Gauge Theory in Two Dimensions: Topological, Geometric and Probabilistic Aspects

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We present a description of two dimensional Yang-Mills gauge theory on the plane and on compact surfaces, examining the topological, geometric and probabilistic aspects.

Keywords: Yang-Mills, Gauge Theory, QCD, Large-N

1. Introduction

Two dimensional Yang-Mills theory has proved to be a surprisingly rich model, despite, or possibly because of, its simplicity and tractability both in classical and quantum forms. The purpose of this article is to give a largely self-contained introduction to classical and quantum Yang-Mills theory on the plane and on compact surfaces, along with its relationship to Chern-Simons theory, illustrating some of the directions of current and recent research activity.

2. Yang-Mills Gauge Theory

The physical concept of a gauge field is mathematically modeled by the notion of a connection on a principal bundle. In this section we shall go over a rapid account of the differential geometric notions describing a gauge field (for a full account, see, for instance, Bleecker[10]).

Consider a smooth manifold M, to be thought of as spacetime. Let G be a Lie group, viewed as the group of symmetries of a particle field. The latter may be thought of, locally, as a function on M with values in a vector space E on which there is a representation ρ of G; a change of 'gauge' (analogous to a change of coordinates) alters ψ by multiplication by $\rho(g)$, where g is a 'local' gauge transformation, i.e. a function on M with values in G. To deal with such fields in a unified way, it is best to introduce a principal G-bundle over M. This is a manifold P, with a smooth surjection

$$\pi : P \to M$$

and a smooth right action of G:

$$P \times G \to P : (p, g) \mapsto R_g p = pg,$$

such that π is locally trivial, i.e. each point of M has a neighborhood U for which there is a diffeomeorphism

$$\phi : U \times G \to \pi^{-1}(U)$$

satisfying

$$\pi\phi(u, g) = u, \qquad \phi(u, gh) = \phi(u, g)h, \qquad \text{for all } u \in U, \text{ and } g, h \in G.$$

It will be convenient for later use to note here that a bundle over M is often specified by an indexing set I (which may have a structure, rather than just be an abstract set), an open covering $\{U_\alpha\}_{\alpha\in I}$ of M, and for each $\alpha, \beta \in I$ for which $U_\alpha \cap U_\beta \neq \emptyset$, a diffeomorphism

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

such that

$$\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x) \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{1}$$

The principal bundle P may be recovered or constructed from this data by taking the set $\cup_{\alpha\in I}\{\alpha\} \times U_\alpha \times G$ and identifying (α, x, g) with (β, y, h) if $x = y \in U_\alpha \cap U_\beta$ and $\phi_{\beta\alpha}(x)g = h$. The elements of P are the equivalence classes $[\alpha, x, g]$, and the map $\pi : P \to M : [\alpha, x, g] \to x$ is the bundle projection and $[\alpha, x, g]k = [\alpha, x, gk]$ specifies the right G-action on P. The map $\phi_\alpha : U_\alpha \times G \to P : (x, g) \mapsto [\alpha, x, g]$ is a local trivialization. This construction is traditional (see, for instance, Steenrod[53]), but lends itself to an interesting application in the context of Yang-Mills as we shall see later.

A particle field is then described by a function $\psi : P \to E$, where E is as before, satisfying the equivariance property

$$\psi(pg) = \rho(g^{-1})\psi(p) \tag{2}$$

which is physically interpreted as the gauge transformation behavior of the field ψ. In terms of local trivializations, the value of the field over a point $x \in M$ would be described by an equivalence class $[\alpha, x, v]$ with $v = \psi([\alpha, x, e]) \in E$, and (α, x, v) declared equivalent to (β, y, w) if $y = x \in$

$U_\alpha \cap U_\beta$ and $w = \rho\big(\phi_{\beta\alpha}(x)\big)v$. An equivalent point of view is to consider the space E_P of all equivalence classes $[p, v] \in P \times E$, with $[p, v] = [pg, \rho(g)^{-1}v]$ for all $(g, p, v) \in G \times P \times E$, which is a vector bundle $E_P \to M : [p, v] \mapsto \pi(p)$, and then ψ corresponds to the section of this bundle given by $M \to E_P : x \mapsto [p, \psi(p)]$ for any $p \in \pi^{-1}(x)$. These traditional considerations will be useful in an unorthodox context later in defining the Chern-Simons action (45).

The interaction of the particle field with a gauge 'force' field is described through a Lagrangian, which involves derivatives of ψ. This derivative is a 'covariant derivative', with a behavior under gauge transformations controlled through a field ω which is the gauge field. Mathematically, ω is a *connection* on P, i.e. a smooth 1-form on P with values in the Lie algebra LG of G such that

$$R_g^*\omega = \mathrm{Ad}(g^{-1})\omega \quad \text{and} \quad \omega(pH) = H \tag{3}$$

for all $p \in P$ and $H \in LG$, where

$$pH = \frac{d}{dt}\Big|_{t=0} p \cdot \exp(tH).$$

The tangent space T_pP splits into the *vertical subspace* $\ker d\pi_p$ and the horizontal subspace $\ker \omega_p$:

$$T_pP = V_p \oplus H_p^\omega, \quad \text{where } V_p = \ker d\pi_p \text{ and } H_p^\omega = \ker \omega_p. \tag{4}$$

A path in P is said to be horizontal, or *parallel*, with respect to ω, if its tangent vector is horizontal at every point. Thus, if

$$c : [a, b] \to M$$

is a piecewise smooth path, and u a point on the initial fiber $\pi^{-1}(c(a))$, then there is a unique piecewise smooth path $\tilde{c}_u : [a, b] \to P$ such that

$$\pi \circ \tilde{c}_u = c, \qquad \tilde{c}_u(a) = u,$$

and $\tilde{c}_u$ is composed of horizontal pieces. The path $\tilde{c}_u$ is called the *horizontal lift* of c through u, and $\tilde{c}_u(t)$ is the *parallel transport* of u along c up to time t.

