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The Spectral Transform in the Semiclassical Limit of a Finite Discrete NLS Chain

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Abstract: The linear spectral problem associated with the inverse solution of a finite discrete nonlinear Schrödinger chain is studied in the semiclassical limit. The discrete spectral problem is a recursion relation for a vector quantity, with boundary conditions, depending on initial data and a spectral parameter. WKB analysis is performed and then interpreted for the case that the quantities in the chain are less than one in modulus. In this case, the spectrum lies on the unit circle and an asymptotic density is obtained. The density is supported by known facts about the discrete spectra, numerical results, and rigorous results concerning the asymptotics of the solution of the spectral boundary-value problem. In addition, the norming constants in the spectral transform are positive in this special case, and a proposed asymptotic norming exponent is corroborated by numerical data.

1 Introduction

This article examines the spectral transform associated with an inverse solution of a finite defocusing discrete nonlinear Schrödinger (DNLS) system of ordinary differential equations in the semiclassical limit. The problem possesses a dichotomy of behavior depending on initial data characterized by the unitarity or non-unitarity of the linear spectral problem. Formal, rigorous, and numerical results lead to an understanding of the asymptotics of the unitary case. The non-unitary case is not addressed and is as yet not understood. In the unitary case, the spectrum of eigenvalues lies on the unit circle of the complex plane, and in the semiclassical limit, the dimension of the linear problem is unbounded and we seek an asymptotic density of eigenvalues. Naive WKB analysis leads to a candidate for this density, which is then confirmed by numerical calculations, comparison with properties of the spectrum of the discrete problem, and rigorous asymptotics of the unitary eigenvalue problem. In addition, the proposed density has been applied successfully in [S] to the study of the semiclassical limit of the solution of the DNLS system. In the WKB analysis, the discrete index in the system of ODEs approaches a continuous variable and the typical intervals of "oscillatory" and "exponential" behavior of the solution arise. The density, as usual, involves an integral over an oscillatory interval. A candidate for the asymptotics of the associated norming constant has been proposed in [S] in light of analysis there of the semiclassical limit of the inverse spectral solution. The candidate, as is typical in such asymptotic problems, involves an integral over the exponential intervals for a special class of data, and it was chosen to provide the correct results in that analysis. It is not understood how it may arise directly from asymptotic analysis. In this article, however, it is corroborated by numerical results and by comparison with properties of the norming constant for the discrete system.

Previous work on continuum limits of discrete systems solvable by inverse methods and the asymptotic (WKB) analysis of the associated linear spectral problem includes, most notably, the analysis by Deift and McLaughlin [DM] of a continuum limit of the Toda lattice. Using candidates arising from formal WKB analysis, they rigorously established the asymptotics of the solutions and the spectral density and norming constants. The analysis included turning-point analysis between

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oscillatory and exponential intervals and the matching of solutions. These results were preceded by rigorous results by Geronimo and Smith [GS] on asymptotic solutions to second-order recursion relations.

In the present problem, we deal with two coupled first-order recursion relations. Away from a turning point, the recursion can be decoupled (diagonalized) by a change of variable at each site, but only to leading order. At higher order, this decoupling is inherent and makes a study of the asymptotics difficult. It is examined in detail in Section 5. There, rigorous asymptotics for a fixed spectral value are obtained for oscillatory and exponential intervals away from any turning points. At a turning point, the recursion relation is not diagonalizable, but assumes a triangular form. Turning-point analysis remains unsolved, though a significant effort has been made to try to match solutions in a diagonal setting to those in a triangular setting.

Akin to the WKB analysis in this article but for the non-unitary case is the non-self-adjoint Zakharov-Shabat eigenvalue problem in the semiclassical limit studied by Bronski [B]. A word in comparison with the results in this paper will be said later on.

2 The Spectral Problem

The (defocusing) discrete nonlinear Schrödinger (DNLS) system

$$i\dot{Q}_n + Q_{n-1} - 2Q_n + Q_{n+1} - |Q_n|^2(Q_{n-1} + Q_{n+1}) = 0$$

is transformed under the change of dependent variable $Q_n \mapsto Q_n e^{-2it}$ into the system

$$i\dot{Q}_n + (1 - |Q_n|^2)(Q_{n-1} + Q_{n+1}) = 0. \tag{1}$$

If one puts

$$|Q_0(0)| = |Q_N(0)| = 1 \tag{2}$$

into (1), then Q_0 and Q_N are constant in time and a finite subchain becomes detached from the rest of the chain. One then has a finite system of ordinary differential equations for $Q_1 \dots Q_{N-1}$. This system is solvable by an inverse spectral method [V].

In the semiclassical limit of the finite system, one considers initial data of the form

$$Q_n(0) = q(n\epsilon)\exp\left(\tfrac{i}{\epsilon}\phi(n\epsilon)\right), \tag{3}$$

in which q and ϕ are fixed functions on the real unit interval such that $q(0)=q(1)=1$ and $\epsilon = 1/N$, and considers the limiting behavior of the modulus and phase as ϵ tends to zero. As we will see, the WKB analysis lends itself to a meaningful interpretation with regard to the asymptotic distribution of eigenvalues in this special case. However, if the condition $|Q_n|<1$ is violated, there is no satisfactory interpretation (so far). The reason for this is that the spectrum, in the case $|Q_n|<1$, is constrained to the unit circle of the complex plane, whereas otherwise such a constraint is not known.

We now discuss the eigenvalue problem associated with the inverse spectral solution for the finite discrete system (1, 2). Let $\{Q_n\}_{n=0}^N$ be given such that $|Q_0| = |Q_N| = 1$ and normalized such that $Q_0 = 1$ (observe that multiplying all Q_n by a common unitary number preserves solutions). Denote Q_N by ξ:

$$\xi = Q_N, \quad |\xi| = 1.$$

Let z be an arbitrary complex parameter, and define the matrices

$$U_n(z) = \begin{bmatrix} z & \bar{Q}_n \\ Q_n & z^{-1} \end{bmatrix},$$

and the resulting "transfer matrices"

$$T_n(z) = \begin{bmatrix} \xi^{\frac{1}{2}} & 0 \\ 0 & \bar{\xi}^{\frac{1}{2}} \end{bmatrix} U_N(z) \ldots U_n(z).$$

The eigenvalues in the spectral transform are the roots of the trace of T_1 as a function of z. We denote

$$J(z) = \operatorname{tr} T_1(z).$$

Let $F(z)$ denote the upper left entry of $T_1(z)$. The coefficients in the partial-fraction decomposition of F/J are the norming constants in the spectral transform. One shows that

$$J(z) = \xi^{\frac{1}{2}} z^{-N} \prod_{k=1}^{N} (z^2 - z_k^2), \qquad \text{(eigenvalues } z_k)$$

$$\frac{F(z)}{J(z)} = z^2 \sum_{k=1}^{N} \frac{W_k}{(z^2 - z_k^2)}. \qquad \text{(norming constants } W_k)$$

The following Proposition lists a number of facts about the spectral problem. We use the notation $\hat{f}(z) := \overline{f(\bar{z}^{-1})}$.

Proposition 1 Facts on the spectral problem

1. *On the transfer matrices:*

   (a) *$T_n(z)$ has the form* $\begin{bmatrix} F_n(z) & z\hat{F}_n(z) \\ z^{-1}F_n(z) & \hat{F}_n(z) \end{bmatrix}$*, and* $F_{n-1}(z) = zF_n(z) + zQ_{n-1}\hat{F}_n(z)$.

   (b) *$F_n(z)$ is a Laurent polynomial in z that is either even or odd and whose first and last terms are $\xi^{\frac{1}{2}} z^{N-n+1}$ and $\bar{\xi}^{\frac{1}{2}} Q_n z^{-(N-n+1)+2}$.*

   (c) $J(z) = \xi^{\frac{1}{2}} z^{-N} \prod_{k=1}^{N} (z^2 - z_k^2)$.

   (d) *The roots of J are equal to the eigenvalues of the following boundary-value problem for the discrete evolution of a complex vector $\mathbf{u}_n$ in $\mathbb{C}^2$:*

$$\mathbf{u}_{n+1}(z) = U_n(z)\mathbf{u}_n(z); \quad \mathbf{u}_0(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \mathbf{u}_{N+1}(z) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (4)$$

   (e) *If $|z|=1$, then z is an eigenvalue if and only if $F_1(z) = F(z)$ is purely imaginary.*

2. *On the spectrum:*

(a) There are $2N$ eigenvalues, counting multiplicities.

(b) The eigenvalues exist in plus-minus pairs.

(c) If z is an eigenvalue, then so is $\bar{z}^{-1}$.

(d) If the values of Q_n are all real, then the eigenvalues exist in conjugate pairs.

(e) If $\{Q_n\}_{n=0}^N$ has spectrum $\{\pm z_k\}_{k=1}^N$, then, for any real constant χ, $\{Q_n e^{in\chi}\}_{n=0}^N$ has spectrum $\{\pm z_k e^{-i\chi/2}\}_{k=1}^N$.

(f) If $|Q_n| < 1$ for $n = 1, \ldots, N-1$, then $|z_k| = 1$ for $k = 1, \ldots, N$ and the eigenvalues are distinct.

3. On the norming constants:

(a) $\sum_{k=1}^{N} W_k = 1$.

(b) If $\bar{z}_k^{-1} = z_l$ then $W_l = \overline{W}_k$.

(c) If $|Q_n| < 1$ for $n = 1, \ldots, N-1$, then the norming constants W_k are real and positive and

$$W_k = \frac{G_k := |F(z_k)|}{\prod_{k' \neq k} |z_{k'}^2 - z_k^2|}. \qquad (|Q_n| < 1)$$

(d) If the Q_n are all real and $z_{k'} = \bar{z}_k$, then $W_{k'} = \bar{W}_k$. In particular, if $|Q_n| < 1$ for $n = 1, \ldots, N-1$, then $W_{k'} = W_k > 0$ and $G_{k'} = G_k$.

(e) Using the notation in (2e), if $\{Q_n\}_{n=0}^N$ has norming constants $\{W_k\}$ and $z_{k'} = z_k e^{-i\chi/2}$, then $W_{k'} = W_k$.

(f) If $|Q_n| < 1$ for $n = 1, \ldots, N-1$ and the G_k are all equal, then the Q_n have the property that $Q_{N-n} = \xi \bar{Q}_n$.

PROOF. To prove (1a) and (1b), we first calculate

$$T_N(z) = \begin{bmatrix} \xi^{\frac{1}{2}} & 0 \\ 0 & \bar{\xi}^{\frac{1}{2}} \end{bmatrix} U_N(z) = \begin{bmatrix} \xi^{\frac{1}{2}} z & \bar{\xi}^{\frac{1}{2}} \\ \xi^{\frac{1}{2}} & \bar{\xi}^{\frac{1}{2}} z^{-1} \end{bmatrix}$$

and see that the statements hold for $n = N$. Next, we write out $T_{n-1} = T_n U_{n-1}$:

$$T_{n-1}(z) = \begin{bmatrix} F_n(z) & z\hat{F}_n(z) \\ z^{-1}F_n(z) & \hat{F}_n(z) \end{bmatrix} \begin{bmatrix} z & \bar{Q}_{n-1} \\ Q_{n-1} & z^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} zF_n(z) + zQ_{n-1}\hat{F}_n(z) & \bar{Q}_{n-1}F_n(z) + \hat{F}_n(z) \\ F_n(z) + Q_{n-1}\hat{F}_n(z) & z^{-1}\bar{Q}_{n-1}F_n(z) + z^{-1}\hat{F}_n(z) \end{bmatrix},$$

and see that the statements in (1a) and (1b) hold for all relevant values of n.

Statement (1c) is evident from the form of $F_1 = F$ and the fact that $J = F + \hat{F}$.

The boundary-value problem (4) in (1d) is

$$\begin{bmatrix} z & \bar{\xi} \\ \xi & z^{-1} \end{bmatrix} U_{N-1} \cdots U_1 \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$[\,\xi \quad z^{-1}\,]\, U_{N-1} \cdots U_1 \begin{bmatrix} z \\ 1 \end{bmatrix} = 0,$$

which is equivalent to the condition

$$\mathrm{tr}\left(\begin{bmatrix} z \\ 1 \end{bmatrix} [\,\xi \quad z^{-1}\,]\, U_{N-1} \cdots U_1 \right) = 0. \tag{5}$$

Calculating

$$\begin{bmatrix} z \\ 1 \end{bmatrix} [\,\xi \quad z^{-1}\,] = \xi^{\frac{1}{2}} \begin{bmatrix} \xi^{\frac{1}{2}} & 0 \\ 0 & \bar{\xi}^{\frac{1}{2}} \end{bmatrix} U_N$$

shows that the left side of (5) is $\xi^{\frac{1}{2}} J(z)$.

To prove (1e), suppose that $z^{-1} = \bar{z}$ and observe that $J(z) = F_1(z) + \hat{F}_1(z) = F_1(z) + \overline{F_1(z)}$, which is equal to 0 if and only if $Re\, F(z) = 0$.

Statements (2a) and (2b) follow from the form of J in (1c). (2c) holds because $\hat{J} = J$, and (2d) holds because the coefficients of J are real whenever the Q_n are real.

To prove (2e) and (3e), let $F^{(1)}(z)$ be the upper left entry of the transfer matrix for the data $\{Q_n\}$, and let $F^{(2)}(z)$ be the corresponding function for the data $\{Q_n e^{in\chi}\}$. Defining

$$A_n = \begin{bmatrix} e^{i(\frac{n}{2}+\frac{1}{4})\chi} & 0 \\ 0 & e^{-i(\frac{n}{2}+\frac{1}{4})\chi} \end{bmatrix},$$

one computes

$$A_n \begin{bmatrix} z & \bar{Q}_n e^{-in\chi} \\ Q_n e^{in\chi} & z^{-1} \end{bmatrix} A_{n-1}^{-1} = \begin{bmatrix} z e^{i\frac{\chi}{2}} & \bar{Q}_n \\ Q_n & z^{-1} e^{-i\frac{\chi}{2}} \end{bmatrix}.$$

The two transfer matrices are then related as follows:

$$\begin{bmatrix} F^{(2)}(z) & z\hat{F}^{(2)}(z) \\ z^{-1} F^{(2)}(z) & \hat{F}^{(2)}(z) \end{bmatrix} = \begin{bmatrix} \xi^{\frac{1}{2}} e^{i\frac{N}{2}\chi} & 0 \\ 0 & \bar{\xi}^{\frac{1}{2}} e^{-i\frac{N}{2}\chi} \end{bmatrix} \prod_{n=1}^{N} \begin{bmatrix} z & \bar{Q}_n e^{-in\chi} \\ Q_n e^{in\chi} & z^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\frac{N}{2}\chi} & 0 \\ 0 & e^{-i\frac{N}{2}\chi} \end{bmatrix} A_N^{-1} \begin{bmatrix} \xi^{\frac{1}{2}} & 0 \\ 0 & \bar{\xi}^{\frac{1}{2}} \end{bmatrix} \left(\prod_{n=1}^{N} \begin{bmatrix} z e^{i\frac{\chi}{2}} & \bar{Q}_n \\ Q_n & z^{-1} e^{-i\frac{\chi}{2}} \end{bmatrix} \right) A_0$$

$$= \begin{bmatrix} e^{-i\frac{\chi}{4}} & 0 \\ 0 & e^{i\frac{\chi}{4}} \end{bmatrix} \begin{bmatrix} F^{(1)}(w) & w\hat{F}^{(1)}(w) \\ w^{-1} F^{(1)}(w) & \hat{F}^{(1)}(w) \end{bmatrix} \begin{bmatrix} e^{i\frac{\chi}{4}} & 0 \\ 0 & e^{-i\frac{\chi}{4}} \end{bmatrix}$$

$$= \begin{bmatrix} F^{(1)}(w) & z\hat{F}^{(1)}(w) \\ z^{-1}F^{(1)}(w) & \hat{F}^{(1)}(w) \end{bmatrix},$$

in which $w = ze^{i\frac{\chi}{2}}$. Thus, $F^{(2)}(z) = F^{(1)}(ze^{i\frac{\chi}{2}})$. (2e) and (3e) follow from this fact.

(2f) is proved in the Appendix to [S].

(3a) is evident from the form of F and J and the definition of W_k.

(3b) is a result of the following calculations:

$$\begin{aligned}
\frac{F(z)}{J(z)} &= z^2 \sum_{k=1}^{N} \frac{W_k}{z^2 - z_k^2} = z^2 \sum_{k=1}^{N} \frac{C_k}{z^2 - \bar{z}_k^{-2}}, \\
\frac{\hat{F}(z)}{J(z)} = \left(\frac{F(z)}{J(z)}\right)^{\wedge} &= z^{-2} \sum_{k=1}^{N} \frac{\bar{C}_k}{z^{-2} - z_k^2} = \sum_{k=1}^{N} \frac{-z_k^{-2}\bar{C}_k}{z^2 - z_k^{-2}}, \\
1 = \frac{F(z)}{J(z)} + \frac{\hat{F}(z)}{J(z)} &= \sum_{k=1}^{N} \frac{z^2 H_k - z_k^{-2}\bar{C}_k}{z^2 - z_k^{-2}}; \\
\implies & \quad z_k^{-2} H_k - z_k^{-2}\bar{C}_k = 0 \\
\implies & \quad H_k = \bar{C}_k.
\end{aligned}$$

(3c): That the W_k are real follows from (2f) and (3b). That they are positive is nontrivial and is proved in the Appendix to [S]. Since they are positive, the relation between W_k and G_k follows from the definition of W_k.

