ON THE EQUIVARIANT K-THEORY OF THE NILPOTENT CONE IN THE GENERAL LINEAR GROUP

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Abstract. Let G be a simple complex algebraic group. Lusztig and Vogan have conjectured the existence of a natural bijection between the set of dominant integral weights of G, and the set of pairs consisting of a nilpotent orbit and a finite-dimensional irreducible representation of the isotropy group of the orbit. This conjecture was established by the author for $G = GL(n, \mathbb{C})$, and by Bezrukavnikov for any G, in slightly different contexts. In this paper, we show that these two bijections for $GL(n, \mathbb{C})$ coincide.

1. Introduction

Let G be a connected complex reductive Lie group, $\mathfrak{g}$ its Lie algebra, and T a maximal torus. Let Λ^+ be the set of dominant weights of G with respect to some chosen Borel subgroup B containing T. Finally, let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, and let Ω be the set of pairs

$$\left\{ (C, \mathcal{E}) \;\middle|\; \begin{array}{c} C \subset \mathcal{N} \text{ a nilpotent orbit,} \\ \text{and } \mathcal{E} \text{ an irreducible } G\text{-equivariant vector bundle on } C \end{array} \right\}.$$

(If we fix an element $e \in C$, the vector bundles $\mathcal{E}$ correspond to irreducible representations of the centralizer G^e.) Lusztig and Vogan have independently conjectured the existence of a natural bijection between Λ^+ and Ω. In [1], this bijection was established for $G = GL(n, \mathbb{C})$ using combinatorial methods to carry out calculations in the Grothendieck group of equivariant coherent sheaves on $\mathcal{N}$. A related bijection appears in the work of Xi in the context of two-sided cells in the affine Hecke algebra of type A_n [7].

Subsequently, the result was established for general G by Bezrukavnikov [3], by a consideration of two different possible t-structures on the bounded derived category of equivariant coherent sheaves on $\mathcal{N}$. Let $\mathcal{D}$ denote this latter category. On the one hand, Deligne's theory of perverse coherent sheaves (see [2]) yields one t-structure on $\mathcal{D}$. The irreducible objects in the heart of this t-structure are certain intersection cohomology complexes naturally indexed by Ω. These objects, denoted $\mathrm{IC}_{C,\mathcal{E}}$, are obtained as the "middle extension" $j_{!*}(\mathcal{E}[-\frac{1}{2}\dim C])$, where $j : C \hookrightarrow \mathcal{N}$ is the inclusion. On the other hand, given a weight $\lambda \in \Lambda^+$, we obtain an object A_λ of $\mathcal{D}$ as follows. Let $\mathcal{L}_\lambda$ be the line bundle over the flag manifold G/B corresponding to λ, and let $\mathfrak{n}$ be the nilradical of the Lie algebra of B. Consider the diagram

$$G/B \xleftarrow{p} G \times_B \mathfrak{n} \xrightarrow{\pi} \mathcal{N}, \tag{1}$$

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where p is the obvious projection, and π is the map $(g, x) \mapsto g \cdot x$. We set $A_\lambda = R\pi_* p^* \mathcal{L}_\lambda$. Bezrukavnikov showed that $\mathcal{D}$ has the structure of a quasi-hereditary category, and that the $\{A_\lambda\}$ constitute a quasi-exceptional set for it, with respect to any total order on Λ^+ compatible with the standard partial order. (See [3] for full details.) Now, this situation gives rise to a new t-structure on $\mathcal{D}$ with the property that the irreducible objects of its heart are naturally in bijection with the set of quasi-exceptional objects in $\mathcal{D}$. Finally, Bezrukavnikov showed that these two t-structures coincide, so one naturally obtains a bijection between Ω and Λ^+.

In this paper, we will show that the combinatorially defined bijection of [1] coincides with that established by Bezrukavnikov. Primarily, this consists of showing that certain preliminary calculations of [1] carried out in the category of equivariant coherent sheaves can be duplicated in the derived category with formally similar results. These calculations are performed in Section 2. The consequence of this is that the same combinatorial methods can be used to obtain the bijection, whether in the non-derived setting of [1], or in the category $\mathcal{D}$, as in the present paper. The main result is stated and proved at the end of Section 3. The remainder of the paper is devoted to the development of the required combinatorial methods and the construction of the algorithms.

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2. Resolutions of nilpotent orbits

We begin by introducing some notation and reviewing Bezrukavnikov's result in detail. Recall that $\mathcal{D}$ is the bounded derived category of G-equivariant coherent sheaves on $\mathcal{N}$. Let $\mathcal{A}$ denote the bounded derived category of algebraic G-modules, and let $K(\mathcal{D})$ and $K(\mathcal{A})$, respectively, denote the Grothendieck groups of these two categories. For any object X in either of these categories, we denote its class in the Grothendieck group by $[X]$. With this terminology, we have the following.

Theorem 2.1 (Bezrukavikov [3])**.** *$K(\mathcal{D})$ is a free abelian group, and each of the sets $\{[A_\lambda] \mid \lambda \in \Lambda^+\}$ and $\{[\mathrm{IC}_{C,\mathcal{E}}] \mid (C, \mathcal{E}) \in \Omega\}$ form a basis for it. In addition, there is a unique bijection $\eta : \Lambda^+ \to \Omega$ such that*

$$[\mathrm{IC}_{\eta(\lambda)}] \in \mathrm{span}\{[A_\mu] \mid \mu \leq \lambda\}$$

for each λ. Moreover, when $[\mathrm{IC}_{\eta(\lambda)}]$ is expressed as a linear combination of the $\{[A_\mu]\}$, $[A_\lambda]$ occurs with coefficient ± 1.

We will refer to the two bases described in this theorem as the "A-basis" and the "IC-basis," respectively. We also recall at this stage that the construction of A_λ in Section 1 makes sense for any weight λ, not merely the dominant ones. Since we will occasionally need to make use of such objectss, we take a moment now to recall the relationship between such complexes and those coming from dominant weights.

Proposition 2.2. *Let $\lambda \in \Lambda^+$, and let w be an element of the Weyl group of G. We have $[A_{w\lambda}] - [A_\lambda] \in \mathrm{span}\{[A_\mu] \mid \mu \in \Lambda^+,\ \mu < \lambda\}$.*

Henceforth, G will always denote the group $GL(n, \mathbb{C})$. In this section, we will construct certain analogues of A_λ with B replaced by a parabolic subgroup, in the hope that these complexes may serve as an intermediary of sorts between the $\{[A_\lambda]\}$ and the $\{[\mathrm{IC}_{C,\mathcal{E}}]\}$. Recall that every nilpotent orbit for G is Richardson, so given an orbit $C \subset \mathcal{N}$, we can choose a parabolic subgroup P (with Levi decomposition LU and Lie algebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$) such that C is the Richardson orbit associated to P. This means, in particular, that C meets $\mathfrak{u}$ in a dense open subset, and, in addition, that P acts transitively on $C \cap \mathfrak{u}$. We fix an element $e \in C \cap \mathfrak{u}$, and let G^e denote the centralizer of e in G. Recall that we have $G^e \subset P$.

Consider the following analogue of (1):

$$G/P \xleftarrow{p} G \times_P \mathfrak{u} \xrightarrow{\pi} \overline{C} \overset{i}{\hookrightarrow} \mathcal{N} \tag{2}$$

Given a weight λ that is dominant for L, let V_λ (or V_λ^L, if there is ambiguity) be the irreducible L-representation of highest weight λ. We can regard V_λ as a P-module by letting U act trivially; this, in turn, gives rise to a G-equivariant vector bundle over G/P. Let $\mathcal{V}_\lambda$ (or $\mathcal{V}_\lambda^L$) be the sheaf of sections of this vector bundle. We define A_λ^L to be the complex $Ri_* R\pi_* p^* \mathcal{V}_\lambda^L$.

Lemma 2.3. *The map π is proper and birational; in particular, it is an isomorphism when restricted to $\pi^{-1}(C)$.*

Proof. Define a map $m : G \times_P \mathfrak{u} \hookrightarrow G/P \times \mathcal{N}$ by $m(g, x) = (gP, g \cdot x)$. This map is injective, and its image in $G/P \times \mathcal{N}$ is closed, so m is proper. The map π is simply the composition of m with the projection $G/P \times \mathcal{N} \to \mathcal{N}$, which is also proper, so π is proper.

Define a map $q : C \to G \times_P \mathfrak{u}$ by $q(x) = (g, e)$, where $g \in G$ is any element such that $g \cdot e = x$. Of course, g is only well-defined up to right-multiplication by an element of G^e, but since $G^e \subset P$, we obtain that (g, e) is a well-defined point of $G \times_P \mathfrak{u}$. It is clear that q is a right inverse to $\pi|_{\pi^{-1}(C)}$. To show that it is also a left inverse, we would need to establish that the image of q is exactly $\pi^{-1}(C)$. But since $C \cap \mathfrak{u}$ is a single P-orbit, we know that for any $(h, x) \in \pi^{-1}(C)$, there is a $y \in P$ such that $x = y \cdot e$, so $(h, x) = (hy, e) = q(hy \cdot e)$. We conclude that $\pi|_{\pi^{-1}(C)}$ is an isomorphism, so π itself is birational. □

Proposition 2.4. *Let $\mathcal{W}$ be the sheaf of sections of the G-equivariant vector bundle over C arising from V_λ regarded as a G^e-representation. Then $A_\lambda^L|_C \simeq \mathcal{W}$.*

Proof. The pullback of the vector bundle $G \times_P V_\lambda \to G/P$ via p is a new G-equivariant vector bundle over $G \times_P \mathfrak{u}$ whose fiber is V_λ. Since π is an isomorphism over C, the restriction of $R\pi_* p^* \mathcal{V}_\lambda$ to C is again the sheaf of sections of a vector bundle with fibre V_λ. Now, G-equivariant vector bundles over any space with a transitive G-action correspond to representations of the isotropy group, so the vector bundle to which A_λ^L corresponds must coincide with that obtained by regarding V_λ as a G^e-representation. □

Corollary 2.5. *In the situation of Proposition 2.4, suppose that $\mathcal{E}$ is an irreducible G-equivariant subbundle of $\mathcal{W}$. Then, when $[A_\lambda^L]$ is expressed in the* IC*-basis, $[\mathrm{IC}_{C,\mathcal{E}}]$ occurs with nonzero coefficient.*

Proof. It is clear that $[A_\lambda^L]$ can be expressed in the form

$$[A_\lambda^L] = \sum_{C' \subset \overline{C}} n_{C',\mathcal{E}'} [\mathrm{IC}_{C',\mathcal{E}'}],$$

since A^L_λ is supported on $\overline{C}$. Let $j : C \hookrightarrow \mathcal{N}$ denote the inclusion map. Now, $j^*\mathrm{IC}_{C',\mathcal{E}'}$ vanishes for any $C' \subset \overline{C}$ with $C' \neq C$, so we obtain

$$[\mathcal{W}] = [j^* A^L_\lambda] = \sum n_{C,\mathcal{E}'}[j^*\mathrm{IC}_{C,\mathcal{E}'}]$$

in the appropriate Grothendieck group. But now, all the complexes in the above equation are just vector bundles, and those on appearing on the right-hand side are irreducible and pairwise inequivalent. The hypothesis of the statement to be proved, then, says exactly that for some particular $\mathcal{E}$, we have $n_{C,\mathcal{E}} \neq 0$, as desired. □

Proposition 2.6. *In the Grothendieck group $K(\mathcal{A})$, we have*

$$[R\Gamma(\mathcal{N}, A^L_\lambda)] = \sum_{w\in W_L} (-1)^{l(w)}[\mathrm{Ind}_T^G \mathbb{C}_{\lambda+\rho_L-w\rho_L}],$$

where W_L is the Weyl group of L, and ρ_L is half the sum of its positive roots.

