Short-Time Behavior of the Exciton-Polariton Equations

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Abstract. In the exciton-polariton system, a linear dispersive photon field is coupled to a nonlinear exciton field. Short-time analysis of the lossless system shows that, when the photon field is excited, the time required for that field to exhibit nonlinear effects is longer than the time required for the nonlinear Schrödinger equation, in which the photon field itself is nonlinear[1].

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1. Short-time behavior of the exciton-polariton system

The lossless unforced exciton-polariton system is a quantum-mechanical system involving a linear dispersive photon wave-function $\phi(\mathbf{x},t)$ and a nonlinear nondispersive exciton wave-function $\psi(\mathbf{x},t)$ of spatial coordinates $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}$:

$$\begin{aligned} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= (\omega_0 + g|\psi|^2)\psi + \gamma\phi\,. \end{aligned} \tag{1.1}$$

For physical discussions of these equations, the reader is referred to [1, 2, 5], among many other references.

The fact that the dispersive term $-\Delta\phi$ and the nonlinear term $g|\psi|^2\psi$ involve different fields results in fundamental differences between the exciton-polariton system (EP) and the nonlinear Schrödinger (NLS) equation

$$i\phi_t = -\Delta\phi + g|\phi|^2\phi\,, \tag{1.2}$$

in which both terms involve a single field ϕ. The NLS equation is Galilean-invariant, whereas the EP system is not; and NLS admits a frequency-scalable

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"ground state", whereas the structure of stationary harmonic solutions of EP is complicated [3]. This communication addresses a fundamental difference in the short-time behavior of these two systems.

We take the point of view that the photon field is excited and measured by the observer and that the exciton field is hidden from the observer. Thus we impose initial conditions

$$\begin{aligned} \phi(\mathbf{x},0) &= \phi_0(\mathbf{x}) \\ \psi(\mathbf{x},0) &= 0 \end{aligned} \tag{1.3}$$

and ask, up to what time can nonlinear effects observed in the photon field through its coupling to the exciton field be considered to be negligible?

At first, the effect of the exciton field on the photon field is altogether negligible and the exciton evolves essentially linearly under the influence of the photon. This is described by the approximate system

$$\begin{aligned} i\phi_t &= -\Delta\phi \\ i\psi_t &= \omega_0\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation A)} \tag{1.4}$$

After some time, the exciton field grows sufficiently large so that its effect on the photon field becomes non-negligible, but the nonlinear effects remain negligible for a longer period of time. The photon acts as if it were coupled to a linear exciton field:

$$\begin{aligned} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= \omega_0\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation B)} \tag{1.5}$$

At a later time, the nonlinear effects imparted by the exciton field are observed significantly in the photon field and the linear Approximation B is no longer acceptable.

Theorem 1.1 makes these assertions precise. The deviation of an approximation $\tilde{\phi}$ to the true photon field ϕ is considered to be negligible if the relative error $\|\tilde{\phi}-\phi\|_{H^s(\mathbb{R}^n)}/\|\phi\|_{H^s(\mathbb{R}^n)}$ is less than a small number ϵ, which is allowed to tend to zero. Our main result is that the deviation of the photon field of the linear polariton system from that of the nonlinear one is negligible up to time $t = C\epsilon^{1/5}$. This result is in contrast to the nonlinear Schrödinger equation, for which nonlinear effects are negligible only up to time $C\epsilon$.

Theorem 1.1. *Let* $(\phi(t),\psi(t))$ *be a solution of the polariton system* (1.1) *in the interval* $0 \le t \le T$*, with each field being a continuous function of* t *with values in* $H^s(\mathbb{R}^n)$ *with* $s > n/2$*. Let* C_1 *and* C_2 *be real numbers, and for all* ϵ *such that* $C_2\epsilon^{1/5} \le T$*, let* $(\tilde{\phi}(t),\tilde{\psi}(t))$ *be a solution of the equations*

$$(\tilde{\phi}(t),\tilde{\psi}(t)) \;\textit{satisfies}\; \begin{cases} \textit{approx. A (1.4)} & \textit{for} \quad 0 \le t \le C_1\epsilon^{1/2} \\ \textit{approx. B (1.5)} & \textit{for} \quad C_1\epsilon^{1/2} \le t \le C_2\epsilon^{1/5} \end{cases}$$