The point $\tilde{c}_u(b)$ lies over the end point $c(b)$. If c is a loop then there is a unique element h in G for which

$$\tilde{c}_u(b) = \tilde{c}_u(a)h.$$

This h is the *holonomy* of ω around the loop c, beginning at u, and we denote this

$$h_u(c; \omega).$$

The property (3) implies that

$$h_{ug}(c;\omega) = g^{-1}h_u(c;\omega)g. \tag{5}$$

In all cases of interest, the gauge group G is a matrix group, and we have then the *Wilson loop observable*

$$\mathrm{Tr}\big(h(c;\omega)\big) \tag{6}$$

where we have dropped the initial point u as it does not affect the value of the trace of the holonomy.

If ρ is a representation of G on a vector space E, then there is induced in the obvious way an 'action' of the Lie algebra LG on E, and this allows us to multiply, or 'wedge', E-valued forms and LG-valued forms. If η is an E-valued k-form on P then the *covariant derivative* $D^\omega\eta$ is the E-valued $(k+1)$–form on P given by

$$D^\omega\eta = d\eta + \omega \wedge \eta \tag{7}$$

The holonomy around a small loop is, roughly, the integral of the *curvature* of ω over the region enclosed by the loop. More technically, the curvature Ω^ω is the LG-valued 2-form given by

$$\Omega^\omega = D^\omega\omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \tag{8}$$

(see discussion following (26) below for explanation of notation). This LG-valued 2-form is 0 when evaluated on a pair of vectors at least one of which is vertical, and is equivariant:

$$R_g^*\Omega^\omega = \mathrm{Ad}(g^{-1})\Omega^\omega. \tag{9}$$

The connection ω is said to be *flat* if its curvature is 0, and in this case holonomies around null-homotopic loops are the identity.

Now consider an Ad-invariant metric $\langle\cdot,\cdot\rangle$ on LG. This, along with a metric on M, induces a metric $\tilde{g}$ on the bundle P in the natural way. Then we can form the curvature 'squared':

$$\langle\Omega^\omega, \Omega^\omega\rangle = \tilde{g}(\Omega^\omega, \Omega^\omega)$$

which, by equivariance of the curvature form and the Ad-invariance of the metric on LG, descends to a well-defined function on the base manifold M. The Yang-Mills action functional is

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2g^2}\int_M \langle\Omega^\omega, \Omega^\omega\rangle\, d\mathrm{vol} \tag{10}$$

where the integration is with respect to the volume induced by the metric on M. The parameter g is a physical quantity which we will refer to as the coupling constant. The Yang-Mills equations are the variational equations for this action.

A *gauge transformation* is a diffeomorphism

$$\phi : P \to P$$

which preserves each fiber and is G-equivariant:

$$\pi \circ \phi = \phi \quad \text{and} \quad R_g \circ \phi = \phi \circ R_g \quad \text{for all } g \in G.$$

The gauge transformation ϕ is specified uniquely through the function $g_\phi : P \to G$, given by

$$\phi(p) = pg_\phi(p), \quad \text{for all } p \in P, \tag{11}$$

which satisfies the equivariance condition

$$g_\phi(ph) = h^{-1}g_\phi(p)h, \quad \text{for all } p \in P \text{ and } h \in G.$$

Conversely, if a smooth function $g : P \to G$ satisfies $g(ph) = h^{-1}g(p)h$ for all $p \in P$ and $h \in G$, then the map

$$\phi_g : p \mapsto pg(p) \tag{12}$$

is a gauge transformation. The group of all gauge transformations is usually denoted $\mathcal{G}$; note that the group law of composition corresponds to pointwise multiplication: $g_{\phi\circ\tau} = g_\phi g_\tau$. If $o \in M$ is a basepoint on M, it is often convenient to consider $\mathcal{G}_o$, the subgroup of $\mathcal{G}$, which acts as identity on $\pi^{-1}(o)$. The group $\mathcal{G}$ acts on the infinite dimensional affine space $\mathcal{A}$ of all connections by pullbacks:

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A} : (\omega, \phi) \mapsto \phi^*\omega = \omega^{g_\phi} \stackrel{\text{def}}{=} \mathrm{Ad}\big(g_\phi^{-1}\big)\omega + g_\phi^{-1}dg_\phi \tag{13}$$

The Yang-Mills action functional and physical observables such as the Wilson loop observables are all gauge invariant.

Gauge groups of interest in physics are products of groups such as $U(N)$ and $SU(N)$, for $N \in \{1, 2, 3\}$. In the case of the electromagnetic field, $G = U(1)$ and the connection form is $\omega = i\frac{e}{\hbar}A$, where A is the electromagnetic potential, $\hbar$ is Planck's constant divided by 2π, and e is the charge of the particle (electron) to which the field is coupled. The curvature Ω^ω descends to an ordinary 2-form on spacetime, and corresponds to $\frac{e}{\hbar}$ times the electromagnetic field strength form F.

