A MATROID EXTENSION RESULT

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ABSTRACT. Adding elements to matroids can be fraught with difficulty. In the Vámos matroid V_8, there are four pairs X_1, X_2, X_3, and X_4 that partition $E(V_8)$ such that $(X_1 \cup X_2, X_3 \cup X_4)$ is a 3-separation while exactly three of the local connectivities $\sqcap(X_1, X_3)$, $\sqcap(X_1, X_4)$, $\sqcap(X_2, X_3)$, and $\sqcap(X_2, X_4)$ are one, with the fourth being zero. As is well known, there is no extension of V_8 by a non-loop element p such that $X_j \cup p$ is a circuit for all j. This paper proves that a matroid can be extended by a fixed element in the guts of a 3-separation provided no Vámos-like structure is present.

1. Introduction

The terminology here will follow [5]. For subsets X and Y of the ground set of a matroid M, the *local connectivity* $\sqcap(X, Y)$ between X and Y is defined by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Consider the Vámos matroid, V_8, the rank-4 paving matroid on $\{a_1, a'_1, a_2, a'_2, b_1, b'_1, b_2, b'_2\}$ having as its non-spanning circuits $\{a_1, a'_1, a_2, a'_2\}$, $\{a_1, a'_1, b_1, b'_1\}$, $\{a_1, a'_1, b_2, b'_2\}$, $\{a_2, a'_2, b_1, b'_1\}$, and $\{b_1, b'_1, b_2, b'_2\}$. Then $(\{a_1, a'_1, a_2, a'_2\}, \{b_1, b'_1, b_2, b'_2\})$ is an exact 3-separation (A, B) of V_8. For each i in $\{1, 2\}$, we have the following local connectivity conditions: $\sqcap(\{a_i, a'_i\}, B) = 1 = \sqcap(\{b_i, b'_i\}, A)$. Moreover, $\sqcap(\{a_1, a'_1\}, \{b_1, b'_1\}) = 1$. In this situation, it is tempting to try to extend V_8 by a rank-one element p that lies on the lines spanned by $\{a_1, a'_1\}$ and $\{b_1, b'_1\}$. But this cannot be done. The proof that this extension is impossible depends on the fact that exactly three of the four local connectivities $\sqcap(\{a_i, a'_i\}, \{b_j, b'_j\})$ for $\{i, j\} \subseteq \{1, 2\}$ are equal to one. The purpose of this paper is to show that this condition is the sole impediment to being able to add a point p to the guts of a 3-separation (A, B) of a matroid so that each of $A \cup p$ and $B \cup p$ contains a circuit containing p whose local connectivity with the other side is one.

Frequently in matroid theory, one wants to add an element p to a matroid M in such a way that p is in a certain special position. In general, the only possible such extension may be the trivial one in which p is added as a loop. This paper considers a natural type of non-trivial extension where p is added to the guts of an exact 3-separation (A, B) of M so that each of A and B has a subset that imposes a specific constraint on the placement of p. The main theorem characterizes exactly when such an extension is possible. An *A-strand* in M is a minimal subset A' of A for which $\sqcap(A', B) = 1$. A *B-strand* is defined symmetrically. A *strand* is an A-strand or a B-strand. The following is the main result of the paper.

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Theorem 1.1. *Let (A, B) be an exact 3-separation in a matroid M. Assume there is an A-strand A_0 and a B-strand B_0 such that $\sqcap(A_0, B_0) = 1$. Then M has an extension by an element p in which $A_0 \cup p$ and $B_0 \cup p$ are circuits if and only if M has no A-strand A_1 and B-strand B_1 distinct from A_0 and B_0, respectively, such that exactly two of $\sqcap(A_0, B_1)$, $\sqcap(A_1, B_0)$, and $\sqcap(A_1, B_1)$ are one. Moreover, when M has an extension by p in which $A_0 \cup p$ and $B_0 \cup p$ are circuits, this extension is unique.*

It is natural to consider performing multiple extensions of the type in the last theorem. In Section 4, we prove the following result along with a natural generalization of it that allows for arbitrarily many extensions.

Theorem 1.2. *Let (X, Y, Z) be a partition of the ground set of a matroid M where Y may be empty. Let $(X, Y \cup Z)$ and $(X \cup Y, Z)$ be exact 3-separations of M. Assume there is an X-strand X_0 and a $(Y \cup Z)$-strand Y_0 of M such that $\sqcap(X_0, Y_0) = 1$ and M has an extension by an element p so that $X_0 \cup p$ and $Y_0 \cup p$ are circuits. Assume there is an $(X \cup Y)$-strand Y_1 and a Z-strand Z_1 of M such that $\sqcap(Y_1, Z_1) = 1$ and M has an extension by an element q so that $Y_1 \cup q$ and $Z_1 \cup q$ are circuits. Then M has a unique extension by the elements p and q such that $X_0 \cup p, Y_0 \cup p, Y_1 \cup q$, and $Z_1 \cup q$ are circuits.*

The next section introduces some terminology and proves some basic lemmas. The proof of Theorem 1.1 will be given in Section 3.

2. Preliminaries

For sets X and Y in a matroid M, we say that $\{X, Y\}$ is *modular pair* if $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$, that is, $\sqcap(X, Y) = 0$. The following result will be useful. We omit the straightforward proof.

Lemma 2.1. *Let $\{U, V\}$ be a modular pair of sets in a matroid. If $w \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$, then $w \in \mathrm{cl}(U \cap V)$.*

Next we note some basic properties of local connectivity that will be used often. The first is a restatement of [5, Lemmas 8.2.3]; the second is [6, Lemma 2.4].

Lemma 2.2. *In a matroid M, let X_1, X_2, Y_1, and Y_2 be subsets of $E(M)$ with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. Then*

$$\sqcap(Y_1, Y_2) \leq \sqcap(X_1, X_2).$$

Lemma 2.3. *For subsets P, Q, and R of a matroid M,*

$$\sqcap(P \cup Q, R) + \sqcap(P, Q) = \sqcap(P \cup R, Q) + \sqcap(P, R).$$

Elements e and f in a matroid M are *clones* if the map that interchanges e and f while fixing every other element is an automorphism of M. Elements g and h of M are *independent clones* if they are clones and $\{g, h\}$ is independent in M. An element z of M is *fixed* in M if there is no extension M' of M by an element z' such that z and z' are independent clones of M'. The next result follows from Theorem 6.1 and Lemma 6.3 of Geelen, Gerards, and Whittle [3] (see also [1]).

Lemma 2.4. *Let (A, B) be an exact 3-separation of a matroid M. Then there is a unique extension M' of M by an element x' such that $x' \in \mathrm{cl}_{M'}(A) \cap \mathrm{cl}_{M'}(B)$ and x' is not fixed in M'.*

The statement of the last lemma matches that of [2, Lemma 7.9] except that the former adds the requirement that the extension M' be unique. Although uniqueness is essentially implicit in the latter, we include the proof here for completeness.

Proof of Lemma 2.4. It is proved in [3, Lemma 6.3] that the set $\mathcal{F}$ of flats F of M such that $A - F$ is a separator of M/F is a modular cut of M and, moreover [3, (6.3.1)], that $\mathcal{F}$ is the unique minimal modular cut of M containing $\{\mathrm{cl}(A), \mathrm{cl}(B)\}$. Corresponding to $\mathcal{F}$, there is an extension M' of M by the element x', and $x' \in \mathrm{cl}_{M'}(A) \cap \mathrm{cl}_{M'}(B)$. Thus $(A, B \cup x')$ is an exact 3-separation of M'. Hence M' has an extension M'' by an element x'' for which the corresponding modular cut $\mathcal{F}'$ consists of the flats F' of M' such that $\lambda_{M'/F'}(A - F') = 0$, that is, such that $A - F'$ is a separator of M'/F'.

We show next [1, Lemma 2.2.7] that

2.4.1. *x' and x'' are independent clones in M''.*

If $F' \in \mathcal{F}'$, then $r(A \cup F') + r(B \cup x' \cup F') = r(M') + r(F')$. As $r(A) + r(B \cup x') = r(M') + 2$, we deduce that $r(F') \geq 2$, so $\mathrm{cl}_{M'}(\{x'\}) \notin \mathcal{F}'$. Hence $x'' \notin \mathrm{cl}_{M''}(\{x'\})$, so $\{x', x''\}$ is independent.

To show that x' and x'' are clones, it suffices by [4, Proposition 4.9] to show that a cyclic flat X of M'' contains x' if and only if it contains x''. We omit the straightforward details of this check.