Proof. Since the derived functor of global sections is unchanged by derived push-forwards, we have

$$R\Gamma(\mathcal{N}, A^L_\lambda) \simeq R\Gamma(G\times_P \mathfrak{u}, p^*\mathcal{V}_\lambda) \simeq R\Gamma(G/P, Rp_*p^*\mathcal{V}_\lambda). \tag{3}$$

Now, the projection from the total space of a vector bundle to its base is an affine morphism. Since p is affine, p_* is exact on quasicoherent sheaves, so we can replace "Rp_*" above by simply "p_*." Finally, we claim that $p_*p^*\mathcal{V}_\lambda$ is the sheaf associated to the infinite-dimensional vector bundle

$$G\times_P (S(\mathfrak{u}^*)\otimes V_\lambda) \to G/P,$$

where $S(\mathfrak{u}^*)$ denotes the symmetric algebra on $\mathfrak{u}^*$. Indeed, because $\mathcal{V}_\lambda$ is a locally free sheaf of finite rank, we have the "projection formula"

$$p_*p^*\mathcal{V}_\lambda \simeq p_*\mathcal{O}_{G\times_P\mathfrak{u}} \otimes \mathcal{V}_\lambda,$$

where $\mathcal{O}_{G\times_P\mathfrak{u}}$ is the structure sheaf of $G\times_P\mathfrak{u}$. Moreover, because $p : G\times_P\mathfrak{u} \to G/P$ is in fact a vector bundle, it is well-known that $p_*\mathcal{O}_{G\times_P\mathfrak{u}}$ is the sheaf associated to the vector bundle of regular functions on the fibres of p. (See, for example, [5, Ex. II.5.18].) That is, $p_*\mathcal{O}_{G\times_P\mathfrak{u}}$ is the sheaf arising from the vector bundle $G\times_P S(\mathfrak{u}^*) \to G/P$. We conclude that

$$R\Gamma(\mathcal{N}, A^L_\lambda) \simeq R\Gamma(G/P, G\times_P (S(\mathfrak{u}^*)\otimes V_\lambda)). \tag{4}$$

We now apply an analogue of [6, Lemma 2.1], which says that for any finite-dimensional L-module F, we have the following equality of virtual G-modules:

$$\sum(-1)^i[H^i(G/P, G\times_P (S(\mathfrak{u}^*)\otimes F))] = [\mathrm{Ind}_L^G F].$$

Although that lemma is stated in a nonderived setting, it is readily seen that a parallel argument establishes that

$$[R\Gamma(G/P, G\times_P (S(\mathfrak{u}^*)\otimes F))] = [\mathrm{Ind}_L^G F] \tag{5}$$

in $K(\mathcal{A})$.

It remains to calculate $[\mathrm{Ind}_L^G V_\lambda]$ in terms of various $[\mathrm{Ind}_T^G \mathbb{C}_\mu]$'s. This turns out to follow from an elementary computation of weight multiplicities. We can combine

Steinberg's formula for multiplicities in a tensor product with Kostant's formula for multiplicities of weights to obtain the first of the following equalities:

$$\begin{aligned}\dim \operatorname{Hom}_L(V_\mu \otimes V_\nu, V_\lambda) &= \sum_{w \in W_L} (-1)^{l(w)} \dim \operatorname{Hom}_T(\mathbb{C}_{\lambda+\rho_L - w(\mu+\rho_L)}, V_\nu) \\ &= \sum_{w \in W_L} (-1)^{l(w)} \dim \operatorname{Hom}_L(V_\nu, \operatorname{Ind}_T^G \mathbb{C}_{\lambda+\rho_L-w(\mu+\rho_L)}).\end{aligned}$$

The second equality comes from Frobenius reciprocity. Now, let us take $\mu = 0$. The first and last members of the above equation tell us that we have an equality of virtual L-modules

$$V_\lambda \simeq \sum_{w \in W_L} (-1)^{l(w)} \operatorname{Ind}_T^L \mathbb{C}_{\lambda+\rho_L - w\rho_L}.$$

The proposition follows by combining this with (4) and (5). □

One special case of the preceding result is that in which $L = T$. Then, the Weyl group is trivial, so we get

$$[R\Gamma(\mathcal{N}, A^T_\lambda)] = [\operatorname{Ind}_T^G \mathbb{C}_\lambda].$$

Since the $[A^T_\lambda]$ form a basis of $K(\mathcal{D})$, and since the $[\operatorname{Ind}_T^G \mathbb{C}_\lambda]$ are linearly independent elements of $K(\mathcal{A})$, we see that the map $K(\mathcal{D}) \to K(\mathcal{A})$ induced by $R\Gamma$ is injective. Therefore, Proposition 2.6 implies the following.

Corollary 2.7. *In $K(\mathcal{D})$, we have*

$$[A^L_\lambda] = \sum_{w \in W_L} (-1)^w [A^T_{\lambda+\rho_L-w\rho_L}].$$

3. Computing with parabolic subgroups

This section is devoted primarily to loose, speculative discussion based on the results of the preceding section. The idea is to lay an intuitive groundwork for the work carried out in the remainder of the paper. At the end of the section, however, we will make some precise statements about the bijection whose existence is to be established, and we will prove that it coincides with Bezrukavnikov's.

Our hope, in view of Corollaries 2.5 and 2.7, is to use the $[A^L_\lambda]$'s as a sort of intermediary between the $[\mathrm{IC}_{C,\mathcal{E}}]$'s and the $[A^T_\lambda]$'s. In particular, given an A^L_λ, suppose we take $\mathcal{E}$ to be the vector bundle on C arising from the G^e-representation whose highest weight is the restriction of λ. (Here we are using the fact that all the groups G^e are connected in $GL(n,\mathbb{C})$, so their irreducible representations are parametrized by highest weights.) Corollary 2.5 says that $[\mathrm{IC}_{C,\mathcal{E}}]$ occurs with nonzero coefficient when $[A^L_\lambda]$ is expressed in the IC-basis.

Let Φ be the set of pairs $\{(L,\lambda)\}$ up to G-conjugacy, where L is a Levi subgroup of G containing T, and λ is a dominant weight for L. Let $\kappa : \Phi \to \Omega$ be the map sending (L,λ) to the pair $(C,\mathcal{E})$, where C is the Richardson class obtained from L, and $\mathcal{E}$ is as in the previous paragraph.

Now, Theorem 2.1 tells us that for a given pair $(C,\mathcal{E})$, $\eta^{-1}(C,\mathcal{E})$ is the largest weight μ (with respect to the usual partial order) such that $[A_\mu]$ occurs with nonzero coefficient in $[\mathrm{IC}_{C,\mathcal{E}}]$. An examination of the formula in Corollary 2.7 reveals that the largest weight μ such that $[A_\mu]$ occurs in $[A^L_\lambda]$ is given by $\mu = \lambda + 2\rho_L$. We therefore define a map $\tau : \Phi \to \Lambda^+$ by

$$\tau(L, \lambda) = \text{the unique dominant weight in the } W_G\text{-orbit of } \lambda + 2\rho_L.$$

The maps we have defined,

$$\Lambda^+ \xleftarrow{\tau} \Phi \xrightarrow{\kappa} \Omega, \tag{6}$$

are far from being bijections. Nevertheless, we will eventually see with relative ease that they are at least both surjections. Given a pair $(L, \lambda) \in \Phi$, we know that $[\mathrm{IC}_{\kappa(L,\lambda)}]$ occurs when $[A^L_\lambda]$ is written in the IC-basis, and $[A_{\tau(L,\lambda)}]$ occurs when it is written in the A-basis.

The hope is to try to zoom in on a certain subset of Φ such that the restrictions of both τ and κ to this subset turn out to be bijections. How might we characterize this subset? Let $\mu \in \Lambda^+$ be such that $\eta(\mu) = \kappa(L, \lambda) = (C, \mathcal{E})$. Since $[A^L_\lambda]$ contains a contribution from $[\mathrm{IC}_{C,\mathcal{E}}]$, it ought to have a term $[A_\mu]$ as well, but this latter term might get cancelled by a contribution from some other $[\mathrm{IC}_{C',\mathcal{E}'}]$ corresponding to a weight larger than μ. We can try to prevent such larger terms from occuring by trying to choose λ in such a way as to make $\tau(L, \lambda)$ as small as possible to begin with. Sections 4–8 are devoted to establishing, in slightly different language, the following fact.

Claim 3.1. *There exists a set $\Phi^\circ \subset \Phi$ with the property that the restricted maps $\kappa : \Phi^\circ \to \Omega$ and $\tau : \Phi^\circ \to \Omega$ are both bijections. Moreover, for a fixed pair $(C, \mathcal{E}) \in \Omega$, and all $(L, \lambda) \in \kappa^{-1}(C, \mathcal{E})$, the function $(L, \lambda) \mapsto \|\tau(L, \lambda)\|^2$ takes its minimal value on the unique pair $(L, \lambda) \in \kappa^{-1}(C, \mathcal{E}) \cap \Phi^\circ$.*

The composition $\kappa \circ \tau^{-1}$ then gives a bijection $\Lambda^+ \to \Omega$. Let us denote this bijection γ.

Theorem 3.2. *Assume that Claim 3.1 holds. Then the bijection $\gamma : \Lambda^+ \to \Omega$ coincides with Bezrukavikov's bijection $\eta : \Lambda^+ \to \Omega$.*

Proof. Given $\lambda \in \Lambda^+$, suppose that $(L, \mu) \in \Phi^\circ$ is the pair with the property that $\tau(L, \mu) = \lambda$. Let B_λ denote the complex A^L_μ. We have

1. $[B_\lambda] \in \mathrm{span}\{[A_{\lambda'}] \mid \lambda' \le \lambda\}$, and $[A_\lambda]$ occurs in $[B_\lambda]$ with coefficient ± 1.
2. $[\mathrm{IC}_{\gamma(\lambda)}]$ occurs with nonzero coefficient in $[B_\lambda]$.

We will prove by induction with respect to the standard partial order on Λ^+ that $\eta(\mu) = \gamma(\mu)$ for all $\mu \in \Lambda^+$. When $\lambda = 0$, the above properties give us that

$$[\mathrm{IC}_{\eta(0)}] = \pm[A_0] = \pm[B_0] = n_0[\mathrm{IC}_{\gamma(0)}] + \sum n_{C',\mathcal{E}'}[\mathrm{IC}_{C',\mathcal{E}'}],$$

with $n_0 \neq 0$. But then, comparing the leftmost and rightmost parts of the equation shows that in fact $n_0 = 1$, and all other $n_{C',\mathcal{E}'}$ are 0. We conclude that $\eta(0) = \gamma(0)$.

For the inductive step, property (2) gives us that

$$[B_\lambda] = n_0[\mathrm{IC}_{\gamma(\lambda)}] + \sum_i n_i[\mathrm{IC}_{C_i,\mathcal{E}_i}],$$

with $n_0 \neq 0$. On the other hand, property (1), together with Theorem 2.1, gives us that

$$[B_\lambda] = \pm[A_\lambda] + \sum_{\lambda' < \lambda} m_{\lambda'}[A_{\lambda'}] = \pm[\mathrm{IC}_{\eta(\lambda)}] + \sum_{\lambda' < \lambda} k_{\lambda'}[\mathrm{IC}_{\eta(\lambda')}]. \tag{7}$$

Comparing these two equations, we see that $[\mathrm{IC}_{\gamma(\lambda)}]$ must occur somewhere among the terms in the rightmost expression in (7). Now, we know by the inductive hypothesis that $\eta(\lambda') = \gamma(\lambda')$ for all $\lambda' < \lambda$. Since γ is a bijection, this means in particular that $\gamma(\lambda) \neq \eta(\lambda')$ for any $\lambda' < \lambda$. The only remaining term in (7) is $[\mathrm{IC}_{\eta(\lambda)}]$, so we conclude that $\eta(\lambda) = \gamma(\lambda)$, as desired. □

4. Weight diagrams

In this section, we introduce the computational machinery required to construct the bijection γ. Partitions play a major role in this matter, so we begin with some notations for them. We write partitions sometimes as $\alpha = [\alpha_1 \geq \cdots \geq \alpha_l]$ and sometimes as $\alpha = [k_1^{a_1}, \ldots, k_l^{a_l}]$, where $k_1 > \cdots > k_l$, and a_i is the multiplicity of k_i.

