Ternary paving matroids

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Abstract

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Acketa has determined all binary paving matroids. This paper specifies all ternary paving matroids. There are precisely four minor-maximal 3-connected such matroids: $S(5, 6, 12)$, $PG(2, 3)$, the real affine cube, and one other 8-element self-dual matroid.

1. Introduction

A *paving matroid* is a matroid in which no circuit has size less than the rank of the matroid. Acketa [1] has determined precisely which binary matroids are paving. The purpose of this paper is to solve the corresponding problem for ternary matroids. The solution presented to this problem is in two parts. First, in Section 2, all paving matroids that are not 3-connected are determined and this information is used to specify all such ternary matroids. Then, in Section 3, all 3–connected ternary paving matroids are characterized. The technique used there is the same as was used to prove the main result of [6]. The preliminaries needed to justify this technique were presented in the introduction of [6] and will not be repeated here.

Most of the matroid terminology used here will follow Welsh [9]. We remark, however, that our definition of a paving matroid follows Acketa [1] and differs slightly from that of Welsh in that he also requires such matroids to have rank at least two. The ground set and rank of the matroid M will be denoted by $E(M)$ and $r(M)$, respectively. If $T \subseteq E(M)$, we shall denote the deletion of T from M by $M\backslash T$ or $M \mid (E(M) - T)$, and the contraction of T from M by M/T. Flats of M of ranks one, two, and three will be called *points*, *lines*, and *planes*. An *n-circuit* of M is a circuit having n elements. If $e \in E(M)$, we call M an *extension* of $M\backslash e$ and a *coextension* of M/e. For matroids M_1 and M_2 whose ground sets have exactly one common element, $P(M_1, M_2)$ will denote their parallel connection and p will

denote the basepoint of the connection. We shall assume familiarity with the basic properties of this operation as discussed in [3].

Next we introduce some particular matroids that will play an important role in this paper. The well-known Steiner system $S(5, 6, 12)$ gives rise to a matroid on the twelve elements of the system, the hyperplanes of which are the blocks of the system. This matroid, which we shall also denote by $S(5, 6, 12)$, is discussed in some detail in [6]. As noted there, the matroid $S(5, 6, 12)$ is ternary, identically self-dual, and has a 5-transitive automorphism group.

We shall denote by T_8 and R_8 the matroids that are represented by the following matrices over GF(3):

$$
\left[\quad I_4 \quad \left| \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \right. \right],
\qquad
\begin{matrix} 1 & 2 & 3 & 4 & & 5 & 6 & 7 & 8 \\ & & & & & & & & \end{matrix}
\left[\quad I_4 \quad \left| \begin{matrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{matrix} \right. \right].
$$

Evidently both these matroids are isomorphic to their duals. Moreover, R_8 has the real affine cube as its Euclidean representation, the labelling being as in Fig. 1 with the planes in R_8 being the six faces of the cube together with the six diagonal planes such as $\{1, 2, 7, 8\}$. It is not difficult to check that R_8 is representable over a field F if and only if the characteristic of F is not two. On the other hand, T_8 is representable over a field F if and only if F has characteristic three. Indeed, one can show that T_8 is a minor-minimal matroid not representable over F for all fields F whose characteristic is not two or three. R_8 has a transitive automorphism group, every single-element contraction being isomorphic to the non-Fano matroid, F_7^-. In contrast, the automorphism group of T_8 has the two obvious orbits.

The next result, which will be proved in Section 3, contains the most difficult part of the characterization of ternary paving matroids. From it and Corollary 2.3 at the end of the next section, one can easily determine all such matroids.

Theorem 1.1. *The 3-connected ternary paving matroids are precisely the 3-connected minors of* $\mathrm{PG}(2, 3)$, $S(5, 6, 12)$, R_8, *and* T_8.
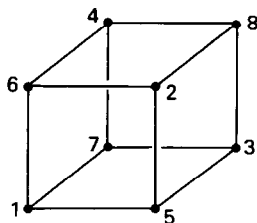
Fig. 1.

We close this section with an elementary characterization of paving matroids which will be useful in the proofs of the main results. The straightforward proof of the next lemma is omitted.

