Problem Set 1

1. Prove the following statements (I did some of these in class). $\mathcal{F}$ and $\mathcal{G}$ are sheaves unless otherwise specified.

   (a) Let $\{V_i\}$ be an open cover of U, and let $s, t \in \mathcal{F}(U)$. If $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$. (In particular, if $s|_{V_i} = 0$ for all i, then $s = 0$.)

   (b) A section is determined by its germs. That is, if $s, t \in \mathcal{F}(U)$ are sections such that $s_x = t_x$ for all $x \in U$, then $s = t$.

   (c) Suppose $\mathcal{F}$ is presheaf, $\mathcal{G}$ is a sheaf, and $\mathcal{F} \subset \mathcal{G}$. Define a subsheaf $\mathcal{F}' \subset \mathcal{G}$ by
   $$\mathcal{F}'(U) = \{s \in \mathcal{G}(U) \mid \text{there is a covering } \{V_i\} \text{ of } U \text{ such that } s|_{V_i} \in \mathcal{F}(V_i) \text{ for all } i\}.$$
   Then $\mathcal{F}^+ \simeq \mathcal{F}'$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}^+ \simeq \mathcal{F}$.

   (d) Let $\mathcal{F}$ be a presheaf. The stalks of $\mathcal{F}^+$ are isomorphic to those of $\mathcal{F}$.

   (e) Given a morphism $f : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, there is a unique morphism $f^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $f = f^+ \circ \iota$, where ι is the canonical morphism $\mathcal{F} \to \mathcal{F}^+$.

   (f) A morphism $f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U. (*Warning*: the corresponding statement for surjective morphisms is not true.)

   (g) A morphism $f : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective, an isomorphism) if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, an isomorphism) for all $x \in X$.

2. Let $X = \mathbb{C}$. Let $\underline{\mathbb{Z}}$ be the constant sheaf on X with stalk $\mathbb{Z}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the sheaves on X defined as follows:
   $$\mathcal{F}(U) = \{\text{continuous functions } U \to \mathbb{C}\} \qquad \text{and} \qquad \mathcal{G}(U) = \{\text{continuous functions } U \to \mathbb{C} \smallsetminus \{0\}\}.$$
   (Note that $\mathcal{G}(U)$ must be regarded as an abelian group with respect to multiplication, not addition.) Next, we define morphisms $f : \underline{\mathbb{Z}} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{G}$ as follows:
   $$f_U(s) = 2\pi i s \qquad \text{where } s \in \underline{\mathbb{Z}}(U) \text{ is a locally constant function } U \to \mathbb{Z}$$
   $$g_U(s) = e^s \qquad \text{where } s \in \mathcal{F}(U) \text{ is a continuous function } U \to \mathbb{C}$$
   Show that
   $$0 \to \underline{\mathbb{Z}} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \to 0$$
   is a short exact sequence of sheaves. Also, show that the presheaf-image of $g : \mathcal{F} \to \mathcal{G}$ is not a sheaf.

3. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be an exact sequence of sheaves on X, and let $U \subset X$ be an open set. Show that
   $$0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H})$$
   is exact. In other words, $\Gamma(U, \cdot)$ is a left-exact functor from the category of sheaves on X to the category of abelian groups. Show by example that $\Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H})$ need not be surjective even if $\mathcal{G} \to \mathcal{H}$ is. (*Hint*: Consider the morphism $g : \mathcal{F} \to \mathcal{G}$ of the previous problem.)

4. Let $\mathcal{H}$ be a subsheaf of $\mathcal{F}$. How should one define the quotient sheaf $\mathcal{F}/\mathcal{H}$? (There is an obvious first guess you can make; is that object actually a sheaf, or only a presheaf that needs to be sheafified?) Prove the "first isomorphism theorem": given a morphism $f : \mathcal{F} \to \mathcal{G}$, we have $\mathcal{F}/\ker f \simeq \operatorname{im} f$.

Problem Set 2

January 30, 2007

From this problem set on, the following assumptions are in effect:

- All sheaves are sheaves of complex vector spaces unless otherwise specified.
- All topological spaces are locally path-connected, semilocally simply connected, locally compact, second-countable, and Hausdorff. Unless otherwise specified, they are also path-connected.

In problems that ask you to "identify" a sheaf, you should either show that the sheaf is isomorphic to some sheaf we have discussed in class, or give as explicit a description as you can of sections of the sheaf over a typical connected open set.

1. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $\mathcal{Q}$ be the square-root sheaf on X.
   (a) Let $f : X \to X$ be the map $f(z) = z^2$. Identify $f_*\mathcal{Q}$ and $f^{-1}\mathcal{Q}$.
   (b) Show that $\operatorname{Hom}(\mathcal{Q}, \underline{\mathbb{C}}) = 0$. Identify the sheaf $\mathcal{H}om(\mathcal{Q}, \underline{\mathbb{C}})$. (It's not the zero sheaf.)
   (c) Identify $\mathcal{Q} \otimes \mathcal{Q}$. Also, show explicitly in this example that the presheaf tensor product $\mathcal{Q} \otimes_{\text{ps}} \mathcal{Q}$ is not a sheaf.
   (d) Let $g : X \to X$ be the map $g(z) = z^3$. Identify $g_*\underline{\mathbb{C}}$. (*Hint*: The answer is $\underline{\mathbb{C}} \oplus \mathcal{F} \oplus \mathcal{G}$, where $\mathcal{F}$ is the "cube-root sheaf" given by
   $$\mathcal{F}(U) = \{\mathbb{C}\text{-linear combinations of functions } k : U \to \mathbb{C} \text{ such that } k(z)^3 = z\},$$
   and $\mathcal{G}$ is another locally constant sheaf that you'll have to identify by yourself.)
   (e) Identify $\mathcal{H}om(\mathcal{F}, \underline{\mathbb{C}})$. (*Hint*: The answer is the sheaf $\mathcal{G}$ that appears in the previous question. If you have already answered that question, then you know what $\mathcal{G}$ is, and you just have to show that $\mathcal{H}om(\mathcal{Q}, \underline{\mathbb{C}})$ is isomorphic to it. Otherwise, you can use this problem to help you answer the previous one.)

2. Let $j : U \hookrightarrow X$ be the inclusion of an open set. If $\mathcal{F}$ is a sheaf on U and $\mathcal{G}$ is a sheaf on X, show that $\operatorname{Hom}(j_!\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}(\mathcal{F}, j^{-1}\mathcal{G})$.

3. (Hartshorne, Exercise 1.19(c)) Let $U \subset X$ be an open set, let $Z = X \smallsetminus U$ be its complement, and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Given a sheaf $\mathcal{F}$ on X, show that there is a short exact sequence of sheaves
$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

4. Let $\mathfrak{Sh}(X)$ be the category of sheaves on X, and let $\mathfrak{PSh}(X)$ be the category of presheaves on X. Let $I : \mathfrak{Sh}(X) \to \mathfrak{PSh}(X)$ be the inclusion functor (note that $\mathfrak{Sh}(X)$ is a subcategory of $\mathfrak{PSh}(X)$). We also have the sheafification functor ${}^+ : \mathfrak{PSh}(X) \to \mathfrak{Sh}(X)$. Show that $({}^+, I)$ is an adjoint pair of functors.

Problem Set 3

February 1, 2007

In this problem set (and henceforth in the course), the following slight abuse of language will be made: if $\rho : \pi_1(X, x_0) \to GL(E)$ is a representation of $\pi_1(X, x_0)$ on the vector space E, we will call E itself "the representation." (Thus, "Let E be a representation of $\pi_1(X, x_0)$" means "Let E be a complex vector space, and suppose there is a representation $\pi_1(X, x_0) \to GL(E)$ of $\pi_1(X, x_0)$ on E.")

1. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be sheaves of abelian groups on X. Prove that

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \qquad \text{and} \qquad \mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

   by using the corresponding facts for abelian groups.

2. Problem 2 of Problem Set 2 asked you to show that $(j_!, j^{-1})$ is an adjoint pair, where $j : U \hookrightarrow X$ is an open inclusion. State and prove a sheaf-Hom version of that theorem. (Note that it does not make sense to say $\mathcal{H}om_X(j_!\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om_U(\mathcal{F}, j^{-1}\mathcal{G})$.)

3. Show that there is an equivalence of categories

$$(\text{local systems on } X) \overset{\sim}{\longleftrightarrow} (\text{representations of } \pi_1(X, x_0)).$$

   In other words: In class, we have shown that there is a bijection between isomorphism classes of local systems and isomorphism classes of representations of $\pi_1(X, x_0)$. Now, let E and F be representations of $\pi_1(X, x_0)$, and let $\mathcal{E}$ and $\mathcal{F}$ be the corresponding local systems on X. Given a morphism $\phi : \mathcal{E} \to \mathcal{F}$, associate to it an equivariant linear transformation (also called an "intertwining operator") $S : E \to F$. Conversely, given an equivariant linear transformation $T : E \to F$, construct an associated morphism $\psi : \mathcal{E} \to \mathcal{F}$. Finally, show that the two assignments $\phi \mapsto S$ and $T \mapsto \psi$ are inverse to each other.

4. If $\mathcal{E}$ and $\mathcal{F}$ are local systems, show that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$ are as well. What are their ranks in terms of $\operatorname{rank} \mathcal{E}$ and $\operatorname{rank} \mathcal{F}$? (The **rank** of a local system is the dimension of any of its stalks.)

5. In the setting of the previous problem, let E and F be the representations of $\pi_1(X, x_0)$ corresponding to $\mathcal{E}$ and $\mathcal{F}$. There is a natural way to regard the vector spaces $\mathrm{Hom}(E, F)$ and $E \otimes F$ as representations of $\pi_1(X, x_0)$ as well. (Ask me if you can't figure out how yourself.) Show that the local systems corresponding to these representations are precisely $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$.

6. Let E and F be two nonisomorphic irreducible representations of $\pi_1(X, x_0)$. (A representation is **irreducible** if contains no nontrivial subspace that is stable under the action of $\pi_1(X, x_0)$.) Show that $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$. (Note, however, that the sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is not, in general, the zero sheaf.)