(3d): Because of (2d), one can write

$$\frac{F(z)}{J(z)} = z^2 \sum_{k=1}^{N} \frac{W_k}{z^2 - z_k^2} = z^2 \sum_{k=1}^{N} \frac{D_k}{z^2 - \bar{z}_k^2}.$$

Since the coefficients of F and J are real,

$$\frac{\overline{F(z)}}{\overline{J(z)}} = \bar{z}^2 \sum_{k=1}^{N} \frac{W_k}{\bar{z}^2 - z_k^2}.$$

Conjugating once more,

$$\frac{F(z)}{J(z)} = z^2 \sum_{k=1}^{N} \frac{\bar{W}_k}{z^2 - \bar{z}_k^2}.$$

One concludes that $D_k = \bar{W}_k$.

(3f): The formulas for reconstructing the values of Q_n from the spectral data are as follows (see [V] or [S]):

$$Q_n = (-1)^n \frac{\tilde{\Delta}_n}{\Delta_n}, \quad n = 1, \ldots, N-1,$$

$$1 - |Q_n|^2 = \frac{\Delta_{n-1}\Delta_{n+1}}{\Delta_n^2}, \quad n = 1, \ldots, N-1,$$

in which, in the case that $|Q_n| < 1$ for $n = 1, \ldots, N-1$,

$$\Delta_n = \sum_{s \in S_n^N} \prod_{k \in s} G_k \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1},$$

$$\tilde{\Delta}_n = \sum_{s \in S_n^N} \prod_{k \in s} z_k^{-2} G_k \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1},$$

where S_n^N denotes the set of all order-n subsets of the set of integers $\{1, \ldots, N\}$. If $G_k = G$ for $k = 1, \ldots, N$, then the positive functions Δ_n can be written as follows:

$$\Delta_n = G^n \sum_{s \in S_n^N} \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1},$$

$$\Delta_{N-n} = G^{N-n} \sum_{s \in S_n^N} \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1}.$$

To rewrite the complex functions $\tilde{\Delta}_n$, we use the fact that $\xi = (-1)^N \prod_{k=1}^N z_k^{-2}$, which follows from the fact that the coefficient of the lowest-order term of $J(z)$ is equal to ξ^{-1}:

$$\tilde{\Delta}_n = G^n \sum_{s \in S_n^N} \prod_{l \in s} z_l^{-2} \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1},$$

$$\tilde{\Delta}_{N-n} = G^{N-n} \sum_{s \in S_n^N} \prod_{l \notin s} z_l^{-2} \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1}$$

$$= (-1)^N G^{N-n} \xi \sum_{s \in S_n^N} \overline{\prod_{l \in s} z_l^{-2}} \prod_{j \in s,\, i \notin s} |z_i^2 - z_j^2|^{-1}.$$

Using these expressions in the formula (for reconstructing $\{Q_n\}$, the proposed properties of these data are verified. $\triangle$

3 The Asymptotics of the Spectral Transform

We consider the eigenvalue condition in the semiclassical limit. The dependence on the spectral parameter will usually be suppressed. Let continuous functions q and ϕ be given such that q has two continuous derivatives and ϕ has three continuous derivatives and

$$\begin{aligned}
&q : [0,1] \to [0,1], \\
&0 \le q(x) < 1 \quad \text{for} \quad x \in (0,1); \\
&\phi : [0,1] \to \mathbb{R}, \\
&\phi(0) = 0;
\end{aligned}$$

and put $Q_n = q(n\epsilon)\exp(\frac{i}{\epsilon}\phi(n\epsilon))$. The eigenvalue condition is (4), in which

$$U_n = \begin{bmatrix} z & q(n\epsilon)\exp\left(-\frac{i}{\epsilon}\phi(n\epsilon)\right) \\ q(n\epsilon)\exp\left(\frac{i}{\epsilon}\phi(n\epsilon)\right) & z^{-1} \end{bmatrix}.$$

To make the problem amenable to WKB analysis, we can remove the large exponent from U_n by means of the change of coordinates

$$\mathbf{u}_n = \begin{bmatrix} u_n^1 \\ u_n^2 \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\phi(n\epsilon)}{2\epsilon}} \check{u}_n^1 \\ e^{i\frac{\phi(n\epsilon)}{2\epsilon}} \check{u}_n^2 \end{bmatrix};$$

then the vectors $\check{\mathbf{u}}_n = \begin{bmatrix} \check{u}_n^1 & \check{u}_n^2 \end{bmatrix}^t$ satisfy

$$\begin{bmatrix} \check{u}_{n+1}^1 \\ \check{u}_{n+1}^2 \end{bmatrix} = \check{U}_n \begin{bmatrix} \check{u}_n^1 \\ \check{u}_n^2 \end{bmatrix}, \tag{6}$$

in which

$$\check{U}_n = \begin{bmatrix} z e^{i\frac{\psi_n}{2}} & q_n e^{i\frac{\psi_n}{2}} \\ q_n e^{-i\frac{\psi_n}{2}} & z^{-1} e^{-i\frac{\psi_n}{2}} \end{bmatrix} \tag{7}$$

and $\psi_n = \frac{\phi(n\epsilon+\epsilon)-\phi(n\epsilon)}{\epsilon}$ and $q_n = q(n\epsilon)$. Let $\lambda_n^\pm$ be the eigenvalues of $\check{U}_n$ and $\mathbf{p}_n^\pm$ corresponding eigenvectors, and set $\theta_n = \arg\frac{\lambda_n^+}{\lambda_n^-}$. Then the following expansions are valid:

$$\begin{aligned}
\check{U}_n &= \underline{\check{U}}^\epsilon(n\epsilon) & \text{where} \quad \underline{\check{U}}^\epsilon(x) &= \underline{\check{U}}(x) + \epsilon\underline{\check{U}}_1(x) + \mathcal{O}(\epsilon^2), \\
\lambda_n^\pm &= \underline{\lambda}^{\pm\epsilon}(n\epsilon) & \text{where} \quad \underline{\lambda}^{\pm\epsilon}(x) &= \underline{\lambda}^\pm(x) + \epsilon\underline{\lambda}_1^\pm(x) + \mathcal{O}(\epsilon^2), \\
\theta_n &= \underline{\theta}^\epsilon(n\epsilon) & \text{where} \quad \underline{\theta}^\epsilon(x) &= \underline{\theta}(x) + \epsilon\underline{\theta}_1(x) + \mathcal{O}(\epsilon^2), \\
\mathbf{p}_n^\pm &= \underline{\mathbf{p}}^{\pm\epsilon}(n\epsilon) & \text{where} \quad \underline{\mathbf{p}}^{\pm\epsilon}(x) &= \underline{\mathbf{p}}^\pm(x) + \epsilon\underline{\mathbf{p}}_1^\pm + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{8}$$

So the underscore signifies functions of the continuous variable x. $\underline{\lambda}^{\pm\epsilon}(x)$ and $\underline{\mathbf{p}}^{\pm\epsilon}(x)$ are the eigenvalues and eigenvectors of $\underline{\check{U}}^\epsilon(x)$, and $\underline{\theta}^\epsilon(x) = \arg\frac{\underline{\lambda}^{+\epsilon}(x)}{\underline{\lambda}^{-\epsilon}(x)}$. One sees that

$$\underline{\check{U}}(x) = \begin{bmatrix} z e^{i\frac{\phi'(x)}{2}} & q(x) e^{i\frac{\phi'(x)}{2}} \\ q(x) e^{-i\frac{\phi'(x)}{2}} & z^{-1} e^{-i\frac{\phi'(x)}{2}} \end{bmatrix}, \tag{9}$$

and, for unitary spectral values $z = e^{i\eta}$,

$$\lambda_n^\pm = \cos\left(\eta + \frac{\psi_n}{2}\right) \pm \sqrt{q_n^2 - \sin^2\left(\eta + \frac{\psi_n}{2}\right)}\,, \tag{10}$$

$$\underline{\lambda}^\pm(x, e^{i\eta}) = \cos\left(\eta + \frac{\phi'(x)}{2}\right) \pm \sqrt{q(x)^2 - \sin^2\left(\eta + \frac{\phi'(x)}{2}\right)}\,. \tag{11}$$

3.1 WKB anaylsis

We begin the asymptotic analysis with a naive WKB approach to determine the leading-order behavior of the vector $\left[\begin{array}{cc} \check{u}_n^1 & \check{u}_n^2 \end{array}\right]^t$. We consider the approximate problem for vectors $\mathbf{v}_n$ given by

$$\mathbf{v}_{n+1} = \underline{\check{U}}(n\epsilon)\mathbf{v}_n$$

and perform leading-order WKB analysis on the components of $\mathbf{v}_n$ with respect to the basis of eigenvectors $\underline{\mathbf{p}}^{\pm}(n\epsilon)$ using the ansatz

$$\mathbf{v}_n = \exp\left(\tfrac{1}{\epsilon}S_+(n\epsilon)\right)\underline{\mathbf{p}}^+(n\epsilon) + \exp\left(\tfrac{1}{\epsilon}S_-(n\epsilon)\right)\underline{\mathbf{p}}^-(n\epsilon) \tag{12}$$

in which S_+ and S_- are functions of x that are to be determined. We write $\mathbf{v}_{n+1}$ in two ways: On one hand,

$$\begin{aligned}
\mathbf{v}_{n+1} &= \exp\left(\tfrac{1}{\epsilon}\left(S_+(n\epsilon)+\epsilon S_+'(n\epsilon)\right)+\mathcal{O}(\epsilon)\right)\underline{\mathbf{p}}^+(n\epsilon+\epsilon) + \\
&+ \exp\left(\tfrac{1}{\epsilon}\left(S_-(n\epsilon)+\epsilon S_-'(n\epsilon)\right)+\mathcal{O}(\epsilon)\right)\underline{\mathbf{p}}^-(n\epsilon+\epsilon) \\
&= \exp\left(\tfrac{1}{\epsilon}S_+(n\epsilon)\right)\exp\left(S_+'(n\epsilon)\right)(1+\mathcal{O}(\epsilon))\,\underline{\mathbf{p}}^+(n\epsilon+\epsilon) + \\
&+ \exp\left(\tfrac{1}{\epsilon}S_-(n\epsilon)\right)\exp\left(S_-'(n\epsilon)\right)(1+\mathcal{O}(\epsilon))\,\underline{\mathbf{p}}^-(n\epsilon+\epsilon).
\end{aligned}$$

On the other hand, from the evolution of $\mathbf{v}_n$,

$$\begin{aligned}
\mathbf{v}_{n+1} &= \lambda^+(n\epsilon)\exp\left(\tfrac{1}{\epsilon}S_+(n\epsilon)\right)\underline{\mathbf{p}}^+(n\epsilon) + \\
&+ \lambda^-(n\epsilon)\exp\left(\tfrac{1}{\epsilon}S_-(n\epsilon)\right)\underline{\mathbf{p}}^-(n\epsilon) \\
&= \exp\left(\tfrac{1}{\epsilon}S_+(n\epsilon)\right)\lambda^+(n\epsilon)\left(\underline{\mathbf{p}}^+(n\epsilon+\epsilon)+\vec{\mathcal{O}}(\epsilon)\right) + \\
&+ \exp\left(\tfrac{1}{\epsilon}S_-(n\epsilon)\right)\lambda^-(n\epsilon)\left(\underline{\mathbf{p}}^-(n\epsilon+\epsilon)+\vec{\mathcal{O}}(\epsilon)\right).
\end{aligned} \tag{13}$$

Comparing the two representations of $\mathbf{v}_{n+1}$, one obtains the formal result

$$S_\pm'(x) = \log(\underline{\lambda}^\pm(x)), \quad \text{or} \quad S_\pm(x) = \int^x \log(\underline{\lambda}^\pm(y))\,dy.$$

Let us consider the implications of this result in the case that $0 \le q(x) < 1$ for $0 < x < 1$. By Statement (2c) of Proposition 1, this condition constrains the spectrum to the unit circle. Thus, let us put $z = e^{i\eta}$. The ratio of the WKB components of $\mathbf{v}_n$ with respect to an eigenvector basis, which will be relevant in proposing the spectral density, is

$$\mathcal{R}(x,\eta) := \frac{\exp\left(\frac{1}{\epsilon}S_+(x,e^{i\eta})\right)}{\exp\left(\frac{1}{\epsilon}S_-(x,e^{i\eta})\right)} = \exp\left[\frac{1}{\epsilon}\int^x \log\frac{\lambda^+(y,e^{i\eta})}{\lambda^-(y,e^{i\eta})}dy\right].$$

We make some observations about the values of $\underline{\lambda}^\pm$ and this ratio. $\underline{\lambda}^\pm$ are either both real with the same sign or complex conjugates of each other. For a given value of η, x-regions with these different properties are separated from each other by "turning points" x_* for which $q^2(x_*) = \sin^2\left(\eta + \frac{\phi'(x_*)}{2}\right)$. In an x-interval in which $\underline{\lambda}^\pm(x)$ are both real, we find that $\mathcal{R}(x,\eta)$ is a real-valued function of x (plus a complex constant), and in an x-interval in which $\underline{\lambda}^\pm(x)$ are complex conjugate, $\mathcal{R}(x,\eta)$ is a unitary complex function of x (plus complex a constant). Thus the interval $[0,1]$ is divided into "exponential" and "oscillatory" intervals separated by turning points, which depend on the value of η. For generic values of η, the x-values 0 and 1 are endpoints of exponential regions. In summary,

$$q^2(x_*) = \sin^2\left(\eta + \frac{\phi'(x_*)}{2}\right), \qquad \text{(turning point)}$$

$$q^2(x) > \sin^2\left(\eta + \frac{\phi'(x)}{2}\right), \qquad \text{(exponential region)}$$

$$q^2(x) < \sin^2\left(\eta + \frac{\phi'(x)}{2}\right). \qquad \text{(oscillatory region)}$$

Observations

1. The leading-order result is valid in any x-region in which the eigenvectors $\underline{\mathbf{p}}^{\pm}(x)$ are constant, in particular, where $q(x)$ and $\phi'(x)$ are constant.

2. This analysis does not make sense through a region containing a turning point. $\underline{\check{U}}(x)$ is not diagonalizable at a turning point, and the functions $\underline{\lambda}^{\pm}$ cannot be cannonically connected through such a point.

3. Consider equation (13) for the evolution of the WKB ansatz in an exponential interval in which, say, $\underline{\lambda}^+(x) > \underline{\lambda}^-(x) > 0$. Typically, the vector quantity $\exp(\frac{1}{\epsilon}S_+(n\epsilon))\underline{\lambda}^+(n\epsilon)\vec{\mathcal{O}}(\epsilon)$ is much larger than $\exp(\frac{1}{\epsilon}S_-(n\epsilon))\underline{\lambda}^-(n\epsilon)\underline{\mathbf{p}}^-(n\epsilon+\epsilon)$ and contains a component of $\underline{\mathbf{p}}^-(n\epsilon+\epsilon)$. This indicates that the WKB result for the $\underline{\lambda}^-$-component of $\mathbf{v}_n$ is invalid.

4. The difficulty of higher-order asymptotic analysis is in the dependence of the matrices $\check{U}_n$ in the exact spectral problem on adjacent sites. Their expansions in ϵ are complicated and will be examined rigorously in Section 5.

5. The ansatz used by Bronski [B] for the the non-self-adjoint Zakharov-Shabat problem $i\epsilon\mathbf{v}_x = M(x,\lambda)\mathbf{v}$ in the semiclassical limit is $\mathbf{v} = e^{-i\phi(x,\lambda)/\epsilon}(\mathbf{v}^0 + \epsilon\mathbf{v}^1 + \ldots)$. Its formal validity implies that $\mathbf{v}^0$ is an eigenvector of M. In contrast to this, the ansatz chosen in our case leads to an oscillatory ratio of eigenvector coordinates for the solution vector in an oscillatory region, whereas in an exponential region, it indicates that the solution is asymptotic to the eigenvector with the larger eigenvalue.