Lemma 1.2. *Every minor of a paving matroid is a paving matroid.*

Proposition 1.3. *A matroid is paving if and only if it has no minor isomorphic to $U_{2,2} \oplus U_{0,1}$.*

Proof. As $U_{2,2} \oplus U_{0,1}$ has a dependent subset of size less than its rank, it is not paving. Thus, by Lemma 1.2, if a matroid is paving, it has no minor isomorphic to $U_{2,2} \oplus U_{0,1}$. For the converse, suppose that M is a rank-r matroid that is not a paving matroid. Then M has a circuit C of size less than r. Thus $r(M/C) \geq 2$. Let $\{a, b\}$ be independent in M/C and c be an element of C. Then it is straightforward to check that $[M \mid (C \cup \{a, b\})]/(C - c) \cong U_{2,2} \oplus U_{0,1}$. $\square$

2. Paving matroids that are not 3-connected

In this section we determine all paving matroids that are not 3-connected and then use this to specify which such matroids are ternary.

We begin by listing all disconnected paving matroids. The elementary proof of this result is omitted.

Proposition 2.1. *The following is a complete list of all disconnected paving matroids.*
   (i) $U_{n,n}$ *and* $U_{0,n}$ *for* $n \geq 2$;
   (ii) $U_{0,n} \oplus U_{1,n}$ *and* $U_{1,n} \oplus U_{1,m}$ *for* $n \geq 1$ *and* $m \geq 2$;
   (iii) $U_{1,1} \oplus U_{r,n}$ *for* $n \geq r + 1 \geq 3$.

Determining the paving matroids that are connected but not 3-connected is not quite as straightforward.

Proposition 2.2. *The following is a complete list of all connected paving matroids that are not 3-connected.*
   (i) $U_{1,n}$ *and* $U_{n-1,n}$ *for* $n \geq 4$;
   (ii) $P(U_{2,m}, U_{2,n})$ *and* $P(U_{2,m}, U_{2,n}) \backslash p$ *for* $m, n \geq 3$;
   (iii) *all loopless lines having at least three points, at least one of which contains more than one element*;
   (iv) *all matroids of the form* $P(M_1, U_{2,3}) \backslash p$ *where, for some* $k \geq 3$ *and some* $n \geq k + 1$, $M_1 \backslash p \cong U_{k,n}$ *and, in* M_1, p *is a noncoloop that is in no circuits of size less than* k.

Proof. It is straightforward to check that each of the matroids listed under (i)–(iv) is a connected paving matroid that is not 3-connected. To show that the list is complete, suppose that M is such a matroid. Then, as M is connected but not 3-connected, $|E(M)| \geq 4$. By [8, (2.6)], $M = P(M_1, M_2)\backslash p$ for some connected matroids M_1 and M_2 each having at least three elements and nonzero rank. Assume that $r(M_1) \geq r(M_2)$.

Suppose that $r(M_2) = 1$. Then $M_2 \cong U_{1,m}$ for some $m \geq 3$. Thus M has a 2-circuit so $r(M) \leq 2$. Hence $r(M_1)$ is 1 or 2. It is not difficult to check that, in the first case, M satisfies (i), while, in the second case, it satisfies (iii).

We may now assume that $r(M_1) \geq r(M_2) \geq 2$. Suppose also that $r(M_1) \geq 3$ and M_2 is not a circuit. Then contracting all but one element of $E(M_2\backslash p)$ from M leaves a matroid having a $(U_{2,2} \oplus U_{0,1})$-minor. Thus we may assume that $r(M_1) = 2$ or M_2 is a circuit. In the first case, $r(M_2) = 2$ and $r(M) = 3$. Thus M has no 2-circuits and so $M \cong P(U_{2,m}, U_{2,n})$ or $P(U_{2,m}, U_{2,n})\backslash p$ for some $m, n \geq 3$. Now suppose that $r(M_1) \geq 3$ and M_2 is a circuit. Then

$$r(M) = r(M_1) + |E(M_2)| - 2. \tag{1}$$

If M_1 has no circuits avoiding p, then M_1 is a circuit and $M \cong U_{n-1,n}$ for some $n \geq 4$; that is, (i) holds. Thus we may suppose that M_1 has a circuit C avoiding p. As M is paving and C is a circuit of M_1.

$$r(M) \leq |C| \leq r(M_1) + 1.$$

On combining this with (1), we deduce that

$$|E(M_2)| = 3, \qquad |C| = r(M_1) + 1, \quad \text{and} \quad r(M) = r(M_1) + 1.$$

As C was an arbitrarily chosen circuit of $M_1\backslash p$, it follows that $M_1\backslash p \cong U_{k,n}$ for some $k \geq 3$ and some $n \geq k + 1$.