Problem Set 4

February 8, 2007

1. Let $p : Y \to X$ be a finite-to-one covering map. If $\mathcal{E}$ is a local system on Y, show that $p_*\mathcal{E}$ is a local system on X. If p is not finite-to-one, the proof of the previous statement does not go through, but it can be fixed up by imposing an additional condition on $\mathcal{E}$. Find such a condition, and prove that with this extra condition, $p_*\mathcal{E}$ is again a local system. (*Hint*: If Y is compact, no additional condition on $\mathcal{E}$ is needed.)

2. Recall that there is a one-to-one correspondence between covering spaces over X (up to isomorphism) and subgroups of its fundamental group $\pi_1(X, x_0)$ (up to conjugacy. Suppose $p : Y \to X$ is a covering map corresponding to the subgroup $H \subset \pi_1(X, x_0)$. Let $E = \mathbb{C}[\pi_1(X, x_0)/H]$, the vector space of formal linear combinations of cosets of H with complex coefficients. There is an obvious representation of $\pi_1(X, x_0)$ on E. Show that the local system $p_*\underline{\mathbb{C}}$ is the one corresponding to this representation.

3. Show that the category of sheaves of abelian groups on a fixed topological space X is an abelian category.

4. Let $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of functors. Show that S is right-exact and that T is left-exact. (Be sure to do this using the language of categories—you cannot talk about "elements" of kernels and images, because in a general abelian category, the concept of "element" may not make sense.) It immediately follows that the functors

$$\mathrm{Hom}(\mathcal{G}, \cdot), \quad \mathcal{H}om(\mathcal{G}, \cdot), \quad f_*$$

   are left-exact (here $\mathcal{G}$ is a sheaf and $f : X \to Y$ is a continuous map), and

$$\cdot \otimes \mathcal{G}, \quad f^{-1}, \quad j_!$$

   are right-exact (here $j : U \to X$ is an open inclusion). Finally, note that j^{-1} (again, j is an open inclusion) is exact, because it has adjoints on both sides: j_* is its right adjoint, and $j_!$ is its left adjoint.

5. Show that f^{-1} is actually an exact functor for any continuous map $f : X \to Y$, not just an open inclusion.

6. Show that if $j : U \to X$ is an open inclusion, then $j_!$ is an exact functor. (It suffices to show that it is right-exact.)

Problem Set 5

February 15, 2007

1. Gelfand & Manin, *Methods of Homological Algebra*, Exercise II.5.2 (page 119).

2. *Ibid.*, Exercise II.5.5 (page 120). (*Note*: See Exercise II.5.4 for the definition of "$f(y)$.")

3. Let $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow A[1]^\bullet$ be a distinguished triangle (in $K(\mathcal{A})$ or $D(\mathcal{A})$). Show that $g \circ f = 0$.

4. Let $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ be a short exact sequence in $C(\mathcal{A})$. Define a morphism $\theta : \operatorname{cone}^\bullet f \to C^\bullet$ by the matrix $\begin{pmatrix} 0 & g \end{pmatrix}$. Show that θ is indeed a morphism of complexes (*i.e.*, that it commutes with differentials) and that the following diagram commutes:

$$\begin{array}{ccccc}
A^\bullet & \xrightarrow{f} & B^\bullet & \longrightarrow & \operatorname{cone}^\bullet f \\
\| & & \| & & \downarrow{\scriptstyle\theta} \\
A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet
\end{array}$$

   Finally, show that θ is a quasi-isomorphism. Deduce that in $D(\mathcal{A})$ (but not necessarily in $K(\mathcal{A})$), there is a distinguished triangle of the form $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to A[1]^\bullet$.

5. Prove that the octahedral property holds for distinguished triangles in $D(\mathcal{A})$.

6. Given a complex $A^\bullet$, define new complexes $\tau_{\leq 0}A^\bullet$ and $\tau_{\geq 1}A^\bullet$ by

$$(\tau_{\leq 0}A)^i = \begin{cases} A^i & \text{if } i < 0, \\ \ker d_A^0 & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases} \qquad \text{and} \qquad (\tau_{\geq 1}A)^i = \begin{cases} 0 & \text{if } i < 1 \\ \operatorname{cok} d_A^0 & \text{if } i = 1, \\ A^i & \text{if } i > 1. \end{cases}$$

   There is an obvious way to define differentials (so that $\tau_{\leq 0}A^\bullet$ and $\tau_{\geq 1}A^\bullet$ are actually complexes), as well as obvious morphisms $\tau_{\leq 0}A^\bullet \to A^\bullet$ and $A^\bullet \to \tau_{\geq 1}A^\bullet$. Find all of these, and then show that there is a distinguished triangle

$$\tau_{\leq 0}A^\bullet \to A^\bullet \to \tau_{\geq 1}A^\bullet \to \tau_{\leq 0}A[1]^\bullet.$$

   What can you say about the cohomology objects of the three complexes $\tau_{\leq 0}A^\bullet$, $A^\bullet$, and $\tau_{\geq 1}A^\bullet$?