We now need to describe the sets Ω and Λ^+ for G. Dominant weights of G are given by decreasing n-tuples of integers. Nilpotent orbits in G are indexed by partitions. We shall write C_α for the nilpotent orbit labelled by α, and G^α for its isotropy group. If $\alpha = [k_1^{a_1}, \ldots, k_l^{a_l}]$, then G^α has a reductive quotient isomorphic to

$$GL(a_1, \mathbb{C}) \times \cdots \times GL(a_l, \mathbb{C}).$$

Therefore, we identify Ω with the set of pairs

$$\{(\alpha; (\mu_1, \ldots, \mu_l))\},$$

where $\alpha = [k_1^{a_1}, \ldots, k_l^{a_l}]$ is a partition of n, and each μ_i is a decreasing a_i-tuple of integers (*i.e.*, a dominant weight for $GL(a_i, \mathbb{C})$).

The following definition introduces the combinatorial objects that will be our principal tool for building the bijection γ. These objects play a role closely related to that of Φ in Section 3, but they do not parametrize it *per se*.

Definition 4.1. Let $a_1, \cdots, a_l$ be positive integers such that $a_1 + \cdots + a_l = n$, and let α be the partition of n whose parts are $a_1, \ldots, a_l$. (We have *not* assumed that $a_1 \geq \cdots \geq a_l$.) An ordered tuple $X = (X_1, \ldots, X_l)$, where each X_i is an ordered a_i-tuple of integers, is called a *weight diagram of shape-class* α. The X_i's are called the *rows* of X, and the j-th entry in a given row X_i is referred to by the double-subscript notation X_{ij}. The set of all weight diagrams whose shape-class is a partition of n is denoted D_n.

Example 4.2. The tuple $X = ((8, 2), (2, 3, 1, 0), (5, 6), (1, 1, 1))$ is an example of a weight diagram of shape-class $[4, 3, 2^2]$. We typically draw weight diagrams in a manner reminscent of Young tableaux: for instance, we may write

$$X = \begin{array}{|c|c|cc} \hline 8 & 2 & \multicolumn{1}{|c}{} & \\ \hline 2 & 3 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} \\ \hline 5 & 6 & \multicolumn{1}{|c}{} & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{1} & \\ \cline{1-3} \end{array}$$

We do not say that X is "of shape $[4, 3, 2^2]$," as one might when discussing Young tableaux, because in this case the partition does not determine the shape of the diagram.

There is an obvious map $\pi : D_n \to \Lambda^+$, given by taking all the entries of a weight diagram X, writing them in decreasing order, and regarding the resulting n-tuple as a weight of G. There is also a map $h : D_n \to \Phi$, defined as follows. Suppose X

is a weight diagram of shape-class α, and let $\alpha^* = [\alpha_1^* \geq \cdots \geq \alpha_l^*]$ be the transpose partition to α (*i.e.*, α_i^* is the number of parts of α that are greater than or equal to i). Let L be a Levi subgroup isomorphic to

$$GL(\alpha_1^*, \mathbb{C}) \times \cdots \times GL(\alpha_l^*, \mathbb{C}). \tag{8}$$

Finally, let λ be the dominant weight for L whose restriction to $GL(\alpha_i^*, \mathbb{C})$ is the α_i^*-tuple obtained by arranging the entries of the ith column of X in decreasing order. The map h is given by the construction $X \mapsto (L, \lambda)$ that we have just described.

We will require an analogue of the map $\tau : \Phi \to \Lambda^+$ from Section 3. By an abuse of notation, we will generally write $\tau(X)$ for a weight diagram X when we really mean $\tau(h(X))$. In addition, it will be particularly useful to have a description of τ entirely at the level of weight diagrams. To that end, we now introduce a map $E : D_n \to D_n$, defined as follows. EX is a diagram of the same shape as X, whose entries are given by

$$\begin{aligned}(EX)_{ir} = X_{ir} + \#\{X_{jr} \mid X_{jr} < X_{ir}, \text{ or } X_{jr} = X_{ir} \text{ with } j > i\} \\ - \#\{X_{jr} \mid X_{jr} > X_{ir}, \text{ or } X_{jr} = X_{ir} \text{ with } j < i\}.\end{aligned}$$

The key property of E, whose verification is routine, is that $\pi(EX) = \tau(X)$.

Similarly, we shall write $\kappa(X)$ in place of $\kappa(h(X))$. A description of κ at the level of weight diagrams is somewhat more difficult than that for τ. To begin with, suppose that the diagram X is of shape-class α, and that $h(X) = (L, \lambda)$. The group L is as in (8). We need to understand how the Levi factor of G^α sits inside L. Suppose the number k occurs in α with multiplicity a. Then the reductive quotient of G^α contains a factor $GL(a, \mathbb{C})$. According to [4, Theorem 6.1.3], this $GL(a, \mathbb{C})$ is imbedded diagonally across the first k factors of L:

$$GL(a, \mathbb{C}) \hookrightarrow \underbrace{GL(a, \mathbb{C}) \times \cdots \times GL(a, \mathbb{C})}_{k \text{ factors}} \subset GL(\alpha_1^*, \mathbb{C}) \times \cdots \times GL(\alpha_k^*, \mathbb{C}).$$

Now, given a weight for some $GL(\alpha_i^*, \mathbb{C})$, we compute its restriction to the ith copy of $GL(a, \mathbb{C})$ by simply forgetting all but a of its coordinates, and thus obtaining a certain a-tuple. Starting from a weight for L, then, we can in this way obtain weights for each of the k copies of $GL(a, \mathbb{C})$. Then, the restriction of the original weight to the diagonal copy of $GL(a, \mathbb{C})$ is simply the sum of the weights of the k copies of $GL(a, \mathbb{C})$.

Let us now translate this procedure into concise operation on weight diagrams. Suppose that $\alpha = [k_1^{a_1}, \ldots, k_l^{a_l}]$. We now produce an a_i-tuple for each i as follows: suppose $X_{i_1}, \ldots, X_{i_{a_i}}$ are the rows of X of length k_i (note that X must have exactly a_i rows of length k_i). Form the sums $b_i^j = \sum_{m=1}^{k_i} X_{i_j, m}$. Let β_i be the a_i-tuple obtained by rearranging $(b_i^1, \ldots, b_i^{a_i})$ in decreasing order. We have $\kappa(X) = (\alpha, (\beta_1, \ldots, \beta_l))$.

Example 4.3. Let X be the weight diagram of Example 4.2. The following two facts are easily read off from the diagram:

$$\pi(X) = (8, 6, 5, 3, 2, 2, 1, 1, 1, 1, 0)$$
$$h(X) = \big(GL(4) \times GL(4) \times GL(2) \times GL(1), ((8, 5, 2, 1), (6, 3, 2, 1), (1, 1), (0))\big)$$

Next, we compute that

$$EX = \begin{array}{|c|c|c|c|}\cline{1-2} 6 & 9 \\ \hline 1 & 4 & 2 & 0 \\ \hline 11 & 1 \\ \cline{1-3} -2 & -2 & 0 \\ \cline{1-3}\end{array}$$

and therefore

$$\tau(X) = (11, 9, 6, 4, 2, 1, 1, 0, 0, -2, -2).$$

Now, the reductive quotient of G^α is isomorphic to $GL(1) \times GL(1) \times GL(2)$. We obtain

$$\kappa(X) = ([4, 3, 2^2]; (6, 3, (11, 10))).$$

We conclude this section by proving that weight diagrams are indeed useful for computing with the objects introduced in Section 2.

Proposition 4.4. *Let X be a weight diagram, and suppse that we have*

$$h(X) = (L, \mu), \qquad \kappa(X) = (C, \mathcal{E}), \qquad \tau(X) = \lambda.$$

Then, when $[A^L_\mu]$ is expressed in the IC*-basis, $[\mathrm{IC}_{C,\mathcal{E}}]$ occurs with nonzero coefficient, and when it is expressed in the A-basis, $[A_\lambda]$ occurs with coefficient ± 1.*

Proof. The second part of the statement follows immediately from Corollary 2.7. The first part is almost as immediate a consequence of Corollaries 2.5: we only need to check that the G^α-representation E corresponding to the bundle $\mathcal{E}$ does indeed occur in the restriction of V^L_μ to G^α. This is clear: the G^α-submodule generated by the μ-weight space of V^L_μ is a representation whose highest weight is the restriction of μ, which is exactly what E is. □

5. Distinguished weight diagrams

Continuing in the spirit of Section 3, we need to isolate a certain subset of D_n such that the restrictions of τ and κ to this subset turn out to be bijections. This section is devoted to defining this subset explicitly and establishing some basic properties of it. Before that, however, we need some terminology for describing the relative positions and values of entries in a weight diagram.

Definition 5.1. X_{ir} and X_{jr} are said to be *column-consecutive* if $j > i$ and the positions $X_{i+1,r}, \ldots, X_{j-1,r}$ are all empty. X_{jr} is then the *column-successor* of X_{ir}, and X_{ir} is the *column-predecessor* of X_{jr}. An entry is *column-last* if it has no column-successor, and it is *column-first* if it has no column-predecessor.

Definition 5.2. X_{ir} is *lowerable* if either it is column-last, or it has column-successor $X_{i'r}$ with $X_{ir} > X_{i'r}$. It is *raisable* if either it is column-first, or it has column-predecessor $X_{i'r}$ with $X_{ir} < X_{i'r}$.

To *lower* (resp. *raise*) X_{ir}, or X at position ir, is to construct a new diagram of the same shape and entries as X, except that its entry in position ir is obtained by subtracting 1 from (resp. adding 1 to) X_{ir}.

We will very often need to work with pairs of diagrams related by the map E. To avoid always having to give names to both these diagrams, we will make use of the following supplementary terminology.

Definition 5.3. Assume that $X \in D_n$ is in the image of E; say $X = E(Y)$. We say that X_{ir} is *E-lowerable* (resp. *E-raisable*) if Y_{ir} is lowerable (resp. raisable).

Remark 5.4. Note that a diagram is in the image of E if and only if it has the property that any two entries in the same column differ by at least 2. Moreover, if X is in the image E, then an entry X_{ir} is E-lowerable (resp. E-raisable) if and only if $X_{ir} \geq X_{i'r} + 2$ (resp. $X_{ir} \leq X_{i'r} - 2$), where $X_{i'r}$ is its column-successor (resp. column-predecessor).

Let $Y \in D_n$, and assume that Y is in the image of E. We state four properties that Y may have:

- $\mathbf{p_1}(r)$**:** $(r > 1)$ For any s, $1 \leq s < r$, such that $s \equiv r \pmod 2$, we have
  - (a) if r and s are odd and $Y_{is} < Y_{ir}$, then Y_{is} is not E-raisable.
  - (b) if r and s are even and $Y_{is} > Y_{ir}$, then Y_{is} is not E-lowerable.
- $\mathbf{p_2}(r)$**:** $(r > 1)$ For any s, $1 \leq s < r$, we have
  - (a) if $Y_{is} \leq Y_{ir} - 2$, then Y_{is} is not E-raisable.
  - (b) if $Y_{is} \geq Y_{ir} + 2$, then Y_{is} is not E-lowerable.
- $\mathbf{p_3}(r)$**:** $(r > 1)$ If r is odd, every difference $Y_{ir} - Y_{i,r-1}$ is either 0 or $+1$. If r is even, every such difference is either 0 or -1.
- $\mathbf{p_4}(r)$**:** Column r of Y has entries in decreasing order.

Every Y is said to have properties $\mathbf{p_1}(1)$ and $\mathbf{p_3}(1)$, for convenience.

We also define capital-letter versions of these properties: Y is said to have property $\mathbf{P_1}(r)$ if it has all of $\mathbf{p_1}(1), \ldots, \mathbf{p_1}(r)$; the properties $\mathbf{P_3}(r)$ and $\mathbf{P_4}(r)$ are defined similarly. We suppose that the properties $\mathbf{P_1}(0)$, $\mathbf{P_3}(0)$, and $\mathbf{P_4}(0)$ always hold, as do $\mathbf{P_1}(1)$ and $\mathbf{P_3}(1)$.