Now consider the circuits of M_1 containing p, letting C' be such a circuit. Then $(C' - p) \cup (E(M_2) - p)$ is a circuit of M having $|C'| + 1$ elements. Thus $|C'| + 1 \geq r(M)$. But $r(M) = r(M_1) + 1$, so $|C'| \geq r(M_1)$. Therefore every circuit of M_1 containing p has at least k elements and, since p is not a coloop of M_1, we conclude that M satisfies (iv). $\square$

The next result is easily obtained by combining the last two propositions with the excluded-minor characterization of ternary matroids [2, 7].

Corollary 2.3. *The following is a complete list of all the ternary paving matroids that are not 3-connected.*

   (i) $U_{n,n}$, $U_{0,n}$ *for* $n \geq 2$;
   (ii) $U_{0,n} \oplus U_{1,m}$, $U_{1,n} \oplus U_{1,m}$ *for* $n \geq 1$ *and* $m \geq 2$;
   (iii) $U_{1,1} \oplus U_{2,4}$, $U_{1,1} \oplus U_{r,r+1}$ *for* $r \geq 2$;
   (iv) $U_{1,n}$, $U_{n-1,n}$ *for* $n \geq 4$;
   (v) $P(U_{2,m}, U_{2,n})$, $P(U_{2,m}, U_{2,n})\backslash p$ *for* $3 \leq m, n \leq 4$;
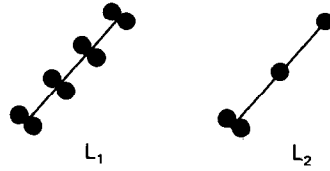
Fig. 2.

(vi) *all loopless lines having three or four points, at least one of which contains more than one element; and*

(vii) *all matroids M such that M* is a loopless line having three or four points, at least one of which contains two elements and none of which contains more than two elements.*

An alternative description of the matroids in (vii) above can be given in terms of the matroids L_1 and L_2 in Fig. 2: M^* is a minor of L_1 that has L_2 as a minor.

3. The 3-connected case

In this section, we shall prove Theorem 1.1 by using Seymour's Splitter Theorem [8, (7.3)]. In particular, we shall construct all 3-connected ternary paving matroids by building up, an element at a time, from a wheel or a whirl through a sequence of 3-connected ternary paving matroids. Much of the potential work here is eliminated by invoking the main theorem of [6], which was proved using the same technique. A key result that underlies this work is Brylawski and Lucas's theorem [4, Corollary 3.3] that ternary matroids are uniquely GF(3)-representable. This means that, when we are dealing with a ternary matroid, we lose no generality in identifying that matroid with the dependence matroid of some particular matrix representation for it.

Proof of Theorem 1.1. It is easy to see, using Proposition 1.3, that all the matroids listed are 3-connected ternary paving matroids. To show that the list is complete, we now let M be an arbitrary such matroid. Suppose first that M has no $M(K_4)$-minor. Then, by Theorem 2.1 of [6], one of the following holds:

(a) M is isomorphic to the rank-r whirl $\mathcal{W}^r$ [9, pp. 80–81] for some $r \geq 2$;

(b) M is isomorphic to a 3-connected minor of $S(5, 6, 12)$; or

(c) M is isomorphic to a certain 8-element rank-4 matroid J.

Now, for $r \geq 4$, $\mathcal{W}^r$ is not a paving matroid because it has a circuit of size less than its rank. For the same reason, J, which has a 3-circuit, is not a paving matroid. On the other hand, both $\mathcal{W}^2$ and $\mathcal{W}^3$ are isomorphic to minors of $S(5, 6, 12)$. We conclude that if M has no $M(K_4)$-minor, then it is isomorphic to a 3-connected minor of $S(5, 6, 12)$.

