results that have been obtained by using the technique that arises naturally from the Splitter Theorem.

In Section 8, the concern is with identifying partial wheels in 3–connected matroids and then breaking these off to leave smaller 3–connected matroids. The results from that section are used in Section 9 to obtain another extension of the Wheels and Whirls Theorem in which the extremal connectivity hypothesis of the latter result is weakened slightly. This result is an example of a number of such extremal results that are known for matroids. Many of these results mimic corresponding results for graphs or have played important roles as lemmas in the derivation of other matroid theorems. Sections 10 and 11 discuss these results for 2– and 3–connected matroids, respectively.

Section 12 examines an alternative definition of n–connectedness for matroids that exactly generalizes the familiar graph concept. Finally, Section 13 looks at some further questions that arise naturally from the Splitter Theorem and also discusses some of the other directions of research in this area. Many of the topics treated in this survey are examined in more detail in Seymour's survey [**84**], the author's book [**62**] (see particularly Chapters 8 and 11), or Truemper's book [**95**]. Both [**62**] and [**95**] have long lists of references that should be used to supplement the list of references for this paper. In particular, each chapter of [**95**] concludes with a section containing historical notes and references to further readings.

3. Some structural results for graphs

The structural results for matroids that will be presented later in the paper were foreshaddowed by certain results for graphs. We now briefly discuss these graph results.

One of the best-known theorems in graph theory is the Kuratowski-Wagner characterization of planar graphs [**38, 102**].

THEOREM 3.1. *A graph G has no $K_{3,3}$-minor and no K_5-minor if and only if G is planar.*

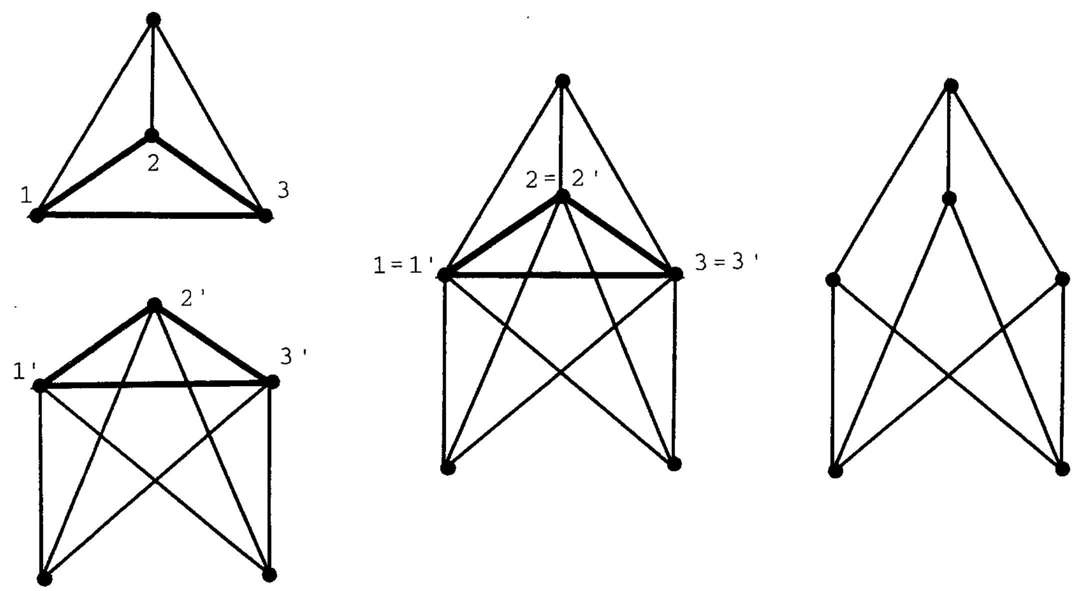

FIGURE 5. An example of a 3–sum of graphs.

Stated in this form, the theorem raises the questions as to whether one can specify all graphs with no $K_{3,3}$-minor and all graphs with no K_5-minor. These questions were answered by D.W. Hall [**31**] and by Wagner [**103**]. Before presenting their answers, we recall the definition of the graph operation of n–sum. An example of a 2–sum was given in Figure 2. More generally, let G_1 and G_2 be graphs each of which has a distinguished K_n-subgraph. To form an *n–sum* of G_1 and G_2, one first pairs the vertices of the chosen K_n-subgraph of G_1 with distinct vertices of the chosen K_n-subgraph of G_2. The paired vertices are then identified, as are the corresponding pairs of edges. Finally, all identified edges are deleted. In Figure 5, for example, it is shown how $K_{3,3}$ can be obtained as a 3–sum of K_4 and $K_5\backslash e$, the last graph being the unique graph that is obtained from K_5 by deleting a single edge. The operation of 0–sum is just disjoint union, while 1–sum involves sticking two graphs together at a vertex. Clearly the 0–sum and all the 1–sums of graphs G_1 and G_2 have identical cycle matroids, namely $M(G_1) \oplus M(G_2)$.

THEOREM 3.2 (D.W. Hall). *A graph G has no $K_{3,3}$-minor if and only if G can be obtained from planar graphs and copies of K_5 by repeatedly applying the operations of* 0*–sum,* 1*–sum, and* 2*–sum.*

The graph V_8 that features in the next theorem is the 4–rung Möbius ladder. It is drawn in two different ways in Figure 6. For comparison, we note that $K_{3,3}$ is the 3–rung Möbius ladder.

THEOREM 3.3 (Wagner). *A graph G has no K_5-minor if and only if G can be obtained from planar graphs and copies of V_8 by repeatedly applying the operations of* 0*–sum,* 1*–sum,* 2*–sum, and* 3*–sum.*

The last two theorems distinguish prominent roles for the graphs K_5 and V_8. Indeed, $M(K_5)$ is a maximal 3–connected member of the class of graphic matroids with no $M(K_{3,3})$-minor, and $M(V_8)$ is a maximal 3–connected member of the class of graphic matroids with no $M(K_5)$-minor. In general, for a minor-closed class $\mathcal{M}$ of matroids, a *splitter* is a member N of $\mathcal{M}$ such that no 3–connected member of $\mathcal{M}$ has N as a proper minor. The last two assertions are that $M(K_5)$ is a splitter

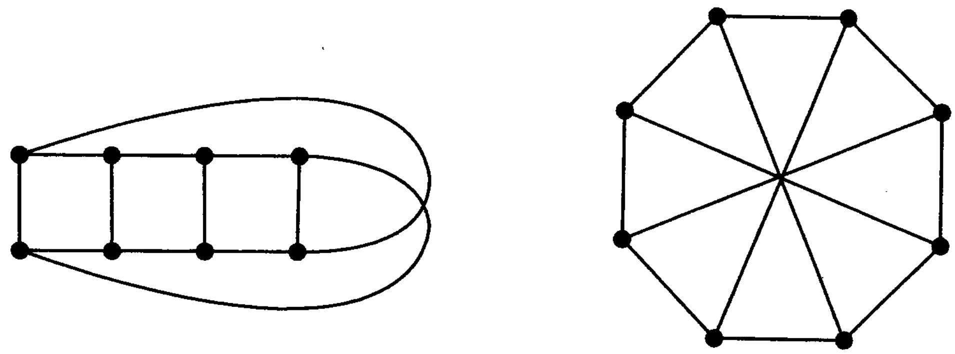

FIGURE 6. Two drawings of the graph V_8.

for the class of graphic matroids with no $M(K_{3,3})$-minor, and $M(V_8)$ is a splitter for the class of graphic matroids with no $M(K_5)$-minor. The task of checking these assertions is potentially immense. The main result of the next section is Seymour's Splitter Theorem, a consequence of which is that, in order to check whether a 3–connected member M is a splitter for a minor-closed class $\mathcal{M}$ of matroids, one needs only to show that M has no 3–connected single-element extensions or coextensions in $\mathcal{M}$. If we accept this assertion, it is easy to see that $M(K_5)$ is a splitter for the class of graphic matroids with no $M(K_{3,3})$-minor. Certainly $M(K_5)$ has no simple graphic single-element extension, and so $M(K_5)$ has no 3–connected such extension. Moreover, the unique graphic coextension of $M(K_5)$ is $M(H_6)$ where H_6 is the graph shown in Figure 7. Since $M(H_6)\backslash e, f \cong M(K_{3,3})$, it follows that every 3–connected single-element graphic coextension of $M(K_5)$ has an $M(K_{3,3})$-minor.

To see that $M(V_8)$ is a splitter for the class of graphic matroids with no $M(K_5)$-minor, we note first that, as V_8 is a cubic graph, $M(V_8)$ has no 3–connected graphic single-element coextension. Moreover, by symmetry, $M(V_8)$ has exactly two non-isomorphic 3–connected graphic single-element extensions, and each of these is easily shown to have an $M(K_5)$-minor.

4. The Splitter Theorem and some applications

A far-reaching extension of Corollary 2.4 is Seymour's Splitter Theorem [**82**], which he proved on the way to his regular matroids decomposition theorem.

THEOREM 4.1 (The Splitter Theorem). *Let M and N be 3–connected matroids such that N is a minor of M, $|E(N)| \geq 4$, and if N is a wheel, then M has no larger wheel as a minor, while if N is a whirl, then M has no larger whirl as a minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids such*

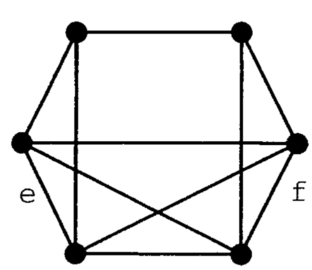

FIGURE 7. H_6.

that $M_0 = M$; $M_n \cong N$; and, for all i in $\{1, 2, \ldots, n\}$, M_i is a single-element deletion or a single-element contraction of M_{i-1}.

The crux of this theorem is that, while removing elements one at a time in going from M to an isomorphic copy of N, one is able to maintain 3–connectedness. It should also be noted here that M_n, while it is isomorphic to N, cannot be guaranteed to be equal to N. The Splitter Theorem was also proved independently by Tan [**85**] for matroids and by Negami [**46**] for graphs. A proof of the theorem, due to C.R. Coullard and L.L. Gardner, may be found in Section 11.1 of [**62**].

Two examples of splitters within classes of graphic matroids were given in the last section. The next proposition gives another example of a splitter. Two matroids that are important in this example are the vector matroids of the following matrices over $GF(2)$:

$$\left[\begin{array}{cccc|cccc} & & & & 0 & 1 & 1 & 1 \\ & I_4 & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 & 1 \\ & & & & 1 & 1 & 1 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc|cccc} & & & & 0 & 1 & 1 & 1 \\ & I_4 & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 & 1 \\ & & & & 1 & 1 & 1 & 1 \end{array}\right].$$

The first of these matroids is the affine geometry $AG(3,2)$; the second is denoted by S_8.

PROPOSITION 4.2 (Seymour [**82**]). *Let $\mathcal{M}$ be the class of binary matroids having no minor isomorphic to the Fano matroid, F_7. Then F_7^* is a splitter for $\mathcal{M}$.*

PROOF. Evidently $F_7^* \in \mathcal{M}$. Moreover, since $F_7 \cong PG(2,2)$, there is no simple binary single-element extension of F_7. Hence F_7^* has no 3–connected single-element coextension in $\mathcal{M}$. Now F_7^* is represented by the matrix A over $GF(2)$ where

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc|ccc} & & & & 0 & 1 & 1 \\ & I_4 & & & 1 & 0 & 1 \\ & & & & 1 & 1 & 0 \\ & & & & 1 & 1 & 1 \end{array}\right] \end{array}.$$

A 3–connected binary single-element extension M of F_7^* can be represented by the matrix that is obtained from A by adjoining the column $(x_1, x_2, x_3, x_4)^T$ where each of x_1, x_2, x_3, and x_4 is in $\{0,1\}$. As M is 3–connected and therefore simple, at least two of x_1, x_2, x_3, and x_4 are non-zero. By symmetry, we may assume that $(x_1, x_2, x_3, x_4)^T$ is one of $(1,1,1,0)^T$, $(1,1,1,1)^T$, $(1,1,0,0)^T$, and $(1,0,0,1)^T$. Adjoining the first of these columns to A gives a representation for $AG(3,2)$. Adjoining the second of the columns to A gives a representation for S_8. Moreover, it is straightforward to check that adjoining any one of the two remaining columns to A also gives a representation for a matroid isomorphic to S_8. Since each of $AG(3,2)$ and S_8 has a representation of the form $[I_4|D]$ where D is symmetric, each is isomorphic to its dual. As each has F_7^* as a minor, each also has F_7 as a minor, and so F_7^* has no 3–connected binary single-element extension in $\mathcal{M}$. Hence F_7^* is a splitter for $\mathcal{M}$. □

The last result can be used in conjunction with Tutte's excluded-minor characterization of regular matroids [**97**] to give the following structural result [**82**].

COROLLARY 4.3. *Every binary matroid that has no F_7-minor can be obtained from regular matroids and copies of F_7^* by a sequence of direct sums and 2–sums.*

The last result used a relatively straightforward application of the Splitter Theorem. Before presenting a more complicated example, we motivate this example. The fundamental role played by wheels and whirls within the class of 3–connected matroids prompts consideration of the structure of a minor-closed class of matroids which avoids some small wheel or some small whirl. The class of matroids for which the smallest whirl $\mathcal{W}^2$ is the unique excluded minor is precisely the class of binary matroids. If we exclude as minors both the smallest whirl $\mathcal{W}^2$ and the smallest 3–connected wheel $M(\mathcal{W}_3)$, then the only non-empty matroids we get are direct sums of series-parallel networks. Here a *series-parallel network* is the cycle matroid of a graph that can be obtained from one of the two connected single-edge graphs by a sequence of the operations of replacing an edge by either two edges in parallel or two edges in series.

Next we shall describe the structure of the class of matroids that have no minor isomorphic to $\mathcal{W}^2$ or $M(\mathcal{W}_4)$, or equivalently, the class of binary matroids with no 4–wheel minor. Since every member of this class that is not 3–connected can be constructed from 3–connected members of the class by direct sums and 2–sums, it suffices to specify the 3–connected members of the class. In general, for a set $\{M_1, M_2, \dots\}$ of matroids, $EX(M_1, M_2, \dots)$ will denote the class of matroids having no minor isomorphic to any of $M_1, M_2, \dots$.