By 2.4.1, x' is not fixed in M'. To establish the uniqueness of the extension M', let N' be an extension of M by a non-fixed element x' that lies in $\mathrm{cl}_{N'}(A) \cap \mathrm{cl}_{N'}(B)$. Since x' is not fixed, N' has an extension N'' by x'' such that x' and x'' are independent clones; and N'' has an extension N''' by x''' such that x'' and x''' are independent clones. In N', let X be a flat such that $r(X - x') = r(X)$. Then $r(X \cup \{x'', x'''\}) = r(X - x')$ in N'''. As $r(A \cup \{x', x'', x'''\}) + r(B \cup \{x', x'', x'''\}) = r(N''') + 2$, it follows that $N'''/(X \cup \{x'', x'''\})$ has $A - (X - x')$ as a separator. Since $N'''/(X \cup \{x'', x'''\})$ has x', x'', and x''' as loops, we deduce that $N'''/(X - x') \backslash \{x', x'', x'''\})$, which equals $M/(X - x')$, has $A - (X - x')$ as a separator. Hence $X - x' \in \mathcal{F}$. Because $\mathcal{F}$ is the unique minimal modular cut of M containing $\{\mathrm{cl}(A), \mathrm{cl}(B)\}$, it follows that $N' = M'$, so M' is unique. □

The following is an immediate consequence of the last lemma.

Corollary 2.5. *Let (A, B) be an exact 3-separation of a matroid M. Then there is a unique extension M'' of M by a pair of independent clones x and y such that $\{x, y\} \subseteq \mathrm{cl}_{M''}(A) \cap \mathrm{cl}_{M''}(B)$.*

In the last result, we shall say that x and y have been *freely added to the guts line* of (A, B).

Lemma 2.6. *Let (A, B) be an exact 3-separation in a matroid M. Assume there is an A-strand A_0 and a B-strand B_0 such that $\sqcap(A_0, B_0) = 1$. Let M'' be the extension of M obtained by freely adding elements x and y to the guts line of (A, B). Then M has an extension by an element p such that $A_0 \cup p$ and $B_0 \cup p$ are circuits if and only if M'' has an extension by p such that $A_0 \cup p$ and $B_0 \cup p$ are circuits.*

Proof. Clearly if M'' has such an extension, then so does M. Conversely, assume that M has an extension M_p by p in which both $A_0 \cup p$ and $B_0 \cup p$ are circuits. As $(A, B \cup p)$ is an exact 3-separation of M_p, we can freely add elements x and y to the guts line of $(A, B \cup p)$ in M_p to get M''_p. Then, in $M''_p \backslash p$, the elements x and y are independent clones that are contained in $\mathrm{cl}_{M''_p \backslash p}(A) \cap \mathrm{cl}_{M''_p \backslash p}(B)$. By Corollary 2.5, $M''_p \backslash p = M''$, so M'' has the desired extension. □

Let (A, B) be an exact 3-separation in a matroid M. Form a bipartite graph G with vertex classes consisting of the set of A-strands and the set of B-strands. An A-strand A' and a B-strand B' are adjacent in G if $\sqcap(A', B') = 1$. A *bunch* of strands is the vertex set of some component of G that has at least one edge. We call a bunch of strands *complete* if the associated bipartite graph is complete. Clearly a bunch of strands that contains a single A-strand or a single B-strand is complete. We call G the *strand graph* of (M, A, B). The following is elementary. In this and the two subsequent lemmas, (A, B) is an exact 3-separation in a matroid M.

Lemma 2.7. *Let Z be a bunch of strands that contains at least two A-strands and at least two B-strands. Then Z is complete if and only if, whenever Z contains A-strands A' and A'' and B-strands B' and B'' such that at least three of $\sqcap(A', B'), \sqcap(A', B''), \sqcap(A'', B')$, and $\sqcap(A'', B'')$ are one, all four are one.*

As all strands are independent, the proof of the next result is straightforward.

Lemma 2.8. *Let A' be an A-strand and B' be a B-strand. Then $\sqcap(A', B') = 1$ if and only if $A' \cup B'$ is a circuit of M.*

Lemma 2.9. *For $A' \subseteq A$ and $B' \subseteq B$, if $A' \cup B'$ is a circuit of M with $A' \neq \emptyset$, and $\sqcap(A, B') = 1$, then B' is a B-strand.*

Proof. Clearly B' contains a B-strand B''. Let A'' be a minimal subset of A such that $\sqcap(A'', B'') = 1$. Then one easily checks that $A'' \cup B''$ is a circuit. Since

$$1 = \sqcap(A'', B'') \leq \sqcap(A' \cup A'', B') \leq \sqcap(A, B') = 1,$$

we see that $\sqcap(A' \cup A'', B') = 1$. Thus $M|(A' \cup A'' \cup B')$ is a 2-sum, with basepoint q say, of matroids M_1 and M_2 having ground sets $B' \cup q$ and $A' \cup A'' \cup q$. Then M_1 has $B' \cup q$ and $B'' \cup q$ as circuits. Thus $B' = B''$. □

Lemma 2.10. *Let X_1, X_2, and Y be sets in a matroid M. If $\{X_1, X_2\}$ is a modular pair, then*

$$\sqcap(X_1, Y) + \sqcap(X_2, Y) \leq \sqcap(X_1 \cup X_2, Y) + \sqcap(X_1 \cap X_2, Y).$$

Proof. By substitution, we see that

$$\begin{aligned} &\sqcap(X_1 \cup X_2, Y) + \sqcap(X_1 \cap X_2, Y) - \sqcap(X_1, Y) - \sqcap(X_2, Y) \\ &\quad = [r(X_1 \cup Y) + r(X_2 \cup Y) - r(X_1 \cup X_2 \cup Y) - r((X_1 \cap X_2) \cup Y)] \\ &\qquad\qquad - [r(X_1) + r(X_2) - r(X_1 \cup X_2) - r(X_1 \cap X_2)]. \end{aligned}$$

The result follows since $\{X_1, X_2\}$ is modular and r is submodular. □

Corollary 2.11. *In a matroid M, suppose X_1, X_2, and Y are sets and $\{X_1, X_2\}$ is a modular pair. If $\sqcap(X_1, Y) = \sqcap(X_1 \cup X_2, Y)$, then $\sqcap(X_2, Y) = \sqcap(X_1 \cap X_2, Y)$.*

Proof. As $\sqcap(X_1 \cap X_2, Y) \leq \sqcap(X_2, Y)$, this is immediate from the last lemma. □

3. The Proof of the Main Result

The purpose of this section is to prove the main theorem.

Proof of Theorem 1.1. Suppose first that M has an extension by an element p such that $A_0 \cup p$ and $B_0 \cup p$ are circuits. Assume also that M has an A-strand A_1 and B-strand B_1 distinct from A_0 and B_0, respectively, such that exactly two of $\sqcap(A_0, B_1)$, $\sqcap(A_1, B_0)$, and $\sqcap(A_1, B_1)$ are one. Without loss of generality, $\sqcap(A_0, B_1) = 1$. By Lemma 2.8, $A_0 \cup B_1$ is a circuit. As $A_0 \cup p$ is also a circuit, for a in A_0, there is a circuit C contained in $(A_0 - a) \cup B_1 \cup p$ that contains p. Since A_0 is an A-strand, $\sqcap(A_0 - a, B) = 0$. As $r(B \cup p) = r(B)$, we deduce that $p \in C \subseteq B_1 \cup p$. Let $C = B_1' \cup p$. Then, by circuit elimination using C and $A_0 \cup p$ with common element p, we deduce that $A_0 \cup B_1'$ contains a circuit that meets both A_0 and B_1'. By Lemma 2.9, B_1' contains a B-strand, so $B_1' = B_1$. Thus $B_1 \cup p$ is a circuit. Recall that $B_0 \cup p$ is also a circuit.

Now $\sqcap(A_1, B_i) = 1$ and $\sqcap(A_1, B_j) = 0$ for some (i, j) in $\{(0, 1), (1, 0)\}$. By Lemma 2.8, $A_1 \cup B_i$ is a circuit. Since $B_i \cup p$ is a circuit, it follows as above that $A_1 \cup p$ is a circuit. Then, as $A_1 \cup p$ and $B_j \cup p$ are circuits, $A_1 \cup B_j$ contains a circuit. As A_1 and B_j are strands, this circuit is $A_1 \cup B_j$. Thus, by Lemma 2.8, $\sqcap(A_1, B_j) = 1$, a contradiction.

