A GAUSSIAN RADON TRANSFORM FOR BANACH SPACES

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ABSTRACT. We develop a Radon transform on Banach spaces using Gaussian measure and prove that if a bounded continuous function on a separable Banach space has zero Gaussian integral over all hyperplanes outside a closed bounded convex set in the Hilbert space corresponding to the Gaussian measure then the function is zero outside this set.

1. Introduction

The traditional Radon transform [10] of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function Rf on the set of all hyperplanes in $\mathbb{R}^n$ given by

$$Rf(P) = \int_P f(x)\,dx, \tag{1.1}$$

for all hyperplanes P in $\mathbb{R}^n$, the integration being with respect to Lebesgue measure on P. Since it is Gaussian measure, rather than any extension of Lebesgue measure, that is central in infinite-dimensional analysis, a natural strategy in developing a Radon transform theory in infinite dimensions would be to use Gaussian measure instead of Lebesgue measure in formulating an appropriate version of (1.1). In section 2 we carry out this program for infinite-dimensional Banach spaces B, defining the Gaussian Radon transform Gf of a function f on B by

$$Gf(P) = \int f\,d\mu_P, \tag{1.2}$$

where μ_P is Gaussian measure, which we will construct precisely, for any hyperplane P in B. (A 'hyperplane' is a translate of a closed linear subspace of codimension one.) This transform was developed in [9] in the context of Hilbert spaces.

An important result concerning the classical Radon transform R is the Helgason support theorem (Helgason [7]): if f is a rapidly decreasing continuous function on $\mathbb{R}^n$ and $Rf(P)$ is 0 on every hyperplane P lying outside a compact convex set K, then f is 0 off K. In Theorem 4.1, which is our main result, we prove the natural analog of this result for the Gaussian Radon transform in Banach spaces.

There are two standard frameworks for Gaussian measures in infinite dimensions: (i) nuclear spaces and their duals [4, 5]; (ii) Abstract Wiener Spaces [6, 8]. (For an extensive account of Gaussian measures in infinite dimensions see the book of Bogachev [3].) We will work within the latter framework, which has become standard for infinite dimensional analysis. Becnel [1] studies the Gaussian Radon transform in the white noise analysis framework, for a class of functions called Hida test functions. The support theorem was proved for Hilbert spaces in [2].

The classical Radon transform in three dimensions has applications in tomography. The infinite-dimensional Gaussian Radon transform is motivated by the task of recovering information about a random variable f, such as a Brownian functional, from certain conditional expectations of f.

2. Definition of the Gaussian Radon Transform

We work with a real separable infinite-dimensional Banach space B. By Gaussian measure on B we mean a Borel probability measure μ on B such that for every $\phi \in B^*$ the distribution of ϕ, as a random variable defined on B, is Gaussian. The general construction of such a measure was given by L. Gross [6].

A norm $|\cdot|$ on a real separable Hilbert space H is said to be a *measurable norm* (following the terminology in [6]) if for any $\epsilon > 0$ there is a finite-dimensional subspace $F_0 \subset H$ such that

$$\text{Gauss}[v \in F_1 : |v| > \epsilon] < \epsilon$$

for every finite-dimensional subspace $F_1 \subset H$ that is orthogonal to F_0, with Gauss denoting standard Gaussian measure on F_1.

Let $|\cdot|$ be a measurable norm on a real separable infinite-dimensional Hilbert space H. We say that a sequence $\{F_n\}_{n \geq 1}$ of closed subspaces of H is *measurably adapted* if it satisfies the following conditions:

(i) $F_1 \subset F_2 \subset \ldots \subset H$

(ii) $F_n \neq F_{n+1}$ for all n and each F_n has finite codimension in F_{n+1}:

$$1 \leq \dim(F_{n+1} \cap F_n^{\perp}) < \infty \qquad (2.1)$$

for all $n \in \{1, 2, 3, \ldots\}$.

(iii) The union $\cup_{n \geq 1} F_n$ is dense in H.

(iv) For every positive integer n:

$$\text{Gauss} \left[v \in F_{n+1} \cap F_n^{\perp} : |v| > 2^{-n}\right] < 2^{-n}, \qquad (2.2)$$

wherein Gauss is standard Gaussian measure on $F_{n+1} \cap F_n^{\perp}$.

Before proceeding to the formally stated results we make some observations concerning subspaces of a Hilbert space H.

For closed subspaces $A \subset F \subset H$, on decomposing F as an orthogonal sum of A and the subspace of F orthogonal to A, we have the relation

$$F = A + (F \cap A^{\perp}),$$

and so, inductively,

$$F_k + (F_{k+1} \cap F_k^{\perp}) + \cdots + (F_{m+1} \cap F_m^{\perp}) = F_{m+1}, \qquad (2.3)$$

as a sum of mutually orthogonal subspaces, for all closed subspaces $F_k \subset F_{k+1} \subset \ldots \subset F_{m+1}$ in H. If $F_1 \subset F_2 \subset \ldots$ are closed subspaces of H whose union is dense in H and $v \in H$ is orthogonal to F_k and to $F_{j+1} \cap F_j^{\perp}$ for all $j \geq k$ then, by (2.3), v is orthogonal to F_{m+1}, for every $m \geq k$, and so $v = 0$; it follows then that

$$F_k^{\perp} = \oplus_{j=k}^{\infty}(F_{j+1} \cap F_j^{\perp}), \qquad (2.4)$$

because, as we have just argued, any $v \in F_k^{\perp}$ that is also orthogonal to each of the subspaces $(F_{j+1} \cap F_j^{\perp}) \subset F_k^{\perp}$, for $j \geq k$, is 0.