The strategy that will be used to find the 3–connected members of $EX(\mathcal{W}^2, M(\mathcal{W}_4))$ is as follows. First we note that all 3–connected matroids with fewer than four elements are trivially in the class. Next we let M be a 3–connected member of the class having four or more elements. Then, by Corollary 2.4, M has an $M(\mathcal{W}_3)$-minor. Moreover, since M has no $M(\mathcal{W}_4)$-minor, it has no minor isomorphic to $M(\mathcal{W}_r)$ for any $r \geq 4$. Thus, by Corollary 2.4, there is a sequence $M_0, M_1, \dots, M_n$ of 3–connected matroids such that $M_n \cong M(\mathcal{W}_3)$; $M_0 = M$; and, for all i in $\{1, 2, \dots, n\}$, M_{i-1} is a single-element extension or a single-element coextension of M_i. Clearly each of $M_0, M_1, \dots, M_n$ is binary. The unique 3–connected binary extension of $M(\mathcal{W}_3)$ is F_7; by duality, the unique 3–connected binary coextension of $M(\mathcal{W}_3)$ is F_7^*. Thus M_{n-1} is F_7 or F_7^*. It now follows, by the proof of Proposition 4.2, that M_2 is $AG(3,2)$ or S_8. The fact that each of $\mathcal{W}^2$ and $M(\mathcal{W}_4)$ is self-dual means that $EX(\mathcal{W}^2, M(\mathcal{W}_4))$ is closed under duality. Moreover, as both $AG(3,2)$ and S_8 are self-dual, either M_{n-2} or M_{n-2}^* is a binary 3–connected extension of S_8 or $AG(3,2)$. To determine the possible such extensions, we take matrices representing S_8 and $AG(3,2)$ over $GF(2)$ and consider what columns can be added to these matrices so as to avoid creating an $M(\mathcal{W}_4)$-minor. We are relying here on the unique representability of binary matroids. Continuing to analyze the sequence $M_0, M_1, \dots, M_n$ in this way, a pattern emerges and, from this, one can formulate and then prove the structure theorem stated below. The details of this proof can be found in [**56**].

Let r be an integer exceeding two and Z_r be the vector matroid of the following matrix over $GF(2)$:

$$\begin{array}{c}
\begin{array}{cccccccccc} a_1 & a_2 & \dots & a_r & b_1 & b_2 & b_3 & \dots & b_r & c_r \end{array}\\
\left[\begin{array}{c|ccccc|c}
 & 0 & 1 & 1 & \cdots & 1 & 1\\
 & 1 & 0 & 1 & \cdots & 1 & 1\\
I_r & 1 & 1 & 0 & \cdots & 1 & 1\\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
 & 1 & 1 & 1 & \cdots & 0 & 1
\end{array}\right].
\end{array}$$

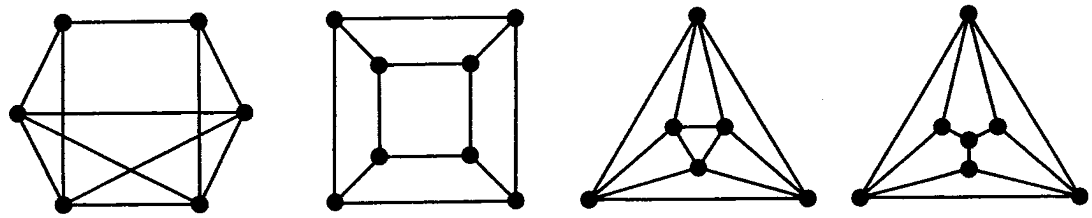

FIGURE 8. The graphs H_6, Q_3, $K_{2,2,2}$, and H_7.

Then Z_r and its minors have the following properties:

(i) $Z_3 \cong F_7$; $Z_4\backslash c_4 \cong AG(3,2)$; and $Z_4\backslash b_4 \cong S_8$;
(ii) $Z_r^* \cong Z_{r+1}\backslash b_{r+1}, c_{r+1}$ for all $r \geq 3$;
(iii) $Z_r\backslash e \cong Z_r\backslash b_r$ for all $e \neq c_r$;
(iv) $Z_r\backslash b_r$ and $Z_r\backslash c_r$ are isomorphic to their duals.

THEOREM 4.4. *Let M be a binary matroid with at least four elements. Then M is 3-connected and has no $M(\mathcal{W}_4)$-minor if and only if $M \cong Z_r$, Z_r^*, $Z_r\backslash b_r$, or $Z_r\backslash c_r$ for some $r \geq 3$.*

Theoretically, the technique used above of building up, an element at a time, from a wheel or a whirl could be applied to find the structure of $EX(\mathcal{W}^2, M(\mathcal{W}_r))$ for any $r \geq 5$. But, even when $r = 5$, the number of possibilities to be considered is large. This makes it much more difficult to detect the patterns in the building-up process, and the class of binary matroids with no $M(\mathcal{W}_5)$-minor has so far defied analysis. However, if one confines attention to the class of graphic matroids with no $M(\mathcal{W}_5)$-minor, the case analysis becomes manageable [**59**].

For $k \geq 3$, consider the graph $K_{3,k}$, labeling its vertex classes V_1 and V_2 where $|V_1| = 3$. Let $K'_{3,k}$, $K''_{3,k}$, and $K'''_{3,k}$ be obtained from $K_{3,k}$ by adding one, two, and three pairwise non-parallel edges joining vertices in V_1. It is straightforward to check that $K'''_{3,k}$ has no $\mathcal{W}_5$-minor. Hence none of $K'''_{3,k}$, $K''_{3,k}$, $K'_{3,k}$, and $K_{3,k}$ has a $\mathcal{W}_5$-minor. Similarly, none of the graphs H_6, Q_3, $K_{2,2,2}$, and H_7 shown in Figure 8 has a $\mathcal{W}_5$-minor.

THEOREM 4.5. *Let G be a graph. Then G is simple and 3-connected having no $\mathcal{W}_5$-minor if and only if*

(i) *G is isomorphic to a simple 3-connected minor of one of H_6, Q_3, $K_{2,2,2}$, and H_7; or*
(ii) *for some $k \geq 3$, G is isomorphic to one of $K_{3,k}$, $K'_{3,k}$, $K''_{3,k}$, and $K'''_{3,k}$.*

When one turns to excluding $M(\mathcal{W}_6)$ as a minor, the number of cases seems unmanageable even in the graphic case. This prompts consideration of the planar graphs with no 6–wheel minor. Gubser [**28**] proved the following structural theorem for this class.

THEOREM 4.6. *Let $\mathcal{G}$ be the class of planar graphs having no $\mathcal{W}_6$-minor. Then every simple 3–connected member of $\mathcal{G}$ is a minor of one of the thirty-eight splitters for $\mathcal{G}$, thirty-six of which have seventeen edges and two of which have sixteen edges.*

The increasing complexity of the case analyses needed in the last three results suggests that a general result concerning exclusion of a wheel minor will not be able to provide such specific structural information as in these three results. The last result provides a hint at what can be proved in general by showing that, for $r = 6$, there are only finitely many simple 3–connected planar graphs with no $\mathcal{W}_r$-minor. A proof that the last assertion is true for all r initiated work of Oporowski, Oxley, and Thomas [**47**] that lead to the following result and the corresponding result for 4–connected graphs.

THEOREM 4.7. *For every integer $n \geq 3$, there is an integer N such that every 3–connected graph with at least N vertices has a minor isomorphic to $\mathcal{W}_n$ or $K_{3,n}$.*

The argument used to prove this theorem relies on results and techniques [**76, 86**] from Robertson and Seymour's graph minors project (see, for example, [**74, 75**]). The reader who is familiar with this work will not be surprised to learn that the bounds obtained on the number N are huge.

Theorem 4.7 gives information about unavoidable minors in large 3–connected graphs. It raises the question as to what can be said for binary matroids or, indeed, for matroids in general. For binary matroids, Ding, Oporowski, Oxley, and Vertigan [**24**] proved the following result.

THEOREM 4.8. *For every integer $n \geq 3$, there is an integer N such that every 3–connected binary matroid with at least N elements has a minor isomorphic to $M(K_{3,n})$, $M^*(K_{3,n})$, $M(\mathcal{W}_n)$, or $Z_n \backslash c_n$.*

The proof of this result treats binary matroids via their matrix representations and depends heavily on certain Ramsey-theoretic results for matrices. Very recently, using new techniques, Ding, Oporowski, Oxley, and Vertigan [**25**] have extended the last result to matroids in general. Before stating this result, we make some observations concerning Z_n. This matroid has the following properties:

(i) the ground set is the union of n lines, $L_1, L_2, \ldots, L_n$, all having three points and passing through a common point, p;
(ii) for all k in $\{1, 2, \ldots, n-1\}$, the union of any k of $L_1, L_2, \ldots, L_n$ has rank $k+1$;
(iii) $r(L_1 \cup L_2 \cup \ldots \cup L_n) = n$.

An arbitrary matroid satisfying these conditions will be called an *n-spike* with *tip* p. It is not difficult to see that Z_n is the unique *binary* n–spike. But it is certainly not the only n–spike. In general, if M is an n–spike with tip p, then

(i) L_i is a circuit of M for all i;
(ii) $(L_i \cup L_j) - p$ is a circuit of M for all distinct i and j;
(iii) every non-spanning circuit of M other than those listed in (i) and (ii) avoids p and contains a unique element from each of $L_1 - p, L_2 - p, \ldots, L_n - p$;
(iv) M/p can be obtained from an n–element circuit by replacing each element by two elements in parallel; and
(v) if $L_i = \{p, x_i, y_i\}$, then each of $M \backslash p / x_i$ and $(M \backslash p \backslash x_i)^*$ is an $(n-1)$-spike with tip y_i.

We are now ready to state the generalization of the last theorem to arbitrary matroids.

THEOREM 4.9. *For every integer $n \geq 3$, there is an integer N such that every 3–connected matroid with at least N elements has a minor isomorphic to $U_{2,n}$, $U_{n-2,n}$, $M(K_{3,n})$, $M^*(K_{3,n})$, $M(\mathcal{W}_n)$, $\mathcal{W}^n$, or an n-spike.*

When one is unable to obtain specific structural information for a minor-closed class of matroids, one can often bound the size of a simple rank-r member of the class. Kung has proved numerous such results and these are surveyed in [**37**]. The following is a result of this type that relates specifically to the problem of excluding wheels from binary matroids.

THEOREM 4.10. *Let $g(r)$ be the maximum number of elements in a simple rank-r member of $EX(\mathcal{W}^2, M(\mathcal{W}_k))$. Then*

$$g(r) - g(r-1) \leq 2^{2k-3} \textit{for all } r.$$

Kung observed that, since both $PG(k-2,2)$ and $PG(k-3,2)$ are members of $EX(\mathcal{W}^2, M(\mathcal{W}_k))$, it follows that $g(k-1) - g(k-2) = 2^{k-2}$. Kung also remarked that it is probable that the bound in Theorem 4.10 can be sharpened to 2^{k-2} for all r.

5. The decomposition of regular matroids

The most significant application of the Splitter Theorem remains its original use in the proof of a decomposition theorem for regular matroids. That result differs from the matroid structural results stated so far in that it allows, in addition to the operations of direct sum and 2–sum, the operation of 3–sum for matroids. An example of the 3–sum operation for graphs is given in Figure 5. For another example of this operation, we refer to Figure 8. In that figure, consider the central triangle in the drawing of $K_{2,2,2}$. The 3–sum of $K_{2,2,2}$ and K_4 across this triangle produces the last graph H_7 in Figure 8. A 3–sum of this type where one of the graphs involved is K_4 is often called a $\Delta - Y$ *exchange*.

The operation that Seymour [**82**] called 3–sum of binary matroids can be derived from a more general matroid operation that involves sticking two matroids together across a common restriction. To ensure that such an operation is well-defined, one needs some additional conditions. Let M_1 and M_2 be matroids such that $M_1|T = M_2|T$ where $T = E(M_1) \cap E(M_2)$. Assume that T is a triangle Δ and that Δ is a modular flat in $\widetilde{M}_1$. This last condition is always satisfied if M_1 is binary. The *generalized parallel connection* $P_\Delta(M_1, M_2)$ of M_1 and M_2 across Δ is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. This operation is a special case of an operation introduced by Brylawski [**15**]. For our particular interests here, it is sufficient to note that if M_1 is binary, then $P_\Delta(M_1, M_2)$ is certainly well-defined. If we delete Δ from $P_\Delta(M_1, M_2)$, we have completed a matroid operation that generalizes the graph operation of 3–sum. If M_1 and M_2 are both binary having more than six elements and Δ does not contain a cocircuit of M_1 or of M_2, then $P_\Delta(M_1, M_2)\backslash\Delta$ is what Seymour called the 3*–sum* of M_1 and M_2. Seymour actually used an equivalent definition of 3–sum taking it to be the matroid on $[E(M_1) \cup E(M_2)] - \Delta$ whose circuits are the minimal non-empty sets of the form $(X_1 \cup X_2) - (X_1 \cap X_2)$ where X_i is a disjoint union of circuits of M_i. We remark that Truemper [**95**] uses the name Δ*–sum* for this operation and uses "3–sum" for a somewhat different operation. It is also interesting to observe that

$$P_\Delta(M(K_4), F_7)\backslash\Delta = F_7^* \text{ and } P_\Delta(M(K_4), F_7^-)\backslash\Delta = (F_7^-)^*.$$

The first of these constructions is illustrated in Figure 9.