Now assume that M has no A-strand A_1 and B-strand B_1 distinct from A_0 and B_0, respectively, such that exactly two of $\sqcap(A_0, B_1)$, $\sqcap(A_1, B_0)$, and $\sqcap(A_1, B_1)$ are one. By Corollary 2.5, M has an extension M'' by a pair of independent clones x and y where $\{x, y\} \subseteq \mathrm{cl}_{M''}(A) \cap \mathrm{cl}_{M''}(B)$. Note that a flat of M'' that contains a circuit containing x or y must contain the line L of M'' that is spanned by $\{x, y\}$.

Clearly $(A \cup x, B \cup y)$ is an exact 3-separation of M''. Moreover, the $(A \cup x)$-strands of M'' consist of the A-strands of M together with $\{x\}$; and the $(B \cup y)$-strands of M'' consist of the B-strands of M'' together with $\{y\}$. It follows that the strand graph of $(M'', A \cup x, B \cup y)$ is obtained from the strand graph of (M, A, B) by adding x and y as isolated vertices.

By Lemma 2.6, to prove that M has the desired extension, we shall show that M'' has an extension by an element p such that $A_0 \cup p$ and $B_0 \cup p$ are circuits. To simplify the notation, from now on, we shall rewrite M'', $A \cup x$, and $B \cup y$ as M, A, and B, respectively. Consider the component of the strand graph of (M, A, B) that contains A_0 and B_0. Call a strand that labels a vertex in this component *special*. For each subset X of $E(M)$, we define $r(X) = r_M(X)$ and

$$r(X \cup p) = \begin{cases} r_M(X) & \text{if } X \text{ contains a special strand or } \mathrm{cl}_M(X) \supseteq L; \\ r_M(X) + 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}$ be the set of subsets F of $E(M)$ such that $r(F \cup p) = r_M(F)$.

We shall complete the proof that M has the desired extension M' by the element p by showing that r is a matroid rank function. Assume the contrary. Then it is straightforward to see that r is not submodular. Thus there are subsets X and Y of $E(M) \cup p$ such that

$$r(X) + r(Y) < r(X \cup Y) + r(X \cap Y). \tag{1}$$

Clearly

$$r(X - p) + r(Y - p) \geq r((X - p) \cup (Y - p)) + r((X - p) \cap (Y - p)). \tag{2}$$

For some α and β in $\{0,1\}$, we have $r(X-p) = r(X)-\alpha$ and $r(Y-p) = r(Y)-\beta$. Then

$$\begin{aligned}
r(X-p) + r(Y-p) &= r(X) - \alpha + r(Y) - \beta \\
&\leq [r(X \cup Y) - \alpha - \beta] + [r(X \cap Y) - 1] \\
&\leq r((X \cup Y) - p) + r((X \cap Y) - p),
\end{aligned}$$

where the last step is immediate if $\alpha + \beta > 0$ and also holds if $\alpha + \beta = 0$. By (2), equality must hold throughout the last chain of inequalities. Thus $r((X\cap Y)-p) = r(X \cap Y) - 1$ so $p \in X \cap Y$. Moreover, $\alpha = 0$ or $\beta = 0$. Thus $r(X - p) = r(X)$ or $r(Y - p) = r(Y)$, so $r((X \cup Y) - p) = r(X \cup Y)$. Hence $\alpha = \beta = 0$. Thus $X - p, Y - p$, and $(X \cup Y) - p$ are in $\mathcal{F}$, but $(X \cap Y) - p \notin \mathcal{F}$. Writing X' and Y' for $X - p$ and $Y - p$, respectively, we see that

$$r(X') + r(Y') = r(X' \cup Y') + r(X' \cap Y'). \tag{3}$$

We now choose a modular pair $\{X', Y'\}$ of subsets of $E(M)$ with $X', Y', X'\cup Y' \in \mathcal{F}$ and $X' \cap Y' \notin \mathcal{F}$ so that $|X' \cup Y'|$ is a minimum. Next we show the following where the closure operator in M has been abbreviated to cl.

3.1.1. *At least one of* $\mathrm{cl}(X')$ *and* $\mathrm{cl}(Y')$ *does not contain* L*.*

Assume both $\mathrm{cl}(X')$ and $\mathrm{cl}(Y')$ contain L. Then $\sqcap(X', L) = 2 = \sqcap(Y', L)$. Thus $\sqcap(X' \cup Y', L) = 2$. Hence, by Corollary 2.11, $\sqcap(X' \cap Y', L) = 2$, so $X' \cap Y' \in \mathcal{F}$, a contradiction. Therefore 3.1.1 holds.

In the proof of the next assertion, it will be useful to recall that, in M, the elements x and y both lie on the line L and neither is fixed.

3.1.2. *Suppose* $A' \subseteq A$ *and* $B' \subseteq B$*. If* $\sqcap(A', B') = 2$*, then* $\sqcap(A' \cup B', L) = 2$*.*

Suppose $\sqcap(A' \cup B', L) \neq 2$. Then $x \notin \mathrm{cl}(A' \cup B')$. Thus $r(A' \cup B' \cup x) = r(A' \cup B') + 1$, so $r(A' \cup x) = r(A') + 1$ and $r(B' \cup x) = r(B') + 1$. Now $x \in \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Thus

$$\begin{aligned}
2 = \sqcap(A', B') &\leq \sqcap(A' \cup x, B' \cup x) \\
&\leq \sqcap(A \cup x, B \cup x) \\
&= \sqcap(A, B) = 2.
\end{aligned}$$

Thus $\sqcap(A' \cup x, B' \cup x) = 2$, so

$$\begin{aligned}
2 &= r(A' \cup x) + r(B' \cup x) - r(A' \cup B' \cup x) \\
&= r(A') + 1 + r(B') + 1 - r(A' \cup B') - 1 \\
&= \sqcap(A', B') + 1 \\
&= 2 + 1.
\end{aligned}$$

This contradiction completes the proof of 3.1.2.

Recall that $\sqcap(A - x, L) = 2 = \sqcap(B - y, L)$, so $\sqcap(A, L) = 2 = \sqcap(B, L)$.

3.1.3. *For* $Z \subseteq B$*,*

$$\sqcap(L, Z) = \sqcap(A, Z) = \sqcap(A \cup L, Z).$$

As $L \subseteq \mathrm{cl}(A)$, we have

$$\begin{aligned}\sqcap(A,Z) &= r(A)+r(Z)-r(A\cup Z)\\ &= r(A\cup L)+r(Z)-r(A\cup L\cup Z)\\ &= \sqcap(A\cup L,Z).\end{aligned}$$

By Lemma 2.3,

$$\sqcap(A\cup L,Z)+\sqcap(A,L)=\sqcap(L\cup Z,A)+\sqcap(L,Z). \tag{4}$$

Now $\sqcap(A,L)=2$, so $2\le \sqcap(A,Z\cup L)\le \sqcap(A,B\cup L)=\sqcap(A,B)=2$. Hence, by (4), $\sqcap(A\cup L,Z)=\sqcap(L,Z)$, so 3.1.3 holds.

3.1.4. *For $Z_A\subseteq A$ and $Z_B\subseteq B$,*

$$\sqcap(Z_A\cup L,Z_B)=\sqcap(L,Z_B).$$

To see this, note that

$$\sqcap(A\cup L,Z_B)\ge \sqcap(Z_A\cup L,Z_B)\ge \sqcap(L,Z_B)=\sqcap(A\cup L,Z_B)$$

where the last equality follows by 3.1.3. Thus 3.1.4 holds.

3.1.5. *For $Z\subseteq E(M)$,*

$$\sqcap(Z,L)+\sqcap(Z\cap A,Z\cap B)=\sqcap(Z\cap A,L)+\sqcap(Z\cap B,L).$$

By Lemma 2.3,

$$\sqcap((Z\cap A)\cup(Z\cap B),L)+\sqcap(Z\cap A,Z\cap B)=\sqcap((Z\cap A)\cup L,Z\cap B)+\sqcap(Z\cap A,L).$$

By 3.1.4, $\sqcap(Z\cap B,L)=\sqcap(Z\cap B,(Z\cap A)\cup L)$. Therefore 3.1.5 holds.

