Toy Model for Density of Bound States

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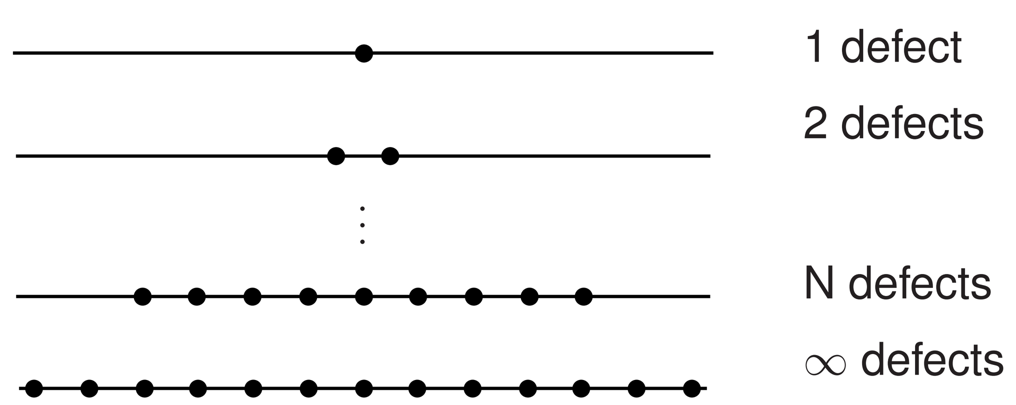
The Bound State Problem

A bound state is a finite-energy localized state of an extended system. Bound states are created at discrete energies by a defect within an otherwise homogeneous medium. As the number of defects increases, the system supports more bound-state frequencies. *We are interested in knowing how these frequencies are distributed and how this distribution depends on the distance L between the defects.*

We analyze a toy model in which bound states can be computed analytically and numerically. It consists of a radiation line with defects.



The state function ϕ for frequency ω satisfies $-\frac{d^2\phi}{dx^2} = \omega\phi(x)$. In each of the intervals separated by the defects, $\phi(x)$ has the form

$$\phi(x) = \begin{cases} Ae^{-\gamma x} + Be^{\gamma x} & \text{for } \omega < 0 \\ Ae^{ikx} + Be^{-ikx} & \text{for } \omega > 0 \end{cases}$$

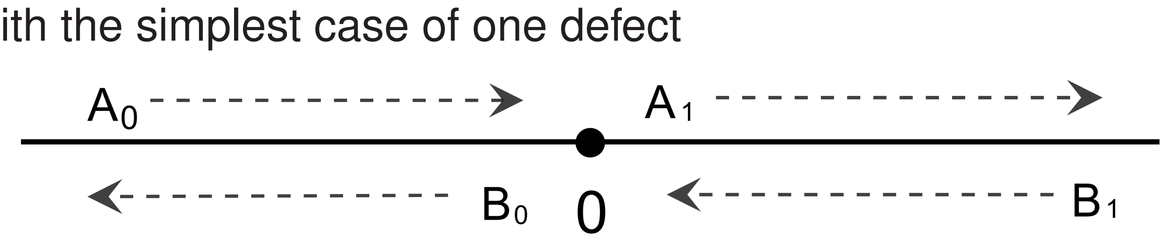
in which $\gamma = -ik$ and $k = \sqrt{\omega}$.

We apply the following *defect conditions* on each defect in our model:

- Continuity at each defect: $\phi(0^-) = \phi(0^+)$
- Jump condition at each defect: $\phi'(0^+) - \phi'(0^-) = -\alpha\phi(0)$

One Defect on a Line

Starting with the simplest case of one defect



The field in this case is as follows.

$$\phi(x) = \begin{cases} A_0 e^{-\gamma(x+L)} + B_0 e^{\gamma x} & \text{for } x \le 0 \\ A_1 e^{-\gamma x} + B_1 e^{\gamma(x-L)} & \text{for } x \ge 0 \end{cases}$$

The bound state of this model occurs when $A_0, B_1 = 0$ and $A_1, B_0 \neq 0$.
To solve for ω's when this occurs, we use the following matrices to relate the field and coefficients on either side of the defect:

- $$\begin{bmatrix} \phi(x) \\ \gamma^{-1}\phi'(x) \end{bmatrix} = \Phi(x)\begin{bmatrix} A_n \\ B_n \end{bmatrix};\ \Phi(x) = \begin{bmatrix} e^{-\gamma(x-(n-1)L)} & e^{\gamma(x-nL)} \\ -e^{-\gamma(x-(n-1)L)} & e^{\gamma(x-nL)} \end{bmatrix}$$
- $$\begin{bmatrix} \phi(x^+) \\ \gamma^{-1}\phi'(x^+) \end{bmatrix} = V_\alpha \begin{bmatrix} \phi(x^-) \\ \gamma^{-1}\phi'(x^-) \end{bmatrix};\ V_\alpha = \begin{bmatrix} 1 & 0 \\ \frac{-\alpha}{\gamma} & 1 \end{bmatrix}$$

$\Phi(x)$ gives our field at x given the coefficients and V_α takes us from one side of a defect to the other.
We create our transfer matrix, T, giving us A_1, B_1 from A_0, B_0.

