# Free Oscillations of Coupled Strings

Lillian Powell

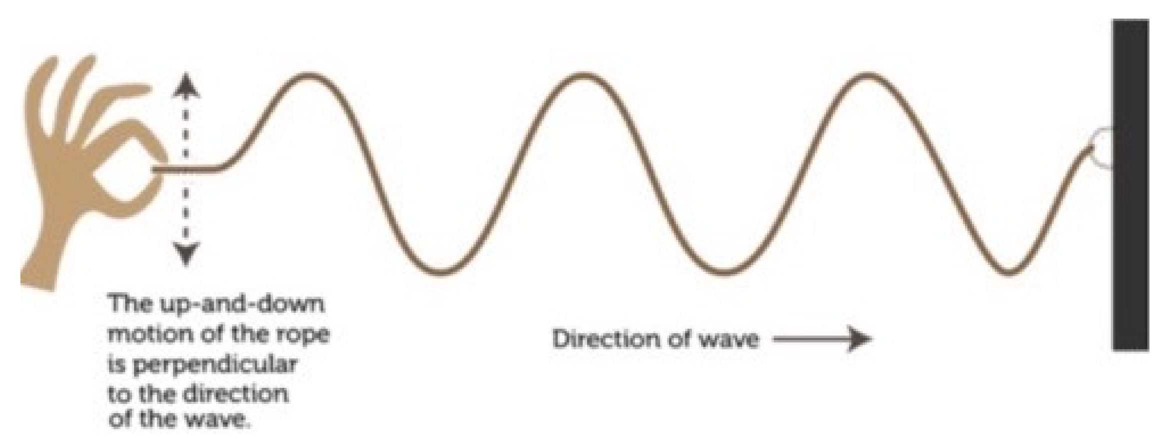
Department of Mathematics, LSU, Baton Rouge; Advisor: Dr. Stephen Shipman

LOUISIANA STATE UNIVERSITY

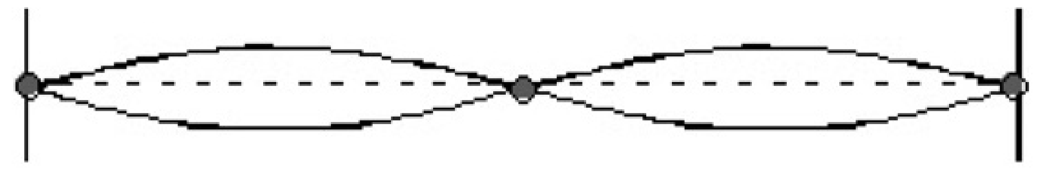
## Abstract

The frequencies at which a string freely oscillates are commonly known as the harmonics, or overtones, of the string. They are simple and easy to compute for single stretched uniform string. But when several strings are coupled to each other at their endpoints, the free oscillations of the configuration of strings becomes more interesting and tricky to compute. Computation of the frequencies amounts to finding the eigenvalues of a coupled system of ordinary differential equations. Even for simple configurations, the results can be unexpected. We calculate the free frequencies and the corresponding vibrational displacement in some explicit examples. Dynamic animations bring these vibrational modes to life!

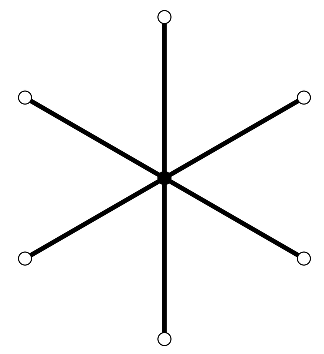
## Physics of free oscillations



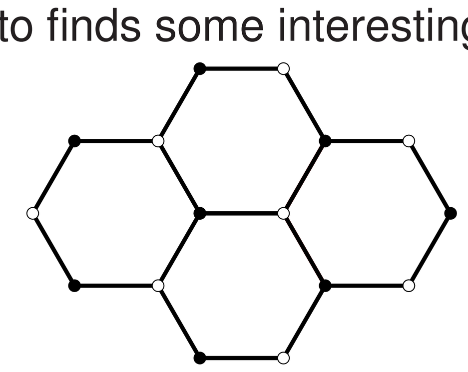

- A **wave** is a disturbance which travels through a medium. When a wave is present in a medium the individual particles of the medium are temporarily displaced from their equilibrium position. Forces act on the particles which cause the displacement and to restore them to their equilibrium position.



- The behavior of single strings may be intuitive given its occurrence in our everyday life. Musicians in particular may be especially familiar with the behavior of strings. As a the string vibrates, the traverse wave travels perpendicular to the oscillations, and it obtains some frequency along with particular harmonics.
- When a transverse wave on a string is fixed at the end point, the reflected wave is inverted from the incident wave, but it is free at the end point, the reflected wave is not inverted from the incident wave. The behavior of these waves depends on influences such as the mass density of the string, tension, the length of the string, and its vertical displacement.