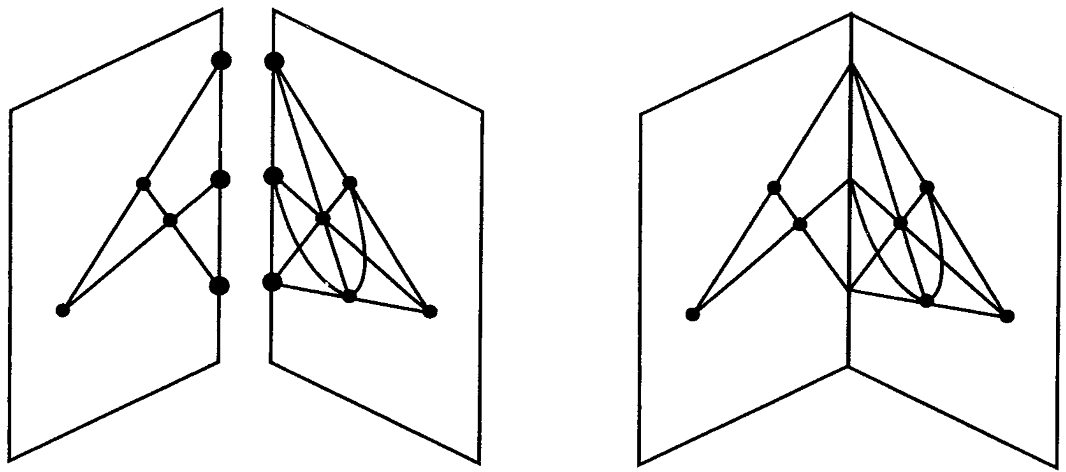

FIGURE 9. A $\Delta - Y$ exchange on F_7 produces F_7^*.

Two matroids that play an important role in the proof of the regular matroids decomposition theorem are R_{10} and R_{12}, which are represented over $GF(2)$ by the matrices A_{10} and A_{12}.

$$A_{10} = \left[\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 \end{array}\right] \qquad A_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 0 \\ I_6 & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

We are now able to state Seymour's theorem [**82**].

THEOREM 5.1. *Every regular matroid M can be constructed by means of direct sums, 2–sums, and 3–sums starting with matroids each of which is isomorphic to a minor of M, and each of which is either graphic, cographic, or isomorphic to R_{10}.*

A key step in the proof of this theorem is the following:

THEOREM 5.2. *Let M be a 3–connected regular matroid. Then either M is graphic or cographic, or M has a minor isomorphic to one of R_{10} and R_{12}.*

This results breaks the rest of the proof of the theorem into two cases: (i) M has an R_{10}–minor; and (ii) M has an R_{12}–minor. The first case is disposed of by showing that R_{10} is a splitter for the class of regular matroids. This is achieved by checking that every simple binary extension of R_{10} has an F_7- or F_7^*-minor. In case (ii), the rest of the argument begins with the observation that R_{12} has an exact 3–separation (Y_1, Y_2) in which Y_1 is the 6–element set that is the union of the only two triangles in R_{12}. The argument proceeds by showing that this exact 3–separation of R_{12} *induces* an exact 3–separation of every regular matroid M with an R_{12}-minor, that is, every such matroid M has an exact 3–separation (Z_1, Z_2) in which $Z_1 \supseteq Y_1$ and $Z_2 \supseteq Y_2$. This exact 3–separation of M implies that M can be decomposed as the 3–sum of two smaller matroids, M_1 and M_2, each of which is isomorphic to a minor of M. To describe the construction of M_1 and M_2, we shall assume that M is simple, for the general case follows easily from this case. We may view M as a restriction of the binary projective geometry $PG(r-1, 2)$ where $r = r(M)$. Then, as $r(Z_1) + r(Z_2) = r + 2$, modularity in the projective geometry implies that the closures of Z_1 and Z_2 in $PG(r-1,2)$ meet in a line L of $PG(r-1,2)$. Now let $M_i = PG(r-1,2)|(Z_i \cup L)$. Then M is the 3–sum of M_1 and M_2 unless L meets Z_1 or Z_2. In the exceptional case, M is again the 3–sum of M_1 and M_2 if we modify M_i by replacing each element of $L \cap Z_i$ by two elements in parallel.

As an example of the process just described, we note that R_{12} itself is the 3–sum of $M^*(K_{3,3})$ and $M(K_5\backslash e)$ where the distinguished triangle in the latter is the one whose vertices are disjoint from the endpoints of e. To show that each of the matroids M_1 and M_2 constructed above is a minor of M requires some effort and we omit the details, which may be found in Seymour's original paper [**82**].

The above discussion allows us to rewrite Theorem 5.2 as follows.

THEOREM 5.3. *Let M be a 3–connected regular matroid. Then*
(i) *M is graphic or cographic; or*
(ii) *$M \cong R_{10}$; or*
(iii) *M has an R_{12}–minor, this minor has an exact 3–separation, and this 3–separation induces an exact 3–separation of M.*

The last theorem provided the model for a general theory of matroid decomposition developed in a sequence of papers by Truemper [**88**]–[**94**] and described in his book [**95**]. The most significant difference between this scheme and the method used in Section 4 is that it incorporates a separation algorithm that efficiently decides whether or not a given k–separation of a minor N of a matroid M induces a k–separation of M.

Among the numerous applications of the regular matroids decomposition theorem is an algorithm that tests in polynomial time whether or not a given real matrix is totally unimodular. For the details of this, the reader is referred to [**77**]; other applications of Theorem 5.1 may also be found in [**84**]. One relatively easy application of the theorem is in extending Theorem 4.5 to describe the structure of all *regular* matroids with no $M(\mathcal{W}_5)$-minor. Indeed, if the columns of the matrix A_{12} representing R_{12} are labeled $1, 2, \ldots, 12$, it is easy to check that $R_{12}/3\backslash 10 \cong M(\mathcal{W}_5)$. The next theorem [**59**] now follows immediately from combining Theorems 4.5 and 5.2.

THEOREM 5.4. *Let M be a regular matroid. Then M is 3–connected and has no $M(\mathcal{W}_5)$-minor if and only if*
(i) *for some $k \geq 3$, M is isomorphic to one of $M(K_{3,k})$, $M(K'_{3,k})$, $M(K''_{3,k})$, or $M(K'''_{3,k})$, or their duals; or*
(ii) *M is isomorphic to a 3–connected minor of one of R_{10}, $M(Q_3)$, $M(K_{2,2,2})$, $M(H_7)$, $M(H_6)$, or $M^*(H_6)$.*

6. Reductions to wheels and whirls

By Corollary 2.4, for every 3–connected matroid M with at least four elements, there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids such that $M_0 = M$; for all i in $\{1, 2, \ldots, n\}$, M_i is a single-element deletion or a single-element contraction of M_{i-1}; and the *end*, M_n, of the sequence is a wheel or a whirl. Indeed, for a given matroid M, there are potentially several such sequences having possibly different ends. We shall call such a sequence a *reduction* of M and say that M has a *unique reduction* if there is just one matroid N such that the end of every reduction of M is isomorphic to N.

For the cycle matroid M of the first graph in Figure 10, one obvious reduction is the sequence $M, M\backslash e$ since the last matroid is isomorphic to $M(\mathcal{W}_5)$. Another reduction, shown in the figure, is $M, M\backslash f, M\backslash f\backslash g, M\backslash f\backslash g/h$. In this case, the last matroid is isomorphic to $M(\mathcal{W}_4)$. Thus the matroid M does not have a unique reduction. A natural problem here is to determine all the 3–connected matroids that do have a unique reduction. This problem was raised for graphic matroids by Hamza Ahmad in a private communication. In this section, we present the answer to this problem which, in light of the theorems presented in the last section, is intriguing.

We begin by noting an extension of the Splitter Theorem due to Coullard [**17**]. This theorem weakens the restriction on how the Splitter Theorem applies to wheels and whirls. A proof of this result may be found in Coullard and Oxley [**18**].

THEOREM 6.1. *Let M and N be 3–connected matroids such that N is a minor of M, $|E(N)| \geq 4$, M is not a wheel or whirl, and if $N \cong \mathcal{W}^2$, then M has no larger whirl as a minor, while if $N \cong M(\mathcal{W}_3)$, then M has no larger wheel as a minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids such that $M_0 = M$; $M_n \cong N$; and, for all i in $\{1, 2, \ldots, n\}$, M_i is a single-element deletion or a single-element contraction of M_{i-1}.*

The next theorem determines all the matroids having a unique reduction.

THEOREM 6.2. *Let M be a 3–connected matroid. Then M has a unique reduction if and only if*

(i) *M is a wheel or a whirl;*
(ii) *M is a binary matroid having no $M(\mathcal{W}_4)$-minor;*
(iii) *M is a regular matroid having no $M(\mathcal{W}_5)$-minor;*
(iv) *M is a ternary matroid having no $M(\mathcal{W}_3)$-minor; or*
(v) *M has no $M(\mathcal{W}_3)$-minor and no $\mathcal{W}^3$-minor.*

The proof of this theorem will use the following extension of Theorem 4.4 [**56**]. The matroid $M(\mathcal{W}_4)$ has exactly three non-isomorphic binary 3–connected single-element extensions, namely $M^*(K_{3,3})$, $M(K_5\backslash e)$, and a matroid denoted P_9. A geometric representation for the last matroid is shown in Figure 11. Evidently P_9 is isomorphic to $P_\Delta(M(K_4), F_7)\backslash p$ where p is an arbitrary element of Δ.

THEOREM 6.3. *Let M be a binary matroid. Then M is 3–connected having no minor isomorphic to P_9 or P_9^* if and only if*

(i) *M is regular and 3–connected; or*
(ii) *for some $r \geq 3$, $M \cong Z_r, Z_r^*, Z_r\backslash b_r$, or $Z_r\backslash c_r$.*

PROOF OF THEOREM 6.2. We distinguish three cases:

(I) M is a wheel or a whirl;

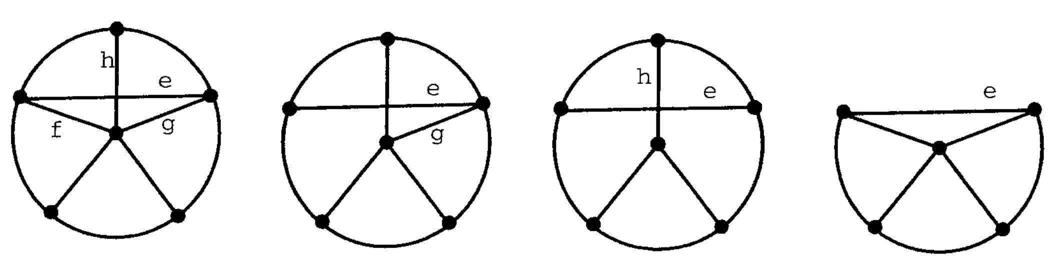

FIGURE 10. A reduction to a 4–wheel.

(II) M is not a wheel or a whirl and has $M(\mathcal{W}_5)$ or $\mathcal{W}^4$ as a minor;
(III) M is not a wheel or a whirl and has neither $M(\mathcal{W}_5)$ nor $\mathcal{W}^4$ as a minor.

In case (I), M clearly has a unique reduction. In case (II), if M has an $M(\mathcal{W}_5)$-minor, then M also has an $M(\mathcal{W}_4)$-minor. Thus, by Theorem 6.1, M has a reduction whose end is isomorphic to $M(\mathcal{W}_5)$ and another whose end is isomorphic to $M(\mathcal{W}_4)$. Hence if M has an $M(\mathcal{W}_5)$-minor, it does not have a unique reduction. Similarly, if M has a $\mathcal{W}^4$-minor, it has reductions to both $\mathcal{W}^3$ and $\mathcal{W}^4$. Thus, in case (II), M does not have a unique reduction.

In case (III), suppose first that M has no $\mathcal{W}^2$-minor. Then M is binary having no $M(\mathcal{W}_5)$-minor. If M has no $M(\mathcal{W}_4)$-minor, then clearly M has a unique reduction. Suppose now that M has an $M(\mathcal{W}_4)$-minor. Then either (a) M has P_9 or P_9^* as a minor; or (b) M has no minor isomorphic to P_9 or P_9^*. In case (a), we may assume, by duality, that M has a P_9-minor. Theorem 4.1 implies that there is a sequence of 3–connected matroids, $M_0, M_1, \dots, M_n$, with each member being a single-element deletion or contraction of its predecessor, $M_0 = M$, and $M_n \cong P_9$. But P_9 is a single-element extension of both $M(\mathcal{W}_4)$ and S_8, the latter is a single-element extension of F_7^* which, in turn, is a single-element coextension of $M(\mathcal{W}_3)$. Thus P_9, and hence M, has reductions to both $M(\mathcal{W}_4)$ and $M(\mathcal{W}_3)$. Therefore, in case (a), M does not have a unique reduction. On the other hand, in case (b), Theorem 6.3 implies that M is regular. Thus M is regular having an $M(\mathcal{W}_4)$-minor but no $M(\mathcal{W}_5)$-minor. Because $M(\mathcal{W}_3)$ has no 3–connected regular single-element extension or coextension, it follows that M has a unique reduction.

To complete the proof in case (III), we need to consider the case when M has a $\mathcal{W}^2$-minor. If M has no $M(\mathcal{W}_3)$-minor and no $\mathcal{W}^3$-minor, then every reduction of M must have its end isomorphic to $\mathcal{W}^2$. Thus, in this case, M has a unique reduction. We may now assume that M has $M(\mathcal{W}_3)$ or $\mathcal{W}^3$ as a minor. In the first case, M has reductions that end in both the largest wheel minor of M and the largest whirl minor of M, so M does not have a unique reduction. Thus we may assume that M has no $M(\mathcal{W}_3)$-minor. Hence M has a $\mathcal{W}^3$-minor. If M has a $U_{2,5}$- or $U_{3,5}$-minor, then M has a reduction to $\mathcal{W}^2$ that goes through such a minor, and M has another reduction to $\mathcal{W}^3$. Thus if M has a $U_{2,5}$- or $U_{3,5}$-minor, then M does not have a unique reduction. We may now suppose that M has no $U_{2,5}$-

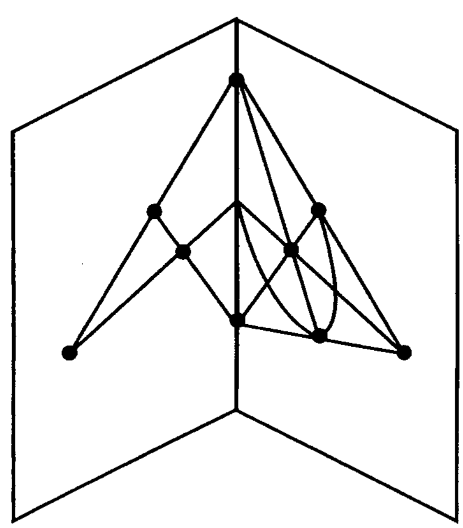

FIGURE 11. P_9.