We may now suppose that M does have an $M(K_4)$-minor. As $M(\mathcal{W}_4)$, the rank-4 wheel, has a minor isomorphic to $U_{2,2} \oplus U_{0,1}$, M has no minor isomorphic to $M(\mathcal{W}_4)$. Thus, by the Splitter Theorem [8, (7.3)], there is a sequence $M_0, M_1, M_2, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong M(K_4)$, $M_n = M$, and, for all i in $\{1, 2, \ldots, n\}$, M_{i-1} is a single-element deletion or a single-element contraction of M_i. Since M is a ternary paving matroid, so too are all of $M_{n-1}, M_{n-2}, \ldots, M_0$.

As Brylawski and Lucas [4, p. 94] have noted, a consequence of their result that ternary matroids are uniquely representable over $GF(3)$ is that the complement of a simple ternary matroid in a ternary projective geometry is well-defined. One easily checks that the complement of $M(K_4)$ in $PG(2,3)$ is isomorphic to F_7^-. Thus $M(K_4)$ has precisely two non-isomorphic 3-connected ternary extensions: F_7^- and the matroid N_7 for which a Euclidean representation is shown in Fig. 3. Hence if M_1 is an extension of M_0, it is isomorphic to F_7^- or N_7. Moreover, as $M(K_4)$ is isomorphic to its dual, it follows from this that if M is a coextension of M_0, it is isomorphic to $(F_7^-)^*$ or N_7^*. As N_7^* has a minor isomorphic to $U_{2,2} \oplus U_{0,1}$, it is not a paving matroid. We conclude that M_1 is isomorphic to one of F_7^-, N_7, and $(F_7^-)^*$.

The next lemma enables us to severely restrict the number of possibilities for the matroids $M_2, M_3, \ldots, M_n$.

Lemma 3.1. *Let N be a 3-connected ternary paving matroid that is a coextension of a rank-3 matroid N'. Then no line of N' has more than three points.*

Proof. Let $N' = N/e$. Suppose that N' has a line that contains four distinct points, a, b, c, and d. As N has no 3-circuits, it follows that $N \mid \{a, b, c, d, e\} \cong U_{3,5}$. But this contradicts the fact that N is ternary. $\square$

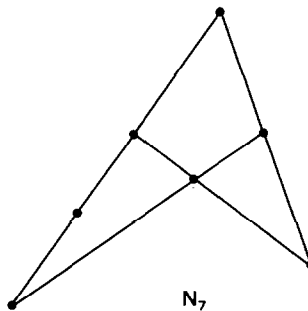By this lemma, if $M_1 \cong N_7$, then $r(M) = 3$ and M is a restriction of $PG(2,3)$.

N_7

Fig. 3.

Now assume that $M_1 \cong (F_7^-)^*$. Then M_1 is represented by the following matrix over GF(3):

$$A = \begin{bmatrix} & & & & 0 & 1 & 1 \\ & & I_4 & & 1 & 0 & 1 \\ & & & & 1 & 1 & 0 \\ & & & & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}.$$

Suppose that M_2 is an extension of M_1. To determine the different possibilities for M_2, we consider the columns that can be adjoined to A to give a matrix representing a ternary paving matroid. Since adjoining the negative of a column gives an isomorphic matroid to that obtained by adjoining the column itself, we shall not distinguish a column from its negative here.

Lemma 3.2. *Suppose that* $(x_1, x_2, x_3, x_4)^T$ *is a column that is adjoined to A to give a matrix representing a ternary paving matroid. Then* $(x_1, x_2, x_3, x_4)^T$ *is one of*

$$e_1 = (-1, -1, -1, 1)^T, \qquad e_2 = (1, 1, 1, 0)^T, \qquad e_3 = (1, 1, -1, 0)^T,$$

$$e_4 = (1, -1, 1, 0)^T, \quad and \quad e_5 = (-1, 1, 1, 0)^T.$$

Proof. M_2 is represented by the matrix

$$X = \begin{bmatrix} & & & & 0 & 1 & 1 & x_1 \\ & & I_4 & & 1 & 0 & 1 & x_2 \\ & & & & 1 & 1 & 0 & x_3 \\ & & & & 1 & 1 & 1 & x_4 \end{bmatrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix},$$