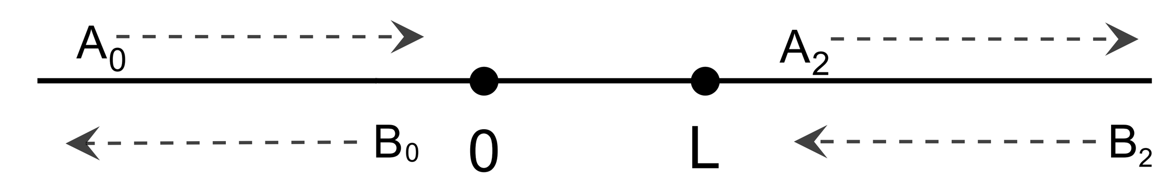
$$T = \Phi^{-1}(0)V_\alpha\Phi(0)\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} e^{-\gamma L}(1+\frac{\alpha}{2\gamma}) & \frac{\alpha}{2\gamma} \\ \frac{-\alpha}{2\gamma} & e^{\gamma L}(1-\frac{\alpha}{2\gamma}) \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

Recall, we are solving for the non-trivial solution of $A_0, B_1 = 0$. This is equivalent to saying the 2,2 entry of our transfer matrix is 0.

The final solution is $\omega = -\frac{\alpha^2}{4}$.

Two Defects on a Line

A second defect is added to our graph, at a distance of L.



The field in this case is as follows.

$$\phi(x) = \begin{cases} A_0 e^{-\gamma(x+L)} + B_0 e^{\gamma x} & \text{for } x \le 0 \\ A_1 e^{-\gamma x} + B_1 e^{\gamma(x-L)} & \text{for } 0 \le x \le L \\ A_2 e^{-\gamma(x-L)} + B_2 e^{\gamma(x-2L)} & \text{for } L \le x \end{cases}$$

The bound state of this model occurs when $A_0, B_2 = 0$ and $A_2, B_0 \neq 0$.
We create our transfer matrix, T, giving us A_2, B_2 from A_0, B_0.

$$T\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \Phi^{-1}(L)V_\alpha\Phi(L)\Phi^{-1}(0)V_\alpha\Phi(0)\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

Our transfer matrix takes a special form of being the square of a *monodromy matrix*, we refer to as M. Our transfer matrix can be written as follows.

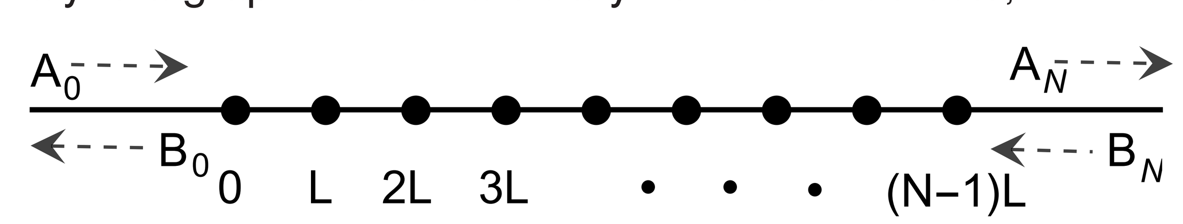
$$T = M^2 = [\Phi^{-1}(x)V_\alpha\Phi(x)]^2 = \begin{bmatrix} e^{-\gamma L}(1+\frac{\alpha}{2\gamma}) & \frac{\alpha}{2\gamma} \\ \frac{-\alpha}{2\gamma} & e^{\gamma L}(1-\frac{\alpha}{2\gamma}) \end{bmatrix}^2$$

Recall, we are solving for the non-trivial solution of $A_0, B_2 = 0$. This is equivalent to saying the 2,2 entry of our transfer matrix is 0.

The system supports two bound states whose frequencies satisfy $\alpha(1 \pm e^{-\sqrt{|\omega|}L}) = 2\sqrt{|\omega|}$.

N Defects on a Line

We now analyze a graph with an arbitrary number of defects, N.