Higher-order WKB analysis

Let us use the refined ansatz

$$\check{\mathbf{u}}_n = \exp\left(\tfrac{1}{\epsilon}S^+(n\epsilon) + S_0^+(n\epsilon) + \mathcal{O}(\epsilon)\right)\mathbf{p}_n^+ + \exp\left(\tfrac{1}{\epsilon}S^-(n\epsilon) + S_0^-(n\epsilon) + \mathcal{O}(\epsilon)\right)\mathbf{p}_n^-$$

and include the order-ϵ change of basis:

$$\mathbf{p}_n^+ = (1 + \epsilon\underline{r}^{11}(n\epsilon) + \mathcal{O}(\epsilon^2))\mathbf{p}_{n+1}^+ + (\epsilon\underline{r}^{21}(n\epsilon) + \mathcal{O}(\epsilon^2))\mathbf{p}_{n+1}^-,$$
$$\mathbf{p}_n^- = (1 + \epsilon\underline{r}^{22}(n\epsilon) + \mathcal{O}(\epsilon^2))\mathbf{p}_{n+1}^- + (\epsilon\underline{r}^{12}(n\epsilon) + \mathcal{O}(\epsilon^2))\mathbf{p}_{n+1}^+,$$

for some functions $\underline{r}^{ij}$ of x. Then, letting $\tilde{u}_n$ represent $\check{\mathbf{u}}_n$ in eigenvector components, one has, for some functions $a^{\pm}$ and $b^{\pm}$ of x,

$$\tilde{u}_{n+1} = \begin{bmatrix} \underline{\lambda}^+(n\epsilon) + \epsilon a^+(n\epsilon) & \epsilon b^-(n\epsilon) \\ \epsilon b^+(n\epsilon) & \underline{\lambda}^-(n\epsilon) + \epsilon a^-(n\epsilon) \end{bmatrix} \tilde{u}_n + \mathcal{O}(\epsilon^2).$$

Inserting the ansatz yields the two dominant balance equations

$$\begin{aligned} &\underline{\lambda}^+(n\epsilon) - e^{S^{+\prime}(n\epsilon)} + \epsilon\left(a^+(n\epsilon) - e^{S^{+\prime}(n\epsilon)}\left(S_0^{+\prime}(n\epsilon) + S^{+\prime\prime}(n\epsilon)\right)\right) + \mathcal{O}(\epsilon^2) \\ &\quad + \left(\epsilon b^-(n\epsilon) + \mathcal{O}(\epsilon^2)\right)\exp\left[\tfrac{1}{\epsilon}\left(S^-(n\epsilon) - S^+(n\epsilon)\right) + \left(S_0^-(n\epsilon) - S_0^+(n\epsilon)\right) + \mathcal{O}(\epsilon)\right] = 0, \end{aligned}$$

$$\begin{aligned} &\underline{\lambda}^-(n\epsilon) - e^{S^{-\prime}(n\epsilon)} + \epsilon\left(a^-(n\epsilon) - e^{S^{-\prime}(n\epsilon)}\left(S_0^{-\prime}(n\epsilon) + S^{-\prime\prime}(n\epsilon)\right)\right) + \mathcal{O}(\epsilon^2) \\ &\quad + \left(\epsilon b^+(n\epsilon) + \mathcal{O}(\epsilon^2)\right)\exp\left[\tfrac{1}{\epsilon}\left(S^+(n\epsilon) - S^-(n\epsilon)\right) + \left(S_0^+(n\epsilon) - S_0^-(n\epsilon)\right) + \mathcal{O}(\epsilon)\right] = 0. \end{aligned}$$

One observes that, if, for example, $|\underline{\lambda}^+| > |\underline{\lambda}^-|$, then, assuming $|S^+| > |S^-|$, one can solve for S^+, S_0^+, ..., but not for S^-, S_0^-, ..., so the WKB ansatz is formally inconsistent in the second component. If $|\underline{\lambda}^+| = |\underline{\lambda}^-|$, then S^+ and S^- can be found as in the crude analysis presented above, but the ansatz is not consistent to higher orders.

3.2 The Spectral Density

We now use the formal WKB result to propose an asymptotic distribution of eigenvalues. Letting $\left[\, c_n^1 \;\; c_n^2 \,\right]$ represent the vector $\check{\mathbf{u}}_n$ with respect to the basis $\{\underline{\mathbf{p}}^{\pm}\}$, the boundary-value problem (4) sets conditions on the quantities $\arg\left(\frac{c_n^1}{c_n^2}\right)$ at $n = 1$ and $n = N$. Since we know that the eigenvalues are unitary, the problem is to specify those values of z, as z traverses the unit circle, for which the total increment of $\arg\left(\frac{c_n^1}{c_n^2}\right)$ is equal to $\arg\left(\frac{c_{N-1}^1}{c_{N-1}^2}\right) - \arg\left(\frac{c_1^1}{c_1^2}\right) + 2\pi k$ for some integer k. We already have the leading order behavior of $\arg\left(\frac{c_n^1}{c_n^2}\right)$: it is constant in an exponential region and equal to $\frac{1}{\epsilon}\int^{n\epsilon} \arg\frac{\underline{\lambda}^+(x)}{\underline{\lambda}^-(x)}\,dx$ in an oscillatory region. Thus the total increment from $n = 1$ to $n = N$ (or $x = 0$ to $x = 1$), to leading order, is $\frac{1}{\epsilon}\int_0^1 \arg\frac{\underline{\lambda}^+(x)}{\underline{\lambda}^-(x)}\,dx$ where the integrand is zero when x is in an exponential region. The asymptotic condition for eigenvalues $z_k = e^{i\eta_k}$ is then

$$\frac{1}{\epsilon}\int_0^1 \arg\frac{\underline{\lambda}^+(x, z_k)}{\underline{\lambda}^-(x, z_k)}dx \sim 2\pi k \quad (\epsilon \to 0).$$

Using the expression (11) for the eigenvalues $\underline{\lambda}^{\pm}(x; e^{i\eta})$, one computes $\arg\frac{\underline{\lambda}^+(x;e^{i\eta})}{\underline{\lambda}^-(x;e^{i\eta})}$ and finds that this condition becomes

$$\Psi(\eta_k) \sim \epsilon k \quad (\epsilon \to 0),$$

where the asymptotic spectral distribution Ψ is defined by

$$\Psi(\eta) = \frac{1}{\pi}\int_0^1 \arctan\, Re\frac{\sqrt{\sin^2\left(\eta + \frac{\phi'(x)}{2}\right) - q(x)^2}}{\cos\left(\eta + \frac{\phi'(x)}{2}\right)}\,dx.$$

To determine the limiting density of eigenvalues, we see that the number of eigenvalues in a η-interval on which Ψ is monotonic is given asymptotically by $1/\epsilon$ times the absolute value of the increment of Ψ over that interval. One calculates that

$$\Psi'(\eta) = \frac{1}{\pi}\int_0^1 Re\frac{\sin\left(\eta+\frac{\phi'(x)}{2}\right)}{\sqrt{\sin^2\left(\eta+\frac{\phi'(x)}{2}\right)-q(x)^2}}\,dx,$$

from which we then obtain the density

$$\rho(\eta) := \frac{1}{\pi}\left|\int_0^1 Re\frac{\sin\left(\eta+\frac{\phi'(x)}{2}\right)}{\sqrt{\sin^2\left(\eta+\frac{\phi'(x)}{2}\right)-q(x)^2}}\,dx\right|, \qquad 0\le\eta\le 2\pi. \tag{14}$$

This means that, for any subinterval $[\eta_1,\eta_2]$ of $[0,2\pi]$,

$$\#[\eta_1,\eta_2]\sim\frac{1}{\epsilon}\int_{\eta_1}^{\eta_2}\rho(\eta)\,d\eta \qquad (\epsilon\to 0),$$

where "#" indicates the number of eigenvalues in the given interval.

One can confirm that the asymptotic analogs of the spectral properties in part 2 of Proposition 1 do hold for this proposed density:

Asymptotic analogs of Proposition 1, part 2

a. The number of eigenvalues should be asymptotically equal to $2/\epsilon$. This is the statement that

$$\int_0^{2\pi}\rho(\eta)\,d\eta = 2.$$

When $\phi'(x)$ is taken to be constant, this is easily verified. In this case, $\Psi(\eta)$ is increasing (resp. decreasing) when $\sin(\eta+\frac{\phi'}{2})$ is positive (resp. negative), and one finds that its total variation is 2. A graph of Ψ for which q is a parabola and $\phi=0$ is shown in Figure 3.

b. The asymptotic analog of the plus-minus parity is that $\rho(\eta)=\rho(\eta+\pi)$.

d. The Q_n being real corresponds to $\phi(x)\equiv 0$. In this case, $\rho(-\eta)=\rho(\eta)$, which is the analog of conjugate parity of eigenvalues. Because of (b), there is then a four-fold spectral symmetry.

e. Multiplying the Q_n all by $e^{in\chi}$ is asymptotically analogous to adding the constant χ to $\phi'(x)$. This does indeed shift the proposed density function by $-\chi/2$, as it should.

The case in which q has a unique local minimum $q_{\rm min}$ and ϕ' is constant provides a simple illustration. For those values of η for which $|\sin(\eta+\frac{\phi'}{2})|>q_{\rm min}$, such data uniquely determine two turning points $x_-(\eta)$ and $x_+(\eta)$, and if $|\sin(\eta+\frac{\phi'}{2})|<q_{\rm min}$, then $\rho(\eta)=0$. This is illustrated in Figure 1. Thus, when $\phi'=0$, the spectrum is asymptotically confined to those values of η for which $|\sin(\eta)|>q_{\rm min}$ and ρ has the additional symmetry $\rho(\pi/2-\eta)=\rho(\eta)$. The spectral intervals are illustrated in Figure 2.

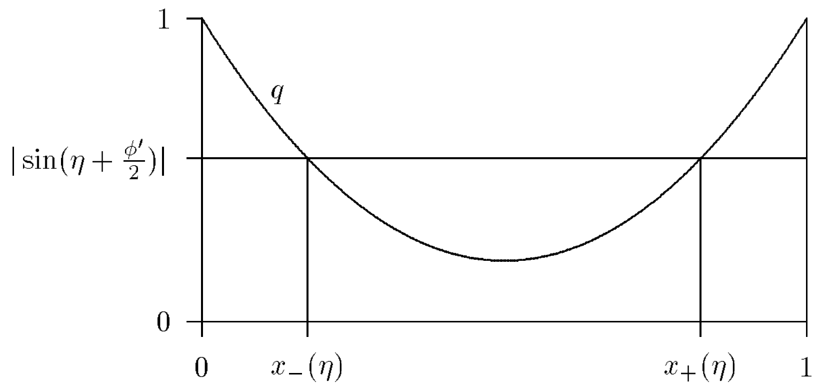


Figure 1: *Determination of* $x_-(\eta)$ *and* $x_+(\eta)$ *when* q *has a unique local minimum and* ϕ' *is constant.*

One can obtain the asymptotic density ρ_2 of squared eigenvalues $e^{i\mu_k} = z_k^2$ by putting $\rho_2(\mu) = \frac{1}{2}\rho(\mu/2)$ so that $\int_0^{2\pi} \rho_2(\mu)d\mu = 1$:

$$\rho_2(\mu) = \frac{1}{2\pi}\left|\int_0^1 Re\frac{\sin\left(\frac{\mu+\phi'(x)}{2}\right)}{\sqrt{\sin^2\left(\frac{\mu+\phi'(x)}{2}\right) - q(x)^2}}dx\right|, \quad 0 \le \mu \le 2\pi. \tag{15}$$

3.3 The Asymptotic Norming Exponent

It can be shown that the norming constants have the following asymptotic behavior in the semi-classical limit:

$$\lim_{\epsilon\to 0} \epsilon \log G_{k_\epsilon} = \mathcal{I}(\eta_*) \qquad \text{if} \qquad z_{k_\epsilon} \to e^{i\eta_*} \quad \text{as} \quad \epsilon \to 0,$$

where $\mathcal{I}$ is a function defined on the support of the asymptotic spectral density and is determined by q and ϕ. In the case that these data give rise to exactly two turning points for values of η in the support of the asymptotic density, a candidate for the asymptotic norming exponent $\mathcal{E}(\mu) = \mathcal{I}(\mu/2)$ corresponding to the squared eigenvalues has been proposed in [S]. Putting $\mu = 2\eta$, one shows that the turning-point condition

$$q(x)^2 - \sin^2\left(\eta + \frac{\phi'(x)}{2}\right) = 0$$

is equivalent to

$$\mu = \alpha(x) + 2k\pi \quad \text{or} \quad \mu = \beta(x) + 2k\pi \quad \text{for some } k$$

where

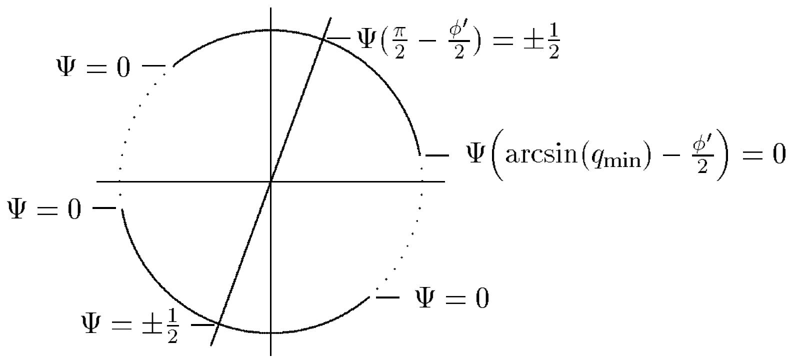


Figure 2: *The asymptotic spectral intervals when q has a unique local minimum and ϕ' is constant.*

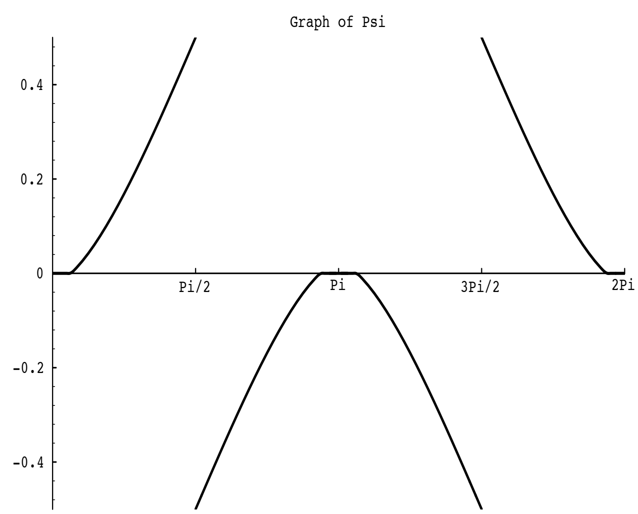


Figure 3: *The distribution Ψ of eigenvalues.*

$$\begin{aligned}\alpha(x) &= 2\arcsin(q(x)) - \phi'(x),\\ \beta(x) &= 2\pi - 2\arcsin(q(x)) - \phi'(x).\end{aligned} \tag{16}$$

One shows that $\beta(x) \geq \alpha(x)$, with equality only at 0 and 1. Graphs of these functions are shown in Examples 8 and 9 of Section 4

The proposed form of the derivative of the asymptotic norming exponent in the variable μ is

$$\frac{d\mathcal{E}}{d\mu} = \frac{1}{2}\left[\pm\int_0^{x-} \pm \int_{x+}^1\right] Re\frac{\sin\left(\frac{\mu+\phi'(x)}{2}\right)}{\sqrt{-\sin^2\left(\frac{\mu+\phi'(x)}{2}\right) + q(x)^2}}\,dx,$$

which is defined in the support of the spectral density. The $\pm$ sign is chosen as illustrated in Example 8 of Section 4. In the variable η, it has the form

$$\frac{d\mathfrak{I}}{d\eta} = \left[\pm\int_0^{x-(\eta)} \pm \int_{x+(\eta)}^1\right] Re\frac{\sin\left(\eta + \frac{\phi'(x)}{2}\right)}{\sqrt{-\sin^2\left(\eta + \frac{\phi'(x)}{2}\right) + q(x)^2}}\,dx. \tag{17}$$

One can compare with this formula the asymptotic analogs of the properties of the norming constants in part 3 of Proposition 1.

Asymptotic analogs of Proposition 1, part 3

d. The Q_n being real corresponds to $\phi(x) \equiv 0$. The symmetry of the norming constant about the angle $\pi/2$ corresponds the the antisymmetry of $d\mathfrak{I}/d\eta$, which is confirmed in the proposed formula.

e. Multiplying the Q_n all by $e^{in\chi}$ is asymptotically analogous to adding the constant χ to $\phi'(x)$, thus shifting the proposed asymptotic norming exponent by $-\chi/2$.

f. The property $Q_{N-n} = \xi\bar{Q}_n$ corresponds to the symmetry of q and ϕ' about $x = 1/2$, and one shows that the candidate for $d\mathfrak{I}/d\eta$ is zero in this case. This corresponds to the converse of item (3f).

A natural candidate for $\mathfrak{I}(\eta)$ may be derived heuristically as follows:

$$\begin{aligned}&\frac{1}{N}\log|F(e^{i\eta})| \sim \frac{1}{N}\log\prod_{n=0}^{N}|U_n(e^{i\eta})| \sim \frac{1}{N}\log\prod_{n=0}^{N}\max|\lambda_n^{\pm}(e^{i\eta})| \sim\\ &\sim \frac{1}{N}\sum_{n=0}^{N}\log\max|\lambda_n^{\pm}(e^{i\eta})| \sim \int_0^1 \log\max|\lambda^{\pm}(x, e^{i\eta})|dx \ =?\ \mathfrak{I}(\eta).\end{aligned}$$

One finds, indeed, that this integral coincides numerically with a limiting upper envelope of the functions $\frac{1}{N}\log|F(e^{i\eta})|$. However, $\frac{1}{N}\log|F(e^{i\eta})|$, from $\eta = 0$ to $\eta = \pi$, has $N-1$ spikes emanating downward from this upper envelope, and the N points $(\eta_k, |F(e^{i\eta_k})|)$ lie at various places along these spikes. This is illustrated in Example 9. Thus, $\frac{1}{N}\log G_k$ is not given by $\int_0^1 \log\max|\lambda^{\pm}(x, e^{i\eta})|dx$. Recall that $\sum_{k=1}^{N} W_k = 1$ and

$$W_k = \frac{G_k}{\sum_{k' \neq k} |z_k^2 - z_{k'}^2|}.$$

So $\frac{1}{N} \log G_k < \frac{1}{N} \sum_{k' \neq k} |z_k^2 - z_{k'}^2|$ for each k. If one calculates numerically the asymptotic form of the right-hand side,

$$\frac{1}{N} \sum_{k' \neq k_N} \log |z_{k_N}^2 - z_{k'}^2| \sim \int_0^\pi \log |e^{2i\eta} - e^{2i\eta'}| \rho(\eta') d\eta' \qquad (N \to \infty)$$

$$\text{if} \quad z_{k_N} \longrightarrow e^{i\eta} \quad \text{as} \quad N \to \infty,$$

one finds that it also coincides with the limiting upper envelope of $F(e^{i\eta})$ (even for values outside the support of ρ).

4 Numerical Results

The numerical calculations in this section compare the proposed asymptotic spectral density $\rho(\eta)$ and norming exponent $\mathcal{I}(\eta)$ defined in (14) and (17) with actual spectral data for various choices of q, ϕ, and N. The following points explain the methods and ideas.