with $\tilde{\phi}$ *and* $\tilde{\psi}$ *being continuous function of* t *with values in* $H^s(\mathbb{R}^n)$*. Let both systems satisfy the initial conditions*

$$(\phi(0),\psi(0)) = (\tilde{\phi}(0),\tilde{\psi}(0)) = (\phi_0, 0), \tag{1.6}$$

with $\|\phi_0\|_{H^s(\mathbb{R}^n)} = \|\phi_0\|_s = M \ne 0$*.*

The relative error in the photon field is bounded by

$$\frac{\|\tilde{\phi}(t)-\phi(t)\|_s}{\|\phi(t)\|_s} \leq K_1\epsilon + O(\epsilon^2) \qquad \text{for} \quad 0 \leq t \leq C_1\epsilon^{1/2} \qquad (\epsilon \to 0),$$

$$\frac{\|\tilde{\phi}(t)-\phi(t)\|_s}{\|\phi(t)\|_s} \leq K_2\epsilon + O(\epsilon^{7/5}) \quad \text{for} \quad C_1\epsilon^{1/2} \leq t \leq C_2\epsilon^{1/5} \qquad (\epsilon \to 0),$$

in which

$$K_1 = \tfrac{1}{2}\gamma^2 C_1^2, \qquad K_2 = \tfrac{1}{2}\gamma^2 C_1^2 + \tfrac{1}{5}|g|K\gamma^3 M^2 C_2^5$$

and K is an absolute constant (defined in the proof below).

The proof of this theorem will be given after existence of solutions and preliminary bounds are established.

2. Existence of solutions to the polariton equations

Theorem 2.1. *Given $0 < r < 1$, $N > 0$, and $\phi_0 \in H^s(\mathbb{R}^n)$ with $s > n/2$, such that $\|\phi_0\|_s \leq rN$, there exists a unique solution to the polariton equations (1.1) subject to $\|\phi\|_{C(I,H^s(\mathbb{R}^n))} \leq N$ and $\|\psi\|_{C(I,H^s(\mathbb{R}^n))} \leq N$ defined for $t \in [0,T]$, where $T = \dfrac{1-r}{2\gamma + |g|\tilde{K}N^2}$ for some constant $\tilde{K}$.*

Proof. The proof is a standard contraction argument. Write $u = (\phi, \psi)^t$, and consider the space

$$E_{N,r} = \left\{ u \in C(I, H^s(\mathbb{R}^n)) \;:\; \|u\|_{C(I,H^s(\mathbb{R}^n))} \leq N, \; \|u_0\|_s \leq rN \right\},$$

with $I = [0,T]$, equipped with the distance $d\,(u_1 - u_2) = \|u_1 - u_2\|_{C(I,H^s(\mathbb{R}^n))}$. $(E_{N,r}, d)$ is a complete metric space. Define a mapping $\Phi : E_{N,r} \to E_{N,r}$ by

$$\Phi(u)(t) = \begin{pmatrix} e^{it\Delta}\phi_0(x) - i\gamma \displaystyle\int_0^t e^{i(t-\tau)\Delta}\psi(\tau)d\tau \\ -i\displaystyle\int_0^t e^{-i\omega_0(t-\tau)}\left(g|\psi|^2\psi(\tau) + \gamma\phi(\tau)\right)d\tau \end{pmatrix}.$$

Minkowski inequalities and the fact that $e^{it\Delta}$ is an isometry in H^s yields

$$\begin{aligned} \|\Phi(u)(t)\|_s &\leq \|\phi_0\|_s + \gamma T\Big(\sup_{\tau\leq T}\|\psi\|_s + \sup_{\tau\leq T}\|\phi\|_s\Big) + |g|KT\sup_{\tau\leq T}\|\psi\|_s^3 \\ &\leq \|\phi_0\|_s + NT\left(2\gamma + |g|KN^2\right). \end{aligned}$$

The constant K for $s > n/2$ is guaranteed by [4, Theorem 3.4]; it relies on the algebra property of $H^s(\mathbb{R}^n)$. For a different constant K', one obtains

$$\|\Phi(u_1) - \Phi(u_2)\|_s \leq T(2\gamma + |g|K'N^2)\Big(\sup_{\tau\leq T}\|\psi_1 - \psi_2\|_s + \sup_{\tau\leq T}\|\phi_1 - \phi_2\|_s\Big).$$