The field in this case is as follows.

$$\phi(x) = \begin{cases} A_0 e^{-\gamma(x+L)} + B_0 e^{\gamma x} & \text{for } x \le 0 \\ A_n e^{-\gamma(x-(n-1)L)} + B_n e^{\gamma(x-nL)} & \text{for } (n-1)L \le x \le nL \\ A_N e^{-\gamma(x-(N-1)L)} + B_N e^{\gamma(x-NL)} & \text{for } (N-1)L \le x \end{cases}$$

The bound state of this model occurs when $A_0, B_N = 0$ and $A_N, B_0 \neq 0$. We create our transfer matrix, T, giving us A_N, B_N from A_0, B_0.

$$T\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = M^N\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} e^{-\gamma L}(1+\frac{\alpha}{2\gamma}) & \frac{\alpha}{2\gamma} \\ \frac{-\alpha}{2\gamma} & e^{\gamma L}(1-\frac{\alpha}{2\gamma}) \end{bmatrix}^N\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_N \\ B_N \end{bmatrix}$$

Recall, we are solving for the non-trivial solution of $A_0, B_N = 0$. This is equivalent to saying the 2,2 entry of our transfer matrix is 0. We solve for this value by diagonalizing our monodromy matrix, M.

$$M^N = P^{-1}D^N P = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & d \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^- \end{bmatrix}^{-N}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

from which we find that $t_{22} = \lambda^{-N}ad - \lambda^N cb = 0$

We substitute in values for a, b, c, and d we get a final equation of

$e^{-\gamma L}(1+\frac{\alpha}{2\gamma})\sin(N\theta) - \sin((N+1)\theta) = 0$

By analyzing the characteristic polynomial, we see its eigenvalues, λ and λ^-, satisfy the property $\lambda\lambda^- = 1$

- Characteristic Polynomial= $\lambda^2 - (e^{\gamma L}(1-\frac{\alpha}{2\gamma}) + e^{-\gamma L}(1+\frac{\alpha}{2\gamma}))\lambda + 1$

By analyzing the trace, we see its eigenvalues satisfy the property $\lambda + \lambda^- \in \mathbb{R}$

- Trace= $e^{\gamma L}(1-\frac{\alpha}{2\gamma}) + e^{-\gamma L}(1+\frac{\alpha}{2\gamma})$

Density of Bound States

From both of these observations can deduce the following.

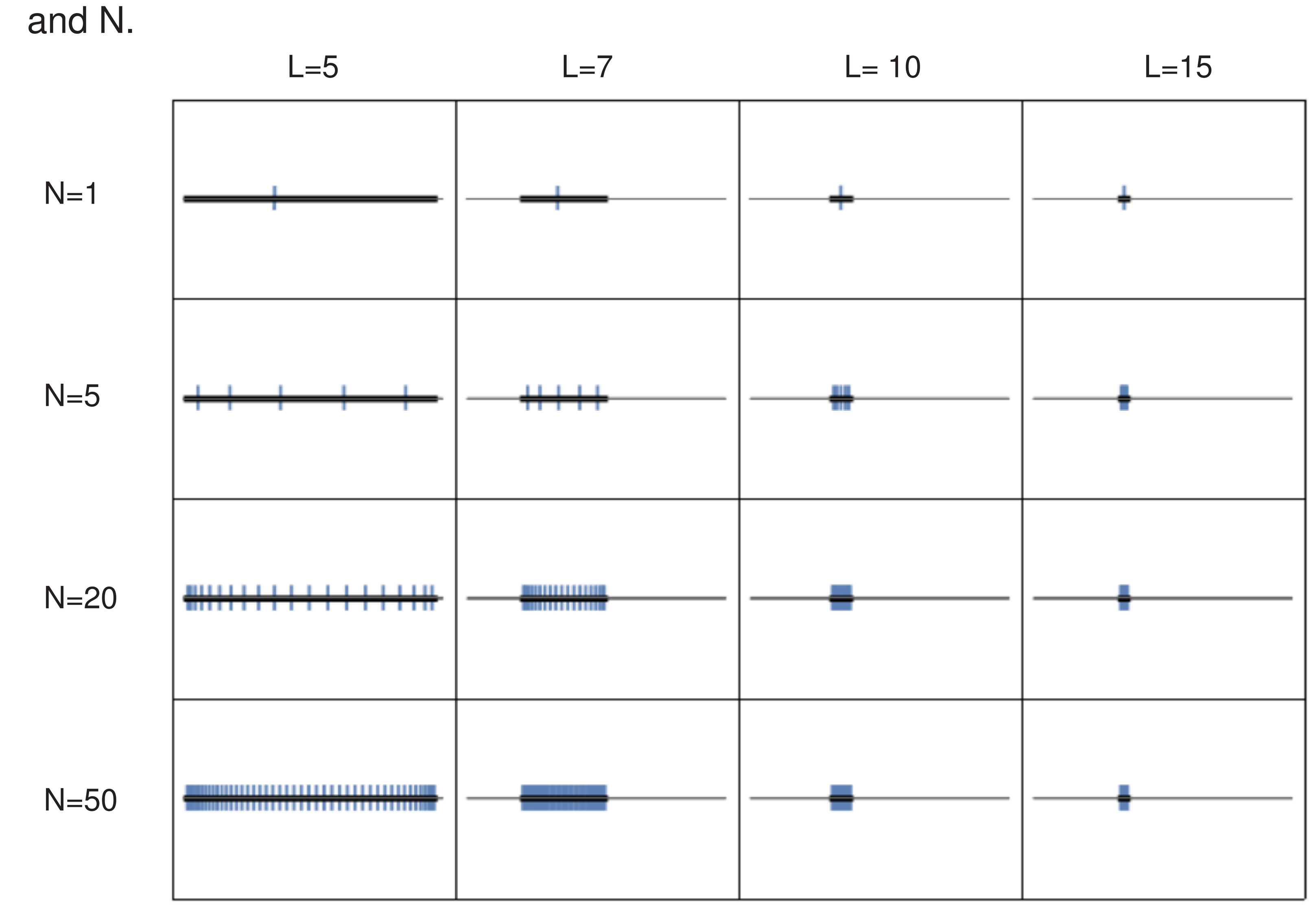
- If $|\,\text{trace}\,| \le 2$ then $\lambda = e^{i\theta}$ and $\lambda^- = e^{-i\theta}$ and that $\lambda + \lambda^- = 2\cos\theta$
- If $|\,\text{trace}\,| \ge 2$ then $\lambda, \lambda^- \in \mathbb{R}$