where each of x_1, x_2, x_3, and x_4 is in $\{0, 1, -1\}$. As M_2 has no circuits of size less than four, at least three of x_1, x_2, x_3, and x_4 are nonzero. Suppose first that all four of them are nonzero. Then, by column scaling, we may assume that $x_4 = 1$. We may also suppose, by symmetry, that (x_1, x_2, x_3) is one of $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$, and $(-1, -1, -1)$. In the first and second cases, $\{1, 5, 8\}$ is a circuit of M_2; a contradiction. In the third case, one easily checks that $M_2/1$ has a 4-point line; a contradiction to Lemma 3.1. Hence if $x_4 \neq 0$, then $(x_1, x_2, x_3, x_4)^T = (-1, -1, -1, 1)^T = e_1$.

Now suppose that one of x_1, x_2, x_3, and x_4 is zero. In the first three cases, it is easy to see that M_2 has a circuit of size at most three. Thus we may assume that $x_4 = 0$. Then $(x_1, x_2, x_3, x_4)^T$ is one of e_2, e_3, e_4, and e_5. $\square$

For i in $\{1, 2, 3, 4, 5\}$, let $(F_7^-)^* + e_i$ be the matroid that is represented by letting 8 equal e_i in the matrix X. Evidently

$$(F_7^-)^* + e_2 \cong T_8 \quad \text{and} \quad (F_7^-)^* + e_3 \cong (F_7^-)^* + e_4 \cong (F_7^-)^* + e_5.$$

Moreover, we have the following.

Lemma 3.3. $(F_7^-)^* + e_2 \cong (F_7^-)^* + e_3$ and $(F_7^-)^* + e_1 \cong R_8$.

Proof. Let $A + e_i$ denote the matrix obtained from A by adjoining column e_i. On pivoting on the last entry of column 7 of $A + e_2$ and then swapping columns 4 and 7, we get the matrix

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 7 & 5 & 6 & 4 & e_2
\end{array}
$$
$$
\left[
\begin{array}{cccc|cccc}
 & & & & -1 & 0 & -1 & 1 \\
 & \multicolumn{1}{c}{I_4} & & & 0 & -1 & -1 & 1 \\
 & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 1 & 0
\end{array}
\right].
$$

Multiplying rows 1 and 2 by -1, and then multiplying columns 1, 2, and 8 by -1 and swapping columns 5 and 6 gives the matrix $A + e_3$ with its columns relabelled. Hence $(F_7^-)^* + e_3 \cong (F_7^-)^* + e_2 \cong T_8$.

Now take $A + e_1$ and pivot on the entry in the bottom right corner and then interchange columns 4 and 8 to get the matrix

$$
\begin{array}{cccccccc}
1 & 2 & 3 & e_1 & 5 & 6 & 7 & 4
\end{array}
$$
$$
\left[
\begin{array}{cccc|cccc}
 & & & & 1 & -1 & -1 & 1 \\
 & \multicolumn{1}{c}{I_4} & & & -1 & 1 & -1 & 1 \\
 & & & & -1 & -1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & 1
\end{array}
\right].
$$

On multiplying row 2 by -1 and then columns 2 and 6 by -1, we get the matrix

$$
\left[
\begin{array}{cccc|cccc}
 & & & & 1 & 1 & -1 & 1 \\
 & \multicolumn{1}{c}{I_4} & & & 1 & 1 & 1 & -1 \\
 & & & & -1 & 1 & 1 & 1 \\
 & & & & 1 & -1 & 1 & 1
\end{array}
\right]
$$

and this clearly represents a matroid isomorphic to R_8. $\square$

Lemma 3.4. Neither $(F_7^-)^* + e_1$ nor $(F_7^-)^* + e_2$ has an extension that is a ternary paving matroid.

Proof. By Lemma 3.2, an extension of $(F_7^-)^* + e_1$ that is a ternary paving matroid must be represented by $A + e_1 + e_2$ or $A + e_1 + e_i$ for some i in $\{3, 4, 5\}$. In the first case, we get a 3-circuit and so there is no such paving matroid. In the second

case, on contracting the element 4 from the resulting matroid N, we get a rank-3 matroid with a 4-point line and so, by Lemma 3.1, N is not a paving matroid. Hence $(F_7^-)^* + e_1$ has no extension that is a ternary paving matroid. A similar argument establishes the same result for $(F_7^-)^* + e_2$. $\square$

In follows immediately from the next lemma that neither $(F_7^-)^* + e_1$ nor $(F_7^-)^* + e_2$ has a coextension that is a ternary paving matroid.