Set $\tilde{K} = \max\{K, K'\}$. Since $T\Big(2\gamma + |g|\tilde{K}N^2\Big) = 1 - r < 1$, Φ is a contraction of $(E_{N,r}, d)$ and thus it has a unique fixed point, which, by the definition of Φ, satisfies the exciton-polariton system. Uniqueness of the solution in $C(I, H^s(\mathbb{R}^n))$ follows from Gronwall's Lemma. □

3. Bounds for solutions to the polariton equations

Assume that (ϕ, ψ) is a solution of the polariton equations as in Theorem 2.1, with initial condition $(\phi(\mathbf{x}, 0), \psi(\mathbf{x}, 0)) = (\phi_0(\mathbf{x}), 0)$ and $\|\phi_0\|_s = rN$.

3.1. Solutions of the polariton equations

The integral form of the system (1.1), namely $\Phi(u) = u$, Minkowski inequalities, and the fact that $e^{it\Delta}$ is an isometry in $H^s(\mathbb{R}^n)$ yield

$$rN - \gamma \int_0^t \|\psi(\tau)\|_s \, d\tau \;\leq\; \|\phi(t)\|_s \;\leq\; rN + \gamma \int_0^t \|\psi(\tau)\|_s \, d\tau \,, \tag{3.1}$$

$$\|\psi(t)\|_s \leq \gamma rN\, t \,+\, \gamma^2 \int_0^t \int_0^\tau \|\psi(\sigma)\|_s \, d\sigma \, d\tau \,+\, |g| K \int_0^t \|\psi(\tau)\|_s^3 \, d\tau \,. \tag{3.2}$$

The constant K is guaranteed by [4, Theorem 3.4]. Hence

$$\sup_{\tau \leq t} \|\psi(t)\|_s \;\leq\; \gamma rN\, t + \tfrac{1}{2}\gamma^2 t^2 \sup_{\tau \leq t} \|\psi(\tau)\|_s + |g| K\, t \sup_{\tau \leq t} \|\psi(\tau)\|_s^3 \,.$$

The last estimate can be written as $P(t, y(t)) \geq 0$, where

$$y(t) = \sup_{\tau \leq t} \|\psi(t)\|_s \quad \text{and} \quad P(t, y) \;:=\; \gamma rN\, t + y \left(|g| K\, t y^2 + \tfrac{1}{2}\gamma^2 t^2 - 1 \right).$$

For each t such that $P(t, y)$ has two positive roots as a function of y, denote these roots by $y_1(t) \leq y_2(t)$. One can show that $y_1(t)$ is increasing in t, with $\lim_{t \to 0} y_1(t) = 0$ and $y_2(t)$ is decreasing with $\lim_{t \to 0} y_2(t) = \infty$. Thus $P(t, y(t)) \geq 0$ is equivalent to $\{\, y(t) \leq y_1(t) \text{ or } y(t) \geq y_2(t) \,\}$. We shall assume from now on that t is small enough so that $y(t) \geq y_2(t)$ is ruled out, so that one has $\sup_{\tau \leq t} \|\psi(\tau)\|_s \leq y_1(t)$, or, equivalently, $\|\psi(t)\|_s \leq y_1(t)$, since $y_1(t)$ is increasing. Hence, (3.2) yields

$$\|\psi(t)\|_s \;\leq\; \gamma rN\, t \,+\, \gamma^2 \int_0^t \int_0^\tau y_1(\sigma) \, d\sigma \, d\tau \,+\, |g| K \int_0^t y_1(\tau)^3 d\tau \,. \tag{3.3}$$

The Taylor expansion of $y_1(t)$ around $t = 0$ is

$$y_1(t) \;=\; \gamma rN\, t + \tfrac{1}{2}\gamma^3 rN\, t^3 + |g| K (\gamma rN)^3\, t^4 + \dots . \tag{3.4}$$