We concern ourselves only with values of $\lambda = e^{i\theta}$ which is the range for which we get propagation throughout the periodic medium.

By analytically diagonalizing the Monodromy matrix we are left with the following two equations to solve simultaneously:

- $e^{-\gamma L}(1+\frac{\alpha}{2\gamma})\sin(N\theta) - \sin((N+1)\theta) = 0$
- $2\cos\theta - e^{\gamma L}(1-\frac{\alpha}{2\gamma}) - e^{-\gamma L}(1+\frac{\alpha}{2\gamma}) = 0$

Solutions for this system of equations are shown below for varying values of L and N.



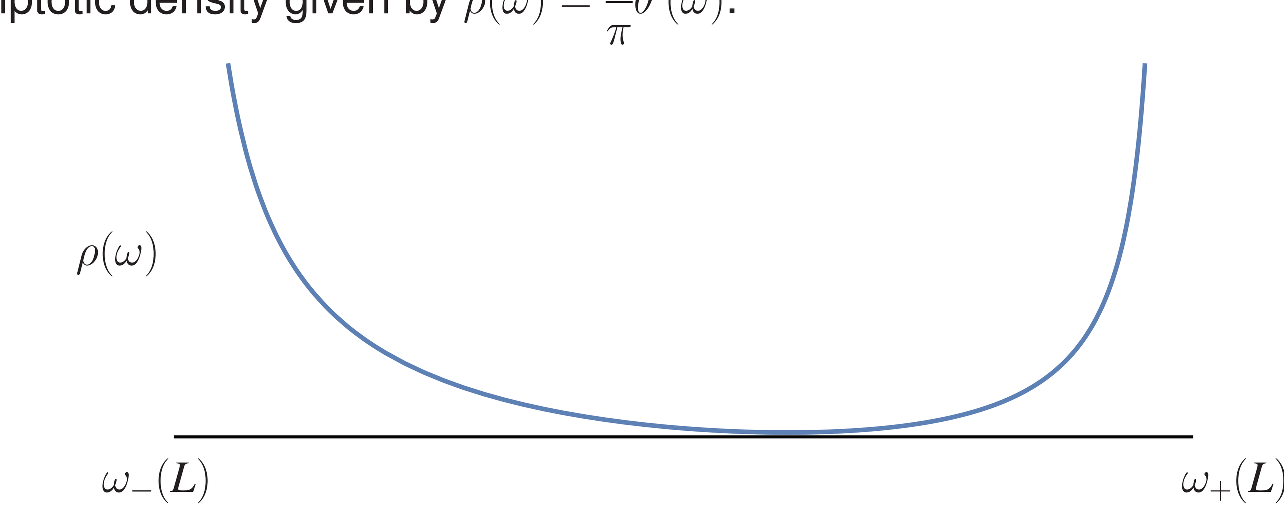
These graphs display bound-state solutions for ω, plotted on the ω axis, ranging from -0.32 to -0.12.

Asymptotic Density of Bound States

Analyzing these results we can clearly note two things.

- As we increase the number of defects, N, the number of discrete frequencies ω that create a bound state increases.
- As we increase the distance, L, between defects, the length of the interval on which these frequencies fall decreases.

As $N \to \infty$, the bound state frequencies, ω, fill a band according to an asymptotic density given by $\rho(\omega) = \frac{1}{\pi}\theta'(\omega)$.



We see that solutions of ω are more dense near the ends of the band of bound state frequencies.

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