Definition 5.5. Given a weight diagram X, let $Y = EX$. The diagram X is said to be *distinguished*, and Y is said to be *E-distinguished*, if Y has properties $\mathbf{P_1}(r)$, $\mathbf{P_2}(r)$, $\mathbf{P_3}(r)$, and $\mathbf{P_4}(r)$ for all r. The set of distinguished weight diagrams is denoted D_n°, and the set of E-distinguished diagrams is denoted $E(D_n^\circ)$.

Example 5.6. Consider the following weight diagrams:

$$R = \begin{array}{cccc} 8 & & & \\ 6 & 6 & 7 & 7 \\ 3 & 3 & 3 & 2 \\ 1 & 1 & 1 & 0 \end{array} \qquad S = \begin{array}{ccccc} 8 & & & & \\ 6 & 6 & 7 & 7 & \\ 3 & 3 & 3 & & \\ 1 & 0 & 1 & 1 & 2 \end{array} \qquad T = \begin{array}{cccc} 8 & & & \\ 6 & 7 & 7 & 6 \\ 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 \end{array} \qquad U = \begin{array}{cccc} 7 & 6 & 7 & 6 \\ 8 & & & \\ 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 \end{array}$$

R fails to have property $\mathbf{p_1}(4)$ because $R_{42} > R_{44}$, yet R_{42} is E-lowerable. S fails to have $\mathbf{p_2}(5)$ because $S_{42} \leq S_{45} - 2$, yet S_{42} is E-raisable. T fails to have $\mathbf{p_3}(2)$ since $T_{22} - T_{21} = 1$ (it is only permitted to be 0 or -1). Finally, U is not even in the image of E, since its first column contains a pair of entries that differ by less than 2.

All the weight diagrams in the preceding example lay in the preimage under π of the weight $\lambda = (8, 7, 7, 6, 6, 3, 3, 3, 2, 1, 1, 1, 0)$. The following E-distinguished weight diagram also lies in $\pi^{-1}(\lambda)$:

$$X = \begin{array}{cccc} 8 & & & \\ 6 & 6 & 7 & 7 \\ 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 \end{array}$$

As a matter of fact, this diagram turns out to be the unique E-distinguished weight diagram in $\pi^{-1}(\lambda)$. Analogously, each fibre of κ contains a unique distinguished weight diagram X. The rest of the paper will be spent establishing these two assertions. Indeed, these two facts give rise to the desired bijection γ via the following diagram, an analogue of (6).

$$\Lambda^+ \xleftarrow{\tau=\pi\circ E} D_n^\circ \xrightarrow{\kappa} \Omega.$$

We can take one step towards establishing this bijection immediately from the above definitions: we can show that κ is injective.

Lemma 5.7. *Let X be a distinguished weight diagram, and suppose the ith row of X has maximal length (among the rows of X). If X' is the diagram obtained from X by deleting its ith row, then X' again distinguished.*

Proof. Since the deleted row had maximal length, every column of X' has exactly one fewer entry than the corresponding column of X. One then computes readily that

$$(EX')_{jr} = \begin{cases} (EX)_{jr} - 1 & \text{if } j \le i-1, \\ (EX)_{j+1,r} + 1 & \text{if } j \ge i. \end{cases}$$

It is clear that EX' satisfies $\mathbf{P_3}(r)$ and $\mathbf{P_4}(r)$ for all r. Now, since the notions of raisability and lowerability depend only on neighboring rows, it is also clear that the conditions of $\mathbf{p_1}(r)$ and $\mathbf{p_2}(r)$ hold on all rows of X' except possibly the $(i-1)$th and ith ones. Furthermore, we only need to check case (b) of each of those conditions on row $i-1$, and case (a) on row i.

We will now check $\mathbf{p_1}(r)$(b) on row $i-1$. Suppose that $s < r$, that s and r are both even, and that $EX'_{i-1,s} > EX'_{i-1,r}$. This means $EX_{i-1,s} > EX_{i-1,r}$, and since X is distinguished, $EX_{i-1,s}$ must not be E-lowerable. That is, $EX_{is} = EX_{i-1,s} - 2$. Now, $\mathbf{p_4}(r)$ implies that $EX_{ir} \le EX_{i-1,r} - 2$, so we conclude that $EX_{is} > EX_{ir}$. Appealing again to the fact that X is distinguished, we see that EX_{is} is not E-lowerable, and therefore that $EX_{i+1,s} = EX_{is} - 2$. Putting all this together, we obtain $EX_{i+1,s} = EX_{i-1,s} - 4$. Then

$$EX'_{is} = EX_{i+1,s} + 1 = EX_{i-1,s} - 3 = EX'_{i-1,s} - 2.$$

That is, $EX_{i-1,s}$ is not E-lowerable, as desired.

The proofs of $\mathbf{p_2}(r)$(b) on row $i-1$ and $\mathbf{p_1}(r)$(a) and $\mathbf{p_2}(r)$(a) on row i are similar; we omit the details. □

In general, deleting a row of nonmaximal length from a distinguished diagram need not yield a distinguished diagram.

Proposition 5.8. *Given $(C, \mathcal{E}) \in \Omega$, there is at most one distinguished weight diagram in $\kappa^{-1}(C, \mathcal{E})$.*

Proof. Suppose that $(C, \mathcal{E})$ corresponds to $(\alpha; (\mu_1, \ldots, \mu_l))$. We proceed by induction on the number of parts of α. The base case, in which α is the empty partition, is trivial. Henceforth, we assume that α is not empty.

Let k be the largest part of α, and suppose that μ_1 is the weight for the part of G^α corresponding to the rows of length k. Let α' be the partition obtained by reducing the multiplicity of r by 1. Suppose that X and Y are two weight diagrams, with $\kappa(X) = \kappa(Y) = (C, \mathcal{E})$, such that X and Y are both distinguished. Let X' and Y' be the diagrams obtained from X and Y, respectively, by omitting the topmost row of length k. Since both X' and Y' have columns in decreasing order, it is clear that the omitted row corresponds to the largest coordinate of μ_1 in both. In other words,

$$\kappa(X') = \kappa(Y') = (\alpha'; (\mu_1', \mu_2, \dots, \mu_l)),$$

where μ_1' is the weight obtained from μ_1 by omitting its largest coordinate. By the preceding lemma, X' and Y' are both distinguished. Then, by the induction hypothesis, we must have that $X' = Y'$.

Suppose that X' and Y' were obtained by omitting the ith row of X and the jth row of Y, respectively. We will begin by showing that $i = j$. Suppose, without loss of generality, that $i < j$. Choose $t \in \{i+1, ..j\}$ such that X_t is a row of maximal length among $X_{i+1}, \dots, X_j$, and let k' be its length. Note that $Y_{t-1} = X_t$, so k' must be less than k (as Y_j is the topmost row of length k in Y). Now, for any $s \le k'$, we have

$$Y_{js} \le Y_{t-1,s} = X_{ts} \le X_{is}. \tag{9}$$

It follows that $EX_{is} \ge EY_{js} + 4$. By $\mathbf{p_3}(k'+1)$, we have

$$EY_{j,k'+1} \le EX_{i,k'+1} - 3. \tag{10}$$

This means $EY_{j,k'+1}$ is E-raisable, since its column-predecessor is the same as that of $X_{i,k'+1}$. Similarly, $X_{j,k'+1}$ is E-lowerable. Applying $\mathbf{p_2}(r)$, we find that.

$$EY_{jr} \le EY_{j,k'+1} + 1 \tag{11}$$

$$EX_{ir} \ge EX_{i,k'+1} - 1 \tag{12}$$

Putting together (10) and (11) yields

$$EY_{jr} \le EX_{ir} - 1 \tag{13}$$

for $k'+1 < r \le k$. Now, the fact that there are no entries beyond column k' in any of the rows $X_{i+1}, \dots, X_j$ implies that

$$EX_{ir} - X_{ir} = EY_{jr} - Y_{jr} \tag{14}$$

for $k' < r \le k$. Combining this with (10) and (10) implies that

$$Y_{jr} \le \begin{cases} X_{jr} - 3 & \text{if } r = k'+1, \\ x_{jr} - 1 & \text{if } k'+1 < r \le k. \end{cases} \tag{15}$$

This inequality, together with (9), implies that the sum of the entries of Y_j is strictly smaller than the corresponding sum for X_i, contradicting the fact that $\kappa(Y) = \kappa(X)$. Therefore, $i = j$.

Finally, it remains to show that $X_i = Y_i$. Since these two rows have the same sum, if they differ at all, they must differ in at least two entries. Assume without loss of generality that for some s and r, we have $X_{is} < Y_{is}$ and $X_{ir} > Y_{ir}$. Note that (14) applies here, so similar inequalities hold for EX and EY. Moreover, it is clear that EX_{is} is E-raisable and EY_{is} is E-lowerable. If $EX_{is} \ge EX_{ir}$, then EY violates $\mathbf{p_2}(r)$, but if $EX_{is} \le EX_{ir} - 2$, then EX violates $\mathbf{p_2}(r)$. The only remaining case is $EX_{is} = EX_{ir} - 1$. Similar reasoning shows that we must also have $EY_{is} = EY_{ir} + 1$. Now, if s and r are both even, then Y violates $\mathbf{p_1}(r)$; if they are both odd, X does. So s and r must be of opposite parity. Assume s is odd and r is even. Then $X_{i,r-1}$ must equal either X_{ir} or $X_{ir}+1$, by $\mathbf{p_3}(r)$. But then, looking at columns s and $r-1$, we see that $\mathbf{p_1}(r-1)$ is violated. Similarly, if s is even and r is odd, then Y violates $\mathbf{p_1}(r-1)$. We conclude that $X_i = Y_i$, and thence that $X = Y$, as desired. $\square$

6. Building distinguished diagrams from weights

The goal of this section is to show that π induces a bijection between the set of E-distinguished weight diagrams and the set of dominant weights.

We do this by giving an algorithm for computing the putative inverse of π: this algorithm builds up a weight diagram column by column from the coordinates of a weight λ. We begin with some terminology for dealing with multisets of integers (*i.e.*, sets whose elements may occur with multiplicity greater than 1).

Definition 6.1. The *length* of a multiset of integers is the number of distinct elements it contains. This is to be distinguished from *size*, the total number of elements.

Definition 6.2. A multiset of integers is called a *clump* if either of the following conditions holds:

(i) The multiset has length 1.
(ii) The multiset has length greater than 1, and for each element of the multiset, there is another element differing from it by exactly 1.

Lemma 6.3. *Any multiset of integers can be written uniquely as a disjoint union of clumps of maximal size.* □

The algorithm, which we call $\mathsf{WtD} : \Lambda^+ \to D_n$, is given below. We start with some weight $\lambda \in \Lambda^+$.

(1) Let $r = 1$, and let σ_1 be the multiset whose elements are the coordinates of λ. Write $\sigma_r = A_{1,1} \amalg \cdots \amalg A_{1,l_1}$, where the $A_{1,i}$ are clumps of maximal size. We are about to start building the first column; below, "r" always refers to the column on which we are currently working.
(2) Write down the distinct values of each clump in decreasing order. Form a set Z_r of (distinct) integers as follows: for each clump of odd length, we include in Z_r the 1st, 3rd, 5th, etc., distinct values of the clump. For each clump of even length, similarly take the odd-index distinct values if r is odd, but if r is even, take the 2nd, 4th, etc., values of each even-length clump. (Another way to think of this is that we always take alternate values from each clump; if r is odd, we must include the largest value in each clump, but if r is even, we must include the smallest.)
(3) Write down the elements of Z_r vertically and in decreasing order. If $r = 1$, we are done—what we have just written down will be the first column of the diagram. Otherwise, we need to worry about what row each entry in the column belongs to. Place each element x of Z_r such that it is adjacent to a spot in column $r - 1$ containing either x or $x + (-1)^r$ (this can always be done uniquely).
(4) Let σ_{r+1} denote the multiset obtained by removing the elements of Z_r from σ_r. If σ_{r+1} is empty, we are finished drawing the diagram; otherwise, partition σ_{r+1} into disjoint maximal-size clumps $A_{r+1,1} \amalg \cdots \amalg A_{r+1,l_r}$. Advance the value of r by 1, and go to step 2.