Therefore, from (3.1), (3.3) and (3.4), we have

$$rN - \tfrac{1}{2}\gamma^2 rN t^2 + O(t^4) \;\leq\; \|\phi(t)\|_s \;\leq\; rN + \tfrac{1}{2}\gamma^2 rN t^2 + O(t^4) \,. \tag{3.5}$$

3.2. Solutions of approximate equation A

Let $(\tilde{\phi}(t), \tilde{\psi}(t))$ be the solution of the approximate system A (1.4) with initial condition $(\tilde{\phi}(0), \tilde{\psi}(0)) = (\phi_0, 0)$, and $(\phi(t), \psi(t))$ be the solution of the true system (1.1). Set $\hat{\phi} := \tilde{\phi} - \phi$ and $\hat{\psi} := \tilde{\psi} - \psi$, so that $(\hat{\phi}(t), \hat{\psi}(t))$ satisfies

$$\left\{ \begin{array}{lcl} i\hat{\phi}_t & = & -\Delta \hat{\phi} + \gamma \psi(t) \\ i\hat{\psi}_t & = & \omega_0 \hat{\psi} + \gamma \hat{\phi} + g|\psi(t)|^2 \psi(t) \end{array} \right. \qquad \left\{ \begin{array}{l} \hat{\phi}(0) = 0 \\ \hat{\psi}(0) = 0 \,. \end{array} \right. \tag{3.6}$$

One obtains the bounds

$$\|\hat{\phi}(t)\|_s \le \gamma \int_0^t \|\psi(\tau)\| d\tau \le \gamma \int_0^t y_1(\tau)\, d\tau, \tag{3.7}$$

$$\|\hat{\psi}(t)\|_s \le \gamma^2 \int_0^t \int_0^\tau y_1(\sigma)\, d\sigma\, d\tau + |g|K \int_0^t y_1(\tau)^3 d\tau\,. \tag{3.8}$$

3.3. Solutions of approximate equation B

Now let $(\tilde{\phi}(t), \tilde{\psi}(t))$ be the solution of the approximate system B (1.5) with arbitrary initial conditions, and set again $\hat{\phi} := \tilde{\phi} - \phi$ and $\hat{\psi} := \tilde{\psi} - \psi$; then $(\hat{\phi}(t), \hat{\psi}(t))$ satisfies

$$\left\{ \begin{array}{lcl} i\hat{\phi}_t & = & -\triangle\hat{\phi} + \gamma\hat{\psi}(t) \\ i\hat{\psi}_t & = & \omega_0\hat{\psi} + \gamma\hat{\phi} + g|\psi(t)|^2\psi(t) \end{array} \right. \qquad \left\{ \begin{array}{l} \hat{\phi}(t_1) = \hat{\phi}_0 \\ \hat{\psi}(t_1) = \hat{\psi}_0 \end{array} \right. , \tag{3.9}$$

and from the integral form of (3.9), one deduces the bounds

$$\|\hat{\phi}(t)\|_s \le \|\hat{\phi}_0\|_s + \gamma \int_{t_1}^t \|\hat{\psi}(\tau)\|_s d\tau \tag{3.10}$$

$$\|\hat{\psi}(t)\|_s \le \|\hat{\psi}_0\|_s + \gamma t \|\hat{\phi}_0\|_s + \gamma^2 \int_{t_1}^t \int_{t_1}^\tau \|\hat{\psi}(\sigma)\|_s d\sigma\, d\tau + |g|K \int_{t_1}^t \|\psi(t)\|_s^3 d\tau.$$

Combining this with (3.4) yields

$$\|\hat{\psi}(t)\|_s \le \left(1 - \tfrac{1}{2}\gamma^2 t^2\right)^{-1} \left(\|\hat{\psi}_0\|_s + \gamma t \|\hat{\phi}_0\|_s + |g|K t\, y_1(t)^3 \right). \tag{3.11}$$

3.4. Proof of Theorem 1.1

For the solutions (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$ in the theorem, define $\hat{\phi} := \tilde{\phi} - \phi$ and $\hat{\psi} := \tilde{\psi} - \psi$, and set $M = rN$.