If $E_1 \subset E_2$ are closed subspaces of H, such that $E_2 = E_1 + E_0$, for some finite-dimensional subspace E_0 then the orthogonal projection

$$E_2 \to E_2 \cap E_1^{\perp},$$

being 0 on E_1, maps E_0 surjectively onto $E_2 \cap E_1^{\perp}$, and so

$$\dim(E_2 \cap E_1^{\perp}) < \infty. \qquad (2.5)$$

In particular,

$$\text{if } E_1^{\perp} \text{ is infinite-dimensional then } E_1^{\perp} \not\subset E_2, \qquad (2.6)$$

which simply says that if E_1 has infinite codimension then H cannot be the sum of E_1 and a finite-dimensional subspace.

The following observation will be useful:

Lemma 2.1. *Suppose $|\cdot|$ is a measurable norm on a separable, infinite-dimensional, real Hilbert space H. Then for any closed subspace $M_0 \subset H$ with $\dim(M_0) = \infty$ there is a measurably adapted sequence $\{F_n\}_{n \geq 1}$ of closed subspaces of H, with $F_1 \supset F_0 \overset{\text{def}}{=} M_0^{\perp}$, and*

$$\dim(F_1 \cap M_0) < \infty.$$

The linear span of the subspaces $F_n \cap F_{n-1}^{\perp}$, for $n \geq 1$, is dense in M_0.

<u>Proof.</u> Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of $M_0 - \{0\}$. Since $|\cdot|$ is a measurable norm on H, there is, for every positive integer n, a finite-dimensional subspace E_n of H such that for any finite-dimensional subspace E orthogonal to E_n we have

$$\text{Gauss}[v \in E : |v| > 2^{-n}] < 2^{-n}. \tag{2.7}$$

Let

$$F_1 = M_0^{\perp} + E_1 + \mathbb{R}d_1.$$

The inclusion $F_1 \supset M_0^{\perp}$ is strict because $d_1 \notin M_0^{\perp}$, and $F_1 \cap M_0$ is a non-zero finite-dimensional subspace (by (2.5)).

Using (2.6), we also see that F_1 does not contain M_0 as a subset. So there exists $n_1 > 1$ such that d_{n_1} is in the non-empty set $M_0 \cap F_1^c$ which is open in M_0. Consider now

$$F_2 = F_1 + E_2 + \mathbb{R}d_2 + \ldots + \mathbb{R}d_{n_1}. \tag{2.8}$$

Then the inclusion $F_1 \subset F_2$ is strict, and $F_2 \cap F_1^{\perp}$ is a non-zero finite-dimensional subspace (by (2.5)) that is orthogonal to F_1, and thus also to E_1. By (2.7) we have

$$\text{Gauss}[v \in F_2 \cap F_1^{\perp} : |v| > 2^{-1}] < 2^{-1}.$$

By the same reasoning F_2, being the sum of $M_0^{\perp}$ and the finite-dimensional space $E_1 + E_2 + \mathbb{R}d_1 + \cdots + \mathbb{R}d_{n_1}$, cannot contain M_0 as a subset; hence there is an $n_2 > n_1$ such that $d_{n_2} \notin F_2$. Let

$$F_3 = F_2 + E_3 + \mathbb{R}d_{n_1+1} + \ldots + \mathbb{R}d_{n_2}.$$

Continuing this process inductively, we obtain a measurably adapted sequence $\{F_n\}_{n \geq 1}$. Since $M_0^{\perp} = F_0 \subset F_1 \subset \ldots F_n$, the linear span of $F_i \cap F_{i-1}^{\perp}$, for $i \in \{1, \ldots, k\}$, is $F_k \cap M_0$ (by (2.3)), and since this contains $\{d_1, \ldots, d_{n_k}\}$ we conclude that the closed linear span of the subspaces $F_n \cap F_{n-1}^{\perp}$, for $n \geq 1$, is M_0. $\boxed{\text{QED}}$

By an *affine subspace* of a vector space V we mean a subset of V that is the translate of a subspace of V. We can express any closed affine subspace of a Hilbert space H in the form

$$M_p = p + M_0,$$

where M_0 is a closed subspace of H and $p \in M_0^{\perp}$; the point p and the subspace $M_0^{\perp}$ are uniquely determined by M_p, with p being the point in M_p closest to 0 and M_0 being then the translate $-p + M_p$.

The following result establishes a specific Gaussian measure that is supported in a closed affine subspace of the Banach space B. The strategy we use in the construction is similar to the one used for constructing the Gaussian measure on an Abstract Wiener Space (in the very convenient formulation described by Stroock [11]). While there are other ways to construct this measure the method we follow will be useful in our later considerations.

Theorem 2.1. *Let B be the Banach space obtained by completing a separable real Hilbert space H with respect to a measurable norm $|\cdot|$. Let $M_p = p + M_0$, where M_0 is a closed subspace of H and $p \in M_0^{\perp}$. Then there is a unique Borel measure μ_{M_p} on B such that*

$$\int_B e^{ix^*} \, d\mu_{M_p} = e^{i\langle x^*, p\rangle - \frac{1}{2}\|x^*_{M_0}\|^2_{H^*}} \tag{2.9}$$

for all $x^* \in B^*$, where $x^*_{M_0}$ denotes the element of H^* that is given by $x^*|M_0$ on M_0 and is 0 on $M_0^{\perp}$. The measure μ_{M_p} is concentrated on the closure $\overline{M_p}$ of M_p in B.