Moving from the classical theory of the gauge field to the quantum theory leads to the consideration of functional integrals of the form

$$\int_{\mathcal{A}} f(\omega) e^{-S_{\mathrm{YM}}(\omega)} D\omega,$$

where f is a gauge invariant function such as the product of traces, in various representations, of holonomies around loops. The integral can be viewed as being over the quotient space $\mathcal{A}/\mathcal{G}$. Here the base manifold M is now a Riemannian manifold rather than Lorentzian (for the latter, the functional integrals are Feynman functional integrals, having an i in the exponent). More specifically, one would like to compute, or at least gain an understanding of, the averages:

$$W(C_1, \ldots C_k) = \frac{1}{Z_g} \int_{\mathcal{A}/\mathcal{G}} \prod_{j=1}^{k} \mathrm{Tr}\left(h(C_j; \omega)\right) e^{-S_{\mathrm{YM}}(\omega)} [D\omega], \tag{14}$$

with $[D\omega]$ denoting the formal 'Lebesgue measure' on $\mathcal{A}$ pushed down to $\mathcal{A}/\mathcal{G}$. Here the traces may be in different representations of the group G. The formal probability measure μ_g on $\mathcal{A}/\mathcal{G}$, or on $\mathcal{A}/\mathcal{G}_o$, given through

$$d\mu_g([\omega]) = \frac{1}{Z_g} e^{-\frac{1}{2g^2}\|\Omega^\omega\|^2_{L^2}} [D\omega], \tag{15}$$

is usually called the *Yang-Mills measure*.

These integrals can be computed exactly when $\dim M = 2$, as we will describe in the following section, and the Yang-Mills measure then has a rigorous definition.

3. Wilson loop integrals in two dimensions

The Yang-Mills action is, on the face of it, quartic in the connection form ω. However, when we pass to the quotient $\mathcal{A}/\mathcal{G}$, a simplification results when the base manifold M is two dimensional. This is most convincingly demonstrated in the case $M = \mathbb{R}^2$. In this case, for any connection ω we can choose, for instance, *radial gauge*, a section

$$s_\omega : \mathbb{R}^2 \to P$$

(a smooth map with $\pi \circ s_\omega(x) = x$ for all points $x \in \mathbb{R}^2$) which maps each radial ray from the origin o into an ω-horizontal curve in P emanating from a chosen initial point $u \in \pi^{-1}(o)$. Then let F^ω be the LG-valued function on $\mathbb{R}^2$ specified by

$$\omega \mapsto s_\omega^* \Omega^\omega = F^\omega d\sigma, \tag{16}$$

where σ is the area 2-form on $\mathbb{R}^2$. Then

$$\omega \mapsto F^{\omega}$$

identifies $\mathcal{A}/\mathcal{G}_o$ with the linear space of smooth LG-valued functions on $\mathbb{R}^2$ and the Yang-Mills measure becomes the well-defined Gaussian measure on the space of functions F given by

$$d\mu_g(F) = \frac{1}{Z_g} e^{-\frac{1}{2g^2}\|F\|^2_{L^2}}\, DF \tag{17}$$

This measure lives on a completion of the Hilbert space of LG-valued L^2 functions on the plane, and the corresponding connections are therefore quite 'rough'. In particular, the differential equation defining parallel transport needs to be reinterpreted as a stochastic differential equation. The holonomy $h(C;\omega)$ (basepoint fixed at u once and for all) is then a G-valued random variable. The Wilson loop expectation values work out explicitly using two facts:

- If C is a piecewise smooth simple closed loop in the plane C then the holonomy $h(C)$ is a G-valued random variable with distribution $Q_{g^2S}(x)dx$, where S is the area enclosed by C, and $Q_t(x)$ is the solution of the heat equation

$$\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2}\Delta Q_t(x), \qquad \lim_{t\downarrow 0}\int_G f(x)Q_t(x)\,dx = f(e),$$

  for all continuous functions f on G, with dx being unit mass Haar measure on G, and Δ is the Laplacian operator on G with respect to the chosen invariant inner product.
- If C_1 and C_2 are simple loops enclosing disjoint planar regions then $h(C_1)$ and $h(C_2)$ are independent random variables.

In the simplest case, for a simple closed loop C in the plane,

$$\int f\big(h(C)\big)\, d\mu_g = \int_G f(x)Q_{g^2S}(x)\,dx \tag{18}$$

with S denoting the area enclosed by C. In particular, for the group $G = U(N)$,

$$W_N(C) = e^{-Ng^2S/2} \tag{19}$$

where

$$W_N(C) = \int \frac{1}{N}\mathrm{Tr}\big(h(C)\big)\, d\mu_g.$$

Now consider the case where $M = \Sigma$, a closed oriented surface with Riemannian structure. We will follow Lévy's development[38] of the discrete Yang-Mills measure. Let $\pi : \tilde{G} \to G : \tilde{x} \mapsto x$ be the universal covering of G. Let $\mathbb{G}$ be a triangulation of Σ, or a graph, with $\mathbb{V}$ the set of vertices, $\mathbb{E}$ the set of (oriented) edges, and $\mathbb{F}$ the set of faces. We assume that each face is diffeomorphic to the unit disk, and the boundary of each face is a simple loop in the graph. Following Lévy,[38] define a discrete connection over $\mathbb{G}$ to be a map $\tilde{h} : \mathbb{E} \to \tilde{G}$ satisfying

$$\pi\big(\tilde{h}(e^{-1})\big) = \pi\big(\tilde{h}(e)\big)^{-1} \qquad \text{for every edge } e \in \mathbb{E} \tag{20}$$