Problem Set 6

March 1, 2007

1. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves. Show that $\underline{\mathrm{Hom}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ (graded tensor product) are well-defined complexes (that is, that $d^2 = 0$). In class, we defined $R\,\mathrm{Hom}$ and $\otimes^L$ as functors only of the second variable: for a fixed $\mathcal{F}^\bullet \in C^-(\mathfrak{Sh}_X)$, we have

$$R\,\mathrm{Hom}(\mathcal{F}^\bullet, -) : D^+(\mathfrak{Sh}_X) \to D^+(\mathfrak{Ab})$$
$$\mathcal{F}^\bullet \overset{L}{\otimes} - : D^-(\mathfrak{Sh}_X) \to D^-(\mathfrak{Sh}_X)$$

Show that $R\,\mathrm{Hom}$ can be regarded as a contravariant functor in its first variable, and that $\otimes^L$ can be regarded as a covariant functors in its first variable. *A priori*, these variables are $C^-(\mathfrak{Sh}_X)$; show that in fact they can be regarded as being in $D^-(\mathfrak{Sh}_X)$. Thus, $R\,\mathrm{Hom}$ and $\otimes^L$ become bifunctors:

$$R\,\mathrm{Hom} : D^-(\mathfrak{Sh}_X) \times D^+(\mathfrak{Sh}_X) \to D^+(\mathfrak{Ab})$$
$$\overset{L}{\otimes} : D^-(\mathfrak{Sh}_X) \times D^-(\mathfrak{Sh}_X) \to D^-(\mathfrak{Sh}_X)$$

2. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor, and suppose that $\mathcal{A}$ has enough injectives, so that RF is defined. Given an object A of $\mathcal{A}$, regard it as a complex with nonzero term only in degree 0 in the obvious way. Show that $H^0(RF(A)) \simeq F(A)$.

3. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor, and suppose that $\mathcal{A}$ has enough injectives, so that RF is defined. If F happens to actually be exact, it gives rise to a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ in a more direct way than the derived-functor construction. Show that this latter functor coincides with RF on $D^+(\mathcal{A})$.

4. Let $A^\bullet$ and $B^\bullet$ be complexes of objects of an abelian category $\mathcal{A}$ with enough injectives. Show that $H^0(R\,\mathrm{Hom}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$.

5. (Extensions) Let A and B be objects of an abelian category $\mathcal{A}$ that has enough injectives. An *extension* of B by A is simply a short exact sequence

$$0 \to A \to E \to B \to 0.$$

Two extensions are *isomorphic* if there exists an isomorphism ϕ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\phi}\downarrow{\scriptstyle\wr} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Show that there is a bijection between isomorphism classes of extensions of B by A and elements of the abelian group $\mathrm{Ext}^1(B, A)$. (*Hint*: Regard A and B as complexes, and first show, using distinguished triangles, that there is a bijection between isomorphism classes of extensions and $\mathrm{Hom}_{D(\mathcal{A})}(B, A[1])$. Then use the preceding exercise.) What extension corresponds to $0 \in \mathrm{Ext}^1(B, A)$?

6. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves on Y, and let $f : X \to Y$ be a continuous map. Show that $f^{-1}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) \simeq (f^{-1}\mathcal{F}^\bullet) \otimes^L (f^{-1}\mathcal{G}^\bullet)$. (You'll need to first prove the corresponding statement in the nonderived setting.)

Problem Set 7

March 13, 2007

1. Let $\mathcal{F}$ be a sheaf on X, and let $\mathcal{G}$ be an injective sheaf on X. Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is flasque. Your proof should be valid in the category of sheaves of abelian groups (in particular, do not assume that $\mathcal{F}$ is flat, and do not cite the result proved in class that states that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is injective if $\mathcal{F}$ is flat and $\mathcal{G}$ is injective).

2. Show that a sheaf $\mathcal{F}$ is injective if and only if $\mathcal{H}om(-,\mathcal{F})$ is an exact functor.

3. (Uniqueness of adjoint functors) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories, and let $G, H : \mathcal{B} \to \mathcal{A}$ be two functors that are both right adjoint to F. Show that $G \simeq H$. *I.e.*, show that there is a rule η that assigns to every object $B \in \mathcal{B}$ a morphism $\eta(B) : G(B) \to H(B)$ in $\mathcal{A}$ such that for every morphism $f : B \to C$ in $\mathcal{B}$, the following square commutes:

$$\begin{array}{ccc} G(B) & \xrightarrow{\eta(B)} & H(B) \\ {\scriptstyle G(f)}\big\downarrow & & \big\downarrow{\scriptstyle H(f)} \\ G(C) & \xrightarrow[\eta(C)]{} & H(C) \end{array}$$

4. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Show that $j^! = j^{-1}$.

5. Let $j : U \hookrightarrow X$ be an inclusion of an open set. Let $Z = X \smallsetminus U$, and let $i : Z \hookrightarrow X$ be the inclusion of Z into X. Define a functor $i^\diamond : \mathfrak{Sh}_X \to \mathfrak{Sh}_Z$ by
$$i^\diamond \mathcal{F} = i^{-1}(\ker(\mathcal{F} \to j_* j^{-1}\mathcal{F}))$$
where $\mathcal{F} \to j_* j^{-1}\mathcal{F}$ is the obvious morphism. Show that $(i_*, i^\diamond)$ is an adjoint pair. Also show that $i^\diamond$ is left-exact, and that $i^! = Ri^\diamond$.