<u>Proof.</u> Suppose first that $\dim M_0 = \infty$. Let $(F_n)_{n\geq 1}$ be a measurably adapted sequence of subspaces of H, with $M_0^{\perp} \subset F_1$ and $\dim(F_1 \cap M_0) < \infty$, as in Lemma 2.1. We choose an orthonormal basis $e_1, \ldots, e_{k_1}$ of $F_1 \cap M_0$, and extend inductively to an orthonormal sequence $e_1, e_2, \ldots \in H$, with $e_{k_{n-1}+1}, \ldots, e_{k_n}$ forming an orthonormal basis of $F_n \cap F_{n-1}^{\perp}$ for every positive integer n, and some $k_0 = 0 \leq k_1 < k_2 < \ldots$. The linear span of $e_1, \ldots, e_{k_n}$ is $F_1 \cap M_0 + F_2 \cap F_1^{\perp} + \ldots + F_n \cap F_{n-1}^{\perp}$, which is $F_n \cap M_0$, and the union of these subspaces is dense in M_0 (by Lemma 2.1). Hence $e_1, e_2, \ldots$ is an orthonormal basis of M_0.

Now let $Z_1, Z_2, \ldots$ be a sequence of independent standard Gaussians, all defined on some common probabilty space $(\Omega, \mathbb{F}, \mathbb{P})$. By the measurably adapted property of $(F_n)_{n\geq 1}$ we have

$$\mathbb{P}\left[\left|Z_{k_{n-1}+1}e_{k_{n-1}+1} + \cdots + Z_{k_n}e_{k_n}\right| > \frac{1}{2^{n-1}}\right] < \frac{1}{2^{n-1}}$$

for every integer $n \geq 1$. Then the Borel-Cantelli lemma implies that the appropriately grouped series

$$Z = \sum_{n=1}^{\infty} Z_n e_n, \tag{2.10}$$

converges in $|\cdot|$-norm $\mathbb{P}$-almost-surely. Moreover, Z takes values in $\overline{M_0}$, the closure of M_0 in B, and for any $x \in B^*$ we have, by continuity of the functional $x^* : B \to \mathbb{R}$,

$$\langle x^*, Z \rangle = \sum_{n=1}^{\infty} \langle x^*, e_n \rangle Z_n, \tag{2.11}$$

which converges in $L^2(\mathbb{P})$ and is a Gaussian variable with mean 0 and variance $\sum_{n=1}^{\infty} \langle x^*, e_n \rangle^2 = \|x_{M_0}^*\|_{H^*}^2$, where $x_{M_0}^* \in H^*$ is the restriction of $x^* | M_0$ on M_0 and 0 on $M_0^{\perp}$.

Let ν be the distribution of Z:

$$\nu(E) = \mathbb{P}\left[Z^{-1}(E)\right] \qquad \text{for all Borel } E \subset B. \tag{2.12}$$

Then x^*, viewed as a random variable defined on (B, ν), is Gaussian with mean 0 and variance $\|x_{M_0}^*\|_{H^*}^2$:

$$\int_B e^{itx^*} \, d\nu = e^{-\frac{t^2}{2} \|x_{M_0}^*\|_{H^*}^2} \tag{2.13}$$

for all $x^* \in B^*$. Finally, for any $p \in M_0^{\perp}$, let μ_{M_p} be the measure specified by

$$\mu_{M_p}(E) = \nu(E - p) \tag{2.14}$$

for all Borel sets $E \subset B$; then

$$\int_B f \, d\mu_{M_p} = \int_B f(w + p) \, d\nu(w), \tag{2.15}$$

whenever either side is defined (it reduces to (2.14) for $f = 1_E$ and the case for a general Borel function follows as usual). Then

$$\int_B e^{itx^*} \, d\mu_{M_p} = \int_B e^{it\langle x^*, w+p \rangle} \, d\nu(w) = e^{it\langle x^*, p \rangle - \frac{t^2}{2} \|x_{M_0}^*\|_{H^*}^2} \tag{2.16}$$

for all $t \in \mathbb{R}$.

If $\dim(M_0) < \infty$, we take

$$Z = Z_1 e_1 + \ldots + Z_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is any orthonormal basis for M_0, and $Z_1, Z_2, \ldots, Z_n$ are independent standard Gaussians on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$,

and define ν and then μ_{M_p} just as above. Then (2.13) holds and hence also (2.16).

By construction, all the values of the B-valued random variable Z given in (2.10) are in the subspace $\overline{M_0}$ and so

$$\mu_{M_p}(\overline{M_p}) = 1. \tag{2.17}$$

That the characteristic function given in (2.16) uniquely specifies the Borel measure μ_{M_p} follows from standard general principles (as sketched in a different context towards the end of the proof of Proposition 3.4) and the fact that the functions $x^* \in B^*$ generate the Borel σ-algebra of the separable Banach space B. $\boxed{\text{QED}}$

In the preceding proof the measure ν satisfying (2.13) is μ_{M_0}. Then the defining equation (2.15) becomes:

$$\int_B f \, d\mu_{p+M_0} = \int_B f(v + p) \, d\mu_{M_0}(v), \tag{2.18}$$

for all $p \in M_0^\perp$ and all bounded Borel functions f on B.

We are now ready to define the Gaussian Radon transform for Banach spaces. As before, let H be an infinite-dimensional separable real Hilbert space H and B the Banach space obtained as completion of H with respect to a measurable norm. For any bounded Borel function f on B the *Gaussian Radon transform* Gf is the function on the set of all hyperplanes in H given by

$$Gf(P) = \int_B f \, d\mu_P \tag{2.19}$$

for all hyperplanes P in H. In the case where H is finite-dimensional, B coincides with H and we define Gf using the standard Gaussian measure on P.