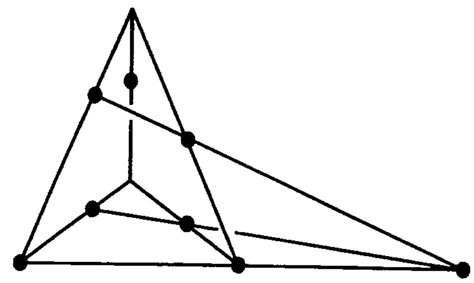

FIGURE 12. J.

or $U_{3,5}$-minor. Since M also has no $M(\mathcal{W}_3)$-minor, M has no F_7- or F_7^*-minor. We conclude that M is a ternary matroid having no $M(\mathcal{W}_3)$-minor and that such matroids have a unique reduction. □

The feature of Theorem 6.2 that seems particularly striking is that, even before the problem was raised, the matroids listed under (ii), (iii), and (iv) had been explicitly described. Moreover, those results were essentially the only known structural results involving exclusion of wheels or whirls. The theorems that specify the matroids listed under (ii) and (iii) were stated above. The matroids listed under (iv) were determined in [**57**]. Before stating that result, we observe that it means that all the matroids having a unique reduction are known explicitly except for those that have no $M(\mathcal{W}_3)$-minor and no $\mathcal{W}^3$-minor. Hence we have the following:

PROBLEM 6.4. *Determine all the matroids that have no $M(\mathcal{W}_3)$-minor and no $\mathcal{W}^3$-minor.*

The class of matroids with no $M(\mathcal{W}_3)$-minor and no $\mathcal{W}^3$-minor includes, for example, all rank-3 matroids in which there are at most two elements that are on more than one non-trivial line. The task of determining all the matroids in this class seems difficult, although it should be noted that the quaternary members of that class were listed in [**57**]. Moreover, Hipp [**32**] proved the following theorem, a proof of which may also be found in [**37**].

THEOREM 6.5. *A rank-r simple matroid M in $EX(\mathcal{W}^3, M(\mathcal{W}_3), U_{2,q+2})$ has at most $q(r-1)+1$ elements. Moreover, equality is attained if and only if M can be formed from $r-1$ copies of $U_{2,q+1}$ by using $r-2$ parallel connections.*

Two special matroids appear in the next result, namely the splitters for the class of ternary matroids with no $M(\mathcal{W}_3)$-minor. One of these matroids is J, the rank-4 self-dual matroid for which a geometric representation is shown in Figure 12. The second such matroid is the vector matroid of the matrix D_{12} over $GF(3)$ where D_{12} is

$$\left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & -1 & -1 & 1 \\ & 1 & 1 & 0 & 1 & -1 & -1 \\ I_6 & 1 & -1 & 1 & 0 & 1 & -1 \\ & 1 & -1 & -1 & 1 & 0 & 1 \\ & 1 & 1 & -1 & -1 & 1 & 0 \end{array}\right].$$

The last matrix and its associated matroid are actually very well known in a slightly different context. The matrix D_{12} is a generator matrix for the ternary Golay code, this being a member of the special class of *perfect codes* (see, for

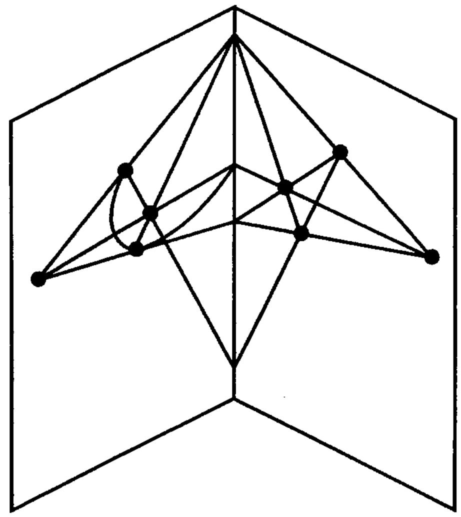

FIGURE 13. T_8.

example, Chapter 20 of [**41**]). Moreover, if E is the set of elements of $M[D_{12}]$ and $\mathcal{H}$ is its set of hyperplanes, then the pair $(E, \mathcal{H})$ is the unique Steiner system $S(5,6,12)$; that is, every member of $\mathcal{H}$ contains exactly six elements and every 5–element subset of E is contained in a unique member of $\mathcal{H}$. The matroid $M[D_{12}]$ has many attractive properties. For instance, both the set of circuits and the set of cocircuits of this matroid equal $\mathcal{H}$. Hence $M[D_{12}]$ is identically self-dual. Moreover, $M[D_{12}]$ has as its automorphism group the Mathieu group, M_{12}, which is 5–transitive; that is, if $(e_1, e_2, \dots, e_5)$ and $(f_1, f_2, \dots, f_5)$ are ordered 5–tuples of distinct elements of $M[D_{12}]$, then there is an automorphism of $M[D_{12}]$ that, for all i, maps e_i to f_i. In particular, if $|X| = 3$, then $M[D_{12}]/X$ is isomorphic to the ternary affine plane, $AG(2,3)$. We shall follow convention in denoting the matroid $M[D_{12}]$ by $S(5,6,12)$.

THEOREM 6.6. *A matroid M is 3–connected, ternary, and has no $M(\mathcal{W}_3)$-minor if and only if M is isomorphic to J, to $\mathcal{W}^r$ for some $r \geq 2$, or to a 3–connected minor of* $S(5,6,12)$.

7. More applications of the Splitter Theorem

The technique that was used to derive Theorems 4.4, 4.5, and 6.5 has also been successfully employed to prove numerous other results. Many of these are noted in Chapter 11 of [**62**]. In this section, we give some examples of such results concentrating on newer results not noted in [**62**].

A matroid is called *paving* if it has no circuits of size less than its rank. For comparison, a matroid is uniform if and only if it has no circuits of size less than or equal to its rank. It is straightforward to show that the class of paving matroids is minor-closed and that the unique excluded minor for the class is $U_{2,2} \oplus U_{0,1}$ [**61**]. It is not difficult to determine all paving matroids that are not 3–connected and these are listed in [**61**]. The following result was proved directly by Acketa [**1**].

THEOREM 7.1. *The 3–connected binary paving matroids are precisely the 3–connected minors of* $AG(3,2)$.

It is straightforward to prove this result using the building-up technique exemplified in Section 4. The same technique can also be used to determine all

3–connected ternary paving matroids although more work is needed for this. Some of the matroids that arise here are familiar or have appeared earlier. Two that we have not yet seen here are R_8 and T_8. The first of these is the real affine cube; the second has the geometric representation shown in Figure 13. These matroids are represented over $GF(3)$ by the matrices

$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array}\right] \text{ and } \left[\begin{array}{c|cccc} & -1 & 1 & 1 & 1 \\ I_4 & 1 & -1 & 1 & 1 \\ & 1 & 1 & -1 & 1 \\ & 1 & 1 & 1 & -1 \end{array}\right],$$

respectively. Evidently both R_8 and T_8 are isomorphic to their duals.

THEOREM 7.2. *The 3–connected ternary paving matroids are precisely the 3–connected minors of $PG(2,3)$, $S(5,6,12)$, R_8, and T_8.*

Evidently there are relatively few 3–connected $GF(q)$-representable paving matroids for $q \in \{2,3\}$. Rajpal [**70**] proved that, in general, there are only finitely many 3–connected $GF(q)$-representable paving matroids by establishing the following result.

PROPOSITION 7.3. *Let M be a rank-r $GF(q)$-representable paving matroid with $1 < r < |E(M)|$. If $r > q$, then*
(i) *M^* is a paving matroid; and*
(ii) *$|E(M)| \leq 4q$ and $r \leq 2q$.*

By using the same technique that was used to prove Theorem 7.2, Rajpal [**71**] was able to determine all quaternary paving matroids. The case-checking required here is considerable and needed the aid of a computer.

THEOREM 7.4. *Every 3–connected quaternary paving matroid is a minor of one of the fifteen splitters for the class of such matroids. These splitters consist of*
(i) *$PG(2,4)$;*
(ii) *eight matroids of rank four of which five have twelve elements and one each have thirteen, fourteen, and sixteen elements;*
(iii) *three matroids of rank five all having ten elements; and*
(iv) *the dual of the Pappus matroid and two other matroids of rank six having ten and twelve elements, respectively.*

One of the graph results that motivated the matroid structural results that we have discussed here is D.W. Hall's Theorem (3.2). A restatement of that result is that every simple 3–connected graph with a K_5-minor must have a $K_{3,3}$-minor, the only exception being K_5 itself. Kingan [**36**] proved an attractive generalization of this result to binary matroids. We denote by T_{12} the vector matroid of the following matrix over $GF(2)$:

$$\left[\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

This matroid has a transitive automorphism group, and T_{12}/e will denote the unique single-element contraction of T_{12}.

THEOREM 7.5. *Let M be a 3-connected binary matroid with an $M(K_5)$-minor. Then either M has an $M(K_{3,3})$- or $M^*(K_{3,3})$-minor, or M is isomorphic to $M(K_5)$, T_{12}, or T_{12}/e.*

The matroid T_{12} has an interesting link with the Petersen graph, P_{10}. Take the 15×12 binary matrix whose rows are indexed by the edges of P_{10} and whose columns are indexed by the 5-cycles of P_{10} with each column being the incidence vector of the corresponding 5-cycle. Then the vector matroid of this matrix is T_{12}.

The matroid T_{12} arises in another interesting context. The classes of binary and ternary matroids are relatively well understood, as is their intersection, the class of regular matroids. The union of the classes of binary and ternary matroids is also minor-closed, but it not known whether the set of excluded minors for this union is finite. The following conjecture is due to Oporowski, Oxley, and Whittle (private communication).

CONJECTURE 7.6. *Let $\mathcal{M}$ be the class of matroids that are binary or ternary. Then the excluded minors for $\mathcal{M}$ are $U_{2,4}\oplus F_7$, $U_{2,4}\oplus F_7^*$, $U_{2,4}\oplus_2 F_7$, $U_{2,4}\oplus_2 F_7^*$, $U_{2,5}$, $U_{3,5}$, and the unique matroids that are obtained by relaxing a circuit-hyperplane in each of $AG(3,2)$ and T_{12}.*

Oporowski, Oxley, and Whittle have proved that the above list contains all the excluded minors for $\mathcal{M}$ with at most twenty-three elements. Moreover, they have shown that, for every remaining excluded minor N, there is a 3-connected binary matroid M whose ground set is the union of two disjoint circuit-hyperplanes such that relaxing one of these produces N and relaxing both produces a ternary matroid. This implies, in particular, that $|E(N)|$ is divisible by 4.

The finiteness problem considered above is a special case of the general question: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are minor-closed classes of matroids, each characterized by a finite set of excluded minors, is $\mathcal{M}_1 \cup \mathcal{M}_2$ also characterized by a finite set of excluded minors? This problem, the *intertwining problem* for matroids, was posed by Brylawski [**16**] and, independently, by several others. It appears as Problem 14.1.8 in [**62**] and was recently answered in the negative by Vertigan [**101**]. While Vertigan gave a large collection of examples of classes $\mathcal{M}_1$ and $\mathcal{M}_2$ for which $\mathcal{M}_1 \cup \mathcal{M}_2$ does not have a finite set of excluded minors yet $\mathcal{M}_1$ and $\mathcal{M}_2$ do, it appears to be very difficult to characterize precisely when this occurs.

In Section 4, we considered sets of unavoidable minors in large 3-connected matroids. The next result, which is much easier to prove, identifies another collection of small unavoidable matroids [**54, 104**]. Geometric representations for the matroids Q_6 and P_6 are shown in Figure 14.

FIGURE 14. Q_6 and P_6.

PROPOSITION 7.7. *Let M be a 3–connected matroid having rank and corank at least three. Then M has a minor isomorphic to one of the matroids $M(\mathcal{W}_3)$, $\mathcal{W}^3$, Q_6, P_6, or $U_{3,6}$.*

This list of five small matroids raises the question of describing the structure of the classes that arise when one excludes certain of these as minors. For instance, it is not difficult to see that $EX(\mathcal{W}^3, Q_6, P_6, U_{3,6})$ consists of those matroids that can be constructed from binary matroids and uniform matroids of rank or corank two by direct sums and 2–sums. We noted above that the problem of describing the members of the class $EX(M(\mathcal{W}_3), \mathcal{W}^3)$ is unsolved. But we should certainly expect to be able to describe those classes that arise when four of $M(\mathcal{W}_3)$, $\mathcal{W}^3$, Q_6, P_6, and $U_{3,6}$ are excluded as minors [**58**]. However, there is one problematic case: the structure of $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$ has yet to be determined, whereas, by contrast, the members of $EX(\mathcal{W}^3, P_6)$ have been specified [**60**]. This raises a general issue that looms over all results of this type, namely its unpredictability. With current methods, it seems impossible to foretell when one is likely to be able to determine the structure of a certain class. One applies the building-up method and hopes that a pattern can be detected before the number of cases explodes. Another interesting feature of this process is that, even when the method produces an answer, knowing the answer helps little in verifying that this answer is correct.