3.1.6. *For $A'\subseteq A$ and $B'\subseteq B$, if $\sqcap(A'\cup B',L)=1=\sqcap(B',L)$ and A' contains a special strand, then B' also contains a special strand.*

Let A_1 be a special strand contained in A'. As $\sqcap(B',L)=1$, it follows by 3.1.3 that $\sqcap(B',A)=1$, so B' contains a B-strand, B_1 say. Now, by 3.1.5,

$$\sqcap(A_1\cup B_1,L)+\sqcap(A_1,B_1)=\sqcap(A_1,L)+\sqcap(B_1,L)=2.$$

As $1=\sqcap(B_1,A)=\sqcap(B_1,L)\le \sqcap(A_1\cup B_1,L)\le \sqcap(A'\cup B',L)=1$, we see that $\sqcap(A_1\cup B_1,L)=1$, so $\sqcap(A_1,B_1)=1$. Thus B_1 is a special B-strand, so 3.1.6 holds.

3.1.7. *Suppose* $\mathrm{cl}(X')\not\supseteq L$. *If $X'\cap A$ contains a strand, then $\sqcap(X'\cap A,L)=1=\sqcap(X',L)$. If, in addition, $X'\cap B$ contains a strand, then $\sqcap(X'\cap B,L)=1=\sqcap(X'\cap A,X'\cap B)$.*

As $\sqcap(X',L)<2$, we know that $\sqcap(X'\cap A,L)\le 1$. Since $X'\cap A$ contains a strand, $\sqcap(X'\cap A,B)\ge 1$, so $\sqcap(X'\cap A,L)\ge 1$. Hence $\sqcap(X'\cap A,L)=1$ and $\sqcap(X',L)=1$. Now suppose $X'\cap B$ also contains a strand. Then $\sqcap(X'\cap B,L)=1$. Moreover, by 3.1.5, $\sqcap(X'\cap A,X'\cap B)=1$. Hence 3.1.7 holds.

Recall that $X',Y'\in\mathcal{F}$.

3.1.8. *Suppose* $\mathrm{cl}(X')\not\supseteq L$. *Then one of the following occurs.*

(i) *Both $X'\cap A$ and $X'\cap B$ contain special strands, $\sqcap(X'\cap A,L)=1=\sqcap(X'\cap B,L)$ and $\sqcap(X'\cap A,X'\cap B)=1$; or*

(ii) *$X'\cap A$ contains a special strand, $X'\cap B$ does not contain a strand, $\sqcap(X'\cap A,L)=1$ and $\sqcap(X'\cap B,L)=0=\sqcap(X'\cap A,X'\cap B)$; or*

(iii) $X' \cap B$ *contains a special strand,* $X' \cap A$ *does not contain a strand,* $\sqcap(X' \cap B, L) = 1$ *and* $\sqcap(X' \cap A, L) = 0 = \sqcap(X' \cap A, X' \cap B)$.

Since $X' \in \mathcal{F}$ but $\mathrm{cl}(X') \not\supseteq L$, at least one of $X' \cap A$ and $X' \cap B$ contains a special strand. Assume $X' \cap A$ does. Then, by 3.1.7, $\sqcap(X' \cap A, L) = 1$ and, if $X' \cap B$ also contains a strand, then $\sqcap(X' \cap B, L) = 1 = \sqcap(X' \cap A, X' \cap B)$. As $\sqcap(X', L) = 1$, it follows by 3.1.6 that $X' \cap B$ contains a special strand, so (i) holds.

Now suppose $X' \cap B$ does not contain a strand. Then, by 3.1.5, $\sqcap(X' \cap A, X' \cap B) = \sqcap(X' \cap B, L)$. If this quantity is 0, then (ii) holds and 3.1.8 is proved. Thus we may assume that $\sqcap(X' \cap A, X' \cap B) = 1 = \sqcap(X' \cap B, L)$. Then, by 3.1.3, $\sqcap(X' \cap B, A) = 1$, so $X' \cap B$ contains a strand, a contradiction. Hence 3.1.8 holds.

Now, for Z in $\{A, B\}$, let

$$\gamma(Z) = r(X' \cap Z) + r(Y' \cap Z) - r((X' \cup Y') \cap Z) - r(X' \cap Y' \cap Z).$$

By submodularity, $\gamma(Z) \geq 0$. Moreover, $\gamma(Z) = 0$ if and only if $\{X' \cap Z, Y' \cap Z\}$ is a modular pair.

3.1.9.

$$\begin{aligned} 0 = \gamma(A) + \gamma(B) \\ &+ [\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) - \sqcap(Y' \cap A, Y' \cap B)] \\ &\qquad - [\sqcap(X' \cap A, X' \cap B) - \sqcap(X' \cap Y' \cap A, X' \cap Y' \cap B)]. \end{aligned}$$

To see this, observe, by (3) that

$$\begin{aligned} 0 &= r(X') + r(Y') - r(X' \cup Y') - r(X' \cap Y') \\ &= [r(X' \cap A) + r(X' \cap B) - \sqcap(X' \cap A, X' \cap B)] \\ &\quad + [r(Y' \cap A) + r(Y' \cap B) - \sqcap(Y' \cap A, Y' \cap B)] \\ &\quad - [r((X' \cup Y') \cap A) + r((X' \cup Y') \cap B) - \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B)] \\ &\quad - [r(X' \cap Y' \cap A) + r(X' \cap Y' \cap B) - \sqcap(X' \cap Y' \cap A, X' \cap Y' \cap B)] \\ &= \gamma(A) + \gamma(B) + [\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) - \sqcap(Y' \cap A, Y' \cap B)] \\ &\quad - [\sqcap(X' \cap A, X' \cap B) - \sqcap(X' \cap Y' \cap A, X' \cap Y' \cap B)]. \end{aligned}$$

By 3.1.1, at most one of $\mathrm{cl}(X')$ and $\mathrm{cl}(Y')$ contains L. We first treat the case when exactly one of $\mathrm{cl}(X')$ and $\mathrm{cl}(Y')$ contains L. In particular, we assume that

3.1.10. $\mathrm{cl}(Y')$ *contains* L *but* $\mathrm{cl}(X')$ *does not.*

By symmetry, we may also assume that (i) or (ii) of 3.1.8 holds. Thus

$$\sqcap(X' \cap A, L) = 1$$

and $X' \cap A$ contains a special strand. Of course, A_0 and B_0 are special strands.

3.1.11. *If* $\{X' \cap A, Y' \cap A\}$ *is a modular pair and* $\sqcap(Y' \cap A, L) = 1$*, then*

(i) $\sqcap((X' \cup Y') \cap A, L) = 2$*; and*
(ii) $\sqcap(L, (X' \cup Y') \cap B) = \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B)$.

Assume $\sqcap((X' \cup Y') \cap A, L) < 2$. Then, since $\sqcap(Y' \cap A, L) = 1$, we see that $\sqcap((X' \cup Y') \cap A, L) = 1$. Now, by 3.1.5,

$$\sqcap(((X' \cup Y') \cap A) \cup B_0, L) + \sqcap((X' \cup Y') \cap A, B_0) = \sqcap((X' \cup Y') \cap A, L) + \sqcap(B_0, L) = 2.$$

Since $X' \cap A$ contains a special strand, $\sqcap((X' \cup Y') \cap A, B_0) \geq 1$. As B_0 is a strand, it follows by 3.1.3 that $\sqcap(((X' \cup Y') \cap A) \cup B_0, L) \geq \sqcap(B_0, L) = \sqcap(B_0, A) = 1$. Thus $\sqcap((X' \cup Y') \cap A, B_0) = 1 = \sqcap(((X' \cup Y') \cap A) \cup B_0, L)$.

By 3.1.5 again,

$$\sqcap((Y' \cap A) \cup B_0, L) + \sqcap(Y' \cap A, B_0) = \sqcap(B_0, L) + \sqcap(Y' \cap A, L) = 2.$$

As each term on the left-hand side is at most one, we deduce that each equals one. Thus, by 3.1.6, $Y' \cap A$ contains a special strand, so $Y' \cap A \in \mathcal{F}$.

As $\sqcap(Y', L) = 2$ and $\sqcap(Y' \cap A, L) = 1$, it follows that $Y' \cap B \neq \emptyset$. Thus $|(X' \cup Y') \cap A| < |X' \cup Y'|$ and we get a contradiction to the choice of $\{X', Y'\}$ since $X' \cap Y' \cap A$, and hence $X' \cap Y'$, is in $\mathcal{F}$. We conclude that $\sqcap((X' \cup Y') \cap A, L) = 2$, so 3.1.11(i) holds.