6. Let $i : \{x\} \hookrightarrow X$ be the inclusion of a point. Show that $i^*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}i^!\mathcal{F}^\bullet$. Do not use the more general version of this fact that was proved in class—the proof of that statement requires this one to be known first. (*Hint*: First, using adjointness of of $Ra_!$ and $a^!$ where $a : X \to \{\text{pt}\}$ is a map to a one-point space, show that
$$R\Gamma(U, \mathbb{D}\mathcal{F}^\bullet) \simeq R\operatorname{Hom}(R\Gamma_c(U, \mathcal{F}^\bullet), \mathbb{C}).$$
Then, show that
$$\varprojlim_{U \ni x} R\Gamma_c(U, \mathcal{F}^\bullet) \simeq Ri^\diamond \mathcal{F}^\bullet.$$
Finally, use the fact that
$$i^*\mathbb{D}\mathcal{F}^\bullet = \varinjlim_{U \ni x} R\Gamma(U, \mathbb{D}\mathcal{F}^\bullet)$$
to complete the proof.)

7. Give an example showing that $f^!$ is not, in a general, a derived functor. (*Hint*: if $f^!$ were a derived functor, it would have to be the derived functor of $H^0 \circ f^!$. Find an example of a map $f : X \to Y$ such that $H^0(f^!\mathcal{F})$ is always 0, but $f^!\mathcal{F}$ is not.)

8. Prove all the statements from the following list that were not proved in class.
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{G}^\bullet, \mathbb{D}\mathcal{F}^\bullet) \qquad Rf_!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_*\mathcal{F}^\bullet$$
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq \mathbb{D}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \qquad Rf_*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_!\mathcal{F}^\bullet$$

Problem Set 8

March 20, 2007

The goal of this problem set is to illustrate that the concept of t-structure is not trivial: the heart of a nontrivial t-structure on the derived category of an abelian category need not be equivalent to the original abelian category.

NOTE: Please do not hand in Problem 1 for credit. (You may of course hand it in if you just want me to look over your solution.)

1. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the category whose objects V are diagrams of complex vector spaces of the form

$$V_1 \to V_2 \leftarrow V_3 \qquad \text{resp.} \qquad V_1 \leftarrow V_2 \to V_3,$$

and in which a morphism $f : V \to W$ is a commutative diagram

$$\begin{array}{ccccc} V_1 & \longrightarrow & V_2 & \longleftarrow & V_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ W_1 & \longrightarrow & W_2 & \longleftarrow & W_3 \end{array} \qquad \text{resp.} \qquad \begin{array}{ccccc} V_1 & \longleftarrow & V_2 & \longrightarrow & V_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ W_1 & \longleftarrow & W_2 & \longrightarrow & W_3 \end{array}$$

Show that up to isomorphism, there are exactly three simple objects of $\mathcal{A}$ (resp. $\mathcal{B}$) are

$$\begin{array}{lcl} E_1 = (\mathbb{C} \to 0 \leftarrow 0) & & F_1 = (\mathbb{C} \leftarrow 0 \to 0) \\ E_2 = (0 \to \mathbb{C} \leftarrow 0) & \text{resp.} & F_2 = (0 \leftarrow \mathbb{C} \to 0) \\ E_3 = (0 \to 0 \leftarrow \mathbb{C}) & & F_3 = (0 \leftarrow 0 \to \mathbb{C}) \end{array}$$

2. Compute $\operatorname{Ext}^1(E_i, E_j)$ and $\operatorname{Ext}^1(F_i, F_j)$. Specifically, show that

$$\dim \operatorname{Ext}^1(E_i, E_j) = \begin{cases} 1 & \text{if there is an arrow from } V_i \text{ to } V_j \text{ in objects of } \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases},$$

and similarly for $\mathcal{B}$. (*Hint*: Use the relationship between Ext^1 and extensions given by Problem 5 of Problem Set 6.) Then, deduce that $\mathcal{A}$ and $\mathcal{B}$ are *not* equivalent categories.

3. In this problem, you will show that although $\mathcal{A}$ and $\mathcal{B}$ are inequivalent abelian categories, their derived categories are equivalent. Let $S : D(\mathcal{A}) \to D(\mathcal{B})$ be the functor that takes a complex $V^\bullet$ (whose ith term is the diagram $V^i = (V_1^i \xrightarrow{f_i} V_2^i \xleftarrow{g_i} V_3^i))$ to the complex whose terms are

$$S(V^\bullet)^i = (\ V_1^i \xleftarrow{(1\ 0\ 0)} V_1^i \oplus V_2^{i-1} \oplus V_3^i \xrightarrow{(0\ 0\ 1)} V_3^i\ )$$

and whose differentials $d^i_{S(V^\bullet)} : S(V^\bullet)^i \to S(V^\bullet)^{i+1}$ are given by

$$\begin{array}{ccccc} V_1^i & \xleftarrow{(1\ 0\ 0)} & V_1^i \oplus V_2^{i-1} \oplus V_3^i & \xrightarrow{(0\ 0\ 1)} & V_3^i \\ {\scriptstyle d_1^i}\downarrow & & \left(\begin{smallmatrix} d_1^i & & \\ f_i & -d_2^{i-1} & g_i \\ & & d_3^i \end{smallmatrix}\right)\downarrow & & {\scriptstyle d_3^i}\downarrow \\ V_1^{i+1} & \xleftarrow[(1\ 0\ 0)]{} & V_1^{i+1} \oplus V_2^i \oplus V_3^{i+1} & \xrightarrow[(0\ 0\ 1)]{} & V_3^{i+1} \end{array}$$

