Perverse Sheaves Quick Reference Guide

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Operations. f^{-1}, Rf_*, $Rf_!$, $f^!$, $\otimes^L$, $R\mathcal{H}om$

$a : X \to \{*\}$ constant map: $R\Gamma = Ra_*$ $\quad R\Gamma_c = Ra_!$

$R\operatorname{Hom} \simeq R\Gamma \circ R\mathcal{H}om$	
Sheaf cohom.:	$H^i(X, \mathcal{F}) := H^i(R\Gamma(\mathcal{F}))$
" w/cpt. supp.:	$H^i_c(X, \mathcal{F}) := H^i(R\Gamma_c(\mathcal{F}))$

Compositions: $(f \circ g)^{-1} \simeq g^{-1} \circ f^{-1}$ $\quad R(f \circ g)_* \simeq Rf_* \circ Rg_*$
$(f \circ g)^! \simeq g^! \circ f^!$ $\quad R(f \circ g)_! \simeq Rf_! \circ Rg_!$

Thm (Local Systems). If X is connected,

$$\{\text{local systems on } X\} \overset{\sim}{\longleftrightarrow} \{\text{repns. of } \pi_1(X, x_0)\}$$

Adjointness Theorems. (also with Hom, $R\operatorname{Hom}$)

$$Rf_* R\mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G}) \simeq R\mathcal{H}om(\mathcal{F}, Rf_*\mathcal{G})$$
$$R\mathcal{H}om(Rf_!\mathcal{F}, \mathcal{G}) \simeq Rf_* R\mathcal{H}om(\mathcal{F}, f^!\mathcal{G})$$
$$R\mathcal{H}om(\mathcal{F} \otimes^L \mathcal{G}, \mathcal{H}) \simeq R\mathcal{H}om(\mathcal{F}, R\mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

Base Change Theorems.

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde g} & X \\ \tilde f \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

In general:	If f proper:
$g^{-1}Rf_! \simeq R\tilde{f}_!\tilde{g}^{-1}$	$g^{-1}Rf_* \simeq R\tilde{f}_*\tilde{g}^{-1}$
$g^!Rf_* \simeq R\tilde{f}_*\tilde{g}^!$	$g^!Rf_! \simeq R\tilde{f}_!\tilde{g}^!$

Verdier Duality. Dualizing complex: $\omega_X := a^!\underline{\mathbb{C}}$.
Proper pullback of dualizing complex: $f^!\omega_Y \simeq \omega_X$.
Duality functor $\mathbb{D} := R\mathcal{H}om(\cdot, \omega_X)$. $\mathbb{D} \circ \mathbb{D} \simeq \mathrm{id}$.

$$\mathbb{D} \circ Rf_* \simeq Rf_! \circ \mathbb{D} \qquad \mathbb{D} \circ f^{-1} \simeq f^! \circ \mathbb{D}$$
$$\mathbb{D} \circ Rf_! \simeq Rf_* \circ \mathbb{D} \qquad \mathbb{D} \circ f^! \simeq f^{-1} \circ \mathbb{D}$$

Thm. (Verdier) $H^{-i}_c(X, \mathbb{D}\mathcal{F})^* \simeq H^i(X, \mathcal{F})$

Oriented Manifolds. If $\dim X = n$, $\omega_X = \underline{\mathbb{C}}_X[n]$.
Let $\mathcal{E}^\vee := R\mathcal{H}om(\mathcal{E}, \underline{\mathbb{C}})$. Then $\mathbb{D}(\mathcal{E}[k]) = \mathcal{E}^\vee[n-k]$.

Cor. (Poincaré) $H^{n-i}_c(X, \mathbb{C})^* \simeq H^i(X, \mathbb{C})$.

Open & Closed Inclusions. $j : U \hookrightarrow X$ open, $Z := X \smallsetminus U$, $i : Z \hookrightarrow X$ closed.