where e^{-1} denotes the edge e with reversed orientation. One should interpret $\tilde{h}(e)$ as the parallel transport along edge e of a continuum connection lifted to $\tilde{G}$ appropriately. Let

$$\mathcal{A}_{\mathbb{G}}$$

be the set of all such conections over $\mathbb{G}$. Note that this is naturally a subset of $\tilde{G}^{\mathbb{E}}$, and indeed can be viewed as $\tilde{G}^{\mathbb{E}_+}$, where $\mathbb{E}_+$ is the set of edges each counted only once with a particular chosen orientation; in particular, we have a unit mass Haar product measure on $\mathcal{A}_{\mathbb{G}}$. Define the *discrete Yang-Mills measure* μ_g^{YM} for the graph $\mathbb{G}$, by requiring that for any continuous function f on $\mathcal{A}_{\mathbb{G}}$, we have

$$\int_{\mathcal{A}_{\mathbb{G}}} f \, d\mu_g^{\mathrm{YM}} = \frac{1}{Z_g} \int f(h) \prod_{F \in \mathbb{F}} Q_{g^2|F|}\big(\tilde{h}(\partial F)\big)\, dh, \tag{21}$$

where $|F|$ is the area of the face F according to the Riemannian metric on Σ, and Z_g a normalizing constant to ensure that $\mu_g^{\mathrm{YM}}(\mathcal{A}_{\mathbb{G}})$ is 1. This is the discrete Yang-Mills measure for connections over all principal $\tilde{G}$-bundles over Σ. However, when G is not simply connected there are different topological classes of bundles, each specified through an element $z \in \ker(\tilde{G} \to G)$. For such z, again following Lévy,[38]

$$\mathcal{A}_{\mathbb{G}}^z = \{\tilde{h} \in \mathcal{A}_{\mathbb{G}} \,:\, \prod_{e \in \mathbb{E}_+} \tilde{h}(e)\tilde{h}(e^{-1}) = z\} \tag{22}$$

corresponds to the set of connections on the principal G-bundle over Σ classified topologically by z. The Yang-Mills measure $\mu_{z,g}^{\mathrm{YM}}$ on $\mathcal{A}_{\mathbb{G}}^z$ is then simply

$$d\mu_{z,g}^{\mathrm{YM}}(\tilde{h}) = c_z 1_{\mathcal{A}_{\mathbb{G}}^z}(\tilde{h}) d\mu_g^{\mathrm{YM}}(\tilde{h}), \tag{23}$$

where c_z is again chosen to normalize the measure to have total mass 1. A key feature of the discrete Yang-Mills measure is that it is unaltered by subdivision of faces (plaquettes), which is why we do not need to index $\mu_g^{\rm YM}$ by the graph $\mathbb{G}$; this invariance was observed by Migdal[42] in the physics literature. Lévy[36,38] constructed a continuum measure from these discrete measures and showed that the continuum measure thus constructed agrees with that constructed in.[46] The continuum construction of the Yang-Mills measure relies on earlier work by Driver[19] and others;[25] a separate approach to the continuum Yang-Mills functional integral in two dimensions was developed by Fine[20,21] (see also Ashtekar et al.[5]).

The normalizing factor which appears in the loop expectation values is given, for a simply connected group G and a closed oriented surface of genus γ, by

$$\int_{G^{2\gamma}} Q_{g^2S}\big(K_\gamma(x)\big)\,dx \tag{24}$$

where K_γ is the product commutator function

$$K_\gamma(a_1, b_1, ..., a_\gamma, b_\gamma) = b_\gamma^{-1}a_\gamma^{-1}b_\gamma a_\gamma ... b_1^{-1}b_1^{-1}b_1 a_1 \tag{25}$$

which plays the role of 'total curvature' of a discrete connection whose holonomies around 2γ standard generators of $\pi_1(\Sigma)$ are given by $a_1, b_1, ..., a_\gamma, b_\gamma$.

4. Yang-Mills on surfaces and Chern-Simons: the symplectic limit

In this section we will describe how Yang-Mills theory on surfaces fits into a hierarchy of topological/geometric field theories in low dimensions. For a detailed development of Chern-Simons theory from the point of view of topological field theory we refer to Freed[22] from which we borrow many ideas, and some notation, here. Most of our discussion below applies to trivial principal bundles (see[23] for non-trivial bundles). In Albeverio et al.,[4] the relationship between the Chern-Simons and Yang-Mills systems was explored using the method of exterior differential systems of Griffiths[24] in the calculus of variations.

One of our purposes here is to also verify that the 'correct' (from the Chern-Simons point of view) inner-product on the Lie algebra of the gauge group $SU(N)$ to use for two-dimensional Yang-Mills is independent of N. This is a small but significant fact when considering the large N limit of the Yang-Mills theory.

4.1. *From four dimensions to three: the Chern-Simons form*

Let $P_W \to W$ be a principal G-bundle over a manifold W. Then for any connection ω on P_W we have the curvature 2-form Ω^ω which gives rise to an $LG \otimes LG$-valued 4-form by wedging

$$\Omega^\omega \wedge \Omega^\omega$$

Now consider a metric $\langle\cdot,\cdot\rangle$ on LG which is Ad-invariant. This produces a 4-form

$$\langle \Omega^\omega \wedge \Omega^\omega \rangle$$

which, by Ad-invariance, descends to a 4-form on W which we denote again by $\langle \Omega^\omega \wedge \Omega^\omega \rangle$. The latter, a Chern-Weil form, is a closed 4-form and specifies a cohomology class in $H^4(W)$ determined by the bundle $P_W \to W$ (independent of the choice of ω).