By Lemma 2.3,

$$\begin{aligned}\sqcap(((X' \cup Y') \cap A) \cup L, (X' \cup Y') \cap B) + \sqcap((X' \cup Y') \cap A, L) \\ = \sqcap(X' \cup Y', L) \\ + \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B).\end{aligned}$$

By (i), $\sqcap((X' \cup Y') \cap A, L) = 2$ and, by 3.1.4, $\sqcap(((X' \cup Y') \cap A) \cup L, (X' \cup Y') \cap B) = \sqcap(L, (X' \cup Y') \cap B)$. Thus

$$\sqcap(((X' \cup Y') \cap A) \cup L, (X' \cup Y') \cap B) = \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B),$$

that is, 3.1.11(ii) holds.

3.1.12. *If $\{X' \cap A, Y' \cap A\}$ is a modular pair, then $\sqcap(Y' \cap A, L) = 1$.*

Assume that $\sqcap(Y' \cap A, L) = 2$. Then, because $X' \cap A$ contains a special strand, all of $X' \cap A, Y' \cap A$, and $(X' \cup Y') \cap A$ are in $\mathcal{F}$. Hence, by the choice of $\{X', Y'\}$, we see that $(X' \cup Y') \cap B = \emptyset$ otherwise $X' \cap Y' \cap A$, and hence $X' \cap Y'$, is in $\mathcal{F}$, a contradiction.

Now $X' \cap A = X'$ and $Y' \cap A = Y'$. As X' contains a special strand, $\sqcap(X', B_0) = 1$. By Lemma 2.3,

$$\sqcap(Y' \cup B_0, L) + \sqcap(Y', B_0) = \sqcap(B_0 \cup L, Y') + \sqcap(B_0, L).$$

Since $\sqcap(Y', L) = 2$, we see using 3.1.4 that $\sqcap(Y' \cup B_0, L) = 2 = \sqcap(B_0 \cup L, Y')$. Thus $\sqcap(Y', B_0) = 1$.

As $1 = \sqcap(X', B_0) \leq \sqcap(X' \cup Y', B_0) \leq \sqcap(A, B_0) = 1$, Corollary 2.11 implies that $\sqcap(X' \cap Y', B_0) = \sqcap(Y', B_0) = 1$. Recall that $X' \cap Y' \subseteq A$. Then, by 3.1.3, $1 = \sqcap(X' \cap Y', B_0) \leq \sqcap(X' \cap Y', L) \leq \sqcap(X', L) = 1$. By 3.1.5, $\sqcap((X' \cap Y') \cup B_0, L) + \sqcap(X' \cap Y', B_0) = \sqcap(X' \cap Y', L) + \sqcap(B_0, L)$, so $\sqcap((X' \cap Y') \cup B_0, L) = 1$. Thus, by 3.1.6, $X' \cap Y'$ contains a special strand, a contradiction. We conclude that $\sqcap(Y' \cap A, L) < 2$.

Now assume that $\sqcap(Y' \cap A, L) = 0$. By 3.1.5,

$$\sqcap(Y', L) + \sqcap(Y' \cap A, Y' \cap B) = \sqcap(Y' \cap A, L) + \sqcap(Y' \cap B, L).$$

Hence $\sqcap(Y' \cap B, L) = 2$ and $\sqcap(Y' \cap A, Y' \cap B) = 0$. By interchanging the terms $\sqcap(Y' \cap A, Y' \cap B)$ and $\sqcap(X' \cap A, X' \cap B)$ in 3.1.9 and observing that each of $\gamma(A)$, $\gamma(B)$, and $[\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) - \sqcap(X' \cap A, X' \cap B)]$ is non-negative, we see that $\{X' \cap B, Y' \cap B\}$ is a modular pair. As $X' \cap A$ contains a special strand, it is non-empty. Therefore $X' \cap B \notin \mathcal{F}$ otherwise $X' \cap Y' \cap B$, and hence $X' \cap Y'$, is in $\mathcal{F}$, a contradiction. Thus $X' \cap B$ does not contain a special strand. Hence (ii) of 3.1.8

holds, so $\sqcap(X' \cap A, X' \cap B) = 0$. Thus, by 3.1.9, $\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) = 0$. By 3.1.5 again,

$$\begin{aligned} 2 &= 2 + \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) \\ &= \sqcap((X' \cup Y') \cap A, L) + \sqcap((X' \cup Y') \cap B, L) \\ &\geq \sqcap(X' \cap A, L) + \sqcap(Y' \cap B, L) \\ &= 1 + 2. \end{aligned}$$

This contradiction implies that $\sqcap(Y' \cap A, L) = 1$, that is, 3.1.12 holds.

3.1.13. *Case (i) of 3.1.8 must hold.*

Assume instead that 3.1.8(ii) holds. Then $\sqcap(X' \cap A, X' \cap B) = 0$, so the second square-bracketed term in 3.1.9 is 0. Since the other square-bracketed term is non-negative as are each of $\gamma(A)$ and $\gamma(B)$, we deduce that $\gamma(A) = 0$. Thus $\{X' \cap A, Y' \cap A\}$ is a modular pair. Hence, by 3.1.12, $\sqcap(Y' \cap A, L) = 1$.

Now, by 3.1.11(ii), $\sqcap(L, (X' \cup Y') \cap B) = \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B)$. Thus, by 3.1.9, as $\sqcap(X' \cap A, X' \cap B) = 0$,

$$\sqcap(L, (X' \cup Y') \cap B) = \sqcap(Y' \cap A, Y' \cap B). \tag{5}$$

By 3.1.5, $\sqcap(Y', L) + \sqcap(Y' \cap A, Y' \cap B) = \sqcap(Y' \cap A, L) + \sqcap(Y' \cap B, L)$, so $2 + \sqcap(Y' \cap A, Y' \cap B) = 1 + \sqcap(Y' \cap B, L)$, that is, $\sqcap(Y' \cap A, Y' \cap B) = \sqcap(Y' \cap B, L) - 1$. Therefore, by (5),

$$\sqcap(L, Y' \cap B) \leq \sqcap(L, (X' \cup Y') \cap B) = \sqcap(Y' \cap A, Y' \cap B) = \sqcap(Y' \cap B, L) - 1,$$

a contradiction. We conclude that 3.1.13 holds.

We now know that both $X' \cap A$ and $X' \cap B$ contain special strands and $\sqcap(X' \cap A, X' \cap B) = 1$. We have symmetry between A and B so, by 3.1.9, we may assume that $\{X' \cap A, Y' \cap A\}$ is a modular pair. Thus, by 3.1.12, $\sqcap(Y' \cap A, L) = 1$.

Next we observe, by 3.1.5, that either

(a) $\sqcap(Y' \cap B, L) = 2$ and $\sqcap(Y' \cap A, Y' \cap B) = 1$; or
(b) $\sqcap(Y' \cap B, L) = 1$ and $\sqcap(Y' \cap A, Y' \cap B) = 0$.

Suppose (a) holds. Then $Y' \cap B$ and $X' \cap B$ are in $\mathcal{F}$. Hence $\{X' \cap B, Y' \cap B\}$ is not a modular pair otherwise we obtain the contradiction that $X' \cap Y' \cap B$, and hence $X' \cap Y'$, is in $\mathcal{F}$. Thus $\gamma(B) \geq 1$. By 3.1.11(ii), $\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) \geq \sqcap(L, Y' \cap B) = 2$. Since $\sqcap(Y' \cap A, Y' \cap B) = 1 = \sqcap(X' \cap A, X' \cap B)$, we obtain a contradiction by 3.1.9. We deduce that (b) holds.

As $\sqcap(Y' \cap A, Y' \cap B) = 0$, using 3.1.9 again with the terms $\sqcap(Y' \cap A, Y' \cap B)$ and $\sqcap(X' \cap A, X' \cap B)$ interchanged, we see, as $\sqcap(X' \cap A, X' \cap B) = 1$, that

$$\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) = 1$$

and $\{X' \cap B, Y' \cap B\}$ is a modular pair. We may now apply 3.1.11 interchanging A and B to get that $\sqcap((X' \cup Y') \cap B, L) = 2$. By 3.1.5,

$$\sqcap(X' \cup Y', L) + \sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) = \sqcap((X' \cup Y') \cap A, L) + \sqcap((X' \cup Y') \cap B, L).$$

It follows that $\sqcap((X' \cup Y') \cap A, (X' \cup Y') \cap B) = 2$. This contradiction completes the proof in the case that 3.1.10 holds.

It remains to treat the case when neither X' nor Y' spans L. Thus we may assume the following.