1. Because of the π-periodicity of the density (see (2b) of Proposition 1 and item (b) on page 12), the computations and plots are restricted to an η-interval of length π.

2. The choices of q and ϕ illustrate the symmetry and shifting properties of Proposition 1, (2d) and (2e), and items (d) and (e) on page 12 and situations with various numbers of turning points.

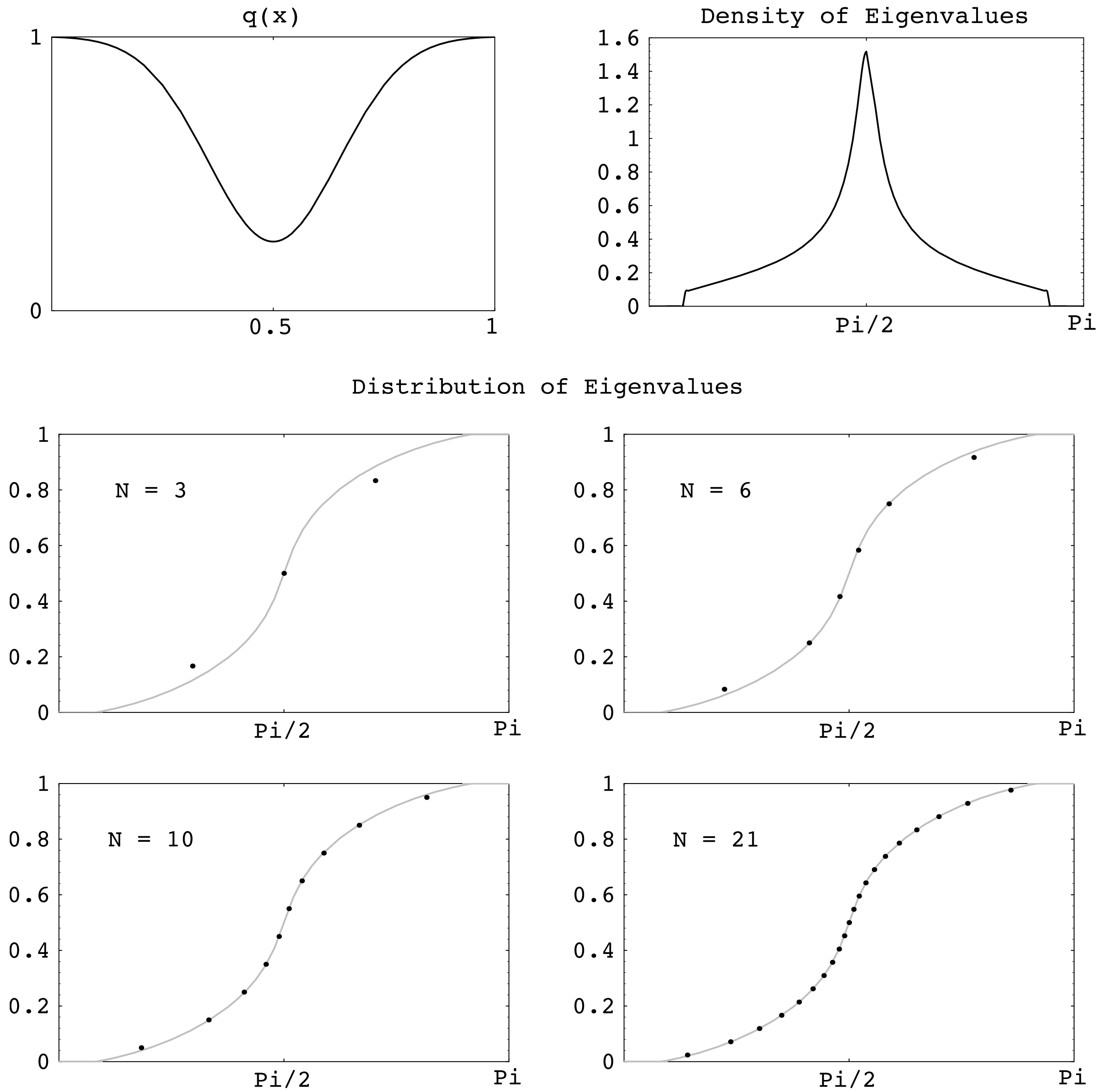
3. Computing the eigenvalues: By (1e) of Proposition 1, $e^{i\eta}$ is an eigenvalue if and only if $Re\,F(e^{i\eta}) = 0$. The roots of $Re\,F(e^{i\eta})$ as a function of η were found by the method of bisection, and values of $F(e^{i\eta})$ were computed using the recursion relation for $F_n(z)$ given in (1a) of Proposition 1.

4. The approximate densities of actual eigenvalues for $1/\epsilon = N$ were obtained by choosing a number $M \ll N$ and, for each consecutive $M+1$ eigenvalues, computing a density value equal to the fraction of eigenvalues M/N in $[0, \pi]$ divided by the range of the $M+1$ values. This density value is plotted against the mean of the $M+1$ values. (There are indeed N eigenvalues in $[0, \pi]$ by statements (2a) and (2b) of Proposition 1.)

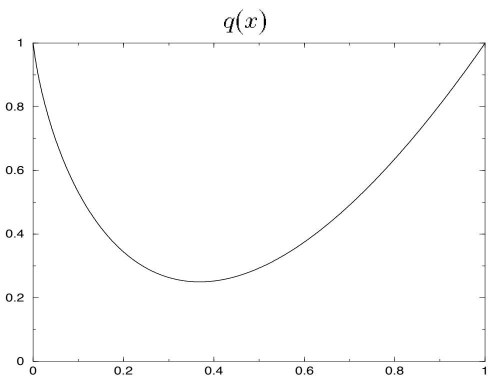
5. The approximate cumulative distrubutions of eigenvalues in Example 1 were obtained by plotting, for $1 \leq k \leq N$, $\frac{k-1/2}{N}$ against the kth eigenvalue.

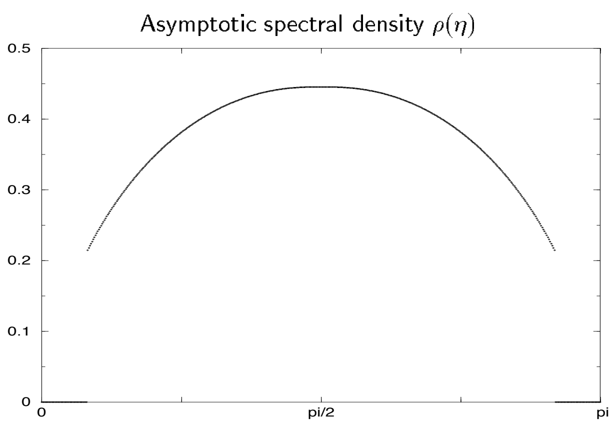
6. In any η-region for which there are two turning points, the proposed asymptotic density is unambiguously confirmed. In regions with more than two turning points, the eigenvalues were either difficult to compute or exhibited less regular behavior in their small-scale distribution. The asymptotic formula, however, still appears to be confirmed.

7. $\mathcal{I}(\eta)$, in the case that there are no more than two turning points, has been compared with actual spectral data as follows: Eigenvalues $e^{i\eta_k}$ for various choices of q, ϕ', and N were obtained, and then difference quotients of $\frac{1}{N}\log|F(e^{i\eta_k})|$ were compared with a plot of $\mathcal{I}'(\eta)$. In the final example, graphs of $\frac{1}{N}\log|F(e^{i\eta})|$, $\int_0^1 \max|\lambda^{\pm}(x, e^{i\eta})|dx$, and $\int_0^\pi |e^{2i\eta} - e^{2i\eta'}|\rho(\eta')d\eta'$ are also shown.



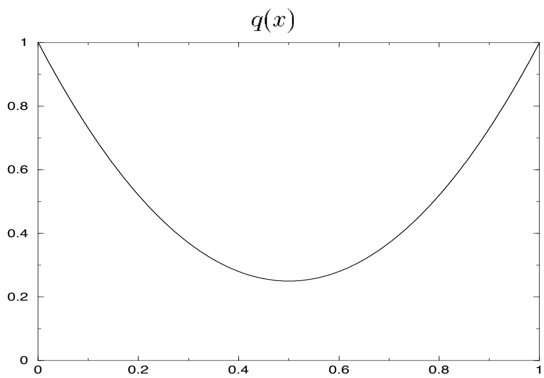
Example 1. *$\phi(x) = 0$ in this example, so the density function $\rho(\eta)$, upper right, is symmetric about $\eta = \pi/2$ (see item (d) of page 12). The grey curve in the lower four graphs is the integral of the asymptotic density. The approximate distribution of actual eigenvalues (see item 5 on page 17), represented by dots, converges rapidly to it. There are only two turning points for each value in the support of the density, which can be visualized as illustrated in Figure 1.*

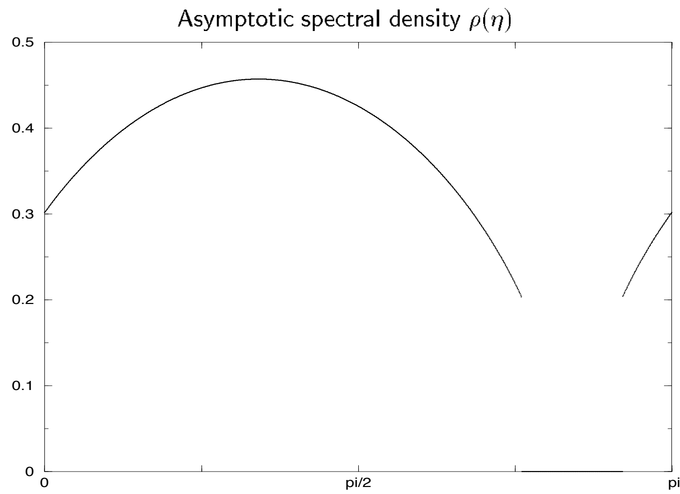


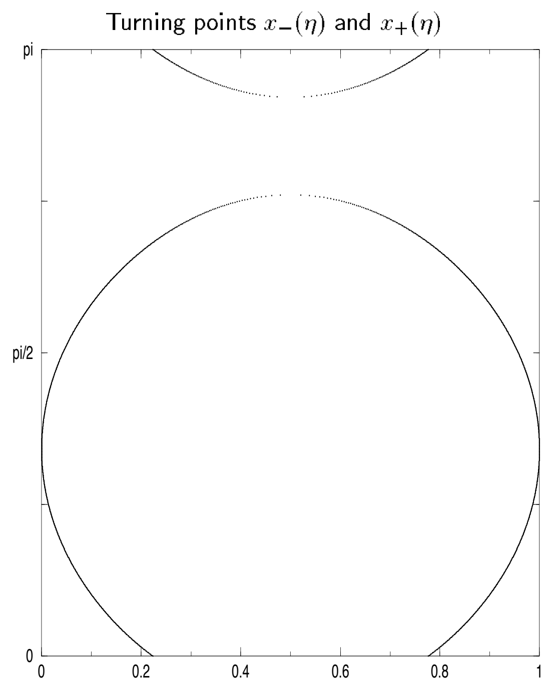


Example 2. *In this example, q has a unique minimum and $\phi(x) = 0$. The data Q_n are therefore real and the spectrum is correspondingly symmetric about $\eta = \pi/2$. The turning points can be demonstrated graphically as illustrated in Figure 1.*

This graph of the asymptotic density is the proposed formula $\rho(\eta)$. The actual eigenvalues have been calculated for $N = 1000$ and the approximate density plotted with $M = 10$ (see item 4). It is not reproduced here because it would be indistinguishable from the graph of ρ.







Example 3. *Here, q is parabolic with a minimum of 0.25 at $x = 0.5$, and $\phi(x) = x$. The asymptotic density ρ, above, is symmetric about the spectral value $\eta = \frac{\pi}{2} - \frac{1}{2}$ because of the constant value of $\phi'(x) = 1$ (see item (e) on page 12). For values of η such that*

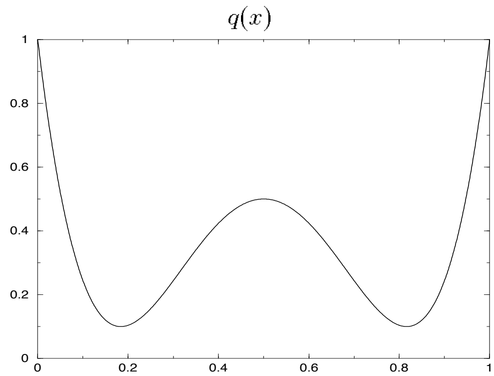
$$\operatorname{asin}(0.25) - \tfrac{1}{2} < \eta < \pi - \operatorname{asin}(0.25) - \tfrac{1}{2},$$

there is an (oscillatory) x-interval for which

$$q(x)^2 - \sin^2\left(\eta + \phi'(x)/2\right) < 0$$

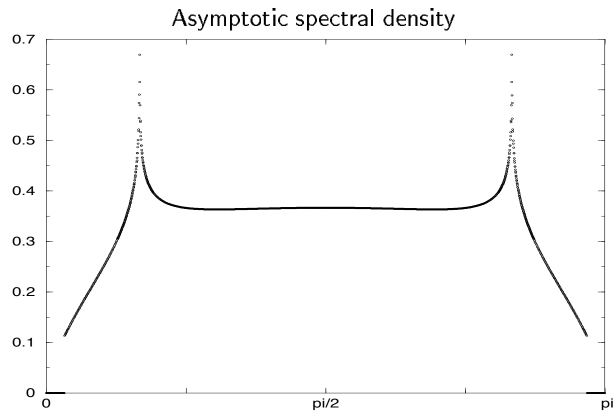
and thus $\rho(\eta) > 0$. This is repeated with a period of π. For other values of η, $\rho(\eta) = 0$. The turning points $x_-(\eta)$ and $x_+(\eta)$ are illustrated to the left for values of η between 0 and π.

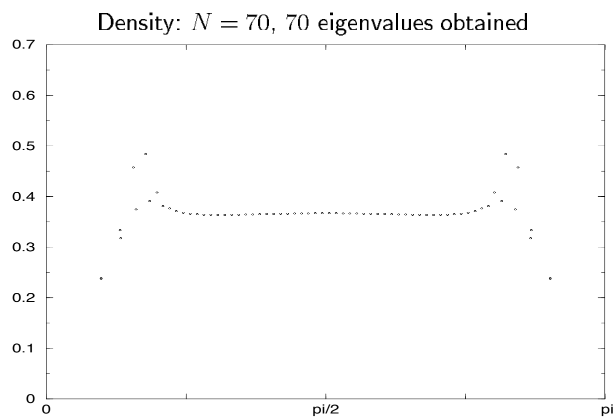
The graph above is a plot of the formula for $\rho(\eta)$. An approximate density obtained by finding actual eigenvalues for $N = 1000$ coincides almost exactly with this graph.

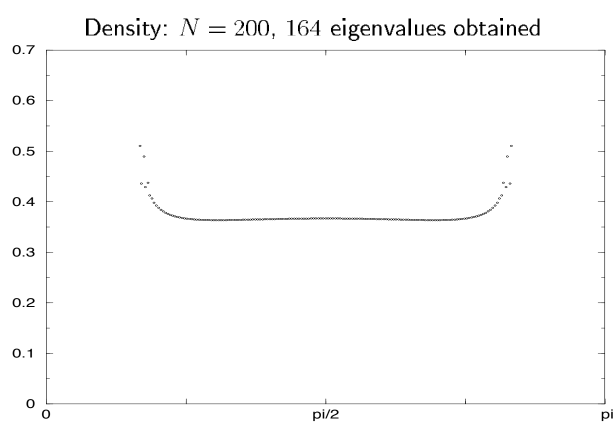


Example 4. *We have taken $\phi(x) = 0$ again here so that the spectrum is symmetric about $\eta = \pi/2$. The turning points can be visualized as demonstrated in Figure 1. The spikes in the proposed asymptotic density function $\rho(\eta)$ occur at the transition from η-values with four turning points to η-values with two turning points.*

It turns out to be very difficult to detect eigenvalues in the η-regions with four turning points. Numerical calculations show that they are grouped in pairs that are extremely far from each other compared with the distance between the two values in a pair. Thus the quick change in the sign of $ImF(e^{i\eta})$ with respect to η can go unnoticed in calculating the eigenvalues (see item 3 on page 17). For $N = 70$, however, all eigenvalues have been obtained and their approximate density graphed to the right with $M = 4$ eigenvalues used per density point. For $N = 200$, the density plot ($M = 7$ eigenvalues used per point) suggests that all the eigenvalues inside the 2-turning-point η-region have been obtained, as it coincides there with the proposed asymptotic density. A resolution of this difficulty with closely spaced pairs of eigenvalues in the case of six turning points is presented in the next example.