8. Essential elements and fans

An element e in a 3–connected matroid M is *essential* if neither $M\backslash e$ nor M/e is 3–connected. The Wheels and Whirls Theorem identified wheels and whirls of rank at least three as being precisely those 3–connected matroids in which every element is essential. In this section, we consider what can be said concerning the local structure about an essential element in a 3–connected matroid. Since a 3–connected matroid with at least four elements is both simple and cosimple, one way for an element to be essential is for it to be in both a triangle and a triad. Indeed, Tutte [**99**] proved the following:

THEOREM 8.1. *An essential element in a 3–connected matroid is in either a triangle or a triad.*

In both wheels and whirls, we have sequences of interlocking triangles and triads. We shall be interested in such sequences in arbitrary 3–connected matroids. For instance, in the cycle matroid of the graph G in Figure 15, the members of the sequence $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \dots, \{a_5, a_6, a_7\}$ are alternately triangles and triads. In general, a non-empty sequence $T_1, T_2, \dots, T_k$ of triangles and triads is a *chain* of *length* k in a matroid M if, for all i in $\{1, 2, \dots, k-1\}$,

(i) exactly one of T_i and T_{i+1} is a triangle;
(ii) $|T_i \cap T_{i+1}| = 2$; and
(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \dots \cup T_i)$ is empty.

Since the only 3–connected matroid with a triangle that is also a triad is $U_{2,4}$, condition (i) here is redundant if M is 3–connected having at least five elements. Evidently $T_1, T_2, \dots, T_k$ is a chain in M if and only if it is a chain in M^*. Moreover, a straightforward induction argument using orthogonality establishes that if $T_1, T_2, \dots, T_k$ is a chain in M, then M has $k+2$ distinct elements $a_1, a_2, \dots, a_{k+2}$ such that $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i in $\{1, 2, \dots, k\}$. The sets $T_1, T_2, \dots, T_k$ are called *links* in the chain.

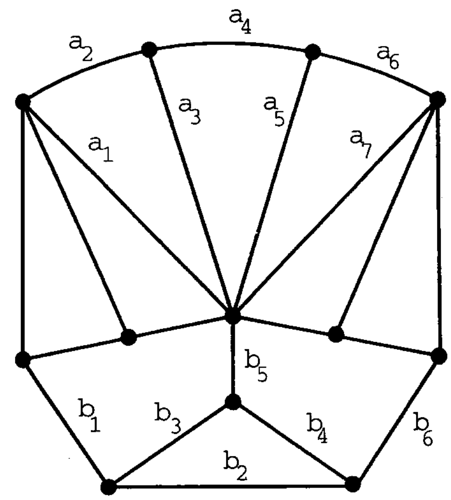

FIGURE 15. A graph G containing several chains.

Although chains can certainly occur in both non-graphic and graphic matroids, we follow Tutte [99] in keeping track of the triangles and triads in a chain by using graphs as in Figure 16(a)–(c). In each case, the chain is $T_1, T_2, \ldots, T_k$ where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$, and every triangle in the graph is a triangle in the chain, while the triads in the chain correspond to circled vertices.

The following result, known as Tutte's Triangle Lemma [99], is an important tool in dealing with chains particularly in extending a given chain.

THEOREM 8.2. *Let $\{x, y, z\}$ be a triangle in a 3–connected matroid M. If neither $M\backslash x$ nor $M\backslash y$ is 3–connected, then x is in a triad with exactly one of y and z.*

Much of our interest is in maximal chains in 3–connected matroids. By extending Tutte's proof of the Wheels and Whirls Theorem, one can show that such a chain has non-essential elements at both ends [65].

THEOREM 8.3. *Let M be a 3–connected matroid with at least four elements and suppose that M is not a wheel or a whirl. Let $T_1, T_2, \ldots, T_k$ be a maximal chain in M. Then the elements of $T_1 \cup T_2 \cup \ldots \cup T_k$ can be labeled so that neither a_1 nor a_{k+2} is essential where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i.*

As an immediate consequence of this, we have the following result [65].

COROLLARY 8.4. *Let M be a 3–connected matroid with at least four elements. Then either M is a wheel or a whirl, or M has at least two non-essential elements.*

A maximal chain $T_1, T_2, \ldots, T_k$ in a 3–connected matroid M other than a wheel or a whirl is called a *fan*. *Type*–1, *type*–2, and *type*–3 *fans* correspond to the chains shown in (a), (b), and (c), respectively, in Figure 16. In that figure, the non-essential elements of the fans have been marked in bold.

Theorem 8.1 established that every essential element e in a 3–connected matroid M is in some chain. Thus, provided M is neither a wheel nor a whirl, e is in some fan. The next result [65] specifies exactly when this fan is unique.

THEOREM 8.5. *Let M be a 3–connected matroid that is not a wheel or a whirl. Suppose that e is an essential element of M. Then e is in a fan. Moreover, this fan is unique unless*

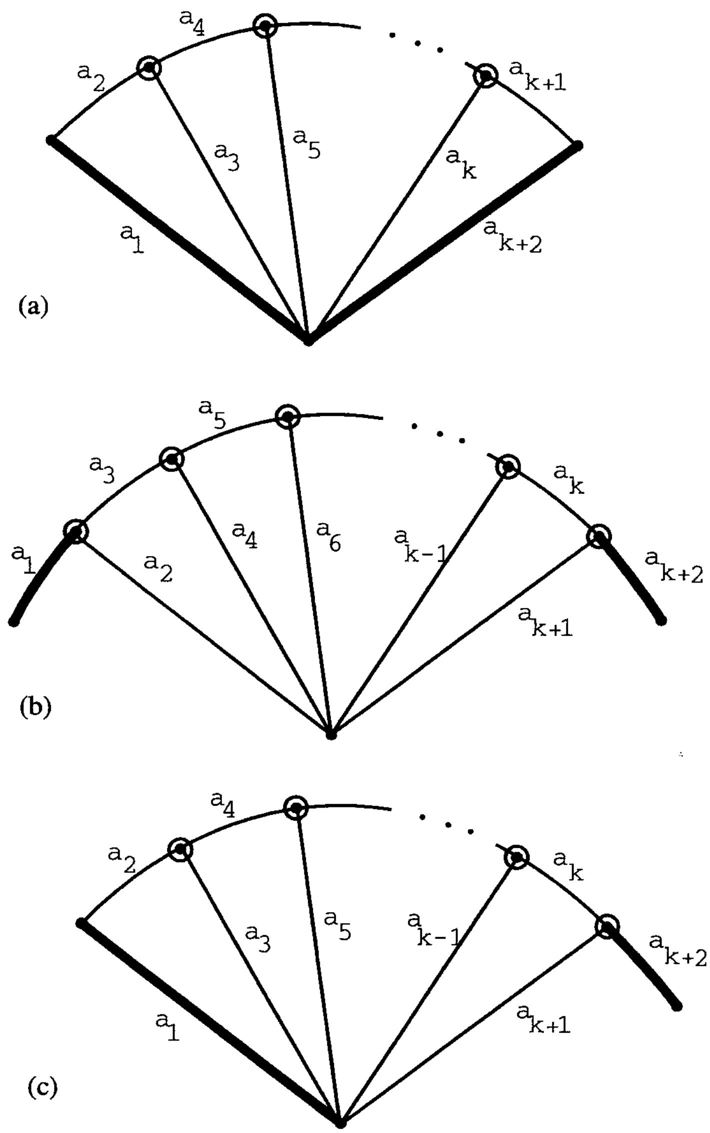

FIGURE 16. The three types of chains.

(a) *every fan containing* e *consists of a single triangle and any two such triangles meet in* $\{e\}$;
(b) *every fan containing* e *consists of a single triad and any two such triads meet in* $\{e\}$;
(c) e *is in exactly three fans; these three fans are of the same type, each has five elements, together they contain a total of six elements; and, depending on whether these fans are of type–1 or type–2, the restriction or contraction, respectively, of* M *to this set of six elements is isomorphic to* $M(K_4)$.

An example of the third possibility above may be found in Figure 15. There, each of b_2, b_3, and b_4 is essential and is in the three fans of the form $T_1, \{b_2, b_3, b_4\}, T_3$ where T_1 and T_3 are any two of $\{b_1, b_2, b_3\}, \{b_3, b_4, b_5\}$, and $\{b_2, b_4, b_6\}$.

When M is graphic, case (b) in Theorem 8.5 can be strengthened to the assertion that e is in exactly two fans each of which is a single triad corresponding to a vertex of degree three. The proof of this fact relies on the following strengthening

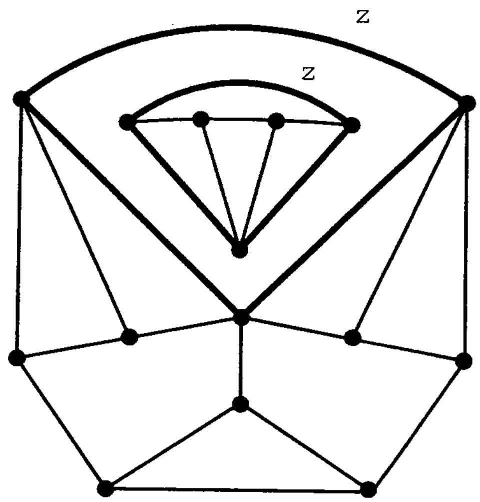

FIGURE 17. Breaking off a wheel; the outer graph is G_1.

of Theorem 8.1 due to Tutte [**98**]: In a 3–connected graphic matroid $M(G)$, every essential element that is not in a triangle meets a degree–3 vertex of G.

Theorem 8.5 implies that the fans in a 3–connected matroid other than a wheel or whirl induce a partition of the set of essential elements.

COROLLARY 8.6. *Let M be a 3–connected matroid that is not a wheel or a whirl. Then there is a partition of the set of essential elements of M such that two elements are in the same class if and only if there is a fan whose ground set contains both.*

Returning to the cycle matroid M of the graph G in Figure 15, we observe that the fan with ground set $\{a_1, a_2, \ldots, a_7\}$ can be viewed as a partial wheel. Indeed, we can break off a wheel from the original matroid leaving a 3–connected matroid. Figure 17 shows two disjoint graphs with one, a 4–wheel, drawn inside the other G_1. If the bold edges are identified in the natural way and then the identified edge z is deleted, we recover the original graph. On the other hand, the graph G_1 can be obtained from G by deleting the edges a_3 and a_5, contracting the edges a_2 and a_4, and then relabeling a_6 as z. The next theorem [**65**] asserts that any 3–connected matroid M having a chain of odd length exceeding two can be constructed by sticking together a wheel and a certain 3–connected minor of M, just as in this example.

THEOREM 8.7. *Let M be a 3–connected matroid and suppose that, for some non-negative integer n, the sequence*

$$\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \ldots, \{y_n, x_n, y_{n+1}\}$$

is a chain in M in which $\{y_0, x_0, y_1\}$ is a triangle. Then

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z$$

where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathcal{W}_{n+2}$ is labeled as in Figure 18; and M_1 is obtained from the matroid $M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots, y_n$ by relabeling x_n as z. Moreover, $\widetilde{M_1}$ is 3–connected. More precisely, either

(i) *M_1 is 3–connected; or*
(ii) *z is in a unique 2–circuit $\{z, h\}$ of M_1, and $M_1\backslash z$ is 3–connected.*

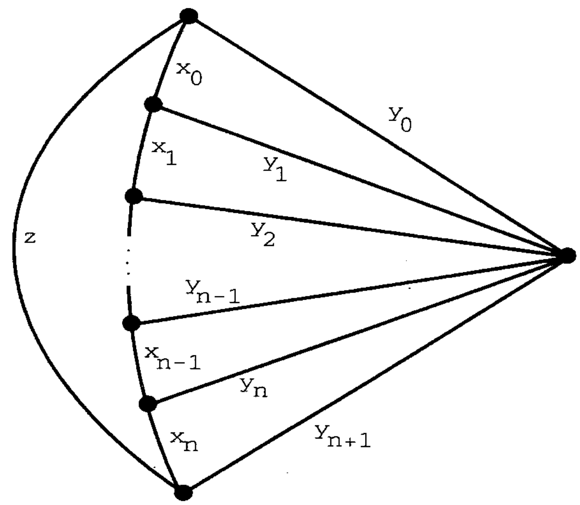

FIGURE 18. A labeled $(n+2)$–wheel.

In the latter case,

$$M = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2)$$

where $\Delta_2 = \{y_0, y_{n+1}, h\}$; $\mathcal{W}_{n+2}$ *is labeled as in Figure 18 with* z *relabeled as* h; *and* M_2 *is* $M_1\backslash z$, *which equals* $M\backslash x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_n$.

An immediate consequence of this theorem is that the restriction of M to $\{x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_{n+1}\}$, the ground set of the chain, is equal to the cycle matroid of the graph shown in Figure 18 with the edge z deleted. Thus our view of a chain as a partial wheel is validated.

The behavior of essential elements when a wheel is broken off as above is determined in [**65**]. In particular, it is shown that, for i in $\{1,2\}$, if M_i is 3–connected, then an element of M_i that is essential in M remains essential in M_i. Non-essential elements behave somewhat less straightforwardly but still predictably.

Theorem 8.7 indicates how one can break off a wheel from a 3–connected matroid having a chain of odd length exceeding two. In fact, that theorem explicitly describes this break off when the chain has a triangle as its first link and hence has a triangle as its last link. If the chain has triads as its first and last links, then one can reduce to the case described in Theorem 8.7 by taking duals. A similar result [**65**] holds for chains of even length although, in this case, it is slightly more difficult to recover a 3–connected matroid in what is left after the break off.