The *Chern-Simons* 3-form $cs(\omega)$ on the bundle space P_W is given by

$$cs(\omega) = \langle \omega \wedge d\omega + \frac{1}{3}\omega \wedge [\omega \wedge \omega] \rangle = \langle \omega \wedge \Omega^\omega - \frac{1}{6}\omega \wedge [\omega \wedge \omega] \rangle \tag{26}$$

Here wedge products of LG-valued forms, and expressions such as $[\omega \wedge \omega]$, may be computed by expressing the forms in terms of a basis of LG and ordinary differential forms. For example, writing ω as $\sum_a \omega^a E_a$, where $\{E_a\}$ is a basis of LG, the 2-form $[\omega \wedge \omega]$, whose value on a pair of vectors (X, Y) is $2[\omega(X), \omega(Y)]$, is $\sum_{a,b} \omega^a \wedge \omega^b [E_a, E_b]$. If LG is realized as a Lie algebra of matrices, then $[\omega \wedge \omega]$ works out to be $2\omega \wedge \omega$, this being computed using matrix multiplication.

The fundamental property[13] of the Chern-Simons form is that its exterior differential is the closed 4-form $\langle \Omega^\omega \wedge \Omega^\omega \rangle$ on the bundle space:

$$dcs(\omega) = \langle \Omega^\omega \wedge \Omega^\omega \rangle \tag{27}$$

Unlike the Chern-Weil form, $cs(\omega)$ does not descend naturally to a form on W, i.e. if $s : W \to P_W$ is a section then $s^*cs(\omega)$ depends on s. If $g : P \to G$ specifies a gauge transformation $p \mapsto pg(p)$ then a lengthy but straightforward computation shows that

$$cs(\omega^g) - cs(\omega) = d\langle \omega \wedge (dg)g^{-1} \rangle - \frac{1}{6}\langle g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg] \rangle \tag{28}$$

If we split a closed oriented 4-manifold W into two 4-manifolds W_1 and W_2, glued along a compact oriented 3-manifold Y, and if P_W admits sections s_1 over W_1 and s over W_2, then

$$\int_W \langle \Omega^\omega \wedge \Omega^\omega \rangle = \int_Y \left(s_1^* cs(\omega) - s^* cs(\omega) \right) \tag{29}$$

Now the sections s_1 and s are related by a gauge transformation g specified through a smooth map

$$\tilde{g} : Y \to G \tag{30}$$

in the sense that (notation as in (11) and (12))

$$s_1(y) = s(y)\tilde{g}(y) = \phi_g(s(y)), \qquad \text{for all } y \in Y. \tag{31}$$

Then, after using Stokes' theorem, the term on the right in (29) works out to

$$-\frac{1}{6}\int_Y \tilde{g}^*\sigma \tag{32}$$

where σ is the 3-form on G given by

$$\sigma = \langle h^{-1}dh \wedge [h^{-1}dh \wedge h^{-1}dh]\rangle, \tag{33}$$

with $h : G \to G$ being the identity map. By choosing the metric on LG appropriately, we can ensure that this quantity is always an integer times (a convenient normalizing factor) $8\pi^2$. For example, if $G = SU(2)$, and the inner-product on LG given by

$$\langle H, K\rangle = -\mathrm{Tr}(HK), \tag{34}$$

computation of the volume of $SU(2)$ shows that

$$\int_{SU(2)} \sigma = -48\pi^2 \tag{35}$$

(The sign on the right just fixes an orientation for $SU(2)$.) This computation can be worked out conveniently through the 2-to-1 parametrization of $SU(2)$ given by $h = k_\phi a_\theta k_\psi$, with $(\phi, \theta, \psi) \in (0, 2\pi) \times (0, \pi) \times (0, 2\pi)$, where

$$k_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

and

$$a_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & i\,\sin\frac{\theta}{2} \\ i\,\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Putting all this together we see that

$$\int_W \left[\frac{1}{8\pi^2}s^*cs(\omega^g) - \frac{1}{8\pi^2}s^*cs(\omega)\right] = -\int_Y \frac{1}{48\pi^2}\tilde{g}^*\sigma \in \mathbb{Z} \tag{36}$$

More generally, we assume that the metric $\langle\cdot,\cdot\rangle$ is such that

$$\frac{1}{8\pi^2}\langle\Omega^\omega \wedge \Omega^\omega\rangle$$

is an integer cohomology class for every closed oriented four-manifold W (this condition can be restated more completely in terms of the classifying space $B\tilde{G}$; see Witten[57]). For instance, for $G = SU(N)$, the properly scaled metric is (according to Witten[57]):

$$\langle H, K\rangle = -\mathrm{Tr}(HK) \tag{37}$$

Let

$$CS(s,\omega) = \frac{1}{8\pi^2}\int_Y s^* cs(\omega) \tag{38}$$

where $s : Y \to P$ is a smooth global section (assumed to exist); the discussions above show that when s is altered, $CS(\omega)$ is changed by an integer. Thus, for any integer $k \in \mathbb{Z}$, the quantity

$$e^{2\pi k i CS(s,\omega)} \in U(1) \tag{39}$$

is independent of the section s, and thus gauge invariant.