3.1.14. *Each of X' and Y' contains a special strand.*

By 3.1.2, $\sqcap(X'\cap A, X'\cap B) \le 1$ and $\sqcap(Y'\cap A, Y'\cap B) \le 1$. If either $\sqcap(X'\cap A, X'\cap B)$ or $\sqcap(Y'\cap A, Y'\cap B)$ is 0, then, by 3.1.9 and symmetry, both $\{X'\cap A, Y'\cap A\}$ and $\{X'\cap B, Y'\cap B\}$ are modular pairs.

3.1.15. $\sqcap(X'\cap A, X'\cap B) = 1 = \sqcap(Y'\cap A, Y'\cap B)$.

Assume $\sqcap(X'\cap A, X'\cap B) = 0$. We know $X'\cap A$ or $X'\cap B$ contains a special strand. Then, by 3.1.7, exactly one of $X'\cap A$ and $X'\cap B$, say $X'\cap A$, contains a strand. Assume $Y'\cap A$ contains a special strand. Then $(X'\cup Y')\cap B = \emptyset$ otherwise $|(X'\cup Y')\cap A| < |X'\cup Y'|$ and we have a contradiction to the choice of $\{X', Y'\}$. We can now write X' and Y' for $X'\cap A$ and $Y'\cap A$.

As $X'\cup Y'$ contains a special strand, $\sqcap(X'\cup Y', B_0) = 1$. Then $M|(X'\cup Y'\cup B_0)$ can be written as a 2-sum with basepoint q of matroids M_1 and M_2 with ground sets $X'\cup Y'\cup q$ and $B_0\cup q$. As each of X' and Y' contains a special strand, it follows that, in M_1, the element q is in the closures of both X' and Y'. Hence, by Lemma 2.1, $q \in \mathrm{cl}_{M_1}(X'\cap Y')$. Thus $M|(X'\cup Y'\cup B_0)$ has a circuit of the form $B_0\cup A_2$ where $A_2 \subseteq X'\cap Y'$. As $1 \le \sqcap(A_2, B) \le \sqcap(X'\cap Y', B) = \sqcap(X'\cap Y', L) \le \sqcap(X', L) = 1$, it follows by Lemma 2.9 that A_2 is a special strand contained in $X'\cap Y'$, a contradiction.

We now know that $Y'\cap A$ does not contain a special strand. Thus $Y'\cap B$ does. Assume $\sqcap(Y'\cap A, L) = 1$. Then, as $Y'\cap B$ contains a special strand, 3.1.6 implies that $Y'\cap A$ also contains a special strand, a contradiction. Thus $\sqcap(Y'\cap A, L) = 0$. As $\sqcap(Y'\cap B, L) = 1$, 3.1.5 implies that $\sqcap(Y'\cap A, Y'\cap B) = 0$. As $X'\cap A$ contains a strand but $X'\cap B$ does not, $\sqcap(X'\cap A, L) = 1$ and $\sqcap(X'\cap B, L) = 0$. Hence, by 3.1.4, $\sqcap(X'\cap A, X'\cap B) = 0$. By 3.1.9, $\sqcap((X'\cup Y')\cap A, (X'\cup Y')\cap B) = 0$. But $X'\cap A$ and $Y'\cap B$ both contain special strands, so $\sqcap(X'\cap A, Y'\cap B) \ge 1$, a contradiction. We conclude that $\sqcap(X'\cap A, X'\cap B) > 0$. By 3.1.2, $\sqcap(X'\cap A, X'\cap B) < 2$. Hence $\sqcap(X'\cap A, X'\cap B) = 1$ and, by symmetry, 3.1.15 follows.

Since $\sqcap(X', L) = 1$, we deduce by 3.1.5 that $\sqcap(X'\cap A, L) = 1 = \sqcap(X'\cap B, L)$. Now $X'\cap A$ or $X'\cap B$, say $X'\cap A$, contains a special strand, say A_1. By 3.1.5 again, $\sqcap(A_1\cup(X'\cap B), L) + \sqcap(A_1, X'\cap B) = \sqcap(A_1, L) + \sqcap(X'\cap B, L)$, so $\sqcap(A_1, X'\cap B) = 1$. Thus M has a circuit $A_1\cup B_1$ for some $B_1 \subseteq X'\cap B$. Now, by 3.1.3,

$$1 = \sqcap(A_1, B_1) \le \sqcap(A, B_1) = \sqcap(L, B_1) \le \sqcap(L, X'\cap B) = 1,$$

so $\sqcap(A, B_1) = 1$. Thus, by Lemma 2.9, B_1 is a B-strand. Since $\sqcap(A_1, B_1) = 1$, B_1 is a special strand. We conclude that $X'\cap A$ and $X'\cap B$ both contain special strands. By symmetry, both $Y'\cap A$ and $Y'\cap B$ also contain special strands.

By 3.1.9, $\{X'\cap A, Y'\cap A\}$ or $\{X'\cap B, Y'\cap B\}$, say the former, is a modular pair. But $|(X'\cup Y')\cap A| < |X'\cup Y'|$. As each of $X'\cap A$, $Y'\cap A$, and $(X'\cup Y')\cap A$ is in $\mathcal{F}$, so is $X'\cap Y'\cap A$. Thus $X'\cap Y' \in \mathcal{F}$. This contradiction completes the proof that r is a matroid rank function, so M has the desired extension.

To finish the proof of the theorem, it remains to show that the extension $M'\backslash x, y$ of $M\backslash x, y$ by p in which $A_0\cup p$ and $B_0\cup p$ are circuits is unique. By the first part of the theorem and Lemma 2.7, the bunch of special strands of M is complete. Let M_p^- be an arbitrary extension of $M\backslash x, y$ by p in which $A_0\cup p$ and $B_0\cup p$ are circuits. Then $(A - x, (B - y)\cup p)$ is an exact 3-separation of M_p^-, so it has an extension M_p by freely adding x and y to the guts line of $(A - x, (B - y)\cup p)$. By Corollary 2.5, $M_p\backslash p = M'\backslash p = M$. To complete the proof of the uniqueness of $M'\backslash x, y$, we shall show that $M_p = M'$.

For a subset X of $E(M)$, we know that, in M', we have $r(X \cup p) = r(X)$ if and only if X contains a special strand, or X spans $\{x, y\}$ in M. Let r_p be the rank function of M_p. We show next that

3.1.16. *$r_p(X \cup p) = r_p(X)$ if $\mathrm{cl}_M(X) \supseteq L$ or if X contains a special strand.*

Suppose $\mathrm{cl}_M(X) \supseteq L$. Then $p \in \mathrm{cl}_{M_p}(A) \cap \mathrm{cl}_{M_p}(B) = \mathrm{cl}_{M_p}(L)$, so $r_p(X \cup p) = r_p(X)$. On the other hand, suppose X contains a special strand. If this strand is A_0 or B_0, then certainly $r_p(X \cup p) = r_p(X)$. Hence, by symmetry, we may assume the special strand contained in X is a B-strand, B_1. We may also assume that $B_1 \cup p$ does not contain a circuit of M_p, otherwise $r_p(X \cup p) = r_p(X)$. As the bunch of special strands of M is complete, $\sqcap(A_0, B_1) = 1$, so $A_0 \cup B_1$ is a circuit of M. As M_p has $A_0 \cup B_1$ and $A_0 \cup p$ as circuits, it has a circuit C that contains p and avoids some element a of A_0. As $B_1 \cup p$ does not contain a circuit, C must contain some element, say a', of A_0. Now, by strong circuit elimination, M_p has a circuit D that contains a' and is contained in $(C \cup B_0) - p$. As $D \cap A \subsetneqq A_0$ and A_0 is a strand, it follows by 3.1.3 that $\sqcap(D \cap A, L) = 0$. But this is a contradiction since D is a circuit meeting both A and B. We conclude that 3.1.16 holds.

To complete the proof that $M_p = M'$, we assume that Z is a minimal subset of $E(M)$ such that $r_p(Z \cup p) = r_p(Z)$ but $\mathrm{cl}_M(Z) \not\supseteq L$ and Z does not contain a special strand. By minimality, $Z \cup p$ must be a circuit of M_p.

3.1.17. $Z \cap \{x, y\} = \emptyset$.

Clearly $\{x, y\} \not\subseteq Z$. Suppose $|Z \cap \{x, y\}| = 1$. Then we may assume that $x \in Z$. As y is a clone of x in M_p, we deduce that $(Z - x) \cup y \cup p$ is a circuit of M_p. Thus $Z \cup y$ contains a circuit of M containing y. Hence $\mathrm{cl}_M(Z) \supseteq L$, a contradiction. Thus 3.1.17 holds.