THEOREM 8.8. *Let* M *be a 3–connected matroid that is not a wheel or a whirl. Suppose that, for some non-negative integer* n, *the sequence*

$$\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \dots, \{y_n, x_n, y_{n+1}\}, \{x_n, y_{n+1}, x_{n+1}\}$$

is a chain in M *in which* $\{y_0, x_0, y_1\}$ *is a triangle. Then*

$$M = P_{\Delta}(M(\mathcal{W}_{n+3}), M_3)\backslash\{z', y'_{n+1}\}$$

where $\Delta = \{y_0, z', y'_{n+1}\}$; $\mathcal{W}_{n+3}$ *is labeled as in Figure 19; and* M_3 *is obtained from the matroid* $M/x_0, x_1, \dots, x_{n-1}\backslash y_1, y_2, \dots, y_n/x_{n+1}$ *by relabeling* x_n *and* y_{n+1} *as* z' *and* y'_{n+1}. *Moreover,* $\widetilde{M_3}$ *is 3–connected. More precisely,*

(i) M_3 *is 3–connected; or*
(ii) z' *is in a unique 2–circuit of* M_3, *and* $M_3\backslash z'$ *is 3–connected; or*
(iii) y'_{n+1} *is in a unique 2–circuit of* M_3, *and* $M_3\backslash y'_{n+1}$ *is 3–connected; or*

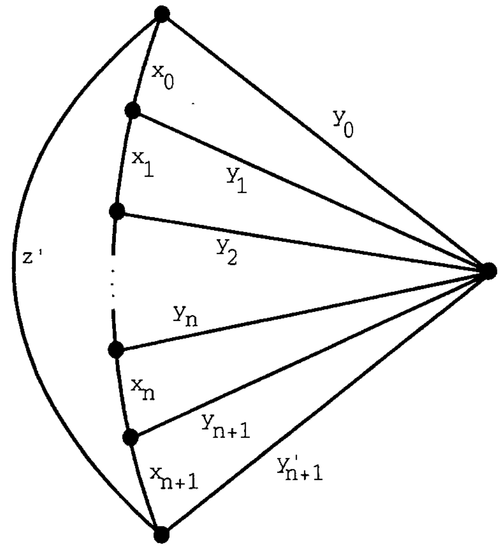

FIGURE 19. A labeled $(n+3)$–wheel.

(iv) *each of z' and y'_{n+1} is in a unique 2–circuit of M_3, and $M_3\backslash z', y'_{n+1}$ is 3–connected.*

By breaking off wheels in the manner described above, one is able to reduce the size of the 3–connected matroid being considered by removing a piece of the matroid whose behavior is well-understood. The reader may be curious that, in the general matroid case, one is always breaking off wheels rather than wheels *or* whirls. This can be explained relatively simply. If Δ is a triangle in a wheel and z is a rim element in this triangle, then the matroid $P_\Delta(M(\mathcal{W}_{n+2}), U_{2,4})\backslash z$ is precisely $\mathcal{W}^{n+2}$. Thus, loosely speaking, attaching wheels and attaching whirls are really the same process with only the points of attachment altering.

9. Matroids with few non-essential elements

A 3–connected matroid with no non-essential elements is a wheel or a whirl. Moreover, by Corollary 8.4, there are no 3–connected matroids with exactly one non-essential element. In this section, following [**66**], we describe the 3–connected matroids that have exactly two non-essential elements.

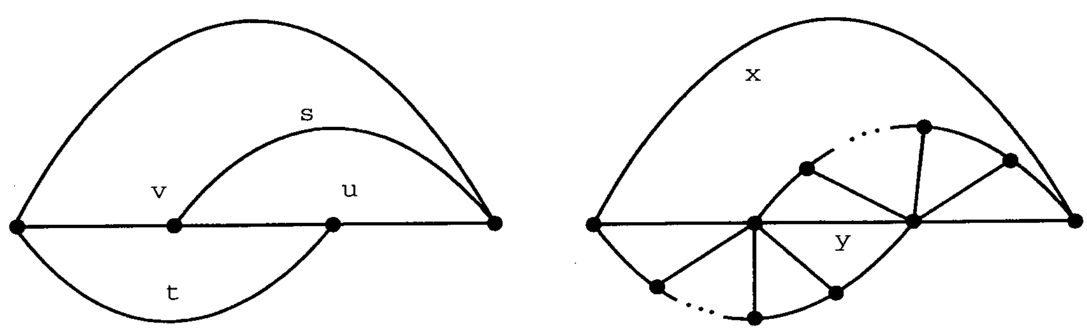

FIGURE 20. Construction of a twisted wheel from K_4.

We know from Theorem 8.3 that if M is a 3–connected matroid other than a wheel or a whirl, then every essential element of M is in a fan the ends of which are non-essential. Thus if M has exactly two non-essential elements, these two elements must occur as the ends of every fan. Therefore M is formed by somehow attaching these fans together across the two non-essential elements. In what follows, we shall describe precisely how these attachments are done. If M is graphic, it is not difficult to find some examples of such attachments. The graph in Figure 20(b) is called a *twisted wheel.* It can be obtained from K_4, drawn as in Figure 20(a), by subdividing each of the edges s and t into at least two edges and then joining each of the newly created vertices to one of u and v as shown. Evidently x and y are the only non-essential elements in the cycle matroid of such a graph. It is clear that a twisted wheel can also be constructed by appropriately joining two type-3 fans with ends x and y.

The graph in Figure 21(b) is an example of a 3–dimensional wheel. In general, a *multidimensional wheel* is constructed as follows: begin with the 3–vertex graph in Figure 21(a) in which u and v are joined by a path u, h, v of length two and by k parallel edges $x_1, x_2, \ldots, x_k$, for some $k \geq 3$. Subdivide each of these parallel edges into at least two edges and, finally, join each newly created vertex to h. Evidently the cycle matroid of the resulting graph has x and y as its only non-essential elements. Moreover, this matroid can be obtained by appropriately joining k type-1 fans with ends x and y. Observe that if each of $x_1, x_2, \ldots, x_k$ is subdivided into exactly two edges, the resulting k–dimensional wheel is isomorphic to $K''_{3,k}$.

Based on Theorem 8.5, it is fairly straightforward to prove that the only graphic matroids with exactly two non-essential elements are those described above.

THEOREM 9.1. *Let M be a 3–connected graphic matroid. Then M has exactly two non-essential elements if and only if M is the cycle matroid of a twisted wheel or a multidimensional wheel.*

There are many non-graphic 3–connected matroids that have exactly two non-essential elements and we now proceed to describe them. Theorem 8.7, which specifies how to break off a wheel, is crucial in deriving this description.

Let M be a 3–connected matroid with exactly two non-essential elements. Then every element of M is in a fan so every element is in a triangle or a triad. Hence,

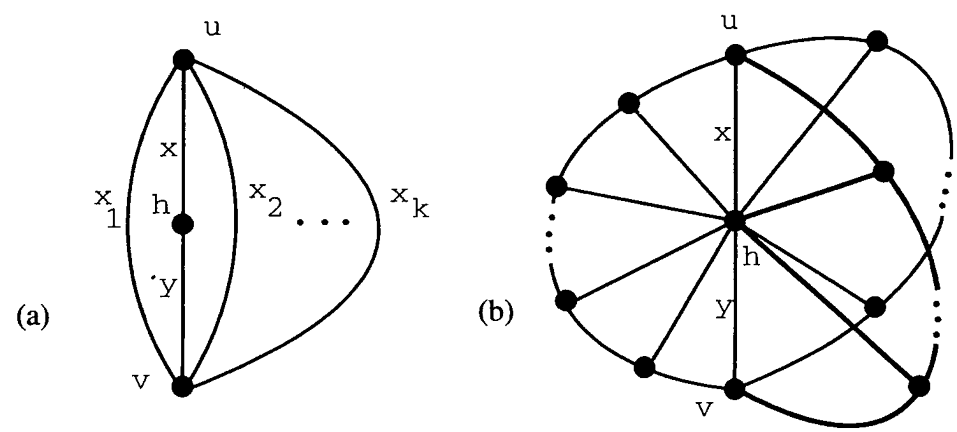

FIGURE 21. Construction of a multidimensional wheel.

for every element x of M, at least one of $M\backslash x$ and M/x fails to be 3–connected. We call an element e of a 3–connected matroid N *deletable* if $N\backslash e$ is 3–connected; e is *contractible* if N/e is 3–connected. In M, we must have one of the following:

(i) both non-essential elements are deletable but not contractible;
(ii) both non-essential elements are contractible but not deletable;
(iii) one non-essential element is deletable but not contractible and the other is contractible but not deletable.

Hence, for example, the cycle matroid of a multidimensional wheel satisfies (i), whereas the cycle matroid of a twisted wheel satisfies (iii). Evidently, in cases (i), (ii), and (iii), every fan of M is of type-1, type-2, or type-3, respectively. Accordingly, in these three cases, we shall refer to M itself as being of *type*-1, *type*-2, or *type*-3. Clearly the class of type-2 matroids coincides with the class of duals of type-1 matroids, so it will suffice to specify the matroids of type-1 and those of type-3.

To see how to construct all type-1 matroids, it is instructive to consider a geometric construction for the cycle matroid of a multidimensional wheel. Begin with a 3–point line $\{x, y, z\}$ and k wheels for some $k \geq 3$. Let $\{x, y, z\}$ also label a triangle in each of these wheels with x and y being spokes. Attach the wheels to the line, one at a time, via generalized parallel connection. Finally, delete the element z to obtain the desired matroid.

THEOREM 9.2. *The class of* 3*–connected matroids that have exactly two non-essential elements each of which is deletable coincides with the class of matroids* M *that are constructed as follows.*

(i) *Let* L *be an* n*–point line for some* $n \geq 3$*, and* x *and* y *be two elements of* L.
(ii) *Let* $N_1, N_2, \ldots, N_k$ *be a collection of wheels of rank at least three such that* $E(L), E(N_1), E(N_2), \ldots, E(N_k)$ *are disjoint and* $k \geq 3$.
(iii) *Let* $\Delta_1, \Delta_2, \ldots, \Delta_k$ *be a collection of triangles in* L *each containing* $\{x, y\}$*, and* $\Delta'_1, \Delta'_2, \ldots, \Delta'_k$ *be triangles in* $N_1, N_2, \ldots, N_k$*, respectively.*
(iv) *For each* i *in* $\{1, 2, \ldots, k\}$*, identify the elements of* Δ'_i *with the elements of* Δ_i *so that* x *and* y *are identified with spokes of* N_i.
(v) *Let* $A_0 = L$ *and, for all* i *in* $\{1, 2, \ldots, k\}$*, let* $A_i = P_{\Delta_i}(N_i, A_{i-1})$.
(vi) *Let* $M = A_k\backslash(L - \{x, y\})$.

It should be noted here that the triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$ in the above construction need not be distinct. We also remark that if (vi) is modified so that one deletes some subset of $L - \{x, y\}$ rather than the whole set, then (i)–(vi) describe the construction for all 3–connected matroids in which every non-essential element is deletable and the set of such elements has rank two.

Like the type-1 matroids, the type-3 matroids are obtained by attaching wheels to a certain root matroid. This root matroid is again a familiar one.

LEMMA 9.3. *For some* $n \geq 3$*, let* N *be an* n*–spike with tip* y *and let* $\{x, y, z\}$ *be a triangle of* N*. Then* $N\backslash z$ *is a* 3*–connected matroid whose set of non-essential elements is contained in* $\{x, y\}$*. Indeed,* y *is deletable and* x *is contractible in* $N\backslash z$ *unless* $n = 3$ *and* $N\backslash z$ *is a wheel or a whirl.*

PROOF. Clearly each element of $E(N\backslash z) - \{x, y\}$ is in both a triangle and a triad. Moreover, y is in a triangle and x is in a triad. It is not difficult to check that $N\backslash z$ is 3–connected, and so the lemma follows. □

THEOREM 9.4. *The class of 3–connected matroids that have exactly two non-essential elements one of which is deletable and one of which is contractible coincides with the non-wheels and non-whirls that are in the class of matroids M that are constructed as follows.*

(i) *Let $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ be the triangles of an n–spike N that contain the tip y of N where $\Delta = \{y, x, z\}$ and $\Delta_i = \{y, x_i, z_i\}$ for all i.*
(ii) *Let $N_0 = N \backslash z$ and, for some $t \leq n-1$, let $N_1, N_2, \ldots, N_t$ be a collection of wheels of rank at least three such that $E(N_0), E(N_1), E(N_2), \ldots, E(N_t)$ are disjoint.*
(iii) *Let $\Delta'_1, \Delta'_2, \ldots, \Delta'_t$ be triangles in $N_1, N_2, \ldots, N_t$, respectively.*
(iv) *For each i in $\{1, 2, \ldots, t\}$, identify the elements of Δ'_i with the elements of Δ_i so that y is identified with a spoke and z_i with a rim element of N_i.*
(v) *Let $R_0 = N_0$ and, for all i in $\{1, 2, \ldots, t\}$, let $R_i = P_{\Delta_i}(N_i, R_{i-1})$.*
(vi) *Let $M = R_t \backslash z_1, z_2, \ldots, z_t$.*

By using the last result together with the modification of Theorem 9.2 discussed immediately following it, it is not difficult to deduce a description of all 3–connected matroids in which the set of non-essential elements is collinear. Moreover, in [**67**], all 3–connected *graphic* matroids with exactly three non-essential elements are determined.

10. Extremal results for 2–connected matroids

The Wheels and Whirls Theorem characterizes the 3–connected matroids M that are extremal in the sense that, for every element e, neither $M \backslash e$ nor M/e is 3–connected. As we have seen, this result is particularly useful in the development of matroid structure theory. It is one of a number of extremal connectivity results that are not only interesting in their own right but have also been used as valuable tools in other areas of matroid theory. In this section and the next, we review such results. Since so many of the results for 3–connected matroids mimic corresponding results for 2–connected matroids, our discussion in this section will focus on the latter results. The matroid results obtained here were strongly influenced by a well-established body of extremal connectivity results for graphs that includes work of Dirac [**26**], Plummer [**68**], Halin [**29, 30**], and Mader [**42, 43, 44**].

One of the most powerful tools in induction arguments for 2–connected matroids is the following result of Tutte [**99**].

THEOREM 10.1. *Let e be an element of a 2–connected matroid M. Then $M \backslash e$ or M/e is 2–connected.*

The following useful extension of this result was proved independently by Brylawski [**14**] and Seymour [**78**].

THEOREM 10.2. *Let N be a 2–connected minor of a 2–connected matroid M and suppose that $e \in E(M) - E(N)$. Then $M \backslash e$ or M/e is 2–connected and has N as a minor.*

Let N be a k–connected minor of a k–connected matroid M. The Splitter Theorem told us that, when $k = 3$, provided N satisfies some very weak restrictions, we can remove elements from M one at a time in *some* order staying k–connected until we arrive at an isomorphic copy of N. The last result tells us that, for $k = 2$ and an *arbitrary* ordering of the elements of $E(M) - E(N)$, we can remove these elements in the specified order staying k–connected until we arrive at N itself. The only aspect of the last process that we do not control is how each element is removed, that is, whether it is deleted or contracted.