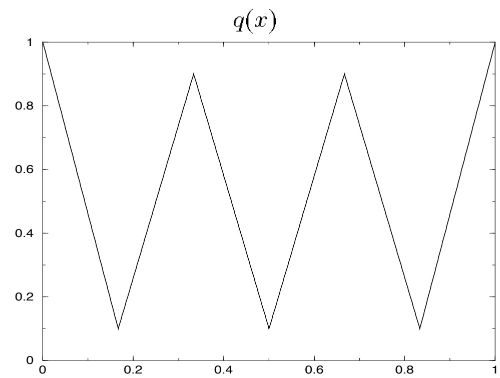


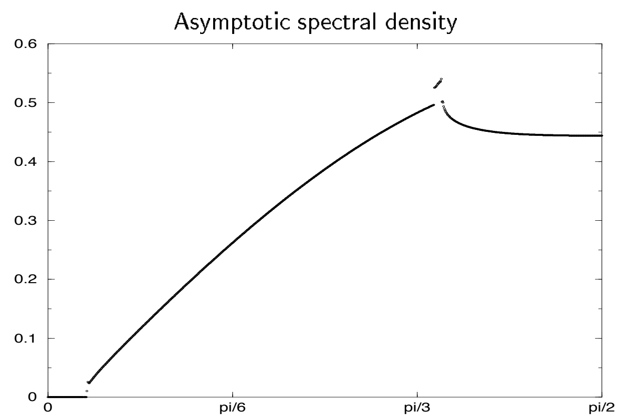


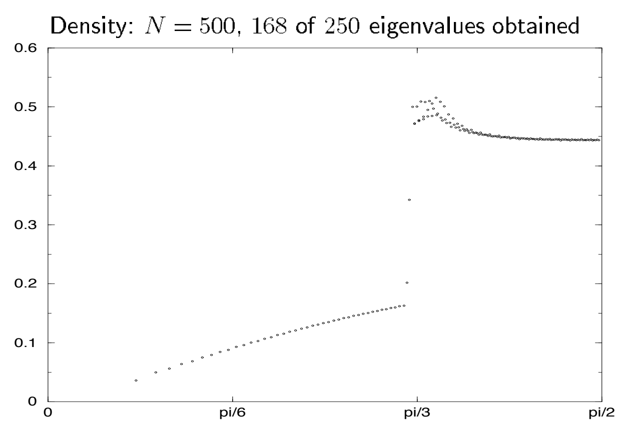


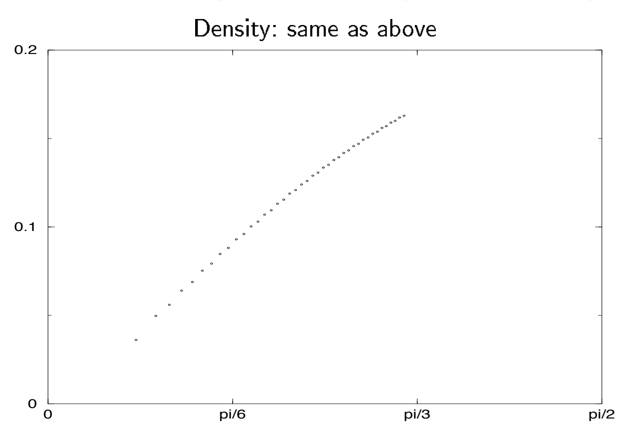
Example 5. *ϕ is taken to be constantly 0 again so that there is spectral symmetry about $\eta = \pi/2$; thus only half a period is plotted. Three eigenvalues per density point were used for the approximate density (middle and bottom, right).*

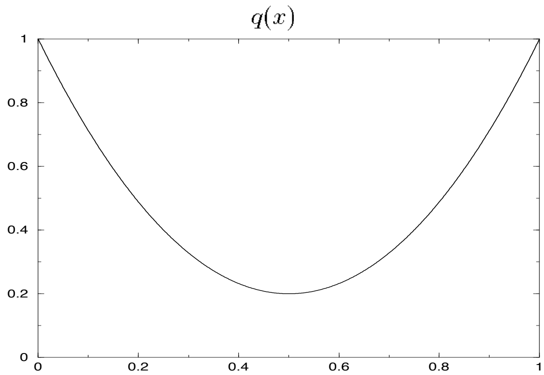
If the eigenvalues are grouped in tight triplets or alternating pairs and singles, then only one-third of the eigenvalues may be detected by the change of sign of $Re F(e^{i\eta})$ (see item 3 on page (see item 3.2, page 12). This is what happens in the region with six turning points in this example. In the middle figure, the asymptotic density seems to be confirmed nicely in the two-turning-point region. The bottom figure shows the same graph with vertical values scaled on the page by a factor of three, and, in the six-turning-point region, these values are quite close to the proposed asymptotic values. Numerical plots of the graph of $Re\, F(e^{i\eta})$ indicate that the eigenvalues in this region occur in triplets and in alternating pairs and singles. The transition between these microscopic behaviors occurs as one value in a triplet gradually moves to the next triplet as one moves along the η-axis. Looking at an example of the graph of $Re\, F(e^{i\eta})$ for $N = 200$, this transition is seen to contribute a few more detectable changes in sign above a third of the number of eigenvalues in the six-turning-point region. This probably also explains the slightly higher-than-expected values also in this example (in the bottom graph near $\pi/3$). The values in a pair become closer together as η moves away from the two-turning-point region toward the end of the support of the asymptotic density and, of course, as N increases. Already when $N = 70$ and $\eta < 0.68$, values in a pair are closer together than 10^{-7} whereas the distance between pairs is around 10^{-1}.

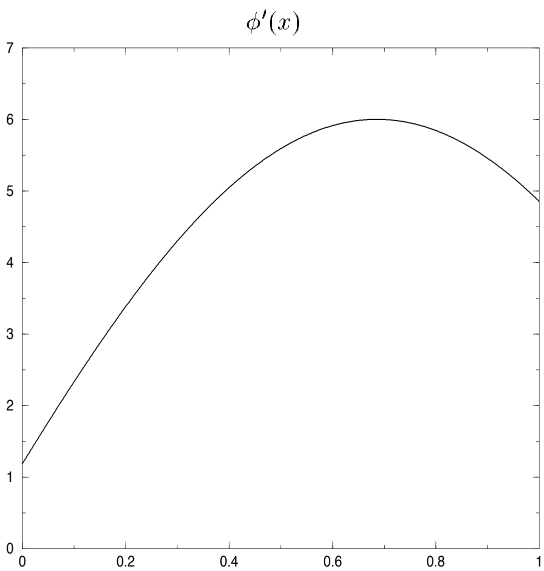


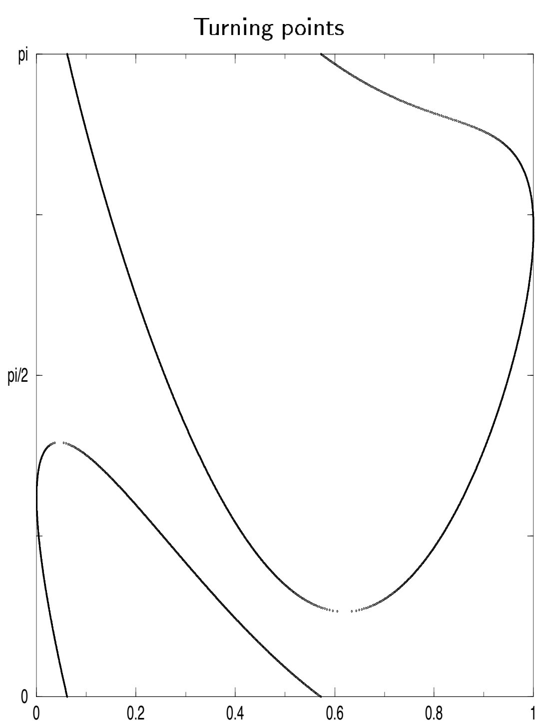


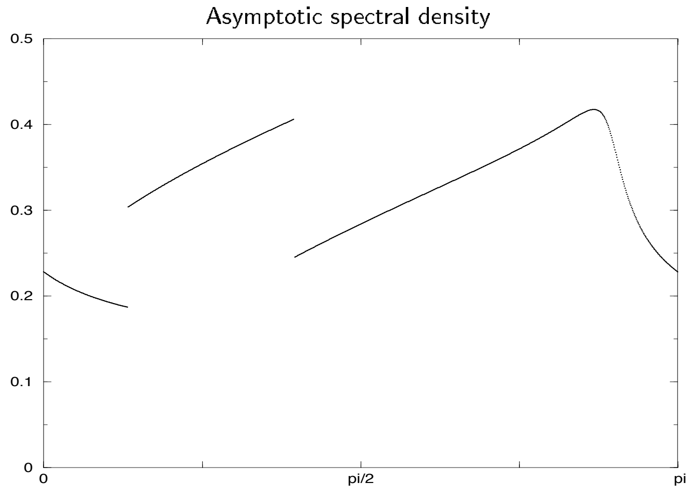


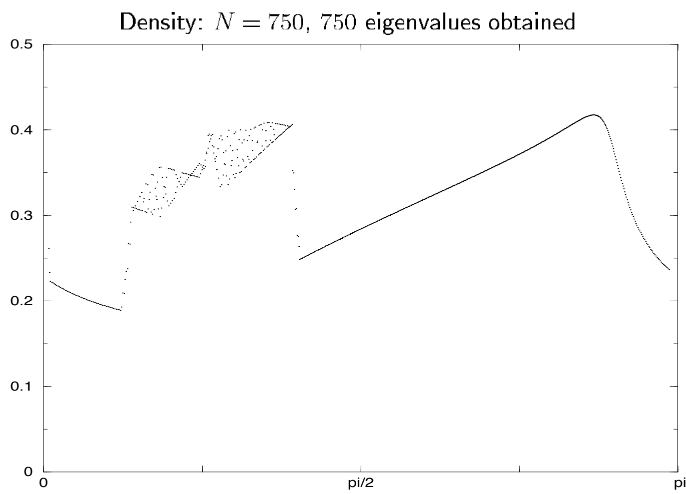




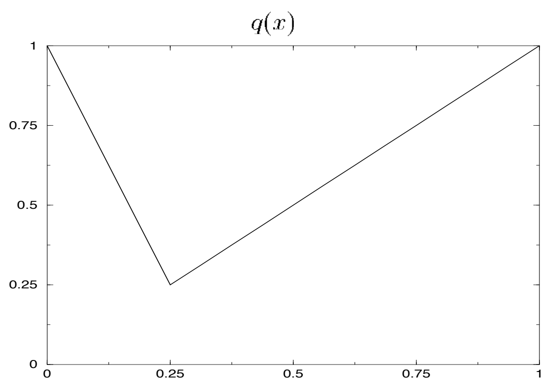


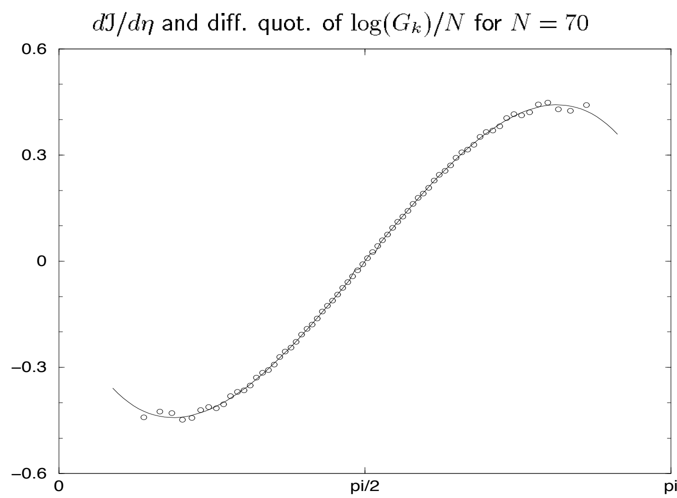




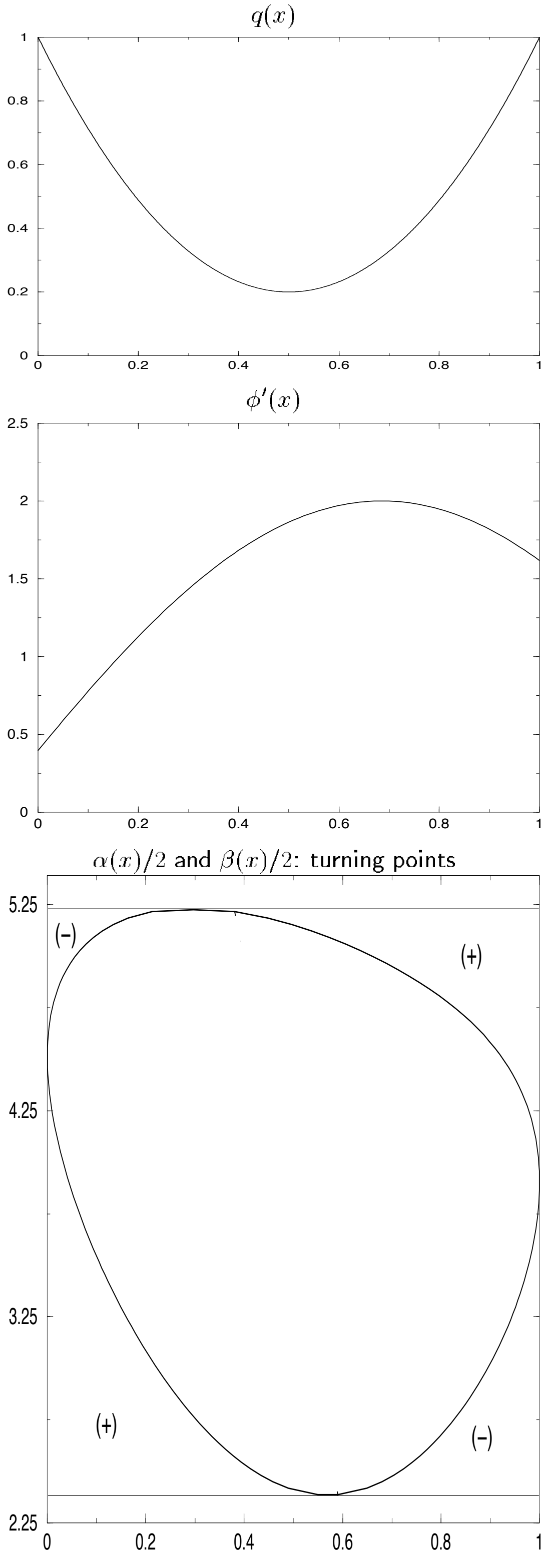


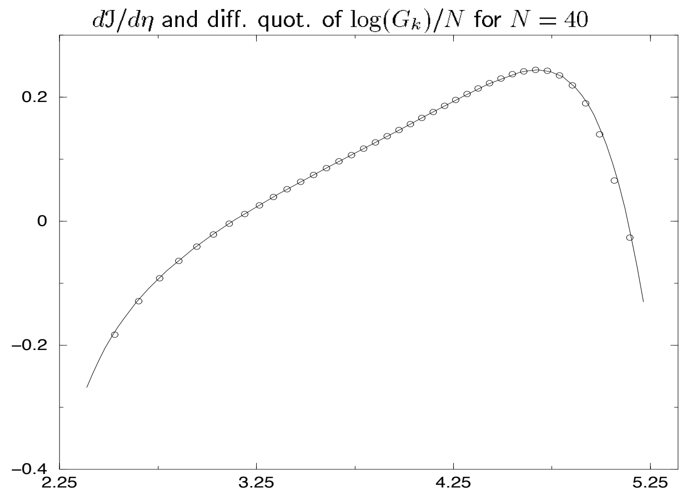
Example 6. *This is an example in which ϕ' is not constant so that the spectral density has no symmetries. The value of the density changes abruptly at the values of η that separate regions with two turning points from those with four. The approximate density for $1/\epsilon = 750$ was obtained using 10 eigenvalues per density point. Two observations about the η-interval with four turning points: The three points where the upper and lower envelopes for the irregularly placed values of the approximate density come together coincide with the graph of the proposed asymptotic density. Using more eigenvalues per density point decreased the deviation from the asymptotic density.*



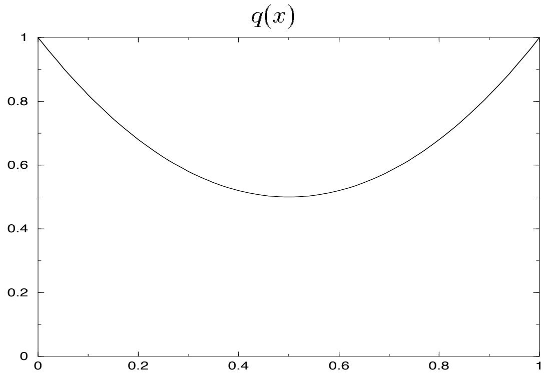


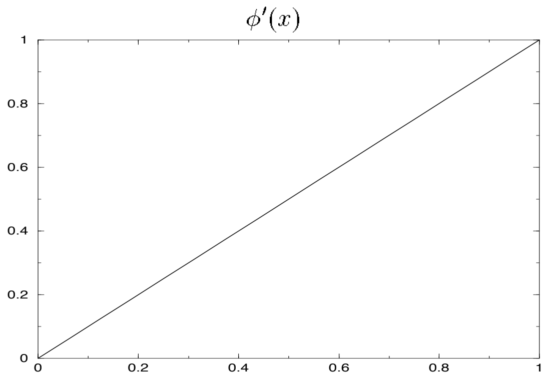
Example 7. *$\phi(x)$ is taken to be zero in this example, so $\mathcal{I}$ is symmetric about $\eta = \pi/2$. The difference quotients of $\frac{1}{N}\log G_k$ represented by circles, corroborate the proposed formula for $\mathcal{I}'$. The error at the endpoints is probably due to the sensitivity of $|F(e^{i\eta})|$ to changes in η, as illustrated in the final example. If $q(x)$ is replaced by the reflection of the graph of $q(x)$ about $x = 1/2$, then the norming exponent is replaced by its negative.*