As we show later in Proposition 5.2 (i), any hyperplane P in B is the $|\cdot|$-closure of a unique hyperplane in H, this being $P \cap H$. Hence we could focus on Gf as a function on the set of hyperplanes in B. However, there are 'more' hyperplanes in H than those obtained from hyperplanes in B (when $\dim H = \infty$) as shown in Proposition 5.2(ii).

3. Supporting Lemmas

In this section we prove results that will be needed in section 4 to establish the support theorem.

Proposition 3.1. *Let B be a Banach space obtained by completing a real separable infinite-dimensional Hilbert space H with respect to a measurable norm $|\cdot|$. For each closed subspace L of H let μ_L be the measure on B given in Theorem 2.1. Let $F_1 \subset F_2 \subset \ldots$ be a measurably adapted sequence of subspaces of H. Then for any $R > 0$ we have*

$$\lim_{n\to\infty} \mu_{F_n^\perp} [v \in B : |v| > R] = 0. \tag{3.1}$$

<u>Proof</u>. Let $u_1, u_2, \ldots$ be an orthonormal sequence in H that is *adapted* to the sequence of subspaces $F_2 \subset F_3 \subset \ldots$ in the sense that there is an increasing sequence of positive integers $n_1 < n_2 < \ldots$ such that $\{u_1, \ldots, u_{n_1}\}$ spans $F_2 \cap F_1^\perp$ and $\{u_{n_{k-1}+1}, \ldots, u_{n_k}\}$ spans $F_{k+1} \cap F_k^\perp$ for all integers $k \geq 2$. The measure $\mu_{F_k^\perp}$ on B is the distribution of the B-valued random variable

$$W_k = \sum_{j=k}^{\infty} \left(\sum_{r=n_{j-1}+1}^{n_j} Z_r u_r \right), \tag{3.2}$$

(with $n_0 = 0$) where $Z_1, Z_2, \ldots$ is a sequence of independent standard Gaussian variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In (3.2) the full sum W_k converges almost surely, and the term

$$S_j = \sum_{r=n_{j-1}+1}^{n_j} Z_r u_r$$

has values in $F_{j+1} \cap F_j^\perp$; its distribution is standard Gaussian on this space because the Z_r's are independent standard Gaussians and the u_r's form an orthonormal basis in this subspace. By the adaptedness criterion (2.2) it follows that

$$\mathbb{P}\left[\left| \sum_{r=n_{j-1}+1}^{n_j} Z_r u_r \right| > \frac{1}{2^j} \right] < \frac{1}{2^j} \tag{3.3}$$

for all $j \in \{1, 2, 3, \ldots\}$.

Now consider any $R > 0$ and choose a positive integer k large enough such that

$$\frac{1}{2^{k-1}} < R.$$

Then

$$\mathbb{P}\left[|W_k| > R\right] \le \mathbb{P}\left[|W_k| > \frac{1}{2^{k-1}}\right]$$

$$\le \sum_{j=k}^{\infty} \mathbb{P}\left[\left|\sum_{r=n_{j-1}+1}^{n_j} Z_r u_r\right| > \frac{1}{2^j}\right], \tag{3.4}$$

for if $\left|\sum_{r=n_{j-1}+1}^{n_j} Z_r u_r\right| \le 1/2^j$ for all $j \ge k$ then $|W_k|$ would be less than or equal to $\sum_{j \ge k} 1/2^j = 1/2^{k-1}$.

Hence by (3.3) we have

$$\mathbb{P}\left[|W_k| > R\right] < \sum_{j=k}^{\infty} 2^{-j} = \frac{1}{2^{k-1}},$$

and this converges to 0 as $k \to \infty$. Since the distribution measure of W_k is $\mu_{F_k^\perp}$, we conclude that the limit (3.1) holds. $\boxed{\text{QED}}$

The following result shows that the value of a continuous function f at a point p can be recovered as a limit of integrals of f over a 'shrinking' sequence of affine subspaces passing through p. An analogous result was proved in [2] for Hilbert spaces.

Proposition 3.2. *Let f be a bounded Borel function on a Banach space B that is obtained by completing a real separable infinite-dimensional Hilbert space H with respect to a measurable norm. Let $F_1 \subset F_2 \subset \ldots$ be a measurably adapted sequence of subspaces of H. Then*

$$\lim_{n \to \infty} \int_B f \, d\mu_{p+F_n^\perp} = f(p) \tag{3.5}$$

if f is continuous at p.

<u>Proof</u>. Using the translation relation (2.18) we have

$$\int_B f \, d\mu_{p+F_n^\perp} - f(p) = \int_B \left(f(v+p) - f(p)\right) d\mu_{F_n^\perp}(v). \tag{3.6}$$

For notational convenience we write μ_n for $\mu_{F_n^\perp}$. Let $\epsilon > 0$. By continuity of f at p there is a positive real number R such that $|f(p+v) - f(p)| < \epsilon$ for all $v \in B$ with $|v| \le R$. Splitting the integral on the right in (3.6) over those v with $|v| \le R$ and those with $|v| > R$, we have

$$\left|\int_B f \, d\mu_{p+F_n^\perp} - f(p)\right| \le 2\|f\|_{\sup}\mu_n[v : |v| > R] + \epsilon. \tag{3.7}$$

As $n \to \infty$ the first term on the right goes to 0 by Proposition 3.1. Since $\epsilon > 0$ is arbitrary it follows that the left side of (3.6) goes to 0 as $n \to \infty$. $\boxed{\text{QED}}$

The following result generalizes a result of [2] to include measurable norms.