On the other hand, given a complex $W^\bullet$ in $D(\mathcal{B})$, with terms $W^i = (W_1^i \xleftarrow{h_i} W_2^i \xrightarrow{k_i} W_3^i)$, let $T(W^\bullet)$ be the complex in $D(\mathcal{A})$ given by

$$T(W^\bullet)^i = (\ W_1^i \xrightarrow{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} W_1^i \oplus W_2^{i+1} \oplus W_3^i \xleftarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} W_3^i\ )$$

with differentials

$$\begin{array}{ccccc}
W_1^i & \xrightarrow{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} & W_1^i \oplus W_2^{i+1} \oplus W_3^i & \xleftarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} & W_3^i \\
\Big\downarrow{\scriptstyle d_1^i} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix}d_1^i & h_i & \\ & -d_2^{i+1} & \\ & k_i & d_3^i\end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle d_3^i} \\
W_1^{i+1} & \xrightarrow[\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)]{} & W_1^{i+1} \oplus W_2^{i+2} \oplus W_3^{i+1} & \xleftarrow[\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)]{} & W_3^{i+1}
\end{array}$$

Show that S and T are equivalences of categories, inverse to one another. (Essentially, you must show that $T(S(V^\bullet))$ is quasi-ismorphic to $V^\bullet$, and that $S(T(W^\bullet))$ is quasi-isomorphic to $W^\bullet$.)

4. By following the standard t-structure on $D(\mathcal{B})$ through T over to $D(\mathcal{A})$, we obtain a nonstandard t-structure on $D(\mathcal{A})$. (It must be nonstandard because its heart is a copy of $\mathcal{B}$, not of $\mathcal{A}$.) Describe this t-structure explicitly.

Problem Set 9

March 22, 2007

In this problem set, you will show that the derived category of the heart of a t-structure on a triangulated category need not be equivalent to the original triangulated category.

Let X be the 2-sphere. Let $\mathcal{C}$ be the full subcategory of $D^b(\mathfrak{Sh}_X)$ consisting of complexes of sheaves $\mathcal{F}$ all of whose cohomology sheaves $H^i(\mathcal{F})$ are constant ordinary sheaves on X.

1. Show that $\mathcal{C}$ inherits from $D^b(\mathfrak{Sh}_X)$ the structure of a triangulated category. (The main thing to show is that any morphism can be completed to a distinguished triangle: if $\mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{C}$, then of course there is a distinguished triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ in $D^b(\mathfrak{Sh}_X)$, but is $\mathcal{H}$ necessarily in $\mathcal{C}$?)

2. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be the t-structure obtained by restricting the standard t-structure on $D^b(\mathfrak{Sh}_X)$ to $\mathcal{C}$:
$$\mathcal{C}^{\leq 0} = \{\mathcal{F} \mid H^i(\mathcal{F}) = 0 \text{ for all } i > 0\},$$
$$\mathcal{C}^{\geq 0} = \{\mathcal{F} \mid H^i(\mathcal{F}) = 0 \text{ for all } i < 0\}.$$
Let $\mathcal{T}$ be the heart of this t-structure. Show that $\mathcal{T}$ is equivalent to the category $\mathfrak{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces.

3. The bounded derived category $D^b(\mathcal{T})$ is also a triangulated category equipped with a t-structure (the standard one). A natural question is: is there an equivalence of categories $\Phi : D^b(\mathcal{T}) \to \mathcal{C}$ that respects all the additional structures on both sides? Such a functor should:
   - be compatible with the shift functors in both categories,
   - take distinguished triangles to distinguished triangles,
   - take $D^b(\mathcal{T})^{\leq 0}$ to $\mathcal{C}^{\leq 0}$ and $D^b(\mathcal{T})^{\geq 0}$ to $\mathcal{C}^{\geq 0}$, and
   - be the identity functor on $\mathcal{T}$.

   Show that there is no such functor. (*Hint*: Consider the constant sheaf $\underline{\mathbb{C}}_X$, which can be regarded as an object of $\mathcal{C}$, of $\mathcal{T}$, or of $D^b(\mathcal{T})$. Show that
$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) \simeq \mathbb{C} \qquad \text{but} \qquad \mathrm{Hom}_{D^b(\mathcal{T})}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) = 0.$$
   To calculate the Hom group in $\mathcal{C}$, you will need to use the following fact about the 2-sphere:
$$H^i(X, \mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$
   You may use this fact without proof. On the other hand, calculating a Hom group in $D^b(\mathcal{T})$ is, by the previous exercise, the same as calculating a Hom group in $D^b(\mathfrak{Vect}_{\mathbb{C}})$.)

Problem Set 10

March 29, 2007

1. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two triangulated categories with t-structures is said to be ***t*-exact** if $F(\mathcal{C}_1^{\leq 0}) \subset \mathcal{C}_2^{\leq 0}$ and $F(\mathcal{C}_1^{\geq 0}) \subset \mathcal{C}_2^{\geq 0}$. Let D_U, D_Z, and D be categories of sheaves as in the theorem on gluing of t-structures. Show that the t-structure on D described in that theorem is the *unique* t-structure on D such that $Ri_* : D_Z \to D$ and $j^{-1} : D \to D_U$ are t-exact functors.