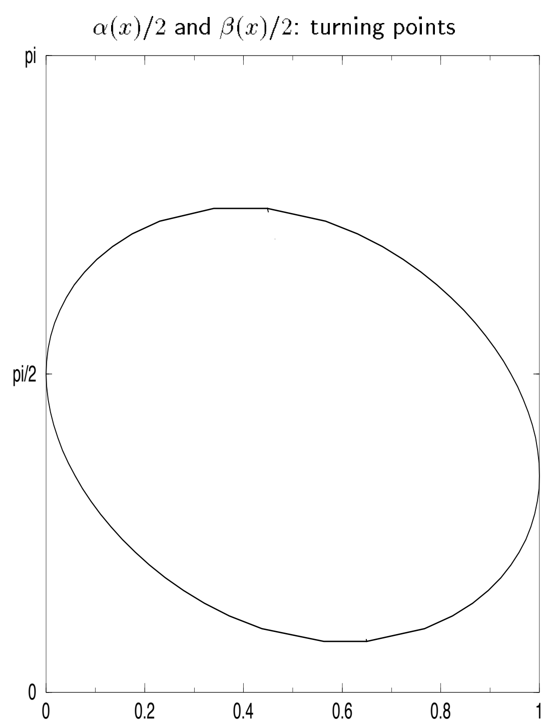


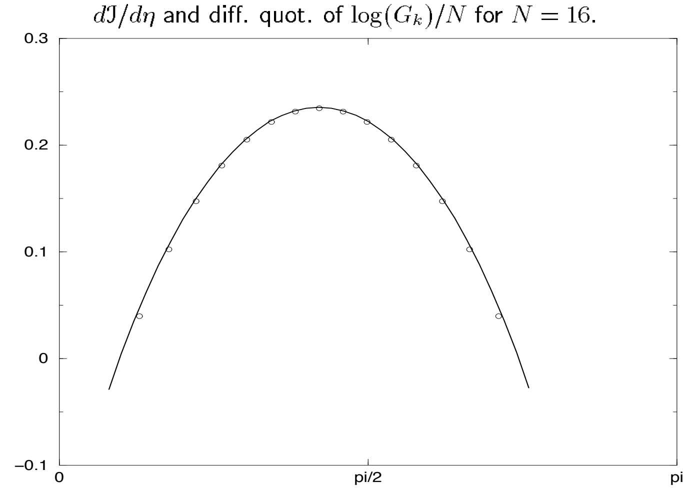


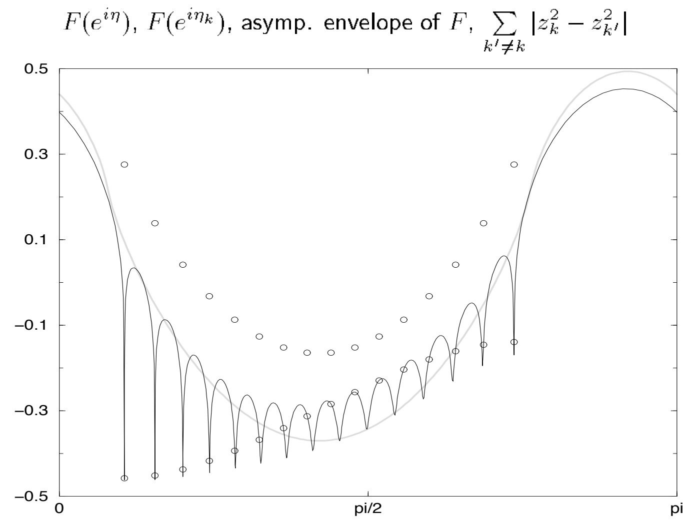
Example 8. *Another confirmation of $\mathfrak{I}$ with no symmetries in η. The $(+)$ and $(-)$ in the lower left figure indicate the sign that is to be taken in the proposed formula (17) for $\mathfrak{I}$.*











Example 9. *$\mathcal{J}'$ is symmetric about $\pi/2 - 1/4$. This is because, if q and ϕ are symmetric about $x = 1/2$, then $\mathcal{J}'$ is symmetric about $\eta = \pi/2$, and the shifting up of ϕ' by $1/2$ produces the shift to the left of the spectral data by half of that. The bottom graph above illustrates several things: The lower string of circles shows the values of G_k, that is, $\frac{1}{N}\log|F(e^{i\eta})|$ (black line) evaluated at the eigenvalues η_k for $N = 16$. Such data are difficult to obtain for large values of N because of the sensitivity of F to changes in η on the spikes. For very large values of N, however, an upper envelope for $F(e^{i\eta})$ can still be calculated, and the limiting values of this envelope as $N \to \infty$ is represented by the grey curve. Also coinciding with the grey curve are the two asymptotic quantities discussed in Subsection 3.3—$\int_0^1 \log\max|\lambda^{\pm}(x, e^{i\eta})|dx$ and $\int_0^{\pi} \log|e^{2i\eta} - e^{2i\eta'}|\rho(\eta')d\eta'$. The upper string of circles are the quantites $\sum_{k'\neq k}|z_k^2 - z_{k'}^2|$ plotted against η_k.*

5 Asymptotics of the Transfer Matrix

In this section, we take a rigorous approach to determining the asymptotic behavior of the transfer matrix over an oscillatory region and over an exponential region and establish some asymptotics of the solution to the linear problem. Let $[a,b]$ be an oscillatory or exponential interval for data $q(x)$ and $\phi(x)$ and spectral value $e^{i\eta}$ whose distance from any turning point is bounded from below. Define $\underline{n} = \lceil a/\epsilon \rceil$ and $\overline{n} = \lfloor b/\epsilon \rfloor$, and let $[c_n^1, c_n^2]^t$ represent the vector $\check{\mathbf{u}}_n(e^{i\eta})$ in the eigenvector basis $\{\mathbf{p}_n^{\pm}(e^{i\eta})\}$ for $\check{U}_n(e^{i\eta})$.

Theorem 2 *Given the notation above,*

1. *Let $[a,b]$ be an oscillatory interval. Then, for each ϵ, there exists a solution $\left[\begin{array}{cc} c_n^1 & c_n^2 \end{array}\right]^t$ such that, if $n\epsilon \in [a,b]$, then*

$$\arg \frac{c_n^1}{c_n^2} = \frac{1}{\epsilon} \int_a^{n\epsilon} \arg \frac{\underline{\lambda}^+(y)}{\underline{\lambda}^-(y)} dy + A(n\epsilon) + \varrho(\epsilon, n\epsilon),$$

$$c_n^{1,2} = A^{\pm}(n\epsilon) \exp\left(\frac{1}{\epsilon} \int_a^{n\epsilon} \underline{\lambda}^{\pm}(y) dy \right) + \varrho_{\pm}(\epsilon, n\epsilon),$$

in which $A(x)$ and $A^{\pm}(x)$ are continuous functions depending on q and ϕ and the choice of eigenvectors and

$$\varrho(\epsilon, x) = o(1) \quad (\epsilon \to 0),$$

$$\varrho_{\pm}(\epsilon, x) = o\left(\exp\left(\frac{1}{\epsilon} \int_a^{x} \underline{\lambda}^{\pm}(y) dy \right) \right) \quad (\epsilon \to 0)$$

uniformly in x.

2. *Let $[a,b]$ be an exponential interval, and suppose that $0 < \lambda_n^- < \lambda_n^+$. Then, for each ϵ, there exists a solution $\left[\begin{array}{cc} c_n^1 & c_n^2 \end{array}\right]^t$ such that, if $n\epsilon \in [a,b]$, then*

$$c_n^1 = B(n\epsilon) \exp\left(\frac{1}{\epsilon} \int_a^{n\epsilon} \underline{\lambda}^+(y) dy \right) + \varrho_1(\epsilon, n\epsilon),$$

$$c_n^2 = \varrho_2(\epsilon, n\epsilon),$$

in which B is determined by q and ϕ and the choice of eigenvectors and depends continuously on its arguments and

$$\varrho_{1,2}(\epsilon, x) = o\left(\exp\left(\frac{1}{\epsilon} \int_a^{x} \underline{\lambda}^+(y) dy \right) \right) \qquad (\epsilon \to 0),$$

uniformly in x.

5.1 Preliminaries

Let $\tilde{U}_n$ be the matrix taking $\begin{bmatrix} c_n^1 & c_n^2 \end{bmatrix}^t$ to $\begin{bmatrix} c_{n+1}^1 & c_{n+1}^2 \end{bmatrix}^t$. Then $\tilde{U}_n = M_n \Lambda_n$ where $\Lambda_n = \mathrm{diag}(\lambda_n^+, \lambda_n^-)$ and M_n is the change-of-basis matrix from $\{\mathbf{p}_n^+, \mathbf{p}_n^-\}$ to $\{\mathbf{p}_{n+1}^+, \mathbf{p}_{n+1}^-\}$. Assuming three continuous derivatives of ϕ and two of q, and using the expansions (8), one computes that $M_n = I + \epsilon R_n$ where the entries r_n^{ij} of R_n have the property that, for some differentiable functions $\underline{r}^{ij}$ of x, $|r_n^{ij} - \underline{r}^{ij}(n\epsilon)| = \mathcal{O}(\epsilon)$ uniformly in $x \in [a,b]$. This means that $r_n^{ij} = \underline{r}^{ij\epsilon}(n\epsilon)$ for some functions $\underline{r}^{ij\epsilon}$ of x such that $\underline{r}^{ij\epsilon}(x) = \underline{r}^{ij}(x) + \mathcal{O}(\epsilon)$ as $\epsilon \to 0$ uniformly in x. We will study the asymptotic behavior of the transfer matrix T^ϵ taking $\begin{bmatrix} c_{\underline{n}}^1 & c_{\underline{n}}^2 \end{bmatrix}$ to $\begin{bmatrix} c_{\overline{n}+1}^1 & c_{\overline{n}+1}^2 \end{bmatrix}$:

$$T^\epsilon := \prod_{n=\underline{n}}^{\overline{n}} \tilde{U}_n = \prod_{n=\underline{n}}^{\overline{n}} (I + \epsilon R_n)\Lambda_n .$$

The multiplication is ordered, factors with a lower index being to the right of factors with a higher index. We will study the case in which $[a,b]$ is contained in an oscillatory x-region and the case in which it is contained in an exponential region. We begin the analysis by bringing this expression for T^ϵ into a form in which its structure and limiting behavior is more transparent. Expanding in powers of ϵ, T^ϵ takes the form

$$T^\epsilon = \sum_{\ell=0}^{L} \epsilon^\ell T_\ell$$

where $L = \overline{n} - \underline{n} + 1$ and

$$T_\ell := \sum_{\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}} \left(\prod_{n=n_\ell+1}^{\overline{n}} \Lambda_n \right) R_{n_\ell} \left(\prod_{n=n_{\ell-1}+1}^{n_\ell} \Lambda_n \right) \cdots R_{n_2} \left(\prod_{n=n_1+1}^{n_2} \Lambda_n \right) R_{n_1} \left(\prod_{n=\underline{n}}^{n_1} \Lambda_n \right)$$

and $T_0 := \prod\limits_{n=\underline{n}}^{\overline{n}} \Lambda_n$. One can bring out a factor on the right, common to each T_ℓ, by using the following formula recursively: For any $i \le n' \le j$,

$$\left(\prod_{n=n'+1}^{j} \Lambda_n \right) R_{n'} \left(\prod_{n=i}^{n'} \Lambda_n \right) = \begin{bmatrix} r_{n'}^{11} & r_{n'}^{12} \prod\limits_{n=n'+1}^{j} \frac{\lambda_n^+}{\lambda_n^-} \\ r_{n'}^{21} \prod\limits_{n=n'+1}^{j} \frac{\lambda_n^-}{\lambda_n^+} & r_{n'}^{22} \end{bmatrix} \prod_{n=i}^{j} \Lambda_n .$$

Setting first (i, n', j) equal to $(n_{\ell-1}+1, n_\ell, \overline{n})$, then $(n_{\ell-2}+1, n_{\ell-1}, \overline{n})$, and so on up to $(\underline{n}, n_1, \overline{n})$, we arrive at the following expression for T_ℓ:

$$T_\ell = \left(\sum_{\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}} \hat{R}_{n_\ell} \cdots \hat{R}_{n_2} \hat{R}_{n_1} \right) \prod_{n=\underline{n}}^{\overline{n}} \Lambda_n ,$$

in which

$$\hat{R}_{n'} := \begin{bmatrix} r_{n'}^{11} & r_{n'}^{12} \prod\limits_{n=n'+1}^{\overline{n}} \frac{\lambda_n^+}{\lambda_n^-} \\ r_{n'}^{21} \prod\limits_{n=n'+1}^{\overline{n}} \frac{\lambda_n^-}{\lambda_n^+} & r_{n'}^{22} \end{bmatrix} .$$

Using the notation

$$P_\ell := \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} \hat{R}_{n_\ell}\cdots\hat{R}_{n_1},$$

we can write

$$T_\ell = P_\ell \prod_{n=\underline{n}}^{\overline{n}} \Lambda_n$$

to obtain the form

$$T^\epsilon = \left(\sum_{\ell=0}^{L} \epsilon^\ell P_\ell\right) \prod_{n=\underline{n}}^{\overline{n}} \Lambda_n.$$

One computes the products $\hat{R}_{n_\ell}\cdots\hat{R}_{n_1}$ (the sums are over n):

$$\hat{R}_{n_2}\hat{R}_{n_1} = \begin{bmatrix} r^{11}_{n_2}r^{11}_{n_1} + r^{12}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{n_2}\frac{\lambda^-_n}{\lambda^+_n} & r^{11}_{n_2}r^{12}_{n_1}\prod_{n_1+1}^{\overline{n}}\frac{\lambda^+_n}{\lambda^-_n} + r^{12}_{n_2}r^{22}_{n_1}\prod_{n_2+1}^{\overline{n}}\frac{\lambda^+_n}{\lambda^-_n} \\ r^{21}_{n_2}r^{11}_{n_1}\prod_{n_2+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} + r^{22}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} & r^{22}_{n_2}r^{22}_{n_1} + r^{21}_{n_2}r^{12}_{n_1}\prod_{n_1+1}^{n_2}\frac{\lambda^+_n}{\lambda^-_n} \end{bmatrix},$$

and the first column of $\hat{R}_{n_3}\hat{R}_{n_2}\hat{R}_{n_1}$ is

$$\begin{bmatrix} r^{11}_{n_3}r^{11}_{n_2}r^{11}_{n_1} + r^{12}_{n_3}r^{21}_{n_2}r^{11}_{n_1}\prod_{n_2+1}^{n_3}\frac{\lambda^-_n}{\lambda^+_n} + r^{11}_{n_3}r^{12}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{n_2}\frac{\lambda^-_n}{\lambda^+_n} + r^{12}_{n_3}r^{22}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{n_3}\frac{\lambda^-_n}{\lambda^+_n} \\ r^{21}_{n_3}r^{11}_{n_2}r^{11}_{n_1}\prod_{n_3+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} + r^{22}_{n_3}r^{21}_{n_2}r^{11}_{n_1}\prod_{n_2+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} + r^{22}_{n_3}r^{22}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} + r^{21}_{n_3}r^{12}_{n_2}r^{21}_{n_1}\prod_{n_1+1}^{n_2}\frac{\lambda^-_n}{\lambda^+_n}\prod_{n_3+1}^{\overline{n}}\frac{\lambda^-_n}{\lambda^+_n} \end{bmatrix}.$$

Inductively, we find that $\hat{R}_{n_\ell}\cdots\hat{R}_{n_1}$ includes the terms $r^{11}_{n_\ell}\cdots r^{11}_{n_1}$ and $r^{22}_{n_\ell}\cdots r^{22}_{n_1}$ in the upper left and lower right entries, respectively. The rest of the terms all contain factors that are products of the form $\prod_{n=n_1'}^{n_2'}\frac{\lambda^\pm_n}{\lambda^\mp_n}$. In the first column, λ^-_n always appears in the numerator, and in the second column, λ^+_n always appears in the numerator. We find then that P_ℓ has the form

$$\begin{aligned} P_\ell &= \begin{bmatrix} \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r^{11}_{n_\ell}\cdots r^{11}_{n_1} & 0 \\ 0 & \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r^{22}_{n_\ell}\cdots r^{22}_{n_1} \end{bmatrix} + \\ &+ \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} \begin{bmatrix} (2^{\ell-1}-1)\text{ terms} & 2^{\ell-1}\text{ terms} \\ 2^{\ell-1}\text{ terms} & (2^{\ell-1}-1)\text{ terms} \end{bmatrix} \end{aligned} \tag{18}$$

(P_0 is the identity matrix) where the "terms" are as described above.

By induction on ℓ, one can prove the following Lemma on the structure of the first column of $\hat{R}_{n_\ell}\cdots\hat{R}_{n_1}$ and a similar lemma for the second column. $\hat{R}_{n_\ell}\cdots\hat{R}_{n_1}$ is assumed to be in simplified form in the sense that factors of the form $\frac{\lambda^+_n}{\lambda^-_n}\frac{\lambda^-_n}{\lambda^+_n}$ are removed.

Lemma 3 *on the first column of* $\hat{R}_{n_\ell} \cdots \hat{R}_{n_1}$.

1. *The first entry contains the term* $r^{11}_{n_\ell} \cdots r^{11}_{n_1}$ *and* $2^{\ell-1} - 1$ *terms with factors of the form* $\prod_{n=m_1+1}^{m_2} \frac{\lambda_n^-}{\lambda_n^+}$ *for* $m_1, m_2 \in \{n_1, \ldots, n_\ell\}$ *(not the empty product). These factors have the following properties:*

   (a) *For any* n*, the factor* $\frac{\lambda_n^-}{\lambda_n^+}$ *occurs with multiplicity at most* 1.

   (b) *For one factor,* $m_2 = n_\ell$.