4.2. *From three dimensions to two: the $U(1)$ bundle over the space of connections on a surface*

Now consider a compact oriented 3-manifold Y with boundary X, a closed oriented 2-manifold. We follow Freed's approach.[22] We assume that G is connected, compact, and simply connected; a consequence is that a principal G-bundle over any manifold of dimension ≤ 3 is necessarily trivial. For any smooth sections $s_1, s : Y \to P$, with $s_1 = s\tilde{g}$, we have on using (28) and notation explained therein,

$$\int_Y s^* cs(\omega^g) - \int_Y s^* cs(\omega) = \int_X \langle s^*\omega \wedge (d\tilde{g})\tilde{g}^{-1}\rangle - \int_Y \frac{1}{6}\tilde{g}^*\sigma \tag{40}$$

Let

$$CS(s,\omega) = \frac{1}{8\pi^2}\int_Y s^* cs(\omega) \tag{41}$$

Then, for any integer k,

$$e^{2\pi k i CS(sg,\omega)} = e^{2\pi k i CS(s,\omega)}\phi_{sg,s}(\omega) \tag{42}$$

where

$$\phi_{s\tilde{g},s}(\omega) = e^{2\pi k i\left[\frac{1}{8\pi^2}\int_X \langle s^*\omega\wedge(d\tilde{g})\tilde{g}^{-1}\rangle - \int_Y \frac{1}{48\pi^2}\tilde{g}^*\sigma\right]} \tag{43}$$

The second term in the exponent on the right is determined, due to integrality of σ, by $\tilde{g}|X$, and is independent of the extension of $\tilde{g}$ to Y. Thus

$\phi_{s\tilde{g},s}(\omega)$ is determined by s, ω, and $\tilde{g}$ on the two-manifold X.

These data specify a principal $U(1)$ bundle over the space $\mathcal{A}_X$ of connections on the bundle $P_X \to X$ (restriction of P over X), as follows. Let I be the set of all smooth sections $s : X \to P_X$. Taking this as indexing set, if $s_1, s \in I$ then, denoting by $\tilde{g} : X \to G$ the function for which $s_1 = s\tilde{g}$, we define $\phi_{s_1,s}$ as above. Thus, $\phi_{s_1,s}(\omega)$ is given by

$$\phi_{s_1,s}(\omega) = e^{2\pi kiCS(s_1,\omega)}e^{-2\pi kiCS(s,\omega)}, \tag{44}$$

where, on the right, the Chern-Simons actions are computed for extensions of $s^*\omega$ and $s^*\omega^g$ over a 3-manifold Y whose boundary is X. If a different 3-manifold Y' is chosen then the value of $\phi_{s_1,s}(\omega)$ remains the same, because it gets multiplied by

$$e^{2\pi kiCS_{Y'\cup -Y}(s_1,\omega)}e^{-2\pi kiCS_{Y'\cup -Y}(s,\omega)},$$

with obvious notation, and we have seen that this is 1. The expression (44) makes it clear that $\{\phi_{s_1,s}\}_{s,s_1\in I}$ satisfies the cocycle condition (1) and thus *specifies a principal* $U(1)$*-bundle over the space* $\mathcal{A}_X$ *of connections on* $P_X \to X$.

Note that the integrality condition on k (which goes back to the integrality property of the inner-product on LG) is what leads to the $U(1)$ bundle.

The principal $U(1)$-bundle constructed along with the natural representation of $U(1)$ on $\mathbb{C}$, yields a line bundle $\mathbb{L}$ over $\mathcal{A}_X$, as described more generally in the context of (2). If Y is a 3-manifold with boundary X then for any connection ω on the bundle over Y, we have a well-defined element

$$e^{2\pi kiCS(\omega)} \stackrel{\text{def}}{=} [s, e^{2\pi kiCS(s,\omega)}] \tag{45}$$

in the $U(1)$-bundle over $\mathcal{A}_X$ in the fiber over $\omega|X$. In this way (following Freed[22]), the *exponentiated Chern-Simons action over an oriented* 3*-manifold* Y *with boundary* X *appears as a section of the line bundle* $\mathbb{L}$ *over* $\mathcal{A}_X$.

4.3. *Connection on the $U(1)$ bundle over the space of connections*

The method of geometric quantization also requires a connection on the $U(1)$-bundle (over phase space). The connection is here generated again using the Chern-Simons action. Let

$$[0,1] \to \mathcal{A}_X : t \mapsto \omega_t$$

be a path of connections, such that $(t,p) \mapsto \omega_t(p)$ is smooth. Then this specifies a connection ω on the bundle

$$[0,1] \times P \to [0,1] \times X$$

in the obvious way (parallel transport in the t direction is trivial). We define parallel transport along the path $t \mapsto \omega_t$ over $\mathcal{A}_X$ geometrically as follows: consider any 3-manifold Y with boundary X, and a principal G-bundle $P_Y \to Y$ with connection $\omega_{0,Y}$ which restricts to the given bundle over X and ω_0, and similarly consider $\omega_{1,Y}$; then parallel-transporting $e^{2\pi kiCS(\omega_{0,Y})}$ along the path will yield

$$e^{2\pi kiCS(\omega_{1,Y})}e^{2\pi kiCS(\tilde{\omega})}$$

where $\tilde{\omega}$ is the connection over $(Y) \cup (X \times [0,1]) \cup (-Y)$, glued along X, obtained by combining ω, $\omega_{0,Y}$ and $\omega_{1,Y}$. In terms of a trivialization of the bundle specified through a section s of P over X, parallel transport is given by multiplication by

$$e^{2\pi kiCS(\tilde{s},\tilde{\omega})} \tag{46}$$

where $\tilde{s}$ is the induced trivialization of $[0,1] \times P \to [0,1] \times X$. Observe that (indicating by the subscript X the differential over X)

$$d\tilde{s}^*\tilde{\omega} = d_X s^*\omega_t + dt \wedge \frac{\partial s^*\omega_t}{\partial t}.$$