There is something to be proved to ensure that Step 3 makes sense: namely, that each element of Z_r can be placed in a unique position next to some entry in column $r - 1$ such that a certain adjacency condition is met. To that end, we make some observations.

First, note that when σ_r is divided into maximal clumps, any two entries differing by only 0 or 1 must end up in the same clump. In other words, entries in different clumps differ by at least 2. Since Z_r is constructed by taking alternate entries in every clump, it follows that any two elements of Z_r must differ by at least 2.

Let $x \in Z_r$, and suppose that we are working on Step 3. We are looking for an entry in the preceding column whose value is either x or $x + (-1)^r$. Since distinct entries in the preceding column differ from one another by at least 2, we know that at most one of x and $x + (-1)^r$ can occur in that column, and if one of them does occur, it occurs only once. In other words, if there is an appropriate entry in the column $r-1$ that meets our adjacency condition, it is unique.

Now we establish that either x or $x + (-1)^r$ actually occurs in the preceding column. Specifically, we assume that x does not occur, and show that $x + (-1)^r$ must occur. Let us rewind the algorithm to the point where we were building column $r-1$. There was some clump $A_{r-1,i}$ to which our element x belonged. Column $r-1$ includes alternate entries from clump $A_{r-1,i}$, so if it does not contain x, it must contain one or both of $x+1$ and $x-1$. If both of these occur, we are finished: one of them is $x + (-1)^r$.

But what if only one of $x \pm 1$ occurs? Suppose that only $x-1$ occurs. This can only happen if $A_{r-1,i}$ has no members equal to $x+1$; since $A_{r-1,i}$ is a clump, it must be that x is the largest value occurring in $A_{r-1,i}$. In constructing Z_{r-1}, we had not taken the largest (i.e first) value in $A_{r-1,i}$, but we had taken the second value. This is only done when the clump has even length and the column index is even. That is, $r-1$ is even, so $x + (-1)^r = x - 1$. Thus $x + (-1)^r$ occurs in column $r-1$.

A similar argument shows that if $x+1$ occurs in the preceding column, but $x-1$ does not, then $r-1$ is odd, and again $x + (-1)^r = x + 1$ occurs in column $r-1$.

We have established that Step 3 of the algorithm WtD makes sense.

Proposition 6.4. *For any* $\lambda \in \Lambda^+$, $\mathsf{WtD}(\lambda) \in E(D_n^\circ)$.

Proof. Let $X = \mathsf{WtD}(\lambda)$. In the course of arguing that Step 3 makes sense, we observed that any two entries in a single column of the resulting diagram differ by at least 2, so X is at least in the image of E. In addition, Step 3 also explicitly decrees that the entries of a column be in decreasing order, and that the differences of horizontally adjacent entries obey the condition of $\mathbf{p_3}(r)$. Thus, $\mathbf{P_3}(r)$ and $\mathbf{P_4}(r)$ automatically hold for all r.

Next, note that because adjacent entries in a given row of $\mathsf{WtD}(\sigma)$ differ by at most 1, those entries must have belonged to the same clump when we first divided σ into clumps. In other words, clumps of σ are unions of rows in $\mathsf{WtD}(\sigma)$. Moreover, the rows that constitute a given clump of σ must be consecutive.

$\mathbf{P_1}(r)$ and $\mathbf{P_2}(r)$ are both consequences of this observation. Suppose, for instance, that in violation of $\mathbf{P_1}(r)$, we have $s < r$, s and r both odd, $X_{is} < X_{ir}$, and X_{is} E-raisable. If $X_{i's}$ is the column-predecessor of X_{is}, then $X_{i's} \geq X_{is} + 3$. That large a difference between column-consecutive entries means that $X_{i's}$ and X_{is} must have come from different clumps of λ_s. We finished taking values from, say, clump $A_{s,j}$ at row i', and X_{is} contains the first value taken from $A_{s,j+1}$. Since s is odd, EX_{is} should be the largest value in $A_{s,j+1}$, but on the other hand, X_{ir} is a value from the same clump, but it is larger than X_{is}, so we have a contradiction.

A similar argument establishes that $\mathbf{P_1}(r)$ holds if r is even. The same type of argument proves that $\mathbf{P_2}(r)$ holds as well, but the stronger inequality in that property means that we do not need to refer to the parity of s in the course of proving it. □

Theorem 6.5. $\mathsf{WtD} : \Lambda^+ \to E(D_n^\circ)$ *is a bijection, and its inverse is given by* π.

Proof. It is clear that $\pi(\mathsf{WtD}(\lambda)) = \lambda$ for any $\lambda \in \Lambda^+$. What we must show now is that for any $X \in E(D_n^\circ)$, we have $\mathsf{WtD}(\pi(X)) = X$. We begin by showing that WtD rebuilds at least the first column of X correctly.

Let $\lambda = \pi(X)$. Since X has $\mathbf{P_3}(r)$ for all r, adjacent entries of X must end up in the same clump of λ. Indeed, as in the proof of Proposition 6.4, we again have that that clumps of λ are unions of consecutive rows of X. We need to prove that the first column of X contains the odd-index values from every clump—that will establish that X and $\mathsf{WtD}(\lambda)$ have the same first column. To this end, we restrict our attention to a sequence of consecutive rows $i_0, i_0+1, \ldots, i_0+l$ whose entries constitute one clump of λ.

First, we show that $X_{i_0+j,1} = X_{i_0+j+1,1} + 2$ for each j. Of course, $X_{i_0+j,1} - X_{i_0+j+1,1}$ is at least 2, but if it were larger, that would mean $X_{i_0+j,1}$ is E-lowerable, and $X_{i_0+j+1,1}$ is E-raisable. Property $\mathbf{P_2}(r)$ implies that all successive entries in row i_0+j of X must differ from $X_{i_0+j+1,1}$ by at least 2, and all successive entries of row i_0+j+1 must differ from $X_{i_0+j,1}$ by at least 2. But if these conditions held, the entries of rows i_0+j and i_0+j+1 could not belong to the same clump. Therefore $X_{i_0+j,1} = X_{i_0+j+1,1} + 2$. It follows that the sequence $X_{i_0,1}, X_{i_0+1,1}, \ldots, X_{i_0+l,1}$ is, in fact, a sequence of alternate values from the clump under examination.

Now, we show that the clump does not contain $X_{i_0,1}+2$ as a value. Suppose this value occurs in row i_0+j. It is larger than $X_{i_0+j,1}$, and since adjacent entries differ by at most 1, the value $X_{i_0,1}$ must occur somewhere in row i_0+j. Then Property $\mathbf{P_2}(r)$ tells us that $X_{i_0,1}+2$ must occur in row i_0+j-1. We iterate this argument and find that $X_{i_0,1}+2$ occurs in each of rows $i_0+j, i_0+j-1, \ldots, i_0, i_0-1$. But that would imply that the entries of rows i_0-1 and i_0 belong to the same clump, a contradiction.

A similar argument shows that $X_{i_0+l,1}-2$ does not occur in the clump either. Thus, the sequence of values $X_{i_0,1}, \ldots, X_{i_0+l,1}$ is a maximal set of alternate values from the clump. We just need to show that these are the *odd*-index values; or, that $X_{i_0,1}$ is the largest value in the clump. We know that $X_{i_0,1}+2$ is not a value in the clump, so we just need to check that $X_{i_0,1}+1$ is not either. If it were, the argument used in the preceeding paragraph would show that it would occur in row i_0. Let r be the leftmost column such that $X_{i_0,r} = X_{i_0,1}+1$. Since adjacent entries differ by at most 1, it must be that $X_{i_0,r} = X_{i_0,r-1}+1$. $\mathbf{P_3}(r)$ then tells us that r must be odd. Now we apply $\mathbf{P_1}(r)$ to columns 1 and r and conclude that $X_{i_0,1}$ is not E-raisable. But if $X_{i_0-1,1} = X_{i_0,1}+2$, and the value $X_{i_0,1}+1$ occurs in row i_0, it would have to be that the entries of rows i_0-1 and i_0 belong to the same clump. Therefore, $X_{i_0,1}$ is actually the largest value in its clump, and the sequence $X_{i_0,1}, \ldots, X_{i_0+l,1}$ is the sequence of odd-index values from this clump.

We have shown that the first column constructed by WtD is the same as the first column of X. The same argument (with appropriate modifications for even-numbered columns) shows that, in general, Z_r as constructed by WtD in Step 2 is the set of entries in column r of X. The one remaining detail to check is that the positioning of entries done in Step 3 is the same as the original positioning in X. The algorithm for WtD does this positioning so as to satisfy $\mathbf{P_3}(r)$; and following the

A
$$\begin{array}{cccccc} a_1 & \cdots & a_{s-1} & a_s & \cdots & a_r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a'_1 & \cdots & a'_{s-1} & a'_s & \cdots & a'_r \end{array} \mapsto \begin{array}{cccccc} a_1 & \cdots & a_{s-1} & a_s - 1 & \cdots & a_r + 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a'_1 & \cdots & a'_{s-1} & a'_s - 1 & \cdots & a'_r + 1 \end{array}$$

B
$$\begin{array}{cccc} a_1 & \cdots & a_{r-1} & a_r \\ a_1 & \cdots & a_{r-1} & \end{array} \mapsto \begin{array}{cccc} a_1 & \cdots & a_{r-1} & \\ a_1 & \cdots & a_{r-1} & a_r \end{array}$$

C
$$\begin{array}{cccc} a_1 & \cdots & a_{r-1} & a_r \\ a_1 & \cdots & a_{r-1} & a'_r \end{array} \mapsto \begin{array}{cccc} a_1 & \cdots & a_{r-1} & a'_r \\ a_1 & \cdots & a_{r-1} & a_r \end{array}$$

TABLE 1. Moves used in the algorithm Dist

definition of WtD, we argued that this positioning can be done uniquely. Since X satisfies $\mathbf{P_3}(r)$, the positioning found in X must be that produced by the algorithm.

Thus, $\mathsf{WtD}(\pi(X)) = X$. □

7. MANIPULATING WEIGHT DIAGRAMS

The next task at hand is to associate a distinguished weight diagram to any given element of Ω. We will not give a direct procedure for building such a diagram, in the spirit of WtD. Rather, we will give an algorithm Dist which takes an arbitrary weight diagram X, and outputs a new weight diagram X' such that $\kappa(X) = \kappa(X')$ and X' is distinguished. This algorithm will be given in terms of certain "moves" that may be performed to modify a weight diagram. In this section, we will investigate a number of technical properties of these moves. The actual definition of Dist will be given in the next section.

The three moves that can be performed in the algorithm Dist are shown in Table 1. The distinct rows shown in the diagram for each move in this table need not actually be consecutive; however, any intermediary rows *must be shorter than* $r - 1$ boxes (shorter than $s - 1$ boxes for move **A**). The rows that are shown in the diagrams may or may not be longer than r boxes.

The algorithm will also make use of the inverse moves $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. (Of course, **C** and $\mathbf{C}^{-1}$ are the same move.) In practice, **B** and **C** look like the same move—they both involve exchanging two rows—but for the sake of proving various facts about the behavior of these moves, it will be convenient to have separate names for them.