2. *The second entry contains* $2^{\ell-1}$ *terms with factors of the form* $\prod_{n=m_1+1}^{m_2} \frac{\lambda_n^-}{\lambda_n^+}$ *for* $m_1, m_2 \in \{n_1, \ldots, n_\ell, \overline{n}\}$ *(not the empty product). These factors have the following properties:*

   (a) *For any* n*, the factor* $\frac{\lambda_n^-}{\lambda_n^+}$ *occurs with multiplicity at most* 1.

   (b) *For one factor,* $m_2 = \overline{n}$.

5.2 Oscillatory Region

Let us now consider the case in which $[a, b]$ is contained in an oscillatory region. The goal is to show that, as ϵ tends to zero, the transfer matrix is asymptotic to a diagonal matrix that depends only on q and ϕ, times $\prod_{n=\underline{n}}^{\overline{n}} \Lambda_n$. The task is to show that, by letting ϵ tend to zero, one can bring the expansion $\sum_{\ell=0}^{L} \epsilon^\ell P_\ell$ into any vicinity of a fixed diagonal matrix. Whereas P_0 is just the identity matrix, it is not obvious that the ϵ^1-term

$$\epsilon P_1 = \epsilon \sum_{\underline{n} \le n_1 \le \overline{n}} \begin{bmatrix} r^{11}_{n_1} & r^{12}_{n_1} \prod_{n=n_1+1}^{\overline{n}} \frac{\lambda_n^+}{\lambda_n^-} \\ r^{21}_{n_1} \prod_{n=n_1+1}^{\overline{n}} \frac{\lambda_n^-}{\lambda_n^+} & r^{22}_{n_1} \end{bmatrix},$$

for example, is tending to a diagonal form. One expects the diagonal entries to converge to the integrals $\int_a^b \underline{r}^{ii}(x)dx$ if the functions q and ϕ are sufficiently smooth. One can apply a naive formal argument to the other entries by replacing the product in, say, the $(1,2)$-entry by its asymptotic form $\exp\left(i\frac{1}{\epsilon}\int_a^x \underline{\theta}(x')dx'\right)$ and replacing the sum by an integral:

$$\epsilon \sum_{\underline{n} \le n_1 \le \overline{n}} r^{12}_{n_1} \prod_{n=n_1+1}^{\overline{n}} \frac{\lambda_n^+}{\lambda_n^-} \quad \longrightarrow \quad \int_a^b \underline{r}^{12}(x) \exp\left(i\frac{1}{\epsilon}\int_x^b \underline{\theta}(x')dx'\right) dx.$$

This formal limit does indeed tend to zero; however, converting what is essentially a Riemann sum into an integral is not so simple because of the fast oscillations in the integrand. Indeed, *the period of the oscillations is at the order of the mesh size* ϵ.

A similar but more complicated situation occurs in the higher-order terms in the expansion of T^ϵ. In an oscillatory region, the "terms" in expression (18) for P_ℓ are, for some index h, of the form

$$r^*_{n_\ell} \cdots r^*_{n_h} \cdots r^*_{n_1}(u) \prod_{n=n_h+1}^{n_{h+1}} \frac{\lambda_n^\pm}{\lambda_n^\mp} \tag{19}$$

where n_{h+1} may be equal to $\overline{n}$ and n_{h-1} may be equal to $\underline{n}$, the asterisk (*) represents any superscript from the set $\{11, 12, 21, 22\}$, and u is a product of expressions that do not depend on n_h and are of the form $\prod_{n=n_1'}^{n_2'} \frac{\lambda_n^\pm}{\lambda_n^\mp}$ and are therefore unitary.

Again, one expects the quantities $\epsilon^\ell \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r_{n_1}^{ii} \cdots r_{n_\ell}^{ii}$ to converge to an ℓ-fold integral over the region $a \le x_1 < \cdots < x_\ell \le b$. Each of the other terms is oscillating in at least one of the variables n_h, and, extending the technique discussed above for the case when $\ell = 1$, one can show that each of these terms converges to zero.

Now, the number of these terms grows exponentially with ℓ, and, as ϵ decreases, the degree L of the expansion of T^ϵ in ϵ increases. One can solve these problems with the observation that the number of terms in a sum over $\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}$ is less than $\binom{N}{\ell}$ (recall that $\epsilon = 1/N$), and, upon multiplying by ϵ^ℓ, one can bound the whole expansion containing the "terms" of T^ϵ by a quantity that tends to zero as $\epsilon \to 0$. The details are in the proof of Proposition 5.

In formulating the lemma, the interval $[a, b]$ must be bounded away from any turning point so that the functions $\underline{r}^{ij}(x)$ are bounded and the function $e^{i\underline{\theta}(x)}$ is bounded away from the real axis. We make the following definitions:

- Let the number σ be such that $|1 - \exp(i\underline{\theta}(x))| > \sigma$ for $x \in [a, b]$.
- If ϕ'' is continuous on $[a, b]$, then $\frac{d\theta}{dx}$ is bounded on $[a, b]$, so there exists a number κ such that

$$|\underline{\theta}(x_2) - \underline{\theta}(x_1)| \le \kappa |x_2 - x_1|$$

  for all $x_1, x_2 \in [a, b]$.

- The difference quotients $\frac{\phi(x+\epsilon)-\phi(x)}{\epsilon}$ converge to $\phi'(x)$ uniformly on $[a, b]$ provided that ϕ'' is continuous. This implies the existence of a number τ such that

$$|\theta_n - \underline{\theta}(n\epsilon)| \le \tau\epsilon$$

  whenever $n\epsilon \in [a, b]$.

- One can verify that the functions $\underline{r}^{ij}(x)$ have continuous derivatives on $[a, b]$, and this implies the existence of a number β such that

$$|\underline{r}^{ij}(x_2) - \underline{r}^{ij}(x_1)| \le \beta |x_2 - x_1|$$

  for $x_1, x_2 \in [a, b]$.

- The existence of a number γ such that, for $i, j \in \{1, 2\}$,

$$|r_n^{ij} - \underline{r}^{ij}(n\epsilon)| < \gamma\epsilon$$

  has already been discussed.

- The continuity of the functions $\underline{r}^{ij}(x)$ and the previous bullet imply the existence of a number α such that, for $x, n\epsilon \in [a, b]$,

$$|\underline{r}^{ij}(x)| < \alpha \quad \text{and} \quad |r_n^{ij}| < \alpha.$$

- Define

$$S_i(a,b) = \int_a^b \underline{r}^{ii}(x)\ dx, \quad \text{for} \quad i = 1,\ 2.$$

The first lemma obtains estimates on the oscillating terms in P_ℓ. One must understand why the sum over $\underline{n} \le n_1 < \cdots < n_\ell \le \overline{n}$ of any one of these terms, multiplied by ϵ^ℓ, tends to zero. Let Υ denote a general one of these quantities:

$$\Upsilon := \epsilon^\ell \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r^*_{n_\ell}\cdots r^*_{n_h}\cdots r^*_{n_1}(u) \prod_{n=n_h+1}^{n_{h+1}} \frac{\lambda^\pm_n}{\lambda^\mp_n}.$$

Lemma 4 *Let $[a,b]$ be an oscillatory interval with positive distance from the set of turning points, and let $\varrho > 0$ be given. Then ϵ' can be chosen sufficiently small such that whenever $\epsilon < \epsilon'$, $|\Upsilon| < \varrho\frac{(\alpha)^{\ell-1}}{(\ell-1)!}$ for any quantity of the type Υ.*

Proof. Denote by $\underline{\theta}(x)$ either $\arg\left(\frac{\underline{\lambda}^+(x)}{\underline{\lambda}^-(x)}\right)$ or $\arg\left(\frac{\underline{\lambda}^-(x)}{\underline{\lambda}^+(x)}\right)$, and by θ_n the corresponding quantity $\arg\left(\frac{\lambda^\pm_n}{\lambda^\mp_n}\right)$. Rewrite Υ as

$$\Upsilon = \epsilon^{\ell-1} \sum_{\underline{n}\le n_1<\ldots<\hat{n}_h<\ldots<n_\ell\le\overline{n}} r^*_{n_\ell}\cdots \hat{r}^*_{n_h}\cdots r^*_{n_1}(u) \left[\epsilon \sum_{n_h=n_{h-1}+1}^{n_{h+1}-1} r^*_{n_h} \exp\left(i \sum_{n=n_h+1}^{n_{h+1}} \theta_n\right)\right].$$

The circumflex marks a removed factor or variable. Since $|r^*_{n_\ell}\cdots\hat{r}^*_{n_h}\cdots r^*_{n_1}(u)| < \alpha^{\ell-1}$ and there are no more than $\binom{N}{\ell-1} < \frac{N^{\ell-1}}{(\ell-1)!}$ terms in the outer sum, we see that

$$|\Upsilon| \ < \ \frac{\alpha^{\ell-1}}{(\ell-1)!} \max_{\underline{n}\le n_1<\ldots<\hat{n}_h<\ldots<n_\ell\le\overline{n}} \left|\epsilon \sum_{n_h=n_{h-1}+1}^{n_{h+1}-1} r^*_{n_h} \exp\left(i \sum_{n=n_h+1}^{n_{h+1}} \theta_n\right)\right|. \tag{20}$$

Let us study a single quantity of the type

$$\Omega := \epsilon \sum_{n_h=n_{h-1}+1}^{n_{h+1}-1} r^*_{n_h} \exp\left(i \sum_{n=n_h+1}^{n_{h+1}} \theta_n\right).$$

Let $\bar{\varrho}$ be given such that $0 < \bar{\varrho} \le \frac{1}{2}$ and $\bar{\varrho}^6 < \frac{1}{2}|b - x_*|$ for all turning points x_*, and assume that $0 < \epsilon < \bar{\varrho}^6$. Let M be a positive number such that $M^{-\frac{13}{2}} < \epsilon < M^{-6} < \bar{\varrho}^6$. The following procedure applies to any one of these quantities Ω. First, if $(n_{h+1} - n_{h-1} - 1)\epsilon < M^{-4}$, then it is clear that $|\Omega| < \alpha M^{-4}$. Otherwise, if $(n_{h+1} - n_{h-1} - 1)\epsilon > M^{-4}$, then we divide the interval $[(n_{h-1}+1)\epsilon, n_{h+1}\epsilon)$ into disjoint subintervals $[\epsilon m_{k-1}, \epsilon m_k)$, $k = 1, ..., K$ where $m_0 = n_{h-1}+1$ and $m_K = n_{h+1}$, such that, if we set $M_k = m_k - m_{k-1}$ for $k = 1, \ldots, K$, then $M^{-4} < M_k\epsilon < 2M^{-4}$. The first part of this inequality implies $K < M^4$. In summary, the conditions are

$$\begin{gathered} M^{-\frac{13}{2}} < \epsilon < M^{-6} < \bar{\varrho}^6, \\ M^{-4} < M_k\epsilon < 2M^{-4}, \\ K < M^4. \end{gathered}$$

Now break Ω into a sum

$$\Omega = \epsilon \sum_{k=1}^{K} \Omega_k, \tag{21}$$

in which

$$\Omega_k := \sum_{n_h=m_{k-1}}^{m_k-1} r^*_{n_h} \exp\left(i \sum_{n=n_h+1}^{n_{h+1}} \theta_n \right).$$

If we define $r^k_{n_h} = r^*_{n_h} \exp\left(i \sum_{n=m_k+1}^{n_{h+1}} \theta_n\right)$, then we can rewrite Ω_k as

$$\Omega_k = \sum_{n_h=m_{k-1}}^{m_k-1} r^k_{n_h} \exp\left(i \sum_{n=n_h+1}^{m_k} \theta_n \right)$$

and compare it with the "constant frequency and amplitude" quantity

$$\overline{\Omega}_k := \sum_{n_h=m_{k-1}}^{m_k-1} r^k_{m_k} \exp[i(m_k - n_h)\underline{\theta}(m_k\epsilon)].$$

First, for any n_h such that $m_{k-1} \le n_h < m_k$,

$$\exp\left(i \sum_{n=n_h+1}^{m_k} \theta_n \right) = \exp[i(m_k - n_h)\underline{\theta}(m_k\epsilon)] \exp\left(i \sum_{n=n_h+1}^{m_k} (\theta_n - \underline{\theta}(m_k\epsilon)) \right)$$

and, from the definitions of κ and τ,

$$|\theta_n - \underline{\theta}(m_k\epsilon)| < M_k\epsilon\kappa + \epsilon\tau$$

whenever $m_{k-1} \le n_h < m_k$, so we get the bound

$$\left| \sum_{n=n_h+1}^{m_k} (\theta_n - \underline{\theta}(m_k\epsilon)) \right| < M_k(M_k\epsilon\kappa + \epsilon\tau).$$

Thus, using only $M_k\epsilon < 2M^{-4}$ and $M^{-\frac{13}{2}} < \epsilon$, we get

$$\left| \exp\left(i \sum_{n=n_h+1}^{m_k} \theta_n \right) - \exp[i(m_k - n_h)\underline{\theta}(m_k\epsilon)] \right| < M_k(M_k\epsilon\kappa + \epsilon\tau) < 4\kappa M^{-\frac{3}{2}} + 2\tau M^{-4} \tag{22}$$

for $m_{k-1} \le n_h < m_k$. Second, by the definition of β,

$$|r^k_{m_k} - r^k_{n_h}| < M_k\epsilon\beta + 2\epsilon\gamma < 2\beta M^{-4} + 2\gamma M^{-6} \tag{23}$$

whenever $m_{k-1} \le n_h < m_k$. Putting (22) and (23) together, we find that whenever $m_{k-1} \le n_h < m_k$,

$$\left| r^k_{n_h} \exp\left(i \sum_{n=n_h+1}^{m_k} \theta_j \right) - r^k_{m_k} \exp[i(m_k - n_h)\underline{\theta}(m_k\epsilon)] \right| < \bar{C} M^{-\frac{3}{2}}$$

for some constant $\bar{C}$ depending only on the functions q and ϕ. Using the same two inequalities as for the bound (22) and the fact that there are M_k elements in the sums Ω_k and $\bar{\Omega}_k$, this estimate implies

$$|\Omega_k - \overline{\Omega}_k| < 2\bar{C}M. \tag{24}$$

To bound the quantities $\overline{\Omega}_k$, we write

$$\overline{\Omega}_k = r^k_{m_k} \sum_{n=1}^{M_k} \exp(in\underline{\theta}(m_k\epsilon)),$$

whence

$$|\bar{\Omega}_k| \le \alpha \left| \frac{\exp(iM_k\underline{\theta}(m_k\epsilon)) - 1}{\exp(i\underline{\theta}(m_k\epsilon)) - 1} \right| < \alpha\frac{4}{\sigma} \tag{25}$$

where σ is as defined above. Combining (24) and (25) with our assumption that $M > 2$ yields

$$|\Omega_k| < \bar{C}''M. \tag{26}$$

Going back to (21) and using that $\epsilon K < M^{-2}$, we get

$$|\Omega| < CM^{-1}$$

where the constant C depends only on q and ϕ. Then going back to (20), we finally obtain the result

$$|\Upsilon| < \frac{\alpha^{\ell-1}}{(\ell-1)!}CM^{-1} < \frac{\alpha^{\ell-1}}{(\ell-1)!}C\bar{\varrho}.$$

The Lemma follows by taking

$$\bar{\varrho} = \min\left\{ \frac{\varrho}{C}, \frac{1}{2}, 2^{-\frac{1}{6}}|b - x_*|^{\frac{1}{6}} : x_* \text{ a turning point} \right\}$$

and $\epsilon' = \bar{\varrho}^6$. $\triangle$

Proposition 5 *Let $[a,b]$ be an oscillatory interval with positive distance from the set of turning points, and let $\varrho > 0$ be given. Then there exists ϵ' sufficiently small such that whenever $\epsilon < \epsilon'$,*

$$T^\epsilon = \begin{bmatrix} e^{S_1(a,b)} + \varrho^{11} & \varrho^{12} \\ \varrho^{21} & e^{S_2(a,b)} + \varrho^{22} \end{bmatrix} \begin{bmatrix} \prod_{n=\underline{n}}^{\overline{n}} \lambda^+_n & 0 \\ 0 & \prod_{n=\underline{n}}^{\overline{n}} \lambda^-_n \end{bmatrix}$$

for some complex numbers ϱ^{ij} such that $|\varrho^{ij}| < \varrho$ for $i, j = 1, 2$.

Remark: The convergence of the left-hand factor to a diagonal form as ϵ tends to zero is not uniform as a or b nears a turning point. However, it is uniform over all x-intervals whose distance from any turning point is bounded below by some positive number. Notice that $S_{ii}(a,b)$ depend on the choice of eigenvectors.