A simple computation then shows

$$CS(\tilde{s},\tilde{\omega}) = -\frac{1}{8\pi^2}\int_{[0,1]}\left(\int_X \left\langle \omega_t \wedge \frac{\partial \omega_t}{\partial t}\right\rangle\right) \wedge dt. \tag{47}$$

Viewing the Lie algebra of $U(1)$ as $i\mathbb{R}$, the parallel transport for a $U(1)$ connection along a path is e^{-P}, where P is the integral of the connection form along the path, we see that the connection form θ on the $U(1)$ bundle over $\mathcal{A}_X$ is given explicitly by

$$\theta|_\omega(A) = 2\pi i \frac{k}{8\pi^2}\int_X \langle \omega \wedge A\rangle, \tag{48}$$

for any connection $\omega \in \mathcal{A}_X$ and any vector A tangent to $\mathcal{A}_X$ at ω (such an A is simply an LG-valued 1-form on P which vanishes on vertical vectors and satisfies $R_g^*A = \mathrm{Ad}(g^{-1})A$ for every $g \in G$). The curvature of this is given by the $i\mathbb{R}$-valued 2-form $\Theta = d\theta$ specified explicitly on $\mathcal{A}_X$ by

$$\Theta(A,B) = A\big(\Theta(B)\big) - B\big(\Theta(A)\big)$$

(where A and B are treated as 'constant' vector fields on the affine space $\mathcal{A}_X$). This yields

$$\Theta(A,B) = 2\pi i \frac{k}{4\pi^2}\int_X \langle A \wedge B\rangle \tag{49}$$

for all $A, B \in T_\omega \mathcal{A}_X$. In keeping with the Bohr-Sommerfeld quantization conditions, we should consider the the symplectic form

$$\frac{1}{2\pi i}\Theta = \frac{k}{4\pi^2}\int_X \langle A \wedge B\rangle \tag{50}$$

This is precisely, with correct scaling factors, the symplectic structure used by Witten [equation (2.29) in[57]] with $k = 1$.

In the context of geometric quantization it is more common to consider the Hermitian line bundle associated to the principal G-bundle over $\mathcal{A}_X$ constructed here, and view the connection as a connection on this line bundle. From this point of view one might as well simply consider the case $k = 1$, since the case of general $k \in \mathbb{Z}$ arises from different representations of $U(1)$, i.e. are tensor powers of the $k = 1$ line bundle (and its conjugate).

4.4. *From Chern-Simons to Yang-Mills on a surface*

The original gauge invariance of $e^{2\pi k i CS(\omega)}$ transfers to an easily-checked gauge invariance of the symplectic structure Θ on the space of connections. Thus, we have the group $\mathcal{G}$ of all gauge transformations acting symplectically on the affine space $\mathcal{A}_X$. As is well known, this action has a moment map:

$$J : \mathcal{A}_X \to (L\mathcal{G})^* : \omega \mapsto \frac{k}{4\pi^2}\Omega^\omega \tag{51}$$

where we have identified the dual of the infinite dimensional Lie algebra $L\mathcal{G}$ with the space of LG-valued Ad-equivariant functions on the bundle space P. This fact is readily checked using Stokes' theorem:

$$\langle J'(\omega)A, H\rangle = \frac{k}{4\pi^2}\int_X \langle (dA + [\omega\wedge A]), H\rangle = \frac{1}{2\pi i}\Theta(A, dH + [\omega, H])$$

The *Yang-Mills* action now can be seen as the norm-squared of the moment map:

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2g^2}\|\frac{4\pi^2}{k}J\|^2 \tag{52}$$

where $\|J\|^2$ is computed as an L^2-norm squared.

We have been discussing Chern-Simons theory in terms of its action, i.e. the integral of the Lagrangian. The Hamiltonian picture works with the *phase space*, i.e. the space of extrema of the Chern-Simons action. A fairly straightforward computation shows that the extrema are flat connections. If we consider the 3-manifold

$$Y = [0, T] \times \Sigma,$$

where Σ is a closed oriented surface, then the phase space, after quotienting out the gauge symmetries, may be identified as the moduli space of flat connections over Σ, which in turn is $J^{-1}(0)/\mathcal{G}$. It is a stratified space, with maximal stratum $\mathcal{M}^0$ which is a symplectic manifold with symplectic structure induced by $\frac{1}{2\pi i}\Theta$. We denote this symplectic structure by $\overline{\Omega}$ when k is set to 1, i.e. it is induced by the symplectic structure on $\mathcal{A}_X$ given by

$$\frac{1}{4\pi^2}\int_X \langle A \wedge B\rangle \tag{53}$$

4.5. *The symplectic limit*

The formal Chern-Simons path integral

$$\int_{\mathcal{A}_Y} e^{2\pi k i CS(\omega)} D\omega$$

is naturally of interest in the quantization of Chern-Simons theory (for progress on a rigorous meaning for Chern-Simons functional integrals see Hahn[29,30]). The path integral may be analyzed in the $k \to \infty$ limit by means of its behavior at the extremal of CS, i.e. on the moduli space of flat connections. This is also what results when we examine the limit of the Yang-Mills measure

$$\frac{1}{N_g} e^{-\frac{1}{2g^2}\|\Omega^\omega\|^2} D\omega,$$

(with N_g a formal normalizing factor) for connections over the surface X, in the limit $g \to 0$.