- The behavior of these waves are significantly more complicated when in combination with several other strings. We will study the behavior of these complex waves on strings to finds some interesting and unexpected results.



## Edge and vertex conditions

**Edge Conditions:**

First we will describe the equation that describes the motion of the displacement of a string.

- $x$ is the distance along the string
- $u(x,t)$ displacement vertically
- $\rho$ is the density of the string
- $\tau$ is the tension

The wave equation governs the displacement of a string:

$$\rho\frac{d^2u}{dt^2} = \tau\frac{d^2u}{dx^2}$$

This applies to each string represented by an edge of the graph.

**Vertex Conditions:**

Let $u_1, \ldots, u_n$ denote the displacements of the $n$ strings connected at a vertex, with $x = 0$ corresponding to the joining vertex.

- At any vertex where multiple strings meet, the displacement of the strings must be equal, so
$$u_1(0,t) = \cdots = u_n(0,t) \qquad \text{(continuity)}$$
- Since there is no point mass at any vertex where strings meet, the net force at each vertex must equal zero. The force exerted by a string is proportional to the derivative of its displacement out of the vertex along the corresponding edge. Therefore, the sum these outward derivatives is zero:
$$\sum_{i=1}^{n}\frac{\partial u_i}{\partial x}(0,t) = 0 \qquad \text{(force balance)}$$

## Free oscillations

We seek displacements of the strings oscillating at frequency $\omega$:

$$u(x,t) = f(x)e^{i\omega t}$$

Inserting this into the wave equation gives the ODE $-\rho\omega^2 f(x) = \tau f''(x)$, or

$$-f''(x) = \left(\frac{\omega}{c}\right)^2 f(x)$$

where the constant

$$c = \sqrt{\tau/\rho}$$

is the speed, or celerity, of waves in the strings.

Putting $u_i(x,t) = f_i(x)e^{i\omega t}$ for the $n$ edges emanating from a vertex, the vertex conditions become

$$f_1(0) = \cdots = f_n(0) \qquad \text{(continuity)}$$
$$\sum_{i=1}^{n} f_i'(0) = 0 \qquad \text{(force balance)}$$

## Eigenvalue problem for graphs

The problem of finding the free oscillatory motions $u(x,t) = f(x)e^{i\omega t}$ on the string network as a whole becomes an eigenvalue problem for the operator $-d^2/dx^2$:

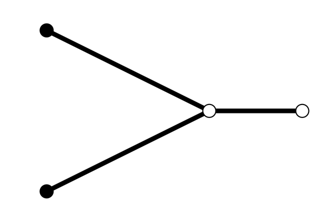
$$-f''(x) = \lambda f(x)$$

where the eigenvalue is related to the frequency and celerity by

$$\lambda = \left(\frac{\omega}{c}\right)^2$$

with all the functions $f$ on the edges being subject to the vertex conditions of continuity of force balance.

## Computation of an example



In this example, two strings of length $1$ are connected to a string of length $1/2$ on a frictionless loop on a rod. The two strings of length $1$ are tied down at the other end, and the string of length $1/2$ is free to move along a frictionless rod.

**Restrictions from ODE and vertex conditions**

1. $-\frac{d^2}{dx^2}f = \lambda f \implies f = a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x)$
2. $f_1(0) = f_2(0) = f_3'(0) = 0 \Rightarrow f_1 = a_1\sin(\sqrt{\lambda}x),\ f_2 = a_2\sin(\sqrt{\lambda}x),\ f_3 = a_3\cos(\sqrt{\lambda}x)$
3. $f_1(1) = f_2(1) = f_3(1/2) \implies a_1\sin(\sqrt{\lambda}) = a_2\sin(\sqrt{\lambda}) = a_3\cos(\sqrt{\lambda}/2)$
4. $f_1'(1) + f_2'(1) + f_3'(1/2) = 0 \implies \sqrt{\lambda}\big((a_1+a_2)\cos\sqrt{\lambda} - a_3\sin(\sqrt{\lambda}/2)\big) = 0$

**Computing eigenvalues $\lambda$ and eigenfunctions**

The first equality in (3) leads to two distinct families of eigenfunctions:

**Case 1:** If $\sin\sqrt{\lambda} \neq 0$

(3) $\implies a_1 = a_2$; then (3,4) becomes

$$\begin{bmatrix} \sin\sqrt{\lambda} & -\cos\sqrt{\frac{\lambda}{2}} \\ 2\cos\sqrt{\lambda} & -\sin\sqrt{\frac{\lambda}{2}} \end{bmatrix}\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvalue condition (EC) is $\det\square = 0$, meaning that $\lambda$ must satisfy

$$\text{(EC)}\quad \sin\sqrt{\lambda}\sin\sqrt{\frac{\lambda}{2}} + 2\cos\sqrt{\lambda}\cos\sqrt{\frac{\lambda}{2}} = 0$$