For $n \geq 2$, an n–connected graph G is *minimally n–connected* if, for all edges e of G, the deletion $G\backslash e$ is not n–connected. One obvious way for the deletion of an edge to destroy n–connectedness is if the edge meets a degree-n vertex. In fact, a minimally n–connected graph must have many vertices of degree n. The following result was proved by Dirac [**26**] for $n = 2$, by Halin [**30**] for $n = 3$, and by Mader [**43**] in general.

THEOREM 10.3. *Let G be a minimally n–connected graph where $n \geq 2$. Then the number of degree-n vertices in G is at least*

$$\frac{(n-1)|V(G)|+2n}{2n-1}.$$

An n–connected matroid M is *minimally n–connected* if, for all elements e of M, the matroid $M\backslash e$ is not n–connected. One potential matroid analogue of the last theorem would be that a minimally n–connected matroid has a lot of n–element cocircuits. For $n \geq 4$, it is not known whether such a result is true. But, if n is 2 or 3, such a result does hold. First we describe what is known for $n = 2$. Murty [**45**], White [**105**], and Seymour [**80**] independently proved that every minimally 2–connected matroid with at least two elements has a 2–cocircuit. This result was later strengthened by Seymour [**81**] when he proved the following result.

PROPOSITION 10.4. *Let M be a 2–connected matroid having at least two elements and let C be a circuit of M such that $M\backslash e$ is not 2–connected for all e in C. Then C contains some 2–cocircuit of M.*

A slight improvement on this result was obtained by Oxley [**51**].

PROPOSITION 10.5. *Let M be a 2–connected matroid having at least two elements. Let f be an element of a circuit C of M such that $M\backslash e$ is not 2–connected for all e in $C - f$. Then $C - f$ contains a 2–cocircuit of M.*

The last result was used to prove the following:

THEOREM 10.6. *Let M be a 2–connected matroid other than a single circuit. Suppose that $A \subseteq E(M)$ and $M\backslash a$ is not 2–connected for all a in A. Then either A is independent in M, or A contains at least $|A| - r(A) + 1$ non-trivial series classes of M.*

As a consequence of this, we are able to show that a minimally 2–connected matroid has a lot of 2–cocircuits [**51**].

COROLLARY 10.7. *Let M be a minimally 2–connected matroid. Then either M is a circuit, or M has at least $r^*(M) + 1$ non-trivial series classes and so has at least $r^*(M) + 1$ pairwise disjoint 2–cocircuits.*

The minimally 2–connected matroids M for which the number of 2–cocircuits is exactly $r^*(M) + 1$ were determined in [**52**]. While the last result maintains the spirit of Theorem 10.3 in the case $n = 2$, the bound obtained is not analogous to the graph bound. The fact that the analogous bound does hold was proved by a different method in [**53**].

THEOREM 10.8. *Let M be a minimally 2-connected matroid having at least four elements. Then the number of pairwise disjoint 2-cocircuits of M is at least $\frac{1}{3}(r(M)+2)$.*

The matroids attaining equality in this theorem were determined in [**52**]. The last result is a matroid theorem that was motivated by a graph result. The next two results [**51**] are graph analogues of two matroid results, Theorem 10.6 and Corollary 10.7. The number of connected components of a graph G is denoted by $\omega(G)$.

THEOREM 10.9. *Let G be a 2-connected loopless graph other than a cycle. Suppose that A is a set of edges of G such that $G\backslash a$ is not 2-connected for all a in A. Then either A is a forest, or $V(A)$ contains at least $|A|-|V(A)|+\omega(G[A])+1$ pairwise non-adjacent vertices having degree two in G.*

COROLLARY 10.10. *A minimally 2-connected graph G having at least four edges has at least $|E(G)|-|V(G)|+2$ pairwise non-adjacent vertices of degree two.*

By combining the last result with Theorem 10.3 in the case $n=2$, one obtains the following result [**51**] after a little additional argument.

THEOREM 10.11. *Let G be a minimally 2-connected graph with at least four edges. Then the number ν_2 of degree-two vertices in G satisfies*

$$\nu_2 \geq \begin{cases} \frac{1}{3}(|V(G)|+5) & \text{for } |E(G)| < \frac{1}{3}(4|V(G)|-2); \\ |E(G)|-|V(G)|+2 & \text{for } \frac{1}{3}(4|V(G)|-2) \leq |E(G)|. \end{cases}$$

The next result [**53**] is obtained by combining Corollary 10.7 and Theorem 10.8.

THEOREM 10.12. *Let M be a minimally 2-connected matroid. Then the number d_2^* of pairwise disjoint 2-cocircuits in M satisfies*

$$d_2^* \geq \begin{cases} \frac{1}{3}(r(M)+2) & \text{for } |E(M)| < \frac{1}{3}(4r(M)-1); \\ r^*(M)+1 & \text{for } \frac{1}{3}(4r(M)-1) \leq |E(M)|. \end{cases}$$

Brylawski [**13**] showed that if a single-element deletion of a 2–connected matroid M is not 2–connected, then M can be written as a series connection of two of its minors. Building on this, we have the following decomposition result [**51**] for minimally 2–connected matroids.

THEOREM 10.13. *A matroid M is minimally 2-connected if and only if M has at least three elements, and either M is 2-connected having every element in a 2-cocircuit, or $M = S((M_1/q_1;p_1),(M_2/q_2;p_2))$ where M_1 and M_2 are minimally 2-connected matroids each of which is isomorphic to a minor of M and has at least five elements, and $\{p_1,q_1\}$ and $\{p_2,q_2\}$ are 2-cocircuits of M_1 and M_2, respectively.*

As one of several applications of the last theorem, we note that it can be used to prove the following result of Murty [**45**] .

COROLLARY 10.14. *For $r \geq 3$, a minimally 2-connected matroid M of rank r has at most $2r-2$ elements, the upper bound being attained if and only if $M \cong M(K_{2,r-1})$.*

The next result is another extremal connectivity result of Seymour [**80**]. He used it as a tool in his proof of the excluded-minor characterization of the class of ternary matroids.

PROPOSITION 10.15. *$U_{2,4}$ is the only 2–connected matroid with more than one element in which no 2–element deletion and no 2–element contraction is 2–connected, but every 1–element deletion and every 1–element contraction is 2–connected.*

Another extremal connectivity lemma is used in Kahn and Seymour's short proof of the excluded-minor theorem for ternary matroids (see [**62**],10.2.4). In [**53**], minor-minimally-connected matroids are considered, these being those 2–connected matroids M such that, for all elements e, either $M\backslash e$ or M/e is not 2–connected. A characterization of such matroids similar to Theorem 10.13 is proved and it is shown that every such matroid must contain a number of 2–element sets that are circuits or cocircuits.

11. Extremal results for 3–connected matroids

In this section, we turn our attention to 3–connected matroids with much of our focus being on which of the results for 2–connected matroids noted in the last section can be generalized. Some examples of extremal connectivity results for n–connected matroids for arbitrary values of n may be found in [**3, 4**].

Although Theorem 10.1 certainly fails if one replaces "2–connected" by "3–connected", the following useful analogue of the theorem was proved by Bixby [**10**].

THEOREM 11.1. *Let e be an element of a 3–connected matroid M. Then $M\backslash e$ or M/e is 3–connected.*

It is well known that if X and Y are subsets of the ground set of a matroid M and both $M|X$ and $M|Y$ are 2–connected, then, provided $X \cap Y$ is non-empty, $M|(X \cup Y)$ is also 2–connected. The following useful generalization of this fact was proved by Oxley and Wu [**64**].

THEOREM 11.2. *Let n be an integer exceeding one and M be a matroid having no circuits with fewer than n elements. If $M|X$ and $M|Y$ are n–connected and the closures of X and Y have at least $n-1$ common elements, then $M|(X \cup Y)$ is n–connected.*

Halin [**30**] made crucial use of the next lemma in his proof that a minimally 3–connected graph G has at least $\frac{1}{5}(2|V(G)|+6)$ vertices of degree three.

LEMMA 11.3. *Every cycle in a minimally 3–connected graph meets at least two vertices of degree three.*

By using the fact that the minimal sets meeting every cycle in a graph G are the cobases of $M(G)$, one can also use this lemma to obtain a second bound on the number of degree-3 vertices in a minimally 3–connected graph. The next result combines these two bounds, identifies the intervals on which each is sharper, and slightly improves Halin's bound on the specified interval.

THEOREM 11.4. *Let G be a minimally 3–connected graph. Then the number ν_3 of degree-three vertices in G satisfies*

$$\nu_3 \geq \begin{cases} \frac{1}{5}(2|V(G)|+7) & \text{for } |E(G)| < \frac{1}{5}(9|V(G)|-3); \\[2ex] \frac{1}{2}(|E(G)|-|V(G)|+3) & \text{for } \frac{1}{5}(9|V(G)|-3) \leq |E(G)|. \end{cases}$$

The following matroid analogue of Lemma 11.3 was proved by Oxley [**51**].

LEMMA 11.5. *Let C be a circuit in a minimally 3–connected matroid M where $|E(M)| \geq 4$. Then M has at least two distinct triads meeting C.*

Using this, one can prove the following analogue of the second bound in Theorem 11.4.

THEOREM 11.6. *A minimally 3–connected matroid M with at least four elements has at least $\frac{1}{2}r^*(M) + 1$ triads.*

A 3–connected matroid M for which every 2–element deletion fails to be 2–connected is easily seen to be minimally 3–connected as long as $|E(M)| \geq 5$. The last result guarantees that such a matroid has many triads but, as Akkari and Oxley [**3**] showed, one can say considerably more.

THEOREM 11.7. *The following statements are equivalent for a matroid M having at least four elements.*

(i) *M is 3–connected and no 2–element deletion from M is 2–connected.*
(ii) *Every pair of elements of M is in a triad.*
(iii) *M and all its 1–element deletions are 2–connected but no 2–element deletion from M is 2–connected.*

Proposition 10.15 follows easily by combining the last result with its dual.

The first bound on ν_3 in Theorem 11.4 and the corresponding results for minimally 2–connected graphs suggest that one may be able to show that a minimally 3–connected matroid M has at least $\frac{2}{5}r(M) + c$ triads for some constant c. However, John Leo (private communication) has found an infinite family of minimally 3–connected non-binary matroids M each of which has only $\frac{1}{4}(r(M) + 6)$ triads.

The next theorem, which was proved by Lemos [**39**], answers a question of Oxley [**51**]. It is a generalization of Proposition 10.5 and shows that the conclusion to Lemma 11.5 holds under a weaker hypothesis.

THEOREM 11.8. *Let C be a circuit in a 3–connected matroid M such that, for all e in C, the matroid $M\backslash e$ is not 3–connected. Then C meets at least two triads of M.*

The last result played a crucial role in the proofs of Theorems 4.8 and 4.9. Using it, one can show that a 3–connected matroid with a k–element circuit has a 3–connected minor of rank at least $k - 1$ that has a spanning circuit. Lemos also noted that the following graph-theoretic analogue of his result is true, and this graph result was also independently proved by Mader [**44**].

THEOREM 11.9. *Let C be a cycle of a simple 3–connected graph G. If $G\backslash e$ is not 3–connected for all e in C, then C meets at least two degree-3 vertices of G.*

Leo [**40**] noted that by combining Theorem 11.8 with the proof technique used to give Theorem 10.6, one can obtain the following analogue of that result.

THEOREM 11.10. *Let M be a 3–connected matroid. Suppose that $A \subseteq E(M)$ and $M\backslash a$ is not 3–connected for all a in A. Then either A is independent, or A meets at least $\frac{1}{2}(|A| - r(A)) + 1$ distinct triads of M.*

Theorems 11.7 and 11.8 suggest consideration of those 3–connected matroids having a circuit such that the deletion of any pair of elements from this circuit produces a matroid that is not 2–connected. The following attractive generalization of Theorem 11.7 was proved by Akkari [**2**].

THEOREM 11.11. *Let C be a circuit in a 3–connected matroid M such that, for all pairs $\{e, f\}$ of distinct elements of C, the matroid $M\backslash e, f$ is not 2–connected. Then either every pair of elements of C is in a triad, or M is a wheel of rank at least four and C is its rim.*

In view of Proposition 10.5 for 2–connected matroids, it is natural to ask whether the corresponding result is true for 3–connected matroids. Leo [**40**] answered this question affirmatively.

THEOREM 11.12. *Let C be a circuit in a 3–connected matroid M and f be an element of C. If $M\backslash e$ is not 3–connected for all e in $C - f$, then M has a triad meeting $C - f$.*

Halin [**29**] proved the following upper bound on the number of edges in a minimally 3–connected graph.

THEOREM 11.13. *Let G be a minimally 3–connected graph. Then*

$$|E(G)| \leq \begin{cases} 2|V(G)| - 2 & \text{if } |V(G)| \leq 6; \\ 3|V(G)| - 9 & \text{if } |V(G)| \geq 7. \end{cases}$$

Moreover, the only graphs attaining equality in these bounds are $\mathcal{W}_m$ for $3 \leq m \leq 6$ and $K_{3,n}$ for $n \geq 4$.

The next result [**49**] shows that precisely the same bounds hold for arbitrary minimally 3–connected matroids.

THEOREM 11.14. *Let M be a minimally 3–connected matroid having at least four elements. Then*

$$|E(M)| \leq \begin{cases} 2r(M) & \text{if } r(M) \leq 5; \\ 3r(M) - 6 & \text{if } r(M) \geq 6. \end{cases}$$

A characterization of the matroids that attain equality in these bounds is given in [**49**]. The only binary matroids attaining equality are the cycle matroids of the graphs that attain equality in the bounds in Theorem 11.13.