2. Suppose we give D_U and D_Z the standard t-structure. Show that the t-structure on D described by the gluing theorem is the standard t-structure. Also, show that the middle-extension functor $j_{!*}$ coincides in this case with the (non-derived) extension-by-zero functor $j_!$.

3. Suppose D_U has the standard t-structure. By shifting the standard t-structure on D_Z and then gluing, can you obtain a t-structure on D for which the middle-extension functor coincides with the non-derived push-forward j_*? How about Rj_*? $Rj_!$? (*Hint*: The answers for j_* and $Rj_!$ are "yes." For Rj_*, it depends on properties of the topological space U. You should find a condition on U under which the answer is "yes.")

4. Let $X = \mathbb{C}$, and let $\mathcal{O}$ be the ordinary sheaf of holomorphic functions on X (in terms of the variable z). Now, let $\mathcal{F}$ be the complex of sheaves on X given by
$$\mathcal{F}^{-1} = \mathcal{F}^0 = \mathcal{O}, \qquad \mathcal{F}^j = 0 \quad \text{if } j \neq -1, 0,$$
with the differential $d^{-1} : \mathcal{F}^{-1} \to \mathcal{F}^0$ given by $d_U^{-1}(s) = z\frac{d}{dz}s$ for $s \in \mathcal{O}(U)$. Show that
$$H^{-1}(\mathcal{F}) \simeq \underline{\mathbb{C}}_X \qquad \text{and} \qquad H^0(\mathcal{F}) \simeq i_*\underline{\mathbb{C}}_{\{0\}}$$
where $i : \{0\} \hookrightarrow X$ is the inclusion map. (That is, $H^0(\mathcal{F})$ is the skyscraper sheaf at the point 0 with stalk $\mathbb{C}$.) Then, show that $\mathcal{F}$ is a perverse sheaf with respect to the stratification $X = (\mathbb{C} \smallsetminus \{0\}) \coprod \{0\}$. (*Hint*: To show that $i^!\mathcal{F} \in D_Z^{\geq 0}$, use adjointness theorems to reduce to the problem of showing that $\mathrm{Hom}(i_*\mathcal{G}, \underline{\mathbb{C}}_X) = 0$ for any ordinary sheaf $\mathcal{G}$ on $Z = \{0\}$.)

5. Find a Jordan–Hölder series for the perverse sheaf $\mathcal{F}$ of the previous exercise. That is, find a sequence of sub-perverse-sheaves $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ such that each quotient $\mathcal{F}_k/\mathcal{F}_{k-1}$ is a simple perverse sheaf. (*Hint*: Form the distinguished triangle
$$\tau_{\leq -1}\mathcal{F} \to \mathcal{F} \to \tau_{\geq 0}\mathcal{F} \to (\tau_{\leq -1}\mathcal{F})[1]$$
with respect to the *standard* t-structure on D, and then take the associated long exact sequence of perverse cohomology sheaves.)

6. We have defined $D_c^b(X)$ to be the subcategory of $D^b(X)$ consisting of complexes of sheaves all of whose cohomology sheaves are constructible. Show by example that this is not the same as the derived category of the category of constructible sheaves. (*Hint*: Consider the sheaf $\mathcal{F}$ of the previous exercise. Show that this sheaf is not quasi-isomorphic to any complex of constructible sheaves.)

Problem Set 11

April 12, 2007

In all of the following problems, X is a stratified space with stratification $\mathcal{S}$, and $p : \mathcal{S} \to \mathbb{Z}$ is a perversity function.

1. Prove that $D^b_c(X)$ is a triangulated category. (All the axioms except one are obvious because they hold in $D^b(X)$. The only thing to show is that given a morphism $f : \mathcal{F} \to \mathcal{G}$, you can extend it to a distinguished triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$. You can of course do that in $D^b(X)$, but is $\mathcal{H}$ necessarily constructible?) (*Hint*: First reduce the problem to the case of one stratum. In that case, be careful: you are dealing with sheaves whose cohomology sheaves are locally constant, but you cannot assume that the sheaves themselves are complexes of locally constant ordinary sheaves.)

2. Let S be a stratum, $\mathcal{E}$ a local system on S, and $\mathcal{F}$ a perverse sheaf with support contained in $\overline{S} \smallsetminus S$. Show that
$$\mathrm{Hom}(\mathrm{IC}(\overline{S}, \mathcal{E}), \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathrm{IC}(\overline{S}, \mathcal{E})) = 0.$$

For the remaining problems, assume that p is a Goresky–MacPherson perversity. That is, $p(S) = \tilde{p}(\dim S)$, where $\tilde{p} : \mathbb{N} \to \mathbb{Z}$ is a function satisfying

$$0 \le \tilde{p}(n) - \tilde{p}(m) \le m - n$$

whenever $n \le m$. Equivalently, $\tilde{p}$ and $\tilde{p}^*$ (where $\tilde{p}^*(n) = -n - \tilde{p}(n)$) are both weakly decreasing.

3. Suppose there is an open dense stratum U in $\mathcal{S}$, and assume that $p(U) < p(S)$ for all strata $S \neq U$. Show that if X is a manifold, then $\mathrm{IC}(\overline{U}, \underline{\mathbb{C}}_U) \simeq \underline{\mathbb{C}}_X[p(U)]$.