Equivalences: $j^{-1} = j^!$ $\quad Ri_* = Ri_!$ $\quad i^! = Ri^\diamond$
Trivial: $i^{-1}Rj_! = 0$ $\quad i^!Rj_* = 0$ $\quad j^{-1}Ri_* = 0$

Distinguished triangles:

$$Rj_!j^{-1}\mathcal{F} \to \mathcal{F} \to Ri_*i^{-1}\mathcal{F} \to Rj_!j^{-1}\mathcal{F}[1]$$
$$Ri_*i^!\mathcal{F} \to \mathcal{F} \to Rj_*j^{-1}\mathcal{F} \to Ri_*i^!\mathcal{F}[1]$$

***t*-Structures.** $\mathcal{C}$ triangulated category with $(\mathcal{C}^{\le 0}, \mathcal{C}^{\ge 0})$
Heart: $\mathcal{H} := \mathcal{C}^{\le 0} \cap \mathcal{C}^{\ge 0}$, an abelian category.
Truncation functors: ${}^t\tau_{\le n} : \mathcal{C} \to \mathcal{C}^{\le n}$, ${}^t\tau_{\ge n} : \mathcal{C} \to \mathcal{C}^{\ge n}$
${}^t\tau_{\le n}(A[m]) = ({}^t\tau_{\le n+m}A)[m]$; ${}^t\tau_{\le n}(A[m]) = ({}^t\tau_{\le n+m}A)[m]$
Truncation distinguished triangle:

$${}^t\tau_{\le n}A \to A \to {}^t\tau_{\ge n+1}A \to ({}^t\tau_{\le n}A)[1]$$

Thm. If $A \in \mathcal{C}^{\le n}$, $\operatorname{Hom}(A, B) \simeq \operatorname{Hom}(A, {}^t\tau_{\le n}B)$.
If $B \in \mathcal{C}^{\ge n}$, $\operatorname{Hom}(A, B) \simeq \operatorname{Hom}({}^t\tau_{\ge n}A, B)$.

t-cohomology: ${}^tH^i := {}^t\tau_{\le i}{}^t\tau_{\ge i} : \mathcal{C} \to \mathcal{H}$.

${}^tH^i(A) \simeq {}^tH^0(A[i])$	
	$\mathcal{C}^{\le n} = \mathcal{C}^{\le 0}[-n] = \{A \mid {}^pH^i(A) = 0\ \forall i > n\}$
	$\mathcal{C}^{\ge n} = \mathcal{C}^{\ge 0}[-n] = \{A \mid {}^pH^i(A) = 0\ \forall i < n\}$

A d.t. $A \to B \to C \to A[1]$ in $\mathcal{C}$ gives LES in $\mathcal{H}$:

$$\cdots \to {}^tH^i(A) \to {}^tH^i(B) \to {}^tH^i(C) \to {}^tH^{i+1}(A) \to \cdots$$

If $A \xrightarrow{f} B \to C \to A[1]$ a d.t. in $\mathcal{C}$ with $A, B \in \mathcal{H}$, then

$$\ker f = {}^pH^{-1}(C), \qquad \operatorname{cok} f = {}^pH^0(C) \qquad \text{in } \mathcal{H}.$$

Ex. $\mathcal{A}$ abelian category, $D(\mathcal{A})$ derived category
Standard t-structure: heart $\simeq \mathcal{A}$,
t-cohomology = ordinary cohomology $H^i : D(\mathcal{A}) \to \mathcal{A}$.

Perverse *t*-Structure. $\mathcal{S}$ GM stratification for X;
$p : \mathcal{S} \to \mathbb{Z}$ perversity. Assume $\exists!$ open stratum S_0.
Inclusions: $i_S : S \hookrightarrow X$ $\quad j_S : S \hookrightarrow \bar{S}$ $\quad i_{\bar S} : \bar{S} \hookrightarrow X$
Constructible: $D^b_c(X) := \{\mathcal{F} \mid \forall S\ H^i(\mathcal{F})|_S \text{ is loc. sys.}\}$

$${}^pD^b_c(X)^{\le 0} = \{\mathcal{F} \mid \forall S\ i_S^{-1}\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\le p(S)}\}$$
$${}^pD^b_c(X)^{\ge 0} = \{\mathcal{F} \mid \forall S\ i_S^!\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\ge p(S)}\}$$

(Use **Gluing Theorem** to show this is a t-structure.)
Heart: $M(X)$ or $M^p(X)$ = cat. of perverse sheaves.
On a single stratum: $M(S)$ = (ordinary sheaves)$[p(S)]$.