In Section 9, we discussed the 3–connected matroids with a small number of non-essential elements. The following result of Wu [**109**] gives interesting information about how the non-essential elements are arranged in a minimally 3–connected matroid.

THEOREM 11.15. *Let M be a minimally 3–connected matroid that is not a wheel or a whirl. Then every largest circuit of M contains a non-essential element of M.*

12. Vertical connectivity

It was noted in Section 2 that the notions of n–connectedness for graphs and matroids, while similar, do not coincide precisely. One difference lies in the fact that, whereas a circuit of size less than n does not prevent a graph from being n–connected, it does prevent its cycle matroid from being n–connected unless the graph has fewer than $2n - 2$ edges. In this section, we shall see that this is the fundamental difference between the graph and matroid concepts. We shall present a slight modification of the definition of n–connectedness which produces a matroid notion that exactly generalizes the graph concept. The cost of making this modification is that one loses invariance under duality with this alternate concept.

For a positive integer k and a matroid M, a partition $\{X, Y\}$ of $E(M)$ is a *vertical k–separation* if

$$min\{r(X), r(Y)\} \geq k;$$

and

$$r(X) + r(Y) - r(M) \leq k - 1.$$

For $2 \leq n \leq r(M)$, the matroid M is *vertically n–connected* provided that, for all k in $\{1, 2, \ldots, n-1\}$, M has no vertical k–separation. Hence M is vertically 2–connected exactly when the matroid obtained by deleting all loops from M is 2–connected. Here "vertical" is used as the adjective corresponding to "vertex". This usage, which was originated by Tutte [**100**], is justified by the following result [**21, 33, 48**].

THEOREM 12.1. *Let G be a connected graph and n be an integer exceeding one. Then G is an n–connected graph if and only if $M(G)$ is a vertically n–connected matroid.*

The next result, a generalization of Proposition 2.1, describes the link between vertical n–connectedness and n–connectedness as defined in Section 2. The latter concept is sometimes called *Tutte n–connectedness.*

THEOREM 12.2. *Let M be a matroid that is not isomorphic to any uniform matroid $U_{r,m}$ with $m \geq 2r - 1$. For all integers n exceeding one, M is n–connected if and only if M is vertically n–connected and has no circuits with fewer than n elements.*

If the dual of a matroid is vertically n–connected, then the matroid itself is called *cyclically n–connected.* The next result notes that Tutte n–connectedness is basically the conjunction of vertical n–connectedness and cyclic n–connectedness.

PROPOSITION 12.3. *Let M be a matroid that is not isomorphic to any uniform matroid $U_{r,m}$ with $2r - 1 \leq m \leq 2r + 1$. For all integers n exceeding one, M is n–connected if and only if M is both vertically and cyclically n–connected.*

Although a wheel $M(\mathcal{W}_r)$ with $r \geq 4$ has no element e such that $M(\mathcal{W}_r)/e$ is 3–connected, for every rim edge f of $\mathcal{W}_r$, the matroid $M(\mathcal{W}_r)/f$ is vertically 3–connected. In general, Cunningham [**21**], and independently Seymour, proved the following result.

PROPOSITION 12.4. *Let M be a non-empty 3–connected matroid. Then M has an element e such that M/e is vertically 3–connected.*

If M is the cycle matroid of a 3–connected graph G, then, by Theorem 12.1, M is vertically 3–connected. Moreover, for every edge e of G, the matroid M/e is vertically 3–connected if and only if the graph G/e is 3–connected. An edge x in a 3–connected graph G is *contractible* if G/x remains 3–connected. Several papers over the last decade (see, for example, [**6, 27, 23, 5**]) have studied the number of contractible edges in 3–connected graphs. Recently Wu [**109**] considered the corresponding problem for matroids. An element x in a 3–connected matroid M is *vertically contractible* if M/x is vertically 3–connected. Proposition 12.4 asserts

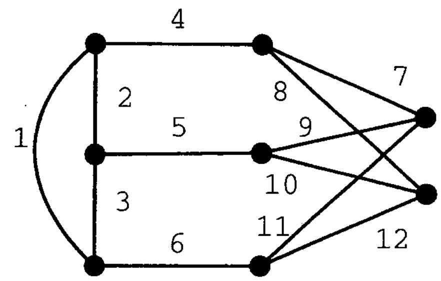

FIGURE 22. G.

that every non-empty 3–connected matroid has at least one vertically contractible element. Wu [**109**] sharpened this result.

PROPOSITION 12.5. *Every 3-connected matroid with at least three elements has at least three vertically contractible elements.*

Wu [**109**] deduced the last result from the following theorem. It is interesting to observe the presence of a familiar class of matroids in this result.

THEOREM 12.6. *Let M be a minimally 3-connected matroid with at least four elements. Then M has at least $max\{\frac{2}{3}|E(M)| - r(M) + 2, 3\}$ vertically contractible elements. Moreover, M has exactly three vertically contractible elements if and only $M \cong M^*(K'''_{3,k})$ for some $k \geq 2$.*

13. Isomorphism versus equality, and roundedness

Let M be the cycle matroid of the graph G in Figure 22, and let $M/\{1,2,3\} = N$. Evidently N is a 3–connected minor of M, which is also 3–connected. The Splitter Theorem guarantees the existence of a sequence M_0, M_1, M_2, M_3 of 3–connected matroids each a single-element deletion or contraction of its predecessor such that $M_0 = M$ and $M_3 \cong N$. Indeed, such a sequence is $M, M/5, M/5/4, M/5/4\backslash 2$. The point that we wish to note here is that $M/5/4\backslash 2$, while it is *isomorphic* to N is not *equal* to N. Moreover, the reader can easily check that there is no sequence M_0, M_1, M_2, M_3 of the required type in which $M_3 = N$. In fact, M has no proper 3–connected minor that has N itself as a proper minor.

One is often interested in maintaining a specific minor rather than just a copy of that minor. Suppose that N_1 is a k–connected minor of a k–connected matroid M_1. If we seek a k–connected minor N_2 of M_1 that has N_1 as a proper minor so that the gap, $|E(N_2) - E(N_1)|$, is as small as possible, then, when $k = 2$, Theorem 10.2 guarantees that N_2 can be found so that $|E(N_2) - E(N_1)| = 1$. Rajan [**69**] gave a family of examples to show that, when $k = 4$, arbitrarily large gaps exist between N_1 and a next largest k–connected minor N_2 of M_1 having N_1 as a minor. But Truemper [**87**] showed that, when $k = 3$, this gap has size at most three. Truemper's result was strengthened slightly by Bixby and Coullard [**11**] who proved the following result.

THEOREM 13.1. *Let N be a 3-connected proper minor of a 3-connected matroid M. Then M has a 3-connected minor M_1 and an element e such that N is a cosimple matroid associated with $M_1\backslash e$ or a simple matroid associated with M_1/e, and $|E(M_1) - E(N)| \leq 3$.*

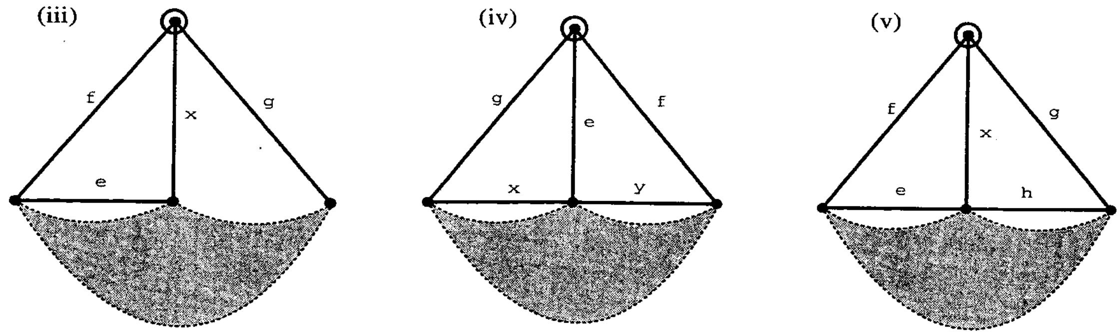

FIGURE 23. A graphic depiction of (iii)–(v) of Theorem 11.2.

Bixby and Coullard [**11**] strengthened the last result when N has no circuits or cocircuits with fewer than four elements. The details of this and many of the other results considered in this section may be found in Section 11.3 of [**62**]. Another very useful result of Bixby and Coullard [**12**] considers a variant of Theorem 13.1 in which one seeks a 3–connected minor M_1 of M that not only has N as a minor but also contains some nominated element e of $E(M) - E(N)$. They show that such a matroid M_1 can be found so that $|E(M_1) - E(N)| \leq 4$. Moreover, their result also contains much very helpful structural information.

THEOREM 13.2. *Let N be a 3–connected minor of a 3–connected matroid M. Suppose that $|E(N)| \geq 4$, $e \in E(M) - E(N)$, and M has no 3–connected proper minor that both uses e and has N as a minor. Then, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of the following holds where $|E(M) - E(N)| = n$.*

(i) *$n = 1$ and $N_1 = M_1\backslash e$.*
(ii) *$n = 2$, $N_1 = M_1\backslash e/f$, and N_1 has an element x such that $\{e, f, x\}$ is a triangle of M_1.*
(iii) *$n = 3$, $N_1 = M_1\backslash e, g/f$, and N_1 has an element x such that $\{e, f, x\}$ is a triangle of M_1 and $\{f, g, x\}$ is a triad of M_1. Moreover, $M_1\backslash e$ is 3–connected.*
(iv) *$n = 3$, $N_1 = M_1\backslash e, g/f = M_1\backslash e, f/g = M_1\backslash f, g/e = M_1\backslash e, f, g$, and $\{e, f, g\}$ is a triad of M_1. Moreover, N_1 has distinct elements x and y such that $\{e, g, x\}$ and $\{e, f, y\}$ are triangles of M_1.*
(v) *$n = 4$, $N_1 = M_1\backslash e, g/f, h$ and N_1 has an element x such that $\{e, f, x\}$ and $\{g, h, x\}$ are triangles of M_1, and $\{f, g, x\}$ is a triad of M_1. Moreover, $M_1\backslash e$ and $M_1\backslash e/f$ are 3–connected.*

Although the last result applies to all matroids and not just graphic ones, Bixby and Coullard use graphs to depict what happens in (iii)–(v) (see Figure 23). Note that a vertex is circled if it corresponds to a known triad in the matroid; all cycles shown are indeed circuits of the matroid; and the shaded part of the diagram corresponds to the rest of the matroid.

Theorem 13.2 has a number of applications that relate to what is called "roundedness" in matroid theory. This subject is concerned with relating certain minors of a matroid to particular elements of the matroid. An example of one of the many such results is the following theorem of Seymour [**83**]. This result, which extends an earlier result of Bixby [**9**] for 2–connected matroids, played an important role in the proof of Kahn's theorem [**34**] that determines precisely when a quaternary matroid is uniquely $GF(4)$–representable.

THEOREM 13.3. *Let M be a 3–connected matroid having a $U_{2,4}$-minor and suppose that e and f are distinct elements of M. Then M has a $U_{2,4}$-minor using $\{e, f\}$.*

The last result, a statement about 3–connected non-binary matroids, was extended to non-graphic matroids by Asano, Nishizeki, and Seymour [**7**] when they proved the following result, a further extension of which was later obtained by Reid [**72**].

THEOREM 13.4. *Let T be a triangle in a 3–connected non-graphic matroid M. Then M has a minor N using T such that N is isomorphic to*

(i) *$M^*(K_{3,3})$ if M is regular;*
(ii) *F_7 if M is binary and non-regular; and*
(iii) *$U_{2,4}$ if M is non-binary.*

A more recent result of the same type is the following theorem [**63**]. The matroid F_7^+ is obtained from the Fano matroid by freely adding an element to one of the lines.

THEOREM 13.5. *Let M be a 3–connected matroid having a $U_{2,5}$-minor and a subset X such that $M|X \cong U_{2,4}$. Then M has a minor N using X such that N is isomorphic to $U_{2,5}$ or F_7^+.*

This theorem is of crucial importance in Oxley, Vertigan, and Whittle's [**63**] proof that, when $q = 5$, a 3–connected $GF(q)$–representable matroid has a bounded number of inequivalent $GF(q)$–representations. The best-possible bound here is six since, for instance, $U_{3,5}$ has six inequivalent $GF(5)$–representations. The existence of inequivalent representations is a major difficulty that arises when dealing with matroid representations. The above result verifies a conjecture of Kahn [**34**] in the case $q = 5$. Kahn had conjectured that the same result holds for all prime powers q, but examples in [**63**] show that this conjecture is false for all $q > 5$.

A 3–connected matroid M is *internally 4–connected* if $min\{|X|, |Y|\} = 3$ for every 3–separation $\{X, Y\}$ of M. Loosely speaking, such a matroid M is 4–connected except that it may have triangles and triads. The following structural result of Tseng and Truemper [**96**] can be used to prove a matroid extension of the edge form of Menger's Theorem, which was originally derived by Seymour [**79**]. A formal statement of the last result and its relationship to Menger's Theorem are described in [**84**] and Section 11.3 of [**62**]. Tseng and Truemper's result is stated here as an example of another matroid structural result that has interesting consequences elsewhere.

THEOREM 13.6. *Let e be an element of a 3–connected, internally 4–connected binary matroid M and suppose that $e \in E(M)$. Then exactly one of the following holds.*

(i) *There is an F_7^*-minor of M using e.*
(ii) *M is regular.*
(iii) *$M \cong F_7$.*

To conclude this section, we note yet another connectivity result that has played a vital role in the proof of a result from another area of matroid theory. The following result is due to Whittle [**107**] and is of central importance in the proofs of his very attractive recent theorems [**107, 108**] that characterize the ternary matroids that are representable over some other field. The result has a more intricate hypothesis than most of the connectivity results noted earlier, so it is probably not surprising that the proof is very difficult.

THEOREM 13.7. *Let M be a 3–connected non-binary matroid having rank at least four. Then M has an independent set $\{a, b, c\}$ such that the simplifications of all of the matroids $M/a, M/b, M/c, M/a, b$, and $M/a, c$ are 3–connected and non-binary.*

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