In order to measure progress towards the goal of making X distinguished, we will use a certain 6-tuple of integers. (This will be of particular use in showing that the algorithm even terminates.) The integer-valued functions that make up this 6-tuple are defined in Table 2. The intent is that these functions take smaller values on diagrams that are closer to the goal. Their definitions are, unfortunately, incredibly opaque. Here is a brief account of what these functions are intended to do.

$$
\begin{aligned}
q_1(X) &= \|EX\|^2\\
q_2(X) &= \sum_{r \text{ even}} \sum_i EX_{ir}\\
q_3(X) &= \sum_{r \text{ odd}} \sum_i r\, EX_{ir} - \sum_{r \text{ even}} \sum_i r\, EX_{ir}\\
q_4(X) &= -\max\{r \mid \mathbf{P_3}(r) \text{ and } \mathbf{P_4}(r) \text{ both hold for } EX\}\\
q_5(X) &= \sum_{\{j \mid X_{jr} \text{ nonempty}\}} \tilde{q}_{5;jr}(X) \qquad \text{where } r = q_4(X)+1\\
q_6(X) &= \sum_{\{j \mid X_{jr} \text{ nonempty}\}} \tilde{q}_{6;jr}(X) \qquad \text{where } r = q_4(X)+1,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{q}_{5;ir}(X) &= \begin{cases} \max\{EX_{ir} - (EX_{i,r-1}+1), EX_{i,r-1} - EX_{ir}\} & \text{if } r \text{ is odd}\\ \max\{EX_{ir} - EX_{i,r-1}, (EX_{i,r-1}-1) - EX_{ir}\} & \text{if } r \text{ is even}\end{cases}\\
\tilde{q}_{6;ir}(X) &= \sum_{\substack{i'<i\\ X_{i'r} \text{ nonempty}}} \max\{0, X_{ir} - X_{i'r}\} + \sum_{\substack{i'>i\\ X_{i'r} \text{ nonempty}}} \max\{0, X_{i'r} - X_{ir}\}
\end{aligned}
$$

TABLE 2. Some integer-valued functions on D_n.

Following the idea of Claim 3.1, we hope that for a given pair $(C, \mathcal{E})$, $\gamma^{-1}(C, \mathcal{E})$ can be computed by choosing a weight diagram X in $\kappa^{-1}(C, \mathcal{E})$ such that $\|EX\|^2$ is minimized. (It turns out that distinguished weight diagrams do indeed have this property.) This motivates the definition of q_1 as our first measure of progress. Now, one implication of $\mathbf{p_1}(r)$ and $\mathbf{p_2}(r)$ is, roughly, that even-numbered columns ought to have smaller entries than odd-numbered columns; q_2 is used to make sure that the entries in even-numbered columns do not become too large. Another implication of these first two properties is that the largest entries ought to appear in the leftmost odd-numbered columns, and the smallest entries in the left-most even-numbered columns. The function q_3 tells us if we have too many excessively large or small entries too far right in the diagram.

Next, q_4 just tells us how many columns on the left-hand side of the diagram satisfy $\mathbf{p_3}(r)$ and $\mathbf{p_4}(r)$. It is negative just because we want progress to be reflected by a decrease in function values. For the leftmost column that does not satisfy $\mathbf{p_3}(r)$ and $\mathbf{p_4}(r)$, q_5 measures directly how far it is from having $\mathbf{p_3}(r)$, and q_6 does the same for $\mathbf{p_4}(r)$.

Remark 7.1. X has $\mathbf{p_3}(r)$ for $r = -q_4(X)+1$ if and only if $q_5(X) = 0$, and it has $\mathbf{p_4}(r)$ for $r = -q_4(X)+1$ if and only if $q_6(X) = 0$.

For $X \in D_n$, we write $\mathbf{A}X$, $\mathbf{B}X$, and $\mathbf{C}X$ to indicate the diagram resulting from performing moves $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ respectively. (The values of the parameters s, r, m, and $i_1, \ldots, i_m$ will be clear from context.) Finally, the symbol $\mathbf{M}$ will be used to stand for any of the moves $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, or their inverses.

The next lemma establishes a vital fact, without which all these moves would be useless.

Lemma 7.2. *For any diagram X and any move $\mathbf{M}$, we have $\kappa(\mathbf{M}X) = \kappa(X)$.*

Proof. Moves $\mathbf{A}$ and $\mathbf{A}^{-1}$ preserve the shape of the diagram, and on some rows they add $+1$ to one entry and -1 to another. Hence the row-sums of X are preserved. In the case of $\mathbf{B}$, $\mathbf{B}^{-1}$, and $\mathbf{C}$, no entries are changed; rather, we merely exchange two rows. In all cases, we see that $\kappa(\mathbf{M}X) = \kappa(X)$. $\square$

Now, we will want to restrict ourselves to making moves that actually result in progress towards the goal of making X distinguished. The following definition captures this notion.

Definition 7.3. $\mathbf{M}X$ is said to be *well-behaved of order* $\geq k$, where $1 \leq k \leq 6$, if there is some k', $k \leq k' \leq 6$, such that

$$q_l(\mathbf{M}X) = q_l(X) \qquad \text{for } l = 0, \dots, k'-1$$

and

$$q_{k'}(\mathbf{M}X) < q_{k'}(X).$$

Lemma 7.4. *Suppose that X has $\mathbf{p_4}(r)$ and that X_{ir} is raisable. If Y is a diagram obtained from X by raising X_{ir}, then EY is equal to EX raised at position ir.*

If X has $\mathbf{p_4}(r)$ and X_{ir} is lowerable, and if Y is X lowered at ir, then EY is EX lowered at ir.

Proof. We prove only the first part of the claim; the second part is proved similarly. E acts on each column of a diagram individually, so EX and EY will certainly agree in every column other than r. Now, in general, the amount E adds to the entries of a column depend only on the order of those entries. In particular, if a column is nonincreasing and of height h, then E adds $h-1$ to the first entry, $h-3$ to the second, *etc.*, down to $-h+1$ to the last entry, irrespective of what those entries actually are.

Because X has $\mathbf{p_4}(r)$, column r is in fact nonincreasing. Moreover, to say X_{ir} is raisable is to say that it is strictly smaller than its column-predecessor, so it follows that column r of Y is still nonincreasing. E acts on both X and Y by adding the same numbers to corresponding entries, so EX and EY differ only where X and Y differ, with $EY_{ir} = EX_{ir} + 1$. $\square$

Proposition 7.5. *Suppose X contains a sequence of rows $i_1, \dots, i_m$ on which $\mathbf{A}$ might be performed, say at columns s and r, with $s < r$. Furthermore, suppose that $q_4(X) \leq -r$, $X_{i_m s}$ is lowerable, $X_{i_1 r}$ is raisable, and $EX_{i_k s} - EX_{i_k r} \geq 2$ for $k = 1, \dots, m$. Then $\|\mathbf{A}X\|^2 < \|X\|^2$. Stated differently, $\mathbf{A}$ is well-behaved on X of order* 1.

Proof. We begin by noting that in any column t of X such that $\mathbf{p_4}(t)$ holds, if X_{jt} is raisable and has column-successor $X_{j't}$, then after raising X_{jt}, position $j't$ will be raisable in the new diagram: if Y is the diagram obtained by raising X_{jt}, then $X_{jt} \geq X_{j't}$ implies $Y_{jt} > Y_{j't}$.

Now, the performance of move $\mathbf{A}$ can be broken down into steps as follows:

1. Raise $X_{i_1 r}$; lower $X_{i_m s}$.
2. Raise $X_{i_2 r}$; lower $X_{i_{m-1} s}$.

$\vdots$

m. Raise $X_{i_m r}$; lower $X_{i_1 s}$.

At each step, the entries being raised and lowered are raisable and lowerable respectively (this is true in Step 1 by assumption, and in each successive step by what we noted above). It follows by Lemma 7.4 that $E\mathbf{A}X_{i_k r} = EX_{i_k r} + 1$ and $E\mathbf{A}X_{i_k s} = EX_{i_k s} - 1$ for each k. Therefore,

$$\begin{aligned}
\|E\mathbf{A}X\|^2 &= \sum_{j,t}(E\mathbf{A}X_{jt})^2 \\
&= \sum_{j,t}(EX_{jt})^2 + \sum_{k=1}^{m}\big((E\mathbf{A}X_{i_k s})^2 + (E\mathbf{A}X_{i_k r})^2 - (EX_{i_k s})^2 - (EX_{i_k r})^2\big) \\
&= \|EX\|^2 + \sum_{k=1}^{m}((-2EX_{i_k s} + 1) + (2EX_{i_k r} + 1)) \\
&= \|EX\|^2 - 2\sum_{k=1}^{m}(EX_{i_k s} - EX_{i_k r} - 1) \qquad (16)
\end{aligned}$$

Since $EX_{i_k s} - EX_{i_k r} \geq 2$, the summation in the last line is strictly positive. Thus $\|E\mathbf{A}X\|^2 < \|EX\|^2$. $\square$

Proposition 7.6. *Assume the conditions of Proposition 7.5, but weaken the requirement on entries in EX to $EX_{i_k s} - EX_{i_k r} \geq 1$. If, in addition, r and s are both even, then $\mathbf{A}$ is well-behaved on X of order ≥ 1.*

Proof. We perform the same computation as in the proof of Proposition 7.5, and we arrive at equation (16). With the weakened inequality in the present assertion, some of the terms in the summation in (16) may be 0, but provided that at least one of them is positive, we still get $\|E\mathbf{A}X\|^2 < \|EX\|^2$.

If, however, $EX_{i_k s} - EX_{i_k r} = 1$ for every k, then that summation will be 0, so we shall have $q_1(\mathbf{A}X) = q_1(X)$. We know by Lemma 7.4 (as argued in the proof of Proposition 7.5) that after move $\mathbf{A}$, m entries in column s of EX are changed by -1, and m entries in column r are changed by $+1$. Thus, the total sum of elements in even-numbered columns is unchanged: $q_2(\mathbf{A}X) = q_2(X)$. Furthermore, we easily compute that q_3 changes by $s - r$, so $q_3(\mathbf{A}X) < q_3(X)$. Hence $\mathbf{A}$ is well-behaved on X, as claimed. $\square$

Proposition 7.7. *Suppose that move $\mathbf{A}$ might be performed on the single row i. Suppose in addition that $q_4(X) = -(r-1)$ and that $q_5(X) \neq 0$; in particular, suppose that $\tilde{q}_{5;ir}(X) > 0$. Furthermore, suppose that $EX_{i,r-1} > EX_{ir}$, and that $X_{jr} = X_{ir}$ implies $j \geq i$. If $s < r$ and EX_{is} is lowerable, and*

(a) *if $EX_{is} - EX_{ir} \geq 2$, then $q_1(\mathbf{A}X) < q_1(X)$.*
(b) *if r is odd, s is even, and $EX_{is} - EX_{ir} = 1$, then $q_1(\mathbf{A}X) = q_1(X)$, but $q_2(\mathbf{A}X) < q_2(X)$.*
(c) *if r and s are both even, and $EX_{is} - EX_{ir} = 1$, then $q_1(\mathbf{A}X) = q_1(X)$ and $q_2(\mathbf{A}X) = q_2(\mathbf{A}X)$, but $q_3(\mathbf{A}X) < q_3(X)$.*

Proof. By the assumption that $q_4(X) = -(r-1)$, we know that $\mathbf{p_4}(s)$ holds, so Lemma 7.4 applies to column s. We do not know enough about column r to say how E acts on it precisely, but the complicated assumptions in the proposition are specifically designed to enable us to compute the difference in size of $E\mathbf{A}X$ and EX regardless.