Proposition 3.3. *Let H be a real, separable, infinite-dimensional Hilbert space, and $|\cdot|$ a measurable norm on H. Let K be a closed convex subset of H, and p a point outside K. Then there is a measurably adapted sequence of finite-dimensional subspaces $F_1 \subset F_2 \subset \ldots \subset H$, with $p \in F_1$, such that $p + F_n^\perp$ is disjoint from K for each $n \in \{1, 2, 3, \ldots\}$. Moreover, p lies outside the orthogonal projection $\mathrm{pr}_{F_n}(K)$ of K onto F_n:*

$$p \notin \mathrm{pr}_{F_n}(K) \qquad \text{for all } n \in \{1, 2, 3, \ldots\}. \tag{3.8}$$

<u>Proof</u>. Let p_0 be the unique point in K closest to p, and u_1 the unit vector along $p - p_0$. Then the hyperplane $p + u_1^\perp$, does not contain any point of K:

$$K \cap (p + u_1^\perp) = \emptyset. \tag{3.9}$$

For, otherwise, there would be some nonzero $w \in u_1^\perp$ with $p + w$ in K, and then in the right angled triangle
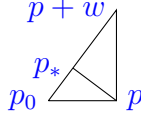
formed by the points p_0, p, and $p + w$ (which has a right angle at the 'vertex' p) there would be a point p_* on the hypotenuse, joining p_0 and $p + w$, and hence lying in the convex set K, that would be closer to p than is p_0. By Lemma 2.1 we can choose a measurably adapted sequence $(F_n)_{n \geq 1}$ with F_1 containing the span of p_0 and u_1.

Next we observe that

$$p + F_n^\perp \subset p + F_1^\perp \subset p + u_1^\perp,$$

and so, using (3.9), we have

$$K \cap (p + F_n^\perp) = \emptyset. \tag{3.10}$$

Since

$$p + u_1^\perp = \{x \in H : \langle x, u_1 \rangle = \langle p, u_1 \rangle\},$$

is disjoint from K, we see that no point in K has inner-product with u_1 equal to $\langle p, u_1 \rangle$. From this it follows that the orthogonal projection of K

on F_n cannot contain p, for if p were $\mathrm{pr}_{F_n}(x)$ for some $x \in K$ then the inner-product $\langle x, u_1 \rangle = \langle \mathrm{pr}_{F_n}(x), u_1 \rangle = \langle p, u_1 \rangle$. This proves (3.8). $\boxed{\text{QED}}$

For any affine subspace Q of H we denote by $Q^\perp$ the orthogonal subspace:

$$Q^\perp = \{x \in H \ : \ \langle x, q_1 - q_2 \rangle = 0 \quad \text{for all } q_1, q_2 \in Q\}. \tag{3.11}$$

A version of the following geometric observation was used in [2]; here we include a proof.

Lemma 3.1. *Let P' be a hyperplane within a finite-dimensional subspace F of a Hilbert space H, and let*

$$P = P' + F^\perp. \tag{3.12}$$

Then

$$P' = P \cap F. \tag{3.13}$$

Moreover, P is a hyperplane in H that is perpendicular to F in the sense that $P^\perp \subset F$.

<u>Proof</u>. Clearly $P' \subset P \cap F$ since P' is given to be a subset of both F and P. Conversely, suppose $x \in P \cap F$; then $x = p' + h$, for some $p' \in P'$ and $h \in F^\perp$, and so $h = x - p' \in F$ from which we conclude that $h = 0$ and hence $x = p' \in P'$. This proves (3.13).

Note that P, being the sum of the finite-dimensional affine space P' and the closed subspace $F^\perp$, is closed in H.

Moreover, if v is any vector orthogonal to P then v is orthogonal to all vectors in $F^\perp$ and hence to $v \in F$; so v is a vector in F orthogonal to P', and since P' is a hyperplane within F this identifies v up to multiplication by a constant. This proves that P is a hyperplane and $P^\perp \subset F$. $\boxed{\text{QED}}$

We will need the following disintegration result:

Proposition 3.4. *Let B be the completion of a real, separable, infinite-dimensional Hilbert space H with respect to a measurable norm. Let F be a finite-dimensional subspace of H, P' a hyperplane within F, and*

$$P = P' + F^\perp. \tag{3.14}$$

For any bounded Borel function f on B, let f_F be the function on F given by

$$f_F(y) = \int f \, d\mu_{y+F^\perp} \tag{3.15}$$

for all $y \in F$. Then

$$Gf(P) = G_F(f_F)(P') \tag{3.16}$$

where Gf is the Gaussian Radon transform of f in B, $G_F(f_F)$ the Gaussian Radon transform, within the finite-dimensional space F, of the function f_F.

The relation (3.16), written out in terms of integrals, is equivalent to

$$\int_B f \, d\mu_P = \int_F \left(\int_B f \, d\mu_{y+F^\perp} \right) d\mu_{P \cap F}(y), \qquad (3.17)$$

where $\mu_{P \cap F}$, the Gaussian measure on the hyperplane $P \cap F$, is the same whether one views $P \cap F$ as being an affine subspace of H or of the subspace $F \subset H$.

<u>Proof.</u> Consider first the special type of function $f = e^{ix^*}$, where $x^* \in B^*$. Then by (2.9) we have:

$$f_F(y) = \int_B e^{ix^*} \, d\mu_{y+F^\perp} = e^{i\langle x^*, y\rangle - \frac{1}{2}\|x^*_{F^\perp}\|^2_{H^*}}.$$

If p_0 is the point of P' closest to 0 then

$$P' = p_0 + P'_0,$$

where $P'_0 = P' - p_0$ is a codimension-one subspace of F. Moreover, p_0 is the point of P closest to 0 and we can write

$$P = p_0 + P_0, \qquad (3.18)$$

where $P_0 = P - p_0$ is a codimension-one subspace of H. Then, recalling (3.14), we have

$$P_0 = P'_0 + F^\perp, \qquad (3.19)$$

with P'_0 and $F^\perp$ being orthogonal.