Before proving this, we derive from it the leading-order asymptotic behavior of $\arg\left(c^1_{\overline{n}+1}/c^2_{\overline{n}+1}\right)$ given the fixed value $\left[\begin{array}{cc} c^1_{\underline{n}} & c^2_{\underline{n}} \end{array}\right]^t$, which is the content of Theorem 2.

$$\arg\left(\frac{c^1_{\overline{n}+1}}{c^2_{\overline{n}+1}}\right) = \arg\left[\frac{c^1_{\underline{n}} e^{S_1(a,b)} \prod_{n=\underline{n}}^{\overline{n}} \lambda^+_n + \varrho^{11} c^1_{\underline{n}} \prod_{n=\underline{n}}^{\overline{n}} \lambda^+_n + \varrho^{12} c^2_{\underline{n}} \prod_{n=\underline{n}}^{\overline{n}} \lambda^-_n}{c^2_{\underline{n}} e^{S_2(a,b)} \prod_{n=\underline{n}}^{\overline{n}} \lambda^-_n + \varrho^{22} c^2_{\underline{n}} \prod_{n=\underline{n}}^{\overline{n}} \lambda^-_n + \varrho^{21} c^1_{\underline{n}} \prod_{n=\underline{n}}^{\overline{n}} \lambda^+_n}\right]$$

$$= \arg\left(\frac{c^1_{\underline{n}} e^{S_1(a,b)}}{c^2_{\underline{n}} e^{S_2(a,b)}}\right) + \sum_{n=\underline{n}}^{\overline{n}} \arg\left(\frac{\lambda^+_n}{\lambda^-_n}\right) + o(1) \qquad (\epsilon \to 0).$$

By the expansion (8) for $\underline{\theta}$,

$$\sum_{n=\underline{n}}^{\overline{n}} \arg\left(\frac{\lambda^+_n}{\lambda^-_n}\right) = \sum_{n=\underline{n}}^{\overline{n}} \theta_n = \sum_{n=\underline{n}}^{\overline{n}} \underline{\theta}^\epsilon(n\epsilon) =$$

$$= \sum_{n=\underline{n}}^{\overline{n}} \left(\underline{\theta}(n\epsilon) + \epsilon \underline{\theta}_1(n\epsilon) + \mathcal{O}(\epsilon^2)\right) = \sum_{n=\underline{n}}^{\overline{n}} \left(\underline{\theta}(n\epsilon) + \epsilon \underline{\theta}_1(n\epsilon)\right) + \mathcal{O}(\epsilon);$$

$$\frac{1}{\epsilon}\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} \underline{\theta}(x)\,dx = \frac{1}{\epsilon}\sum_{n=\underline{n}}^{\overline{n}} \int_{n\epsilon}^{n\epsilon+\epsilon} \underline{\theta}(x)\,dx =$$

$$= \frac{1}{\epsilon}\sum_{n=\underline{n}}^{\overline{n}} \left(\epsilon\underline{\theta}(n\epsilon) + \frac{\epsilon^2}{2}\underline{\theta}'(n\epsilon) + \mathcal{O}(\epsilon^3)\right)$$

$$= \sum_{n=\underline{n}}^{\overline{n}} \left(\underline{\theta}(n\epsilon) + \epsilon\frac{\underline{\theta}'(n\epsilon)}{2}\right) + \mathcal{O}(\epsilon).$$

This gives the asymptotics

$$\sum_{n=\underline{n}}^{\overline{n}} \theta_n = \frac{1}{\epsilon}\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} \underline{\theta}(x)\,dx + \epsilon\sum_{n=\underline{n}}^{\overline{n}} \left(\underline{\theta}_1(n\epsilon) - \frac{\underline{\theta}'(n\epsilon)}{2}\right) + \mathcal{O}(\epsilon)$$

$$= \frac{1}{\epsilon}\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} \underline{\theta}(x)\,dx + \int_a^b \left(\underline{\theta}_1(x) - \frac{\underline{\theta}'(x)}{2}\right) dx + \mathcal{O}(\epsilon),$$

with which we can refine our result:

$$\arg\left(\frac{c^1_{\overline{n}+1}}{c^2_{\overline{n}+1}}\right) = \frac{1}{\epsilon}\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} \underline{\theta}(x)\,dx + \arg\left(\frac{c^1_{\underline{n}}}{c^2_{\underline{n}}}\right) + \int_a^b \left[Im\left(r^{11}(x) - r^{22}(x)\right) + \underline{\theta}_1(x) - \frac{\underline{\theta}'(x)}{2}\right] dx + o(1).$$

Putting

$$\arg\frac{c^1_{\underline{n}}}{c^2_{\underline{n}}} = \frac{1}{\epsilon}\int_a^{\underline{n}\epsilon} \arg\frac{\underline{\lambda}^+(y)}{\underline{\lambda}^-(y)}dy,$$

one finds that

$$\arg\frac{c^1_{\overline{n}+1}}{c^2_{\overline{n}+1}} = \frac{1}{\epsilon}\int_a^{\overline{n}\epsilon+\epsilon} \arg\frac{\underline{\lambda}^+(y)}{\underline{\lambda}^-(y)}dy + A(b) + \mathcal{O}(1) \quad (\epsilon\to 0),$$

where A is a function of x depending on q, ϕ, a, and the choice of eigenvectors. These four objects being fixed, the $\mathcal{O}(1)$ part depends on b and ϵ, and it is uniform in the endpoint b, as long as it is bounded away from any turning point. Thus, given ϵ and $n\epsilon \in [a,b]$, one can reapply this result for the oscillatory interval $[a, n\epsilon-\epsilon]$ and obtain

$$\arg\frac{c^1_{n}}{c^2_{n}} = \frac{1}{\epsilon}\int_a^{n\epsilon} \arg\frac{\underline{\lambda}^+(y)}{\underline{\lambda}^-(y)}dy + A(n\epsilon) + \mathcal{O}(1) \quad (\epsilon\to 0),$$

the asymptotics being uniform throughout $[a,b]$.

In addition, similar considerations imply that for some functions $A^\pm$ of x,

$$c_n^{1,2} \sim \exp\left(\frac{1}{\epsilon}\int_a^{n\epsilon}\underline{\lambda}^\pm(y)dy\right)A^+(n\epsilon) + \mathcal{O}\left(\exp\left(\frac{1}{\epsilon}\int_a^{n\epsilon}\underline{\lambda}^\pm(y)dy\right)\right) \qquad (\epsilon\to 0).$$

Proof. We begin by considering ϵ^ℓ times the diagonal matrix in expression (18) for P_ℓ. The quantities $\epsilon^\ell\sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r^{ii}_{n_1}\cdots r^{ii}_{n_\ell}$ are essentially Riemann sums for the integrals

$$\int\cdots\int_{\mathcal{R}_\ell} \underline{r}^{ii}(x_1)\cdots\underline{r}^{ii}(x_\ell)\; dx_1\cdots dx_\ell$$

in which the integration is over the subregion $\mathcal{R}_\ell$ of $[0,1]^\ell$ described by the inequalities $a < x_1 < \ldots < x_\ell < b$. These integrals are in fact equal to

$$\frac{1}{\ell!}\int_a^b\cdots\int_a^b \underline{r}^{ii}(x_1)\cdots\underline{r}^{ii}(x_\ell)\; dx_1\cdots dx_\ell \;=\; \frac{1}{\ell!}\left(\int_a^b \underline{r}^{ii}(x)\;dx\right)^\ell \;=\; \frac{1}{\ell!}(S_i(a,b))^\ell.$$

The sum over all ℓ should then converge to $e^{S_i(a,b)}$. This can be made rigorous:

We define, for any ϵ, the step functions s^i_ϵ of x as follows: $s^i_\epsilon(x) = r^{ii}_n$ where n is such that $n\epsilon \le x < n\epsilon+\epsilon$. These functions, which are defined on the interval $[\underline{n}\epsilon, \overline{n}\epsilon+\epsilon)$, converge to $\underline{r}^{ii}(x)$ uniformly on $[a,b]$ as $\epsilon\to 0$ as guaranteed by the constants γ and β. In addition, the functions $s^i_\epsilon(x_1)\cdots s^i_\epsilon(x_\ell)$, which are defined on the box $[\underline{n}\epsilon, \overline{n}\epsilon+\epsilon)^\ell$ for $\ell = 1,\ldots,L$, are totally symmetric. Then we write

$$\epsilon^\ell \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r^{ii}_{n_1}\cdots r^{ii}_{n_\ell} = \int_{\mathcal{R}^\epsilon_\ell - E^\epsilon_\ell} s^i_\epsilon(x_1)\cdots s^i_\epsilon(x_\ell)\,dx_1\cdots dx_\ell,$$

in which

$$\mathcal{R}^\epsilon_\ell := \Big\{(x_1,\ldots,x_\ell)\in[\underline{n}\epsilon,\overline{n}\epsilon+\epsilon)^\ell :\; x_1<\ldots<x_\ell\Big\},$$

$$E^\epsilon_\ell := \Big\{(x_1,\ldots,x_\ell)\in\mathcal{R}^\epsilon_\ell :\; n\epsilon\le x_i<x_{i+1}<n\epsilon+\epsilon \;\text{ for some } i \text{ and some } n\Big\}.$$

One can show that the volume of E_ℓ^ϵ is less than $\frac{\epsilon}{(\ell-1)!}$ so that

$$\left|\int_{E_\ell^\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)dx_1\cdots dx_\ell\right| \le \epsilon\frac{\alpha^\ell}{(\ell-1)!}. \tag{27}$$

Also,

$$\begin{aligned}\int_{\mathcal{R}_\ell^\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)\,dx_1\cdots dx_\ell &= \\ = \frac{1}{\ell!}\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon}\cdots\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)\,dx_1\cdots dx_\ell \\ = \frac{1}{\ell!}\left(\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} s_\epsilon^i(x)dx\right)^\ell.\end{aligned}$$

Notice that all of this makes sense for any positive integer ℓ, not only those that are less than $L+1$: indeed, the set E_ℓ^ϵ is equal to $\mathcal{R}_\ell^\epsilon$ for all $\ell > L$. Taking the sum over $\ell = 0,\ldots,L$ and taking the empty sum to be 1 since P_0 is the identity matrix, we have

$$\begin{aligned}\sum_{\ell=0}^{L}\epsilon^\ell \sum_{\underline{n}\le n_1<\ldots<n_\ell\le\overline{n}} r_{n_1}^{ii}\cdots r_{n_\ell}^{ii} &= \\ = \sum_{\ell=0}^{L}\int_{\mathcal{R}_\ell^\epsilon - E_\ell^\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)\,dx_1\cdots dx_\ell \\ = \sum_{\ell=0}^{\infty}\int_{\mathcal{R}_\ell^\epsilon - E_\ell^\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)\,dx_1\cdots dx_\ell \\ = \exp\left(\int_{\underline{n}\epsilon}^{\overline{n}\epsilon+\epsilon} s_\epsilon^i(x)dx\right) - \sum_{\ell=0}^{\infty}\int_{E_\ell^\epsilon} s_\epsilon^i(x_1)\cdots s_\epsilon^i(x_\ell)\,dx_1\cdots dx_\ell.\end{aligned}$$

The second term converges to zero because of the bounds (27), and the first term converges to $\exp\left(\int_a^b \underline{r}^{ii}(x)dx\right) = e^{S_i(a,b)}$, as $\epsilon \to 0$.

Regarding the second summand of $\epsilon^\ell P_\ell$ in equation (18), we see that any one of its entries contains no more that $2^{\ell-1}$ terms of the type Υ, and so by Lemma 4, given $\bar{\varrho} > 0$, the sum over all these terms can be made to be less in modulus than $\bar{\varrho}\frac{(2\alpha)^{\ell-1}}{(\ell-1)!}$ for each ℓ by taking ϵ sufficiently small, and thus, in the sum over all ℓ, this second summand contributes less than $\bar{\varrho}\exp(2\alpha)$ in modulus to any entry of the matrix $\sum_{\ell=0}^{L}\epsilon^\ell P_\ell$. $\triangle$

5.3 Exponential Region

We now turn to an exponential region $[a,b]$. Let us suppose, for the sake of the argument, that $0 < \lambda^-(x) < \lambda^+(x)$ on this interval. Referring to the form of P_ℓ computed on page 29, we observe that each entry of its second column consists of terms containing a factor of the form $\prod_{n=n_\ell}^{\overline{n}}\frac{\lambda_n^-}{\lambda_n^+}$, which are summed over $\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}$, and one expects these to converge to zero as

$\epsilon \to 0$ since $\frac{\lambda_n^-}{\lambda_n^+} < 1$. The first column contains a sum over $\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}$ of terms of the form $\prod_{n=n_h}^{n_{h+1}-1} \frac{\lambda_n^+}{\lambda_n^-}$, but n_{h+1} is never equal to $\overline{n}$ except in the term containing the product $r_{n_1}^{ii} \cdots r_{n_\ell}^{ii}$, which has $\prod_{n=\underline{n}}^{\overline{n}} \frac{\lambda_n^+}{\lambda_n^-}$ as a factor. This suggests that, although both entries of the first column diverge as $\epsilon \to 0$, the upper left entry will dominate. This can be proved rigorously as long as the interval $[a,b]$ is bounded away from any turning point. Suppose that, for some fixed value of s with $0 < s < 1$,

$$\frac{\lambda^-(x)}{\lambda^+(x)} < s \qquad \text{for} \quad x \in [a,b],$$

and let α, β, γ, and $S_i(a,b)$, be defined as before (α, β, and γ depend on s). The following proposition makes precise what is meant by the dominance of the $(1,1)$-entry of T^ϵ.

Proposition 6 *Let $[a,b]$ be an exponential interval with positive distance from the set of turning points, and assume that $0 < \lambda^-(x) < \lambda^+(x)$ for every x in $[a,b]$. Then, for any $\varrho > 0$, ϵ' can be chosen sufficiently small such that*

$$T^\epsilon = \begin{bmatrix} e^{S_1(a,b)} + \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{bmatrix} \prod_{n=\underline{n}}^{\overline{n}} \lambda_n^+$$

whenever $\epsilon < \epsilon'$, for certain numbers ϱ_{ij} with modulus less than ϱ.

Remark: the convergence of the left-hand matrix as ϵ tends to zero is not uniform as a or b nears a turning point since a suitable value of s approaches 1 near a turning point. However, it is uniform over all exponential x-intervals whose distance from any turning point is bounded below by some positive number.

PROOF. Consider the matrix

$$T^\epsilon \prod_{n=\underline{n}}^{\overline{n}} \frac{1}{\lambda_n^+} = \left(\sum_{\ell=0}^{L} \epsilon^\ell P_\ell \right) \begin{bmatrix} 1 & 0 \\ 0 & \prod_{n=\underline{n}}^{\overline{n}} \frac{\lambda_n^-}{\lambda_n^+} \end{bmatrix}.$$

As in the proof of Proposition 5,

$$\sum_{\ell=0}^{L} \epsilon^\ell \sum_{\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}} r_{n_1}^{11} \cdots r_{n_\ell}^{11} \longrightarrow e^{S_1(a,b)} \qquad \text{as } \epsilon \to 0.$$

Now consider the matrices

$$\epsilon^\ell P_\ell \begin{bmatrix} 1 & 0 \\ 0 & \prod_{n=\underline{n}}^{\overline{n}} \frac{\lambda_n^-}{\lambda_n^+} \end{bmatrix} - \begin{bmatrix} \epsilon^\ell \sum_{\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}} r_{n_1}^{11} \cdots r_{n_\ell}^{11} & 0 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

Each entry is a sum of no more than $2^{\ell-1}$ quantities of the form

$$\Upsilon = \epsilon^\ell \sum_{\underline{n} \le n_1 < \ldots < n_\ell \le \overline{n}} r_{n_1}^* \cdots r_{n_\ell}^*(r) \prod_{n=n_h+1}^{n_{h+1}} \frac{\lambda_n^-}{\lambda_n^+}$$

for some h (n_{h+1} may be equal to $\overline{n}$) where r is a product of quantities $\frac{\lambda_n^-}{\lambda_n^+}$ and is therefore less than 1. Υ can be rewritten as

$$\Upsilon = \epsilon^\ell \sum_{\underline{n} \le n_1 < \ldots < \hat{n}_h < \ldots < n_\ell \le \overline{n}} r^*_{n_1} \cdots \hat{r}^*_{n_h} \cdots r^*_{n_\ell}(r) \left(\sum_{n_{h-1} < n_h < n_{h+1}} r^*_{n_h+1} \prod_{n=n_h+1}^{n_{h+1}} \frac{\lambda_n^-}{\lambda_n^+} \right).$$

The circumflex indicates the removal of a factor. An estimate can be obtained for the sum in parentheses:

$$\left| \sum_{n_{h-1} < n_h < n_{h+1}} r^*_{n_h} \prod_{n=n_h+1}^{n_{h+1}} \frac{\lambda_n^-}{\lambda_n^+} \right| < \alpha \sum_{n_{h-1} < n_h < n_{h+1}} s^{n_{h+1}-n_h} < \alpha \frac{s}{1-s}.$$

Since the number of elements in the sum over $\underline{n} \le n_1 < \ldots < \hat{n}_h < \ldots < n_\ell \le \overline{n}$ is less than $\frac{N^{\ell-1}}{(\ell-1)!}$, we obtain the following estimate for Υ:

$$|\Upsilon| < \epsilon^\ell (\alpha)^\ell \left(\frac{s}{1-s} \right) \frac{N^{\ell-1}}{(\ell-1)!} = \epsilon\alpha \left(\frac{s}{1-s} \right) \frac{(\alpha)^{\ell-1}}{(\ell-1)!}.$$

Therefore, any entry of the matrix (28) is bounded by $\epsilon\alpha\left(\frac{s}{1-s}\right)\frac{(2\alpha)^{\ell-1}}{(\ell-1)!}$ and the sum over $\ell = 0, \ldots, L$ (recall that P_0 is the identity matrix) is bounded by $\epsilon\alpha\left(\frac{s}{1-s}\right)\exp(2\alpha)$. This proves that the matrix

$$\left(\sum_{\ell=0}^{L} \epsilon^\ell P_\ell \right) \begin{bmatrix} 1 & 0 \\ 0 & \prod_{n=\underline{n}}^{\overline{n}} \frac{\lambda_n^-}{\lambda_n^+} \end{bmatrix} = T^\epsilon \prod_{n=\underline{n}}^{\overline{n}} \frac{1}{\lambda_n^+}$$

converges to $\begin{bmatrix} e^{S_i(a,b)} & 0 \\ 0 & 0 \end{bmatrix}$ as $\epsilon \to 0$, and the Proposition follows. $\triangle$

Part (b) of Theorem 2 follows from Proposition 6.

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