Let $\|EX_{*r}\|^2$ denote the size of the r-th column of EX. This quantity depends on what the entries in the r-th column are, but it is independent of their order within the column. If the column contains h entries, of which l are strictly greater than X_{ir}, it is easy to check that replacing X_{ir} by $X_{ir}+1$ results $\|EX_{*r}\|^2$ changing by $(X_{ir}+1+(h-1-2l))^2-(X_{ir}+(h-1-2l))^2$, or

$$2\left(X_{ir}+(h-1-2l)\right)+1. \tag{17}$$

By Lemma 7.4, lowering X_{is} changes $\|EX_{*s}\|^2$ by $(EX_{is}-1)^2-(EX_{is})^2$, or

$$-2EX_{is}+1. \tag{18}$$

The net change in $\|EX\|^2$ brought about by performing move $\mathbf{A}$ on X would be the sum of (17) and (18). Now, recalling the definition of E, we can compute EX_{ir} explicitly, thanks to the assumption that $X_{jr}=X_{ir}$ implies $j\geq i$:

$$\begin{aligned} EX_{ir} = X_{ir} &+ \#\{X_{jr} \mid X_{jr} < X_{ir}, \text{ or } X_{jr} = X_{ir} \text{ with } j > i\} \\ &- \#\{X_{jr} \mid X_{jr} > X_{ir}, \text{ or } X_{jr} = X_{ir} \text{ with } j < i\} \\ = X_{ir} &+ (h-(l+1)) - l. \end{aligned}$$

Thus, expression (17) is equal to $2EX_{ir}+1$. We obtain that the sum of (17) and (18) is

$$q_1(\mathbf{A}X)-q_1(X)=2-2(EX_{is}-EX_{ir}). \tag{19}$$

Part (a) of the proposition follows because if $EX_{is}-EX_{ir}\geq 2$, the quantity in (19) is strictly negative. On the other hand, if $EX_{is}-EX_{ir}=1$, the quantity in (19) is 0. If r is odd and s even, lowering X_{is} obviously decreases the sum q_2 of even-column entries, so part (b) is true as well. Finally, if r and s are both odd, q_2 remains unchanged, but q_3 changes by $m(s-r)$: this establishes part (c). □

Proposition 7.8. *Suppose rows i and i' of X are such that $\mathbf{B}$ or $\mathbf{C}$ might be performed on them: in particular, they agree in their first $r-1$ entries, and intervening rows have length less than $r-1$. Suppose furthermore that $q_4(X)=-(r-1)$, and that row i has length at least r.*

- (a) *Suppose X has no entry at position $i'r$. If $\tilde{q}_{5;ir}>0$ and $EX_{ir}<EX_{i,r-1}$, $\mathbf{B}X$ is well-behaved of order ≥ 4.*
- (b) *Suppose that X does have an entry at position $i'r$. If $X_{ir}<X_{i'r}$, then $\mathbf{C}X$ is well-behaved of order ≥ 4.*

Proof. Both of these moves change the shape of the diagram without changing any entries, so q_1 is preserved. Moreover, this shape change is brought about by exchanging rows; the entries in any given column remain the same, albeit possibly rearranged. Hence q_2 and q_3 are preserved as well. Before proceeding, we define two convenient functions:

$$w_{5;r}(X)=\sum_j \tilde{q}_{5;jr}(X) \qquad \text{and} \qquad w_{6;r}(X)=\sum_j \tilde{q}_{6;jr}(X)$$

Provided that $q_4(X)=-(r-1)$, we will of course have $w_{5;r}=q_5$ and $w_{6;r}=q_6$. Let us also note that $EX_{i',r-1}=EX_{i,r-1}-2$.

For part (a) of the proposition, $EX_{i,r-1} - EX_{ir}$ is greater than or equal to 1 (if r is odd) or 2 (if r is even). Move $\mathbf{B}$ does not change the relative position of entry X_{ir} in column r, so $E\mathbf{B}X_{i'r} = EX_{ir}$. We have

$$E\mathbf{B}X_{i',r-1} - E\mathbf{B}X_{i'r} = EX_{i,r-1} - EX_{ir} - 2;$$

we can conclude that

$$\tilde{q}_{5;i'r}(\mathbf{B}X) = \max\{0, \tilde{q}_{5;ir}(X) - 2\}$$

and

$$w_{5;r}(\mathbf{B}X) = \max\{0, w_{5;r}(X) - 2\}.$$

In particular, $w_{5;r}(\mathbf{B}X) < w_{5;r}(X)$. This may mean that $q_4(\mathbf{B}X) = q_4(X)$ and $q_5(\mathbf{B}X) < q_5(X)$, or, if $w_{5;r}(\mathbf{B}X) = 0$ (meaning that $\mathbf{B}X$ has $\mathbf{p_3}(r)$, which X did not), it may be that $q_4(\mathbf{B}X) = -r$ and that $q_5(\mathbf{B}X)$ is unpredictable. In either case, we observe that $\mathbf{B}X$ is well-behaved of order ≥ 4.

For part (b), we break down the argument into three cases:

Case 1. $\tilde{q}_{5;ir}(X) > 0$, and $EX_{ir} > EX_{i,r-1}$. In this case, we must have $EX_{i'r} > EX_{ir} > EX_{i,r-1}$; it is easy to check that

$$\begin{aligned}\tilde{q}_{5;ir}(\mathbf{C}X) &= \tilde{q}_{5;i'r}(X) - 2\\ \tilde{q}_{5;i'r}(\mathbf{C}X) &= \tilde{q}_{5;ir}(X) + 2.\end{aligned}$$

This equations imply that the following holds (of course, we actually have equality, but we write an inequality to accomodate the two cases considered below):

$$\tilde{q}_{5;ir}(\mathbf{C}X) + \tilde{q}_{5;i'r}(\mathbf{C}X) \leq \tilde{q}_{5;ir}(X) + \tilde{q}_{5;i'r}(X). \tag{20}$$

Case 2. $\tilde{q}_{5;ir}(X) = 0$. This time we have

$$\begin{aligned}\tilde{q}_{5;ir}(\mathbf{C}X) &= \tilde{q}_{5;i'r}(X) - 2\\ \tilde{q}_{5;i'r}(\mathbf{C}X) &= 1 \text{ or } 2.\end{aligned}$$

These facts imply that (20) holds here as well.

Case 3. $\tilde{q}_{5;ir}(X) > 0$, and $EX_{ir} < EX_{i,r-1}$. This is the most complicated of the three cases; part of the computation has to be broken down into three sub-cases. We obtain:

$$\begin{aligned}\tilde{q}_{5;ir}(\mathbf{C}X) &= \begin{cases}\max\{0, \tilde{q}_{5;i'r}(X) - 2\} & \text{if } \tilde{q}_{5;i'r} > 0 \text{ and } EX_{i'r} > EX_{i',r-1}\\ 1 \text{ or } 2 & \text{if } \tilde{q}_{5;i'r} = 0\\ \tilde{q}_{5;i'r} + 2 & \text{if } \tilde{q}_{5;i'r} > 0 \text{ and } EX_{i'r} < EX_{i',r-1}\end{cases}\\ \tilde{q}_{5;i'r}(\mathbf{C}X) &= \max\{0, \tilde{q}_{5;ir}(X) - 2\}\end{aligned}$$

Once again, (20) holds.

We finish up the argument almost as we did for part (a), but this time we have only the weaker inequality $w_{5;r}(\mathbf{C}X) \leq w_{5;r}(X)$ following from (20). This gives rise to the possibility that $q_4(\mathbf{C}X) = q_4(X)$ and $q_5(\mathbf{C}X) = q_5(X)$, compelling us to examine the behavior of q_6. But it is clear that move $\mathbf{C}$ brings column r closer to being nonincreasing, in the sense that $w_{6;r}(\mathbf{C}X) < w_{6;r}(X)$. If q_4 and q_5 do not change under move $\mathbf{C}$, then $q_6(\mathbf{C}X) < q_6(X)$. Hence $\mathbf{C}X$ is well-behaved of order ≥ 4. □

The facts in Propositions 7.5, 7.7, and 7.8 are collected and summarized in Table 3.

Conditions	*Move*	*Well-Behavedness*
$q_4(X) \leq -r$.		Order ≥ 1.
$X_{i_m s}$ is lowerable, $X_{i_1 r}$ is raisable, and $EX_{i_k s} - EX_{i_k r} \geq 1$ for $k = 1, \ldots, m$.	$\mathbf{A}$	
$X_{i_1 s}$ is raisable, $X_{i_m r}$ is lowerable, and $EX_{i_k s} - EX_{i_k r} \leq -1$ for $k = 1, \ldots, m$.	$\mathbf{A}^{-1}$	
$m = 1$; $q_4(X) = -(r-1)$. $X_{i_1 s}$ is lowerable. $\tilde{q}_{3;i_1 r}(X) \neq 0$. If $X_{jr} = X_{ir}$, then $j \geq i$ (for $\mathbf{A}$) or $j \leq i$ (for $\mathbf{A}^{-1}$).		Order $= 1$.
$EX_{i_1 s} - EX_{i_1 r} \geq 2$.	$\mathbf{A}$	
$EX_{i_1 s} - EX_{i_1 r} \leq -2$.	$\mathbf{A}^{-1}$	
r odd, s even, and $EX_{i_1 s} - EX_{i_1 r} = 1$.	$\mathbf{A}$	
r even, s odd, and $EX_{i_1 s} - EX_{i_1 r} = -1$.	$\mathbf{A}^{-1}$	
$q_4(X) = -(r-1)$; $\tilde{q}_{5;i_1 r}(X) > 0$		Order ≥ 4.
$EX_{i_1, r-1} > EX_{i_1 r}$.	$\mathbf{B}$	
$EX_{i_1, r-1} < EX_{i_1 r}$.	$\mathbf{B}^{-1}$	
$X_{ir} < X_{i'r}$.	$\mathbf{C}$	

TABLE 3. Well-behavedness of moves under various hypotheses

8. PRODUCING DISTINGUISHED DIAGRAMS VIA MOVES

At this stage, we are ready to put these moves to work for us. The algorithm $\mathsf{Dist} : D_n \to D_n$, shown below, is extremely simple. To start with, X denotes the input weight diagram.

(1) Look down the leftmost column in Table 3 and find a hypothesis satisfied by X (for some choice of s, r, and $i_1, \ldots, i_m$). If no such hypothesis exists, then X is already distinguished, and the algorithm is finished.
(2) Modify X by performing the move corresponding to the hypothesis found in Step 1, and then return to Step 1.

There is some ambiguity if X satisfies more than one of the hypotheses, or if it satisfies some hypothesis for more than one choice of s, r, and $i_1, \ldots, i_m$. Ultimately, we shall see that the final output of the algorithm is independent of these choices, but until we can prove it, we need to make some specific choice. Let us decide to always choose the *first* satisfied hypothesis (in the order in which they are listed in the table), together with the *lowest* choice of vector $(s, r, m, i_1, \ldots, i_m)$ (where these vectors are ordered lexicographically).

We begin with the following fundamental fact about Dist.

Proposition 8.1. *The algorithm* Dist *terminates after a finite number of steps.*

Proof. The assertion is a consequence of the fact that only well-behaved moves are performed while computing Dist. Given a diagram X, let c be the number of columns in X, and consider the map $q : D_n \to \mathbb{N}^6$ defined by

$$q(X) = (q_1(X), q_2(X), q_3(X), c + q_4(X), q_5(X), q_6(X)).$$

(Here, the c in the fourth coordinate appears only to make that coordinate have a nonnegative value.) Give $\mathbb{N}^6$ the lexicographical ordering; then, to say that $\mathbf{M}X$ is well-behaved is to say that $q(\mathbf{M}X) < q(X)$. Furthermore, we evidently have

$$q(X) \geq (0,0,0,0,0,0).$$

Each move performed while computing Dist decreases q, and q is bounded below. Since $\mathbb{N}^6$ is well-ordered, it follows that the algorithm must stop after a finite number of steps. □

Lemma 8.2. *Suppose EX has $\mathbf{P_1}(r)$, $\mathbf{P_3}(r)$, and $\mathbf{P_4}(r)$. If a range of entries $X_{is}, \ldots, X_{ir}$ in a single row is such that none of them is lowerable (resp. raisable), then all their column-successors (resp. column-predecessors) lie in a single row. Moreover, if $s > 1$, then the column-successor (resp. column-predecessor) of $X_{i,s-1}$ lies in that same row as well.*

Proof. We prove the statement in the case that none of the entries in the range is lowerable; the other case is proved similarly. If $s = r$, the statement is trivial. Assume $s < r$, and pick any t such that $s \leq t < t+1 \leq r$. Suppose the column-successor of X_{it} is on row j_1, and that of $X_{i,t+1}$ on row j_2. There are three cases to consider:

Case 1. $EX_{it} = EX_{i,t+1}$. Hence $EX_{j_1 t} = EX_{j_2,t+1}$. But if $j_1 \neq j_2$, then $EX_{j_2,t+1}$ differs from $EX_{j_2 t}$ by at most 1, and $EX_{j_2 t}$ in turn differs from $EX_{j_1 t}$ by at least 2, so, $EX_{j_1 t}$ and $EX_{j_2,t+1}$ must differ by at least 1. Hence $j_1 = j_2$.