Then we have the finite-dimensional Gaussian Radon transform of f_F:

$$\begin{aligned}
G_F f_F(P') &= \int_F e^{i\langle x^*, y\rangle - \frac{1}{2}\|x^*_{F^\perp}\|^2_{H^*}} \, d\mu_{p_0+P'_0}(y) \\
&= e^{-\frac{1}{2}\|x^*_{F^\perp}\|^2_{H^*}} \int e^{i\,x^*|F} \, d\mu_{p_0+P'_0} \\
&= e^{-\frac{1}{2}\|x^*_{F^\perp}\|^2_{H^*}} e^{i\langle x^*,p_0\rangle - \frac{1}{2}\|x^*_{P'_0}\|^2_{H^*}} \qquad (3.20) \\
&= e^{i\langle x^*,p_0\rangle - \frac{1}{2}\left(\|x^*_{P'_0}\|^2_{H^*} + \|x^*_{F^\perp}\|^2_{H^*}\right)} \\
&= e^{i\langle x^*,p_0\rangle - \frac{1}{2}\|x^*_{P_0}\|^2} \quad \left(\text{using (3.19)}\right),
\end{aligned}$$

which is, indeed, equal to $Gf(P)$.

The passage from exponentials to general functions f is routine but we include the details for completeness.

Consider a C^∞ function g on $\mathbb{R}^N$ having compact support. Then g is the Fourier transform of a rapidly decreasing smooth function and so, in particular, it is the Fourier transform of a complex Borel measure ν_g on $\mathbb{R}^N$:

$$g(t) = \int_{\mathbb{R}^N} e^{it \cdot w} \, d\nu_g(w) \qquad \text{for all } t \in \mathbb{R}^N.$$

Then for any $x_1^*, ..., x_N^* \in B^*$, the function $g(x_1^*, ..., x_N^*)$ on B can be expressed as

$$
\begin{aligned}
g(x_1^*, ..., x_N^*)(x) &= \int_{\mathbb{R}^N} e^{it_1 \langle x_1^*, x \rangle + \cdots + it_N \langle x_N^*, x \rangle} \, d\nu_g(t_1, ..., t_N) \\
&= \int_{\mathbb{R}^N} e^{i \langle t_1 x_1^* + \cdots + t_N x_N^*, x \rangle} \, d\nu_g(t_1, ..., t_N).
\end{aligned}
\tag{3.21}
$$

Here the exponent $\langle t_1 x_1^* + \cdots + t_N x_N^*, x \rangle$ is a measurable function of $(x, t) \in B \times \mathbb{R}^N$, with the product of the Borel σ-algebras on B and $\mathbb{R}^N$. We have already proven the disintegration identity (3.17) for f of the form e^{ix^*}. So we can apply Fubini's theorem to conclude that the identity (3.17) holds when f is of the form $g(x_1^*, ..., x_N^*)$.

The indicator function 1_C of a compact cube C in $\mathbb{R}^N$ is the pointwise limit of a uniformly bounded sequence of C^∞ functions of compact support on $\mathbb{R}^N$, and so the result holds also for f of the form $1_C(x_1^*, ..., x_N^*)$, which is the same as $1_{(x_1^*, ..., x_N^*)^{-1}(C)}$. Then, by the Dynkin π-λ theorem it holds for the indicator functions of all sets in the σ-algebra generated by the functions $x^* \in B$, and this is the same as the Borel σ-algebra of B. Then, taking linear combinations and applying monotone convergence, the disintegation formula (3.17) holds for all non-negative, or bounded, Borel functions f on B. $\boxed{\text{QED}}$

Let us, finally, note the following result on convexity:

Proposition 3.5. *If K is a closed, bounded, convex subset of a real separable Hilbert space H, and if $L : H \to V$ is a continuous linear mapping into a real finite-dimensional vector space V, then $L(K)$ is compact and convex.*

<u>Proof</u>. Since K is bounded, there is some $\alpha > 0$ such that $K \subset \alpha D$, where D is the closed unit ball in H. But D is weakly compact, and hence so is αD. Now since K is convex and closed in H, it is weakly closed (by the Hahn-Banach theorem for H). So K, being a weakly closed subset of a weakly compact set, is weakly compact. Finally, L is continuous with respect to the weak topology on H and so $L(K)$ is

compact and convex, being the continuous linear image of a (weakly) compact convex set. $\boxed{\text{QED}}$

4. The Support Theorem

We turn now to proving our main result:

Theorem 4.1. *Let f be a bounded, continuous function on the real, separable Banach space B, which is the completion of a real separable Hilbert space H with respect to a measurable norm $|\cdot|$. Suppose K is a closed, bounded, convex subset of H and suppose that the Gaussian Radon transform Gf of f is 0 on all hyperplanes of H that do not intersect K. Then f is 0 on the complement of K in B.*

<u>Proof</u>. Let p be a point of H outside K. Then by Proposition 3.3 there is a measurably adapted sequence

$$F_1 \subset F_2 \subset \ldots$$

of finite-dimensional subspaces of H, with $p \in F_1$ and with p lying outside the orthogonal projection $\mathrm{pr}_{F_n}(K)$ of K onto F_n:

$$p \notin K_n \stackrel{\text{def}}{=} \mathrm{pr}_{F_n}(K) \tag{4.1}$$

for every positive integer n.

Now let f_n be the function on F_n given by

$$f_n(y) = \int f \, d\mu_{y+F_n^\perp} \quad \text{for all } y \in F_n. \tag{4.2}$$

We show next that f_n is 0 outside K_n.