Eigenfunctions: $[f_1,f_2,f_3]$ in (2) with $[a_1,a_2,a_3] =$

$$\text{const.} \times \left[\sin\frac{\sqrt{\lambda}}{2},\ \sin\frac{\sqrt{\lambda}}{2},\ 2\cos\sqrt{\lambda}\right]$$

**Case 2:** If $\sin\sqrt{\lambda} = 0$

$\implies \sqrt{\lambda} = \pi k$ for $k \in \mathbb{Z}$

**Subcase 2A:** $a_3 = 0$

(4) $\implies a_1 + a_2 = 0$

Eigenfunctions: $[f_1,f_2,f_3]$ in (2) with

$[a_1,a_2,a_3] = \text{const} \times [1,-1,0]$

**Subcase 2B:** $\cos\frac{\sqrt{\lambda}}{2} = 0$

$\implies \sqrt{\lambda} = \pi + 2n\pi,\ n \in \mathbb{Z}$

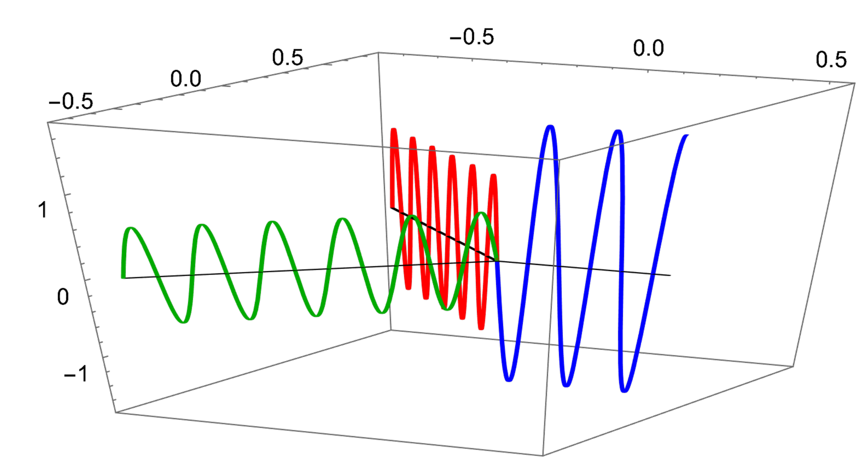
(4) $\implies a_1 + a_2 + (-1)^n a_3 = 0$

Eigenfunctions: $[f_1,f_2,f_3]$ in (2) with

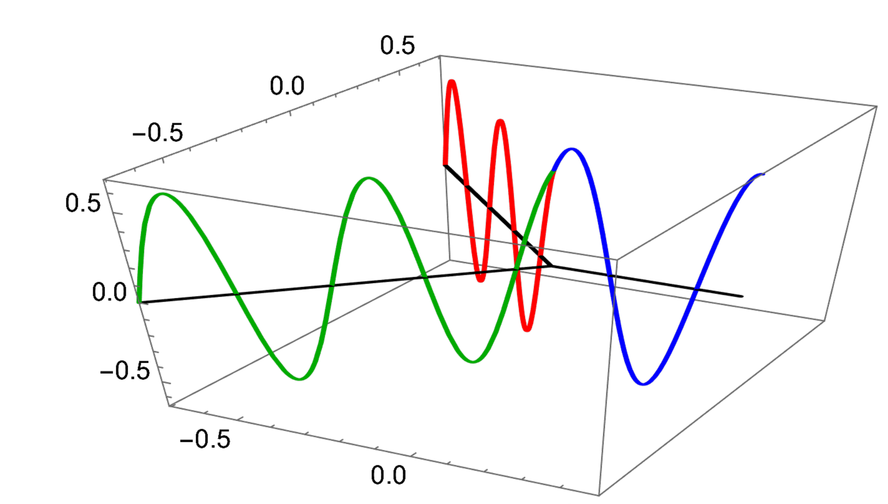
$$[a_1,a_2,a_3] = \text{const} \times [1,-1,0] + \text{const} \times [1,0,(-1)^{n+1}]$$

## Depictions of the eigenfunctions

To compute numerically the waves in our string, which are the eigenfunctions computed above, I used Mathematica®.




**Case 2B.** This is a picture of the displacement of the eleventh mode of this family of free oscillations. For $n = 11$, the eigenvalue is $\lambda = (23\pi)^2$, which means that the vibrational frequency is $\omega = 23\pi c$.




**Case 1.** This is a picture of the displacement of the fifth mode of this family of free oscillations. The eigenvalue $\lambda$ was computed numerically.

## Acknowledgements

We would like to give a huge "thank you" to Dr. Stephen Shipman, for his patience and incredible guidance. And Thank you to my wonderfully loving and supportive family and friends.