4. (a) Suppose $\mathcal{S}$ consists of exactly two strata, U and Z, with $\overline{U} = X$. Assume that $p(U) < p(Z)$. Let $j : U \hookrightarrow X$ be the inclusion map. Show that
$$\mathrm{IC}(\overline{U}, \mathcal{E}) = {}^{\mathrm{std}}\tau_{\le p(Z)-1} Rj_*(\mathcal{E}[\dim U]).$$
   (b) Now, let S be a stratum with the property that $p(S) < p(T)$ for all strata $T \subset \overline{S}$, $T \neq S$. Give a construction of $\mathrm{IC}(\overline{S}, \mathcal{E})$ that uses only derived push-forwards and truncation functors with respect to the *standard* t-structure. (Since the standard truncation functors can be defined by explicit formulas at the level of complexes, this problem shows that intersection cohomology complexes can be defined without using the formalism of t-structures at all.) (*Hint:* Use induction on the number of strata in $\overline{S}$ and imitate part (a).)

5. Let $\mathcal{F}$ be a constructible sheaf. Show that $\mathcal{F} \in {}^pD^b_c(X)^{\le 0}$ if and only if
$$\dim \operatorname{supp} H^{\tilde{p}(k)}(\mathcal{F}) \le k \qquad \text{for all } k,$$
$$H^i(\mathcal{F}) = 0 \qquad \text{for all } i > \tilde{p}(0).$$
   Then show that $\mathcal{F}$ is perverse if and only if the above condition holds for both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$.

Problem Set 12

April 26, 2007

In all of the following problems, X is a stratified space with stratification $\mathcal{S}$, and X has a unique open stratum U. All perverse sheaves are with respect to the middle perversity.

1. Let $f : Y \to X$ be a semismall resolution. Show that $Rf_*\underline{\mathbb{C}}$ is a perverse sheaf.

2. Let $f : Y \to X$ be a small resolution. Show that $Rf_*\underline{\mathbb{C}} \simeq \mathrm{IC}(X, \underline{\mathbb{C}})$.

3. Suppose X has exactly two strata, U and Z (so of course U is open and Z is closed). Suppose $p(U) > p(Z)$ (in particular, this is *not* a Goresky–MacPherson perversity). Show that all perverse sheaves are of the form $j_!\mathcal{E}[p(U)] \oplus i_*\mathcal{F}[p(Z)]$, where $\mathcal{E}$ and $\mathcal{F}$ are local systems on U and Z, respectively, and $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ are the inclusion maps. Thus, there is an equivalence of categories

$$M(X) \overset{\sim}{\longleftrightarrow} \{\text{representations of } \pi_1(U) \times \pi_1(Z)\}.$$

   Thus, perverse sheaves with respect to a non-Goresky–MacPherson perversity are not that interesting—they do not encode topological information about the singularities of X.

4. Let $\mathcal{F}$ be a perverse sheaf, and let S be a stratum that is open in the support of $\mathcal{F}$. Show that $(\mathcal{F}|_S)[-p(S)]$ is a local system (in particular, you must show that it is an ordinary sheaf, not a complex of sheaves). Next, let $\mathcal{E}$ be that local sytem. Show that $\mathrm{IC}(\bar{S}, \mathcal{E})$ occurs as a quotient in the Jordan–Hölder series for $\mathcal{F}$. (That is, show that $\mathrm{IC}(\bar{S}, \mathcal{E})$ is a quotient of some sub-perverse sheaf of $\mathcal{F}$.)

5. Let $\mathcal{F}$ and $\mathcal{G}$ be two perverse sheaves. Show that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i < 0$. (Recall that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = H^i(R\mathcal{H}om(\mathcal{F}, \mathcal{G}))$.) (*Hint:* Use induction on the number of strata and a distinguished triangle associated to open and closed inclusions. It may be useful to recall the facts below.)

$$\begin{aligned} j^{-1}R\mathcal{H}om(\mathcal{F}, \mathcal{G}) &\simeq R\mathcal{H}om(j^{-1}\mathcal{F}, j^{-1}\mathcal{G}) && \text{if } j \text{ is an open inclusion,} \\ f^!R\mathcal{H}om(\mathcal{F}, \mathcal{G}) &\simeq R\mathcal{H}om(f^{-1}\mathcal{F}, f^!\mathcal{G}) && \text{for any map } f. \end{aligned}$$

   Note: This result, called "vanishing of negative local Ext's," is an important step in the proof of a theorem that states that perverse sheaves on open sets can be glued together to form a perverse sheaf on the whole space, just like ordinary sheaves can.

6. Show that $\mathrm{Hom}(\mathrm{IC}(\bar{S}, \mathcal{E}), \mathrm{IC}(\bar{S}, \mathcal{F})) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ (where $\mathcal{E}$ and $\mathcal{F}$ are local systems on S).

7. (Schur's lemma for perverse sheaves) Show that if $\mathcal{E}$ is a simple local system on a connected stratum S, then $\mathrm{Hom}(\mathrm{IC}(\bar{S}, \mathcal{E}), \mathrm{IC}(\bar{S}, \mathcal{E})) \simeq \mathbb{C}$. (*Hint:* First show that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$, and use the preceding exercise.)