Let P' be a hyperplane within the finite-dimensional space F_n. Then

$$P' = P \cap F_n,$$

where P is the hyperplane in H given by

$$P = P' + F_n^\perp.$$

Projecting onto F_n, we have:

$$\mathrm{pr}_{F_n}(P) = P'. \tag{4.3}$$

We have then, from Proposition 3.4, the disintegration formula

$$Gf(P) = G_n(f_n)(P'), \tag{4.4}$$

where G_n is the Gaussian Radon transform within the finite-dimensional subspace F_n.

From our hypothesis, the left side in (4.4) is 0 if P is disjoint from K. From

$$\mathrm{pr}_{F_n}(P \cap K) \subset \mathrm{pr}_{F_n}(P) \cap \mathrm{pr}_{F_n}(K) = P' \cap \mathrm{pr}_{F_n}(K) \quad \text{(using (4.3))}$$

we see that P is disjoint from K if P' is disjoint from $\mathrm{pr}_{F_n}(K)$. Thus $G_n(f_n)(P')$ is zero whenever the hyperplane P' in F_n is disjoint from the set K_n. By Proposition 3.5 K_n is convex and compact. The function f_n is bounded and continuous and so by the Helgason support theorem (for finite dimensional spaces) the fact that $G_n(f_n)$ is 0 on all hyperplanes lying outside K_n implies that f_n is 0 outside K_n. (Note: Helgason's support theorem applies to any continuous function f on a finite-dimensional space $\mathbb{R}^n$ for which $|x|^k f(x)$ is bounded for every positive integer k; this 'rapid decrease' property is provided automatically for bounded functions in our setting by the presence of the the density term $e^{-|x|^2/2}$ in the Gaussian measure.)

From (4.1) we conclude then that

$$f_n(p) = 0$$

for all positive integers n. Then by Proposition 3.2 we have $f(p) = 0$.

Thus f is 0 at all points of H outside K. Since K is weakly compact in H it is also weakly compact, and hence closed, in B, and so f is 0 on $\overline{H} = B$ outside K. $\boxed{\text{QED}}$

5. Affine Subspaces

In this section we explore the relationship between closed affine subspaces of the Banach space B and those in the Hilbert space H that sits as a dense subspace in B.

Let $i : H \to B$ be the continuous inclusion map. Let L be a hyperplane in B given by $\phi^{-1}(c)$ for some $\phi \in B^*$ (the dual space to B) and $c \in \mathbb{R}$. Then

$$L \cap H = (\phi \circ i)^{-1}(c)$$

is a hyperplane in H because $\phi \circ i \in H^*$. As we see in Proposition 5.1 below, $L \cap H$ is a dense subset of L. Since $L \cap H$ is a closed convex subset of the Hilbert space H there is a point $p \in L \cap H$ closest to 0; then $L \cap H$ consists precisely of those points of the form $p + v$ with $v \in \ker(\phi \circ i)$. Hence

$$L \cap H = p + M_0,$$

for some codimension-1 subspace M_0 in H. Then, taking closures in B and noting that translation by p is a homeomorphism $B \to B$, we see

that every hyperplane L in B is of the form

$$L = p + \overline{M_0}, \tag{5.1}$$

for some codimension-1 subspace M_0 of H.

Proposition 5.1. *Suppose X and Y are topological vector spaces, with X being a linear subspace of Y that is dense inside Y. Let $T : Y \to \mathbb{R}^n$ be a surjective continuous linear map, where n is a positive integer. Then $T^{-1}(c) \cap X$ is a dense subset of $T^{-1}(c)$ for every $c \in \mathbb{R}^n$.*

Some of the arguments in the proof are from elementary linear algebra but we present full details so as to be careful with the roles played by the dense subspace X and the full space Y.

<u>Proof</u>. First we show that $T(X) = \mathbb{R}^n$. If $T(X)$ were a proper subspace of $\mathbb{R}^n$ then there would be a nonempty open set $U \subset \mathbb{R}^n$ in the complement of $T(X)$ and then $T^{-1}(U)$ would be a nonempty open subset of Y lying in the complement of X, which is impossible since X is dense in Y.

We can choose $e'_1, \ldots, e'_n \in X$ such that $T(e'_1), \ldots, T(e'_n)$ form a basis, say the standard one, in $\mathbb{R}^n$. Let F be the linear span of $e'_1, \ldots, e'_n$. By construction, T maps a basis of F to a basis of $\mathbb{R}^n$ and so $T|F$ is an isomorphism $F \to \mathbb{R}^n$; the inverse of $T|F$ is the linear map

$$J : \mathbb{R}^n \to F$$

that carries $T(e'_j)$ to e'_j, for each j. Thus

$$T(Jw) = w \qquad \text{for all } w \in \mathbb{R}^n. \tag{5.2}$$

Since $T|F$ is injective we have

$$(\ker T) \cap F = \ker(T|F) = \{0\}.$$

Thus the mapping

$$I : \ker T \oplus F \to Y : (a, b) \mapsto a + b$$

is a linear injection.

Next, for any $y \in Y$ we have $Ty \in \mathbb{R}^n$ and $J(Ty) \in F$, and then

$$y - J(Ty) \in \ker T,$$

which follows on using (5.2). Thus

$$y = y - J(Ty) + J(Ty) = I\big(y - J(Ty), J(Ty)\big) \quad \text{for all } y \in Y, \tag{5.3}$$

which shows that I is surjective.

Thus I is a linear isomorphism. Then by Lemma 5.1 (proved below) I is a homeomorphism as well.