Assume first that $Z \subseteq A$. As $Z \cup p$ and $B_0 \cup p$ are circuits of M_p, we deduce that $Z \cup B_0$ contains a circuit $Z' \cup B'_0$ of M_p, which must also be a circuit of M. Now $1 \leq \sqcap(Z', B'_0) \leq \sqcap(A, B'_0) \leq \sqcap(A, B_0) = 1$. Thus, as B'_0 is a B-strand, $B'_0 = B_0$. Suppose $\sqcap(Z', B) = 1$. Then Z' is an A-strand of M by Lemma 2.9. As $Z' \cup B_0$ is a circuit, Z' is a special strand, a contradiction. We deduce that $\sqcap(Z', B) = 2$, so, by 3.1.3, $\sqcap(Z', L) = 2$. Hence $\mathrm{cl}_M(Z') \supseteq L$, a contradiction.

We may now assume that $Z = A_Z \cup B_Z$ where $A_Z = A \cap Z$ and $B_Z = B \cap Z$, and both A_Z and B_Z are non-empty. We also know that neither A_Z nor B_Z contains a special strand. Both $B_0 \cup p$ and $Z \cup p$ are circuits of M_p. Take a in A_Z. Then M_p has a circuit that contains a and avoids p. This circuit is a circuit of M. Thus $\sqcap(A_Z, B) \geq 1$. As $\sqcap(A_Z, L) \neq 2$, we deduce that $\sqcap(A_Z, B) = 1$. Hence A_Z contains an A-strand A'_Z of M. Likewise, B_Z contains a B-strand B'_Z of M. As $A'_Z \cup B'_Z$ is a proper subset of the circuit $Z \cup p$ of M_p, it follows that $A'_Z \cup B'_Z$ is independent in M. Thus $\sqcap(A'_Z, B'_Z) = 0$. Also, $\sqcap(A'_Z \cup B'_Z, L) \neq 2$ as $\sqcap(Z, L) \neq 2$. Now, by Lemma 2.3,

$$\sqcap(A'_Z \cup B'_Z, L) + \sqcap(A'_Z, B'_Z) = \sqcap(A'_Z \cup L, B'_Z) + \sqcap(A'_Z, L).$$

Since the right-hand side is at least two but the left-hand side is at most one, we have a contradiction. We conclude that $M_p = M'$, so the extension $M' \backslash x, y$ of $M \backslash x, y$ is unique. □

4. Multiple Extensions

In Theorem 1.1, we gave conditions for the existence of a certain extension of a matroid by a fixed element in the guts of an exact 3-separation. We begin this section by proving Theorem 1.2, which allows us to do two extensions of the type in Theorem 1.1. Using Theorem 1.2 will enable us to establish the following more general result.

Theorem 4.1. *Let T be an n-vertex tree whose vertices are labelled by non-empty disjoint sets $X_1, X_2, \dots, X_n$. Let $E = X_1 \cup X_2 \cup \cdots \cup X_n$ and let M be a matroid with ground set E. Suppose that, for every edge e of T, the induced partition (Y_e, Z_e) of E is an exact 3-separation of M and there are Y_e- and Z_e-strands $Y_{e1}, Y_{e2}, \dots, Y_{em_e}$ and $Z_{e1}, Z_{e2}, \dots, Z_{em_e}$ such that, for each k in $\{1, 2, \dots, m_e\}$, the local connectivity $\sqcap(Y_{ek}, Z_{ek}) = 1$ and there is an extension of M by an element p_{ek} in which $Y_{ek} \cup p_{ek}$ and $Z_{ek} \cup p_{ek}$ are circuits. Then M can be extended by $\bigcup_{e \in E(T)} \{p_{e1}, p_{e2}, \dots, p_{em_e}\}$ to produce a matroid M' in which $Y_{ek} \cup p_{ek}$ and $Z_{ek} \cup p_{ek}$ are circuits for all e in $E(T)$ and all k in $\{1, 2, \dots, m_e\}$. Moreover, the matroid M' is unique.*

Proof of Theorem 1.2. By Corollary 2.5, we can freely add elements x and y to the guts line of $(X, Y \cup Z)$ to get a unique extension M'' of M.

4.1.1. *M'' has an extension by p in which $X_0 \cup p$ and $Y_0 \cup p$ are circuits. Moreover, M'' has an extension by q in which $Y_1 \cup q$ and $Z_1 \cup q$ are circuits.*

By Lemma 2.6, M'' has an extension by p in which $X_0 \cup p$ and $Y_0 \cup p$ are circuits. Now M has an extension M_q by q such that $Y_1 \cup q$ and $Z_1 \cup q$ are circuits. As M_q has $(X, Y \cup Z \cup q)$ as an exact 3-separation, by Corollary 2.5, we can freely add elements x and y to the guts line of $(X, Y \cup Z \cup q)$ to get an extension M''_q of M_q. Then, in $M''_q \backslash q$, the elements x and y are independent clones on the guts line of $(X, Y \cup Z)$. The uniqueness of M'' implies that $M''_q \backslash q = M''$. We conclude that 4.1.1 holds.

It will be convenient to work with the elements x and y. Thus, in the argument that follows, we replace M'' by M. This means that we assume that $\{x, y\} \subseteq E(M)$. Indeed, we assume that $\{x, y\} \subseteq Y$ noting that x and y are independent clones in M, and $\{x, y\} \subseteq \mathrm{cl}_M(X) \cap \mathrm{cl}_M((Y - \{x, y\}) \cup Z)$.

Let M_p be the extension of M by the element p such that $X_0 \cup p$ and $Y_0 \cup p$ are circuits. We want to show that M_p has an extension by q in which $Y_1 \cup q$ and $Z_1 \cup q$ are circuits. If such an extension exists, it is unique. Observe that Y_1 is an $(X \cup Y \cup p)$-strand of M_p, that Z_1 is a Z-strand of M_p, and that $\sqcap(Y_1, Z_1) = 1$. We assume that M_p does not have the desired extension by q. Then, by Theorem 1.1, M_p has an $(X \cup Y \cup p)$-strand Y_2 and a Z-strand Z_2 such that exactly two of $\sqcap(Y_1, Z_2)$, $\sqcap(Y_2, Z_1)$, and $\sqcap(Y_2, Z_2)$ are one. Clearly Z_2 is a Z-strand of M. Because M has an extension by q in which $Y_1 \cup q$ and $Z_1 \cup q$ are circuits, it follows that $p \in Y_2$ otherwise $\sqcap(Y_1, Z_2)$, $\sqcap(Y_2, Z_1)$, and $\sqcap(Y_2, Z_2)$ give a violation of Theorem 1.1.

Let L be the line of M_p that is spanned by $\{x, y\}$. Then $p \in L$. Now, for k in $\{1, 2\}$, Lemma 2.8 implies that $\sqcap(Y_2, Z_k) = 1$ if and only if $Y_2 \cup Z_k$ is a circuit of M_p. Hence

4.1.2. *M_p has $Y_2 \cup Z_i$ as a circuit for some i in $\{1, 2\}$.*

We divide the rest of the argument into two cases based on whether or not $Y_2 \cap X$ is empty. Suppose first that $Y_2 \cap X = \emptyset$.

4.1.3. *For k in $\{1,2\}$, the set $Y_2 \cup Z_k$ is a circuit of M_p if and only if $X_0 \cup (Y_2 - p) \cup Z_k$ is a circuit of M.*

As $\sqcap(X_0, Y \cup Z \cup p) = 1$ and M_p has $X_0 \cup p$ as a circuit, $M_p \backslash (X - X_0)$ is the parallel connection with basepoint p of $M_p|(X_0 \cup p)$ and $M_p \backslash X$. Thus 4.1.3 holds.

As $Y_2 \cup Z_i$ is a circuit of M_p, we see that $X_0 \cup (Y_2 - p) \cup Z_i$ is a circuit of M. Next we show that

4.1.4. *$X_0 \cup (Y_2 - p)$ is an $(X \cup Y)$-strand of M; and $\sqcap(X_0 \cup (Y_2 - p), Z_k) = \sqcap(Y_2, Z_k)$ for each k in $\{1,2\}$.*

By Lemma 2.3, we have

$$\sqcap(X_0 \cup (Y_2 - p), Z) + \sqcap(X_0, Y_2 - p) = \sqcap(X_0, (Y_2 - p) \cup Z) + \sqcap(Y_2 - p, Z).$$

As M_p has Y_2 as an $(X \cup Y \cup p)$-strand, $\sqcap(Y_2 - p, Z) = 0$. It follows, since $\sqcap(X_0, L) = 1$, that $\sqcap(X_0 \cup (Y_2 - p), Z) \leq 1$. As M has $X_0 \cup (Y_2 - p) \cup Z_i$ as a circuit, $\sqcap(X_0 \cup (Y_2 - p), Z) = 1$. Thus Lemma 2.9 implies that $X_0 \cup (Y_2 - p)$ is an $(X \cup Y)$-strand of M. The second part of 4.1.4 follows immediately by combining Lemma 2.8 with 4.1.3 since each of $\sqcap(X_0 \cup (Y_2 - p), Z_k)$ and $\sqcap(Y_2, Z_k)$ is in $\{0,1\}$.