Case 2. The column index t is odd, and $EX_{it} = EX_{i,t+1} + 1$. Hence $EX_{j_1 t} = EX_{j_2,t+1} + 1$. Now, $EX_{j_2 t}$ must equal either $EX_{j_2,t+1}$ or $EX_{j_2,t+1} + 1$, but neither of these values differs from $EX_{j_1 t}$ by 2 or more, so it must be that $j_1 = j_2$.

Case 3. The column index t is even, and $EX_{it} = EX_{i,t+1} - 1$. This case is similar to the preceding one.

We see that any two neighboring entries in the range $X_{is}, \ldots, X_{ir}$ must have column-successors on the same row, so the lemma follows. □

Proposition 8.3. *If EX has $\mathbf{p_3}(r)$ and $\mathbf{p_4}(r-1)$, then it also has $\mathbf{p_4}(r)$.*

Proof. Let ir, $i'r$ be column-consecutive positions in column r. Since $i < i'$ and $\mathbf{p_4}(r-1)$ holds, $X_{i,r-1} \geq X_{i',r-1}$. This in turn implies $EX_{i,r-1} - EX_{i',r-1} \geq 2$. By $\mathbf{p_3}(r)$, we know

$$|EX_{ir} - EX_{i,r-1}| \leq 1$$
$$|EX_{i'r} - EX_{i',r-1}| \leq 1.$$

It follows from these inequalities that $EX_{ir} - EX_{i'r} \geq 0$. But no two entries in a single column of EX can be closer than 2: we conclude that $EX_{ir} - EX_{i'r} \geq 2$. From this last inequality we conclude that $X_{ir} - X_{i'r} \geq 0$; hence, column r is nonincreasing, and $\mathbf{p_4}(r)$ holds. □

Corollary 8.4. *Suppose EX has $\mathbf{P_1}(r-1)$, $\mathbf{P_2}(r-1)$, $\mathbf{P_3}(r-1)$, and $\mathbf{P_4}(r-1)$. If it does not have $\mathbf{P_4}(r)$, then it also does not have $\mathbf{P_3}(r)$.* □

Proposition 8.5. *Suppose EX has $\mathbf{P_1}(r-1)$, $\mathbf{P_2}(r-1)$, $\mathbf{P_3}(r-1)$, and $\mathbf{P_4}(r-1)$. If it does not have $\mathbf{P_3}(r)$, then X satisfies some hypothesis in the left-hand column of Table 3.*

Proof. We know that $q_5(X) > 0$; find a row i such that $\tilde{q}_{5;i_0r}(X)$ is maximal. We assume henceforth that $EX_{i_0,r-1} > EX_{i_0r}$; the argument would proceed similarly if the inequality were reversed. Let $i \le i_0$ be the index of the uppermost row whose entry in column r equals X_{i_0r}. In other words, row i has the property that $X_{jr} = X_{ir}$ implies $j \ge i$. It is easy to check that this setup implies

$$EX_{i,r-1} - EX_{ir} \ge EX_{i_0,r-1} - EX_{i_0r},$$

and by maximality, the preceeding line must actually be an equality. The remainder of the argument is a boring and complicated case-by-case analysis. To assist the reader in staying awake while reading it, we give here a road map of the breakdown into cases, although we do not yet define all the symbols contained herein.

(1) r is odd.
  (a) $X_{i',r-1}$ is lowerable.
  (b) $X_{i',r-1}$ is not lowerable.
    (i) $s = 0$.
    (ii) $s > 0$.
      (A) s is even.
      (B) $s = r - 2$.
      (C) s is odd and less then $r - 2$.
(2) r is even.

First, we consider the case where r is odd. Let $i' \ge i$ be the last row such that the consecutive rows $i, i+1, \ldots, i'$ all agree in the first r columns.

If $X_{i',r-1}$ is lowerable, we are done: Proposition 7.7 applies to rows $i, \ldots, i'$, with $s = r - 1$. (In the event that $EX_{i,r-1} - EX_{ir} = 1$, we meet the additional requirement that s be even.)

If $X_{i',r-1}$ is not lowerable, let $X_{i'',r-1}$ be its column-successor. Let s be such that rows i' and i'' of X agree in columns $s+1, s+2, \ldots, r-1$. Using Lemma 7.4, we see that either $s = 0$, or X_{is} $(s > 0)$ is lowerable.

Suppose $s = 0$. If $X_{i''r}$ is empty, we can apply Proposition 7.8a and do move **B** on rows i' and i''. If $X_{i''r}$ is not empty, we must have $X_{i''r} \ge X_{ir}$ in order not to violate the maximality of $\tilde{q}_{5;ir}(X)$; but on the other hand, we cannot have $X_{i''r} = X_{ir}$, since i' is the last row to agree with row i in each of the first r columns. Hence $X_{i''r} > X_{i'r}$, and we can do move **C** by Proposition 7.8b.

On the other hand, if $s > 0$, we know by Lemma 7.4 that $X_{i''s}$ is still the column-successor of $X_{i's}$, and that $EX_{i's} - EX_{i''s} = 3$. Since $X_{i''s}$ is raisable, property $\mathbf{P_2}(r-1)$ tells us the first of the following inequalities, from which we derive the latter ones:

$$\begin{aligned} EX_{i''s} &\ge EX_{i'',r-1} - 1 \\ EX_{i''s} + 3 &\ge EX_{i'',r-1} + 2 \\ EX_{i's} &\ge EX_{i',r-1} \\ EX_{i's} &> EX_{i'r} \end{aligned} \tag{21}$$

Indeed, $EX_{js} > EX_{jr}$ for every $j = i, \ldots, i'$. Now we can almost apply Proposition 7.7 to rows $i, \ldots, i'$ with our specified s and r. If $EX_{i,r-1} - EX_{ir} \ge 2$, then $EX_{is} - EX_{ir} \ge 2$ as well, and the proposition applies. But if that difference is only 1, we need to make sure that s is even. We shall show instead that if s is odd, then $EX_{is} - EX_{ir}$ is necessarily at least 2. Observe that by property $\mathbf{P_3}(r-1)$, we must have $EX_{i's} = EX_{i',s+1} + 1$, and $EX_{i''s} = EX_{i'',s+1}$.

If $s = r-2$, then we are done: since $EX_{i',r-1} - EX_{i'r} \geq 1$ and $EX_{i's} = EX_{i',r-1} + 1$, we conclude $EX_{i's} - EX_{i'r} \geq 2$, and hence $EX_{is} - EX_{ir} \geq 2$, as desired.

If $s < r-2$, then because $X_{i''s}$ is raisable, property $\mathbf{P_1}(r-1)$ tells us that $EX_{i''s} \geq EX_{i'',r-2}$. In turn, property $\mathbf{P_3}(r-1)$ tells us that $EX_{i'',r-2}$ is at least as large as $EX_{i'',r-1} = EX_{i',r-1} - 2$. We compute:

$$\begin{aligned} EX_{i''s} &\geq EX_{i',r-1} - 2 \\ EX_{i''s} + 3 &\geq EX_{i',r-1} + 1 \\ EX_{i's} &> EX_{i',r-1} \end{aligned}$$

This last strict inequality implies that $EX_{i's} - EX_{i'r} \geq 2$, so once again, $EX_{is} - EX_{ir} \geq 2$, as desired.

What happens if r is even? We can repeat the above arguments until the final few steps, where we worried about the possibility that $EX_{is} - EX_{ir} = 1$. But that worry is irrelevant when r is even: now $\tilde{q}_{5;ir} > 0$ means that $EX_{i,r-1} - EX_{ir} \geq 2$, so inequality (21) implies directly that $EX_{is} - EX_{ir} \geq 2$ as well. □

Proposition 8.6. *Suppose EX has $\mathbf{P_1}(r-1)$ and $\mathbf{P_2}(r-1)$, as well as $\mathbf{P_3}(r)$ and $\mathbf{P_4}(r)$. If it does not have $\mathbf{P_2}(r)$, then X satisfies some hypothesis in the left-hand column of Table 3.*

Proof. We assume without loss of generality that there are some positions $i's$, $i'r$ in X such that $EX_{i's} - EX_{i'r} \geq 2$, but such that $EX_{i's}$ is lowerable. (The argument is similar if the inequality is reversed and $EX_{i's}$ is raisable.) Let $i_1, i_2, \ldots, i_m = i'$ be a sequence of row indices that are column-consecutive in column r, and such that $X_{i_1 r}$ is raisable. Then Proposition 7.6 applies, and we can do move **A**. □

Proposition 8.7. *Suppose EX has $\mathbf{P_1}(r-1)$, as well as $\mathbf{P_2}(r)$, $\mathbf{P_3}(r)$, and $\mathbf{P_4}(r)$. If it does not have $\mathbf{P_1}(r)$, then X satisfies some hypothesis in the left-hand column of Table 3.*

Proof. The argument is identical to that in the proof of Proposition 8.6, except that in the first sentence, the inequality is replaced by "$EX_{i's} - EX_{i'r} \geq 1$," and in the last sentence, we apply Proposition 7.5. □

Theorem 8.8. *The map $\kappa : D_n^\circ \to \Omega$ is a bijection. Given any $X \in D_n$, $\mathsf{Dist}(X)$ is the unique weight diagram such that $\mathsf{Dist}(X)$ is distinguished and $\kappa(\mathsf{Dist}(X)) = \kappa(X)$.*

Proof. It is clear that $\mathsf{Dist}(X)$ is distinguished for any X, since Propositions 8.5, 8.6, and 8.7 together say that additional moves can be performed on any diagram not lying in D_n°. It is also obvious that for any pair $(\alpha; (\mu_1, \ldots, \mu_l)) \in \Omega$, there exists a weight diagram X with

$$\kappa(X) = (\alpha; (\mu_1, \ldots, \mu_l)).$$

(For example, one could take X to be a diagram in which the coordinates of the various μ_i's are placed in the first column, and the remainder of the diagram contains only 0's.) By Lemma 7.2, Dist respects the fibres of κ, so we conclude that each fibre of κ contains at least one element of D_n°. But Proposition 5.8 says each fibre contains at most one element of D_n°, so in fact $\kappa \circ E^{-1}$ is a bijection. □

Corollary 8.9 (*cf.* Claim 3.1)**.** *Given $(C,\mathcal{E}) \in \Omega$, let X_0 be the unique distinguished weight diagram in $\kappa^{-1}(C,\mathcal{E})$. Then, for any $X \in \kappa^{-1}(C,\mathcal{E})$, we have $\|\tau(X_0)\|^2 \leq \|\tau(X)\|^2$.*

Proof. From the preceding theorem, we know that $\mathsf{Dist}(X) = X_0$. The result then follows from the fact that the value of q_1 never increases during the operation of Dist. □

We can now assemble everything we have done into the following result, obtained by combining Theorems 3.2, 6.5, and 8.8.

Theorem 8.10. *The map $\gamma : \Lambda^+ \to \Omega$ given by $\lambda \mapsto \kappa(E^{-1}\mathsf{WtD}(\lambda))$ is a bijection; its inverse is given by $(C,\mathcal{E}) \mapsto \pi(E\mathsf{Dist}(X_1))$, where X_1 is any weight diagram in $\kappa^{-1}(C,\mathcal{E})$. Moreover, this bijection has the property that*

$$[\mathrm{IC}_{\gamma(\lambda)}] \in \mathrm{span}\{[A_\mu] \mid \mu \leq \lambda\};$$

i.e., it coincides with the bijection established by Bezrukavnikov.

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