Let
$$\pi_F : Y \to F : y \mapsto y_F$$
be I^{-1} composed with the projection $\ker T \oplus F \to F$, and
$$\pi_K : Y \to \ker T : y \mapsto y_K$$
the corresponding projection on $\ker T$. Thus,
$$y = y_K + y_F \quad \text{for all } y \in Y \tag{5.4}$$
and so $T(y) = T(y_F)$ for all $y \in Y$.

Now consider any $c \in \mathbb{R}^n$ and choose a neighborhood U of some $y \in T^{-1}(c)$. Then, by continuity of I, there is a neighborhood U_K of y_K in $\ker T$ and a neighborhood U_F of y_F in F such that
$$W = U_K + U_F \subset U. \tag{5.5}$$
Now W is open because I is an open mapping, and so it is a neighborhood of y. Since X is dense in Y, the neighborhood W contains some $x \in X$. Then consider
$$x' = x_K + y_F \in U_K + U_F = W.$$
Since $x_F \in F \subset X$ it follows that $x_K = x - x_F$ is also in X. Moreover, $y_F \in F \subset X$, and so $x' = x_K + y_F$ itself is in X:
$$x' \in X.$$
Thus in the neighborhood U of $y \in T^{-1}(c)$ there is an element $x' \in X$ whose F-component is y_F, and so
$$T(x') = T(x_K) + T(y_F) = T(y_F) = T(y) = c.$$
This proves that $T^{-1}(c) \cap X$ is dense in $T^{-1}(c)$. $\boxed{\text{QED}}$

We have used the following observation:

Lemma 5.1. *Let F be a finite dimensional subspace of a topological vector space Y and suppose L is a closed subspace of Y that is a complement of F in the sense that every element of Y is uniquely the sum of an element in F and an element in L. Then the mapping*
$$j : L \oplus F \to Y : (a, b) \mapsto a + b$$
is an isomorphism of topological vector spaces.

Proof. It is clear that j is a linear isomorphism. We prove that j is a homeomorphism. Since addition is continuous in Y it follows that j is continuous. Let us display the inverse p of j as
$$p = j^{-1} : Y \to L \oplus F : y \mapsto \big(\pi_L(y), \pi_F(y)\big).$$

The component π_F is the composite of the continuous projection map Y/L (the quotient topological vector space) and the linear isomorphism $Y/L \to F : y + L \mapsto \pi_F(y)$, and this, being a linear mapping between finite-dimensional spaces, is continuous. Hence π_F is continuous. Next, continuity of π_L follows from observing that

$$Y \to Y : y \mapsto \pi_L(y) = y - \pi_F(y)$$

is continuous and has image the subspace L, and so π_L is continous when L is equipped with the subspace topology from Y. $\boxed{\text{QED}}$

We can now discuss the relationship between codimension-1 subspaces of a space that sits densely inside a larger space:

Proposition 5.2. *Let B a real Banach space, and H a real Hilbert space that is a dense linear subspace of B such that the inclusion map $H \to B$ is continuous.*

  (i) *If L is a codimension-1 closed subspace of B then there is a unique codimension-1 closed subspace M of H such that L is the closure of M in B; the subspace M is $L \cap H$.*
  (ii) *If $f_0 \in H^*$ is nonzero and is continuous with respect to $|\cdot|$ then the closure of $\ker f_0$ in B is a codimension-1 subspace of B; if f_0 is not continuous with repect to $|\cdot|$ then $\ker f_0$ is dense in B.*

Note that this result is non-trivial only in the infinite-dimensional case. <u>Proof</u>. (i) Let L be a codimension-1 closed subspace of B. Then $\dim B/L = 1$. Composing the projection $B \to B/L$ with any isomorphism $B/L \to \mathbb{R}$ produces a non-zero $f \in B^*$ such that $L = \ker f$.

The restriction $f_0 = f|H$, being the composite of f with the continuous inclusion map $H \to B$, is in H^* and is nonzero because $f \neq 0$ and H is dense in B. Then $\ker f_0$ is a codimension-1 subspace of H, and, by Proposition 5.1 applied with $n = 1$ and $c = 0$, the closure of $M = \ker f_0$ is $L = \ker f$. Observe that $\ker f_0$ is the set of all $h \in H$ on which f is 0; thus, $M = L \cap H$.

Now suppose N is a codimension-1 closed subspace of H whose closure in B is the codimension-1 subspace L. In particular, $N \subset L$ and so

$$N \subset H \cap L = M.$$

Since both M and N have codimension 1 in H they must be equal.

(ii) Suppose $f_0 \in H^*$ is continuous with respect to $|\cdot|$; then f_0 extends uniquely to a continuous linear functional f on B. Then by Proposition 5.1, applied with $T = f$, $n = 1$ and $c = 0$, it follows that $\ker f$ is the

closure of $\ker f_0$ in B. Since $f_0 \neq 0$ we have $f \neq 0$ and so $\ker f$ is a codimension-1 subspace of B.

Conversely, suppose $f_0 \in H^*$ is not 0 and $\ker f_0$ is not dense as a subset of B. By the Hahn-Banach theorem there is a nonzero $f_1 \in B^*$ that vanishes on $\overline{\ker f_0}$ (closure in B). Then $\ker f_0 \subset \ker f_1$ and so $\ker f_0$ is contained inside the kernel of $f_1|H$; since $f_1|H$ is nonzero and in H^* the subspace $\ker(f_1|H)$ has codimension 1 in H. Since it contains $\ker f_0$ which is also a codimension-1 subspace of H it follows that $\ker f_0$ and $\ker(f_1|H)$ coincide and so f_0 is a scalar multiple of $f_1|H$. Hence f_0 is continuous with repect to the norm $|\cdot|$. $\boxed{\text{QED}}$

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