Formally, it is clear that the limiting measure, if it is meaningful, should live on those connections where Ω^ω is 0, i.e. the flat connections. Quotienting by gauge transformations yields the moduli space $\mathcal{M}^0$ of flat connections. For a compact oriented surface Σ of genus $\gamma \geq 1$, the fundamental group $\pi_1(\Sigma, o)$, where o is any chosen basepoint, is generated by the homotopy classes of loops $A_1, B_1, ..., A_\gamma, B_\gamma$ subject to the constraint that the word $B_\gamma^{-1}A_\gamma^{-1}B_\gamma A_\gamma \ldots B_1^{-1}A_1^{-1}B_1A_1$ is the identity in homotopy. Considering a (compact, connected,) simply connected gauge group G (so that a principal

G-bundle over Σ is necessarily trivial), each flat connection is specified, up to gauge transformations, by the holonomies around the loops A_i, B_i. In this way, $\mathcal{M}^0$ is then identified with the subset of $G^{2\gamma}$, modulo conjugation by G, consisting of all $(a_1, b_1, ..., a_\gamma, b_\gamma)$ satisfying

$$b_\gamma^{-1} a_\gamma^{-1} b_\gamma a_\gamma \dots b_1^{-1} a_1^{-1} b_1 a_1 = e.$$

Recalling our description of the Yang-Mills measure in terms of the heat kernel Q_t on G, we have the following result:[49]

Theorem 4.1. *Consider a closed, oriented Riemannian two-manifold of genus $\gamma \geq 2$, and assume that G is a compact, connected, simply-connected Lie group, with Lie algebra equipped with an Ad-invariant metric. Let f be a G-conjugation invariant continuous function on $G^{2\gamma}$, and $\tilde{f}$ the induced function on subsets of $G^{2\gamma}/G$. Then*

$$\lim_{t\downarrow 0} \int_{G^{2\gamma}} f(x) Q_t\big(K_\gamma(x)\big)\, dx = \frac{(2\pi)^n}{|Z(G)|\mathrm{vol}(G)]^{2\gamma-2}} \int_{\mathcal{M}^0} \tilde{f}\, d\mathrm{vol}_{\overline{\Omega}}, \qquad (54)$$

where $|Z(G)|$ is the number of elements in the center $Z(G)$ of G, and $\mathrm{vol}_{\overline{\Omega}}$ *is the symplectic volume form $\frac{1}{n!}\overline{\Omega}^n$ on the space $\mathcal{M}^0$ whose dimension is* $2n = (2\gamma - 2)\dim G$.

With $f = 1$ this yields Witten's volume formula (formula (4.72) in[58])

$$\mathrm{vol}_{\overline{\Omega}}\big(\mathcal{M}^0\big) = \frac{|Z(G)|\mathrm{vol}(G)]^{2\gamma-2}}{(2\pi)^n} \sum_\alpha \frac{1}{(\dim \alpha)^{2\gamma-2}} \qquad (55)$$

where the sum is over all non-isomorphic irreducible representations α of G. Specialized to $G = SU(2)$, this gives the symplectic volume of the moduli space of flat $SU(2)$ connections over a closed genus γ surface to be the rational number $\frac{2^{\gamma-1}}{(2\gamma-2)!}(-1)^\gamma B_{2\gamma-2}$, where B_k is the k-th Bernoulli number. (Note that keeping track of all the factors of 2π pays off in reaching this rational number!)

5. Concluding Remarks

We have given an overview of the geometric and topological aspects of two-dimensional Yang-Mills theory and described how they relate to the Yang-Mills probability measure.

Many physical systems involving a parameter N have asymptotic limiting forms as $N \to \infty$, even though such a limit may not have a clear physical meaning. For the case of Yang-Mills gauge theory with gauge group $U(N)$, the limit as $N \to \infty$ (holding g^2N fixed, where g is the coupling constant) has been of particular interest since the path breaking work of ‘t Hooft.[54] We refer to the recent review[50] for more details on the large N limit of Yang-Mills in two dimensions. A key observation is that letting $N \to \infty$, while holding $\tilde{g}^2 = g^2 N$ fixed, yields meaningful finite limits of all Wilson loop expectation values. There is also good reason to believe (see Singer[52]) that a meaningful $N = \infty$ theory also exists, possibly with relevance to Yang-Mills gauge theory in higher dimensions as well. Free probability theory (see, for instance, Voiculescu et al.[55] and Biane[9]) is likely to play a significant role here.

The partition function for $U(N)$ gauge theory on a genus γ surface is the normalizing constant we have come across:

$$Z_\gamma = \sum_\alpha (\dim \alpha)^{2-2\gamma} e^{-\tilde{g}^2 S c_2(\alpha)/(2N)}$$

where the sum is over all distinct irreducible representations α of $U(N)$, which may be viewed as a sum over the corresponding Young tablueaux (which parametrize the irreducble representations), and $c_2(\alpha)$ is the quadratic Casimir for α. This sum may be viewed naturally as a statistical mechanical partition function for a system whose states are given by the Young tableaux. This point of view leads to the study of Schur-Weyl duality for $U(N)$ gauge theory (see, for example,[1]) and to the study of phase transitions in the parameter $\tilde{g}^2 S$ as $N \uparrow \infty$, viewed as a thermodynamic limit.

The references below present a sample of relevant works, and does not aspire to be a comprehensive bibliography.

Acknowledgment. The author thanks the anonymous referee for useful comments. Research support from the U.S. National Science Foundation (Grant DMS-0601141) is gratefully acknowledged.

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