Lemma 3.5. $(F_7^-)^*$ *has no coextension that is a ternary paving matroid.*

Proof. Assume that N_1 *is such a coextension of* $(F_7^-)^*$. Then N_1 can be represented by the matrix

$$\begin{bmatrix} & & | & 0 & 1 & 1 \\ & & | & 1 & 0 & 1 \\ & I_5 & | & 1 & 1 & 0 \\ & & | & 1 & 1 & 1 \\ & & | & y_1 & y_2 & y_3 \end{bmatrix},$$

where each of y_1, y_2, and y_3 is in $\{0, 1, -1\}$. As N_1 has no 4-circuits, none of y_1, y_2, and y_3 is 0. Thus two of y_1, y_2, and y_3 are equal. Two columns containing such an equal pair have three pairs of equal coordinates and so these two columns are contained in a 4-circuit; a contradiction. $\square$

On combining the last three lemmas, we deduce that if $M_1 \cong (F_7^-)^*$, then M is isomorphic to one of $(F_7^-)^*$, T_8, or R_8. It now remains only to check the case when $M_1 \cong F_7^-$. In that case, if M_2 is an extension of M_1, then, as $M(K_4)$ is the complement in $PG(2,3)$ of F_7^-, M_2 is the complement in $PG(2,3)$ of the ternary affine matroid $P(U_{2,3}, U_{2,3})$ and therefore M_2 contains a 4-point line. Thus, by Lemma 3.1, $r(M) = 3$ and M is a restriction of $PG(2,3)$. Hence we may assume that if $M_1 \cong F_7^-$, then M_2 is a coextension of M_1. The next lemma completes the proof of Theorem 1.1.

Lemma 3.6. *If N is a coextension of F_7^- that is a ternary paving matroid, then N is isomorphic to T_8 or R_8.*

Proof. We may assume that N is represented by the matrix

$$\begin{bmatrix} & & | & 0 & 1 & 1 & 1 \\ & I_4 & | & 1 & 0 & 1 & 1 \\ & & | & 1 & 1 & 0 & 1 \\ & & | & z_1 & z_2 & z_3 & z_4 \end{bmatrix},$$

where z_1, z_2, z_3, and z_4 are in $\{0, 1, -1\}$. As N has no 3-circuits, none of z_1, z_2, and z_3 is 0. If $z_4 = 0$, then $N \cong ((F_7^-)^* + e_i)^*$ for some i in $\{2, 3, 4, 5\}$. But, for

each such i, $(F_7^-)^* + e_i$ is isomorphic to the self-dual matroid T_8. Hence if $z_4 = 0$, then $N \cong T_8$. If $z_4 \neq 0$, then, by row scaling, we may assume that $z_4 = 1$. To avoid having a 3-circuit in N, we must have $z_1 = z_2 = z_3 = -1$. Hence $N \cong ((F_7^-)^* + e_1)^* \cong R_8^* \cong R_8$. $\square$

We close this paper with two remarks. Firstly, we note that the same technique that was used to prove Theorem 1.1 can also be used to determine all 3-connected binary paving matroids. In the binary case, however, the argument is considerably shorter. Using this and the results of Section 2, we get an alternative proof of Acketa's characterization [1] of binary paving matroids. Secondly, we have not explicitly listed all 3-connected ternary paving matroids here. Such a list is not difficult to obtain by amalgamating various results already in the literature: each of the six 3-connected matroids with fewer than four elements, $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$, is a ternary paving matroid; the 3-connected minors of $S(5, 6, 12)$ with more than three elements are all the matroids in Table II of [6] except J and $\mathcal{W}^r$ for $r \geq 4$; the 3-connected minors of $PG(2, 3)$ not already listed above can be deduced from looking at their complements (see Table 1 of [5]); and finally, the 3-connected minors of T_8 and R_8 that are not minors of $PG(2, 3)$ are $(F_7^-)^*$, T_8, and R_8.

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