For $t \in [0, t_1]$ with $t_1 = C_1 \epsilon^{1/2}$, (3.7) yields

$$\|\hat{\phi}(t)\|_s \le \gamma \int_0^t y_1(\tau)\, d\tau \le \tfrac{1}{2}\gamma^2 M t^2 + O(t^4) \le \tfrac{1}{2}\gamma^2 M C_1^2 \epsilon + O(\epsilon^2). \tag{3.12}$$

Using (3.5), the relative error is controlled by

$$\frac{\|\hat{\phi}(t)\|_s}{\|\phi(t)\|_s} \le \tfrac{1}{2}\gamma^2 C_1^2 \epsilon + O(\epsilon^2).$$

For $t \in [t_1, t_2]$ with $t_2 = C_2 \epsilon^{1/5}$, (3.4), (3.8), and (3.12) give initial bounds

$$\|\hat{\psi}(t_1)\|_s \le \tfrac{1}{6}\gamma^3 M t_1^3 + O(t_1^4) \le \tfrac{1}{6}\gamma^3 M C_1^3 \epsilon^{3/2} + O(\epsilon^2),$$

$$\|\hat{\phi}(t_1)\|_s \le \tfrac{1}{2}\gamma M C_1^2 \epsilon + O(\epsilon^2).$$

Using these in (3.11) yields

$$\|\hat{\psi}(t)\|_s \le \left(1 + O(t^2)\right) \times \left[\tfrac{1}{6}\gamma^2 M C_1^3 \epsilon^{3/2} + O(\epsilon^2) + \gamma t\left(\tfrac{1}{2}\gamma M C_1^2 \epsilon + O(\epsilon^2)\right) + |g|K(\gamma M)^3 t^4 + O(t^6)\right],$$

and then inserting this into (3.10) gives

$$\|\hat{\phi}(t)\|_s \leq \tfrac{1}{2}\gamma^2 M C_1^2 \epsilon + O(\epsilon^2) + \left(1+\epsilon^{1/5}\right)\times$$
$$\left[\tfrac{1}{6}\gamma^2 M C_1^3 \epsilon^{3/2} t + O(\epsilon^2)t + \tfrac{1}{4}\gamma M C_1^2 \epsilon t^2 + O(\epsilon^2)t + \tfrac{1}{5}|g|K(\gamma M)^3 t^5 + O(t^7)\right].$$

In view of $t \leq C_2\epsilon^{1/5}$, the first four terms in the brackets are $O(\epsilon^{17/10})$, $O(\epsilon^{11/5})$, $O(\epsilon^{7/5})$, and $O(\epsilon^{11/5})$, and the last one is $O(\epsilon^{7/5})$. Therefore

$$\|\hat{\phi}(t)\|_s \;\leq\; \left(\tfrac{1}{2}\gamma^2 r N C_1^2 + \tfrac{1}{5}|g|K(\gamma r N)^3 C_2^5\right)\epsilon + O(\epsilon^{7/5})\,. \tag{3.13}$$

The relative error is obtained from this and (3.5),

$$\frac{\|\hat{\phi}(t)\|_s}{\|\phi(t)\|_s} \;\leq\; \left(\tfrac{1}{2}\gamma^2 C_1^2 + \tfrac{1}{5}|g|K\gamma^3 (rN)^2 C_2^5\right)\epsilon + O(\epsilon^{7/5}).$$

4. Final remark

The analysis above uses strictly $H^s(\mathbb{R}^n)$ estimates and triangle inequalities and does not address whether the time $t = C\epsilon^{1/5}$ is sharp. A future communication will include a comparison between EP and NLS for initial photon data of order ϵ^α and for nonlinearities not just of order 3 but of any power greater than 1.

References

[1] Iacopo Carusotto and Cristiano Ciuti. Probing microcavity polariton superfluidity through resonant rayleigh scattering. *Phys. Rev. Lett.*, 93(16):166401–1–4, Oct. 2004.

[2] B. Deveaud, editor. *The Physics of Semiconductor Microcavities.* Wiley-VCH, Weinheim, 2007.

[3] Stavros Komineas, Stephen P. Shipman, and Stephanos Venakides, Lossless Polariton Solitons. *Physica D*, 316 (2016) 43-56.

[4] Felipe Linares and Gustavo Ponce. *Introduction to nonlinear dispersive equations.* Springer, 2014.

[5] Francesca Maria Marchetti and Marzena H. Szymanska. Vortices in polariton OPO superfluids. Preprint 2011.

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