Thm. (Perverse Duality) $\mathbb{D}M^p(X) = M^{p^*}(X)$
$\mathbb{D}{}^pD^b_c(X)^{\le k} = {}^{p^*}D^b_c(X)^{\ge -k}$ $\quad \mathbb{D}{}^pD^b_c(X)^{\ge k} = {}^{p^*}D^b_c(X)^{\le -k}$

IC-complexes. $\mathcal{E}$ local system on S. $\mathrm{IC}(\bar{S}, \mathcal{E})$ is the *unique* perverse sheaf such that $i_S^{-1}\mathrm{IC}(\bar{S}, \mathcal{E}) \simeq \mathcal{E}[p(S)]$,

$$i_T^{-1}\mathrm{IC}(\bar{S}, \mathcal{E}) \in {}^pD^b_c(T)^{\le -1} = {}^{\mathrm{std}}D^b_c(T)^{\le p(T)-1}$$
$$i_T^!\mathrm{IC}(\bar{S}, \mathcal{E}) \in {}^pD^b_c(T)^{\ge 1} = {}^{\mathrm{std}}D^b_c(T)^{\ge p(T)+1}$$

$\forall T \subset \bar{S} \smallsetminus S$. If $T \not\subset \bar{S}$, $i_T^{-1}\mathrm{IC}(\bar{S}, \mathcal{E}) = i_T^!\mathrm{IC}(\bar{S}, \mathcal{E}) = 0$.
Also: $\mathrm{IC}(\bar{S}, \mathcal{E})|_{\bar S} = {}^p\tau^-_{\le 0}Rj_{S*}\mathcal{E}[p(S)] = {}^p\tau^+_{\ge 0}Rj_{S!}\mathcal{E}[p(S)]$
(Use **Middle-Extension Theorem** to define $\mathrm{IC}(\bar{S}, \mathcal{E})$.)

Thm. $\mathbb{D}\mathrm{IC}^p(\bar{S}, \mathcal{E}) = \mathrm{IC}^{p^*}(\bar{S}, \mathcal{E}^\vee)$

Cor. $H^{-i}_c(X, \mathrm{IC}^p(\bar{S}_0, \underline{\mathbb{C}})) \simeq H^i(X, \mathrm{IC}^{p^*}(\bar{S}_0, \underline{\mathbb{C}}))$

Thm. $\mathcal{F} \in M(X)$ simple $\Leftrightarrow \mathcal{F} \simeq \mathrm{IC}(\bar{S}, \mathcal{E})$, $\mathcal{E}$ simple.

GM perversities. Henceforth, assume p is GM.

Thm. For all $\mathcal{F} \in M(X)$ and $T \subset \bar{S} \smallsetminus S$,

$$H^i(\mathcal{F})|_S = 0 \text{ unless } p(S_0) \le i \le p(S)$$
$$H^i(\mathrm{IC}(\bar{S}, \mathcal{E}))|_T = 0 \text{ unless } p(S) \le i < p(T)$$

Thm. If X is a manifold, $\mathrm{IC}(X, \underline{\mathbb{C}}) \simeq \underline{\mathbb{C}}[p(S_0)]$.

(Semi)small maps. Y manifold; $f : Y \to X$ proper and $f|_{Y_0} : Y_0 \to S_0$ covering map, where $Y_0 := f^{-1}(S_0)$.

If $\forall x \in S$, $\forall S \neq S_0$, $\begin{cases} \dim f^{-1}(x) < p(S) - p(S_0), & f \text{ is small.} \\ \dim f^{-1}(x) \le p(S) - p(S_0), & f \text{ is semismall.} \end{cases}$

Thm. If f semismall, $Rf_*\underline{\mathbb{C}}[p(S_0)]$ is perverse.
If f small, $Rf_*\underline{\mathbb{C}}[p(S_0)] \simeq \mathrm{IC}(X, f_*\underline{\mathbb{C}}_{Y_0})$.

Decomposition Thm. If $f : Y \to X$ is a semismall projective morphism of complex algebraic varieties, there are vector spaces $V_{S,\mathcal{E}}$ such that for middle perversity,

$$Rf_*\underline{\mathbb{C}}[\tfrac{1}{2}\dim Y] \simeq \bigoplus \mathrm{IC}(\bar{S}, \mathcal{E}) \otimes^L \underline{V_{S,\mathcal{E}}}$$