Now $\sqcap(Y_1, Z_1) = 1$ and exactly two of $\sqcap(Y_1, Z_2)$, $\sqcap(Y_2, Z_1)$, and $\sqcap(Y_2, Z_2)$ are one. Thus, by 4.1.4, exactly two of $\sqcap(Y_1, Z_2)$, $\sqcap(X_0 \cup (Y_2 - p), Z_1)$, and $\sqcap(X_0 \cup (Y_2 - p), Z_2)$ are one. Hence the $(X \cup Y)$-strand $X_0 \cup (Y_2 - p)$ and the Z-strand Z_2 of M contradict Theorem 1.1. This completes the argument when $Y_2 \cap X = \emptyset$.

We may now assume that $Y_2 \cap X \neq \emptyset$. We show first that

4.1.5. *$\sqcap(Y_2, L) = 2$ and $\sqcap(Y_2 \cap X, L) = 1$.*

Since $p \in Y_2 \cap L$, we see that $\sqcap(Y_2, L) \geq 1$. From the circuit $Y_2 \cup Z_i$, we deduce that $\sqcap(Y_2 \cap X, (Y_2 - X) \cup Z_i) = 1$, so $\sqcap(Y_2 \cap X, Y \cup Z) \geq 1$. Hence $\sqcap(Y_2 \cap X, L) \geq 1$. As $p \in Y_2 - X$, it follows that $Y_2 \cap X$ does not span p otherwise $(Y_2 \cap X) \cup p$ contains a circuit that is properly contained in $Y_2 \cup Z_i$.

As Y_2 is independent, $\sqcap(Y_2 \cap X, Y_2 - X) = 0$. Thus, by Lemma 2.3,

$$\begin{aligned} 2 \geq \sqcap(Y_2, L) &= \sqcap((Y_2 \cap X) \cup (Y_2 - X), L) + \sqcap(Y_2 \cap X, Y_2 - X) \\ &= \sqcap(Y_2 \cap X, (Y_2 - X) \cup L) + \sqcap(Y_2 - X, L) \\ &\geq \sqcap(Y_2 \cap X, L) + \sqcap(Y_2 - X, L) \\ &\geq 1 + 1, \end{aligned}$$

where the last inequality follows because $p \in Y_2 - X$. We deduce that 4.1.5 holds.

Since $Y_2 \cup Z_i$ is a circuit, it now follows from Lemma 2.9 that

4.1.6. *$Y_2 \cap X$ is an X-strand of M_p.*

Because $\sqcap(Y_2 \cap X, L) = 1$, we see that $M_p \backslash (X - Y_2)$ has $(Y_2 \cap X, E(M_p) - X)$ as a 2-separation. As $Y_2 \cup Z_i$ is a circuit of M_p, the elements of $Y_2 \cap X$ are in series in $M_p \backslash (X - Y_2)$. Pick u in $Y_2 \cap X$ and contract the elements of $(Y_2 \cap X) - u$ from $M_p \backslash (X - Y_2)$. In this matroid M'_p, the set $\{u, p\} \cup (Y_2 - (X \cup p)) \cup Z_i$ is a circuit. Restricting M'_p to the union of this circuit and $\{x, y\}$, we see that the resulting matroid is the 2-sum with basepoint w of a 5-point line $\{u, p, x, y, w\}$ and a circuit $w \cup (Y_2 - (X \cup p)) \cup Z_i$. We deduce that

4.1.7. *$(Y_2 - (X \cup p)) \cup Z_i \cup \{x, y\}$ and $(Y_2 \cap X) \cup \{x, y\}$ are circuits of M.*

We show next that

4.1.8. $\sqcap((Y_2 - (X \cup p)) \cup \{x, y\}, Z) = 1$.

As $\{x, y\}$ spans p, we see that $\sqcap((Y_2 - (X \cup p)) \cup \{x, y\}, Z) = \sqcap((Y_2 - X) \cup \{x, y\}, Z)$. By Lemma 2.3,

$$\sqcap((Y_2 - X) \cup \{x, y\}, Z) + \sqcap(Y_2 - X, \{x, y\}) = \sqcap(\{x, y\}, (Y_2 - X) \cup Z) + \sqcap(Y_2 - X, Z).$$

As Y_2 is an $(X \cup Y \cup p)$-strand meeting X, the right-hand side is at most 2. Now $\sqcap(Y_2 - X, \{x, y\}) \geq 1$, so $\sqcap((Y_2 - X) \cup \{x, y\}, Z) \leq 1$. The result follows by 4.1.7.

By Lemma 2.9, 4.1.7, and 4.1.8, we deduce that

4.1.9. $(Y_2 - (X \cup p)) \cup \{x, y\}$ *is an* $(X \cup Y)$*-strand of* M.

We now apply Theorem 1.1. Suppose that $\sqcap(Y_1, Z_2) = 0$. Then, for each k in $\{1, 2\}$, we have $\sqcap(Y_2, Z_k) = 1$, so $Y_2 \cup Z_k$ is a circuit of M_p. Hence, replacing Z_i by Z_k in the proof of 4.1.7 gives that $(Y_2 - (X \cup p)) \cup \{x, y\} \cup Z_k$ is a circuit of M. Thus $\sqcap((Y_2 - (X \cup p)) \cup \{x, y\}, Z_k) = 1$ and the $(X \cup Y)$-strand $(Y_2 - (X \cup p)) \cup \{x, y\}$ and the Z-strand Z_2 yield a contradiction to Theorem 1.1 in M.

We may now assume that $\sqcap(Y_1, Z_2) = 1$. As $\sqcap(Y_2, Z_i) = 1$, we have $\sqcap(Y_2, Z_j) = 0$ where $\{i, j\} = \{1, 2\}$. Now $(Y_2 - (X \cup p)) \cup Z_i \cup \{x, y\}$ is a circuit of M, so, by Theorem 1.1, $\sqcap((Y_2 - (X \cup p)) \cup \{x, y\}, Z_j) = 1$. Hence, by Lemma 2.8, M has $(Y_2 - (X \cup p)) \cup \{x, y\} \cup Z_j$ as a circuit. Thus

4.1.10. $r((Y_2 - (X \cup p)) \cup Z_j \cup \{x, y\}) = |Y_2 - X| + |Z_j|$.

Now

$$\begin{aligned}
r(Y_2 \cup Z_j) &= r(Y_2 \cup Z_j \cup \{x, y\}) \text{ as } \sqcap(Y_2, L) = 2;\\
&= r((Y_2 - p) \cup Z_j \cup \{x, y\})\\
&= r((Y_2 \cap X) \cup (Y_2 - (X \cup p)) \cup Z_j \cup \{x, y\})\\
&\leq r(Y_2 \cap X) + r((Y_2 - (X \cup p)) \cup Z_j \cup \{x, y\}) - 1\\
&\qquad\qquad \text{as } \sqcap(Y_2 \cap X, L) = 1;\\
&\leq |Y_2 \cap X| + (|Y_2 - X| + |Z_j|) - 1 \text{ by 4.1.10};\\
&= |Y_2| + |Z_j| - 1.
\end{aligned}$$

Thus $\sqcap(Y_2, Z_j) \geq 1$, a contradiction. □

Proof of Theorem 4.1. Take some edge f of T. We can extend M by p_{f1} to get a matroid M_1 in which $Y_{f1} \cup p_{f1}$ and $Z_{f1} \cup p_{f1}$ are circuits. By Theorem 1.2, for all pairs (e, k) other than $(f, 1)$ for which $e \in E(T)$ and $k \in \{1, 2, \ldots, m_e\}$, we can extend M_1 by p_{ek} so that $Y_{ek} \cup p_{ek}$ and $Z_{ek} \cup p_{ek}$ are circuits. We now repeat this process using M_1 in place of M. Continuing in this way, it is clear that we will obtain the required result. At each stage of the process, the matroid we obtain is unique, so the matroid obtained at the conclusion of the process is unique. □

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