POWER SERIES FOR WAVES IN MICRO-RESONATOR ARRAYS

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Abstract – **We obtain convergent power-series representations for Bloch waves and their complex dispersion relation in an infinite periodic array of micro-resonators, which exhibits artificial bulk magnetism. The small parameter η of expansion is the ratio of period to wavelength, and the high conductivity is inversely proportional to η. The coefficient fields are solutions to an infinite sequence of period-cell problems.**[1]

I. INTRODUCTION

Periodic arrays of micro-resonators consisting of metal shells or rings were introduced by Pendry, *et. al.*, [6] for the purpose of creating bulk magnetism at desired frequencies. When the period of the array is small compared to the observable modulation of the fields, the key element in the creation of magnetism is high conductivity in the resonators. Even if the component materials are magnetically neutral, the balance of small cell-to-wavelength ratio η and high conductivity produces a magnetic flux through the rings at the microscopic level, thus creating magnetic dipoles, and the effect is the emergence of bulk magnetic activity. Rigorous two-scale analysis has been carried out for a 2D model by Kohn and Shipman [5] and for a 3D model by Bouchitté and Schweizer [1].

An analysis of the limit of vanishing η, often called the quasi-static limit when the operating frequency is nonzero, begins with a formal power series expansion of the electromagnetic fields, which are assumed to depend on a macroscopic variable $\mathbf{x}$ and a microscopic variable $\mathbf{y} \propto \mathbf{x}/\eta$,

$$u(\mathbf{x}) = u_0(\mathbf{x},\mathbf{y}) + \eta u_1(\mathbf{x},\mathbf{y}) + \eta^2 u_2(\mathbf{x},\mathbf{y}) + \cdots .$$

Typically, the first term or two suffice for the calculation of the bulk properties of a sample of the homogenized medium, and rigorous analysis utilizes the theory of two-scale convergence. The effects on the fields due to boundaries or interfaces between the sample and another medium are more subtle and can be large compared to η. Thus one does not normally expect the power series to converge for nonzero values of η or even to be an asymptotic expansion for fields in the medium. But if one considers an infinite periodically micro-structured medium without boundaries and assumes that the field has the form of a Bloch wave, it turns out that the power series do converge within some radius $\eta < R > 0$. Thus one has a representation for fields near but not at the quasi-static limit, which is valuable for the investigation of the properties of the metamaterial in its own right, and in particular the contribution of higher-order multipoles to its bulk characteristics. A demonstration of the convergence for a two-dimensional array of micro-resonators is the objective of this paper.

The approach is to expand the waves and their dispersion relation in power series in η and obtain an infinite sequence of unit-cell problems coupling the coefficients of the field and the dispersion relation. One must then prove that this infinite sequence has a solution and that the power series converge, for sufficiently small but nonzero η, to the Bloch waves and their dispersion relation for structures near, but not at, the homogenization limit. The convergence is shown by obtaining recursive bounds on the fields and then using a system of coupled generating functions to prove that the sequence is bounded by a geometric series. Convergent series solutions have been obtained recently by Fortes, Lipton, and Shipman [4, 3] for high-contrast photonic and plasmonic crystals and earlier by Bruno [2] in a static problem of high-contrast conductivity.

II. MATHEMATICAL MODEL AND FORMAL POWER SERIES

Following [5], we model a 2D micro-resonator array by periodically dispersed cylindrical shells whose cross section in the $\mathbf{x}$-plane consists of copies, scaled by the period d, of the boundary ∂P of a region P in the unit cube $Q = [0,1]^2$, with complement $P^c = Q \backslash P$ (see Figure). In order to focus attention on the role of high conductivity in the resonators, we set $\epsilon = 1$ and $\mu = 1$ in the host material and consider magnetically polarized fields. The harmonic Maxwell equations reduce to the following system for the out-of-plane component $h(\mathbf{x})$ of the magnetic field and the in-plane electric field $E(\mathbf{x})$, in which $\hat{k}$ is the out-of-plane normal vector, and $[h]$ is the interior-to-exterior jump

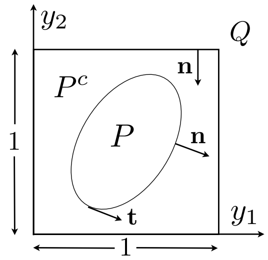
[1]This material is based upon work supported by the National Science Foundation under Grant No. 0807325.

of h across the shells:

$$\begin{array}{ll} \nabla\cdot\nabla h + (\frac{\omega}{c})^2 h = 0, & E = \frac{c}{i\omega}\hat{k}\times\nabla h, \\ [h] := h_{\mathrm{e}} - h_{\mathrm{i}} = -\sigma E\cdot t = -j, & E\cdot t \text{ continuous across } d(\partial P + (m,n)),\ m,n\in\mathbb{Z}. \end{array} \tag{1}$$

Here, σ is a complex conductivity, accounting for conduction and capacitance in the resonators, and j is the complex magnitude of the current flowing around the resonator. We assume a Bloch-wave form of h, namely, that h is a plane wave distorted periodically by the micro-geometry,

$$\begin{array}{l} h(\mathbf{x}) = e^{ik\boldsymbol{\kappa}\cdot\mathbf{x}}u(\mathbf{x}/d), \quad |\boldsymbol{\kappa}| = 1, \\ u(\mathbf{y}) \text{ has period cell } Q = [0,1]^2. \end{array}$$



Using this ansatz in the system (1) with the definitions

$$\begin{array}{l} \eta = kd, \quad \mathbf{x} = d\cdot\mathbf{y}, \\ \sigma^{-1} = \frac{d}{L}b \quad (L \text{ a fixed length and } b \text{ a constant with } \operatorname{Im} b \geq 0), \end{array}$$

yields a system for the periodic factor u. The small-η analysis, including solution of the formal sequence of cell problems and obtention of bounds on the coefficient fields, is most conveniently approached through the weak formulation of the system (1), which is reduced, by virtue of the Bloch-wave form, to a problem in the unit cell Q. This formulation is posed in the Hilbert space $H^1_{\text{per}}(P^c)\oplus H^1(P)$, consisting of pairs of functions $v = (v_{\mathrm{e}}, v_{\mathrm{i}})$ with square-integrable value and gradient, in which the first component has support in P^c and satisfies periodic conditions on the boundary ∂Q of Q, the second has support in P, and the jump $[v] = v_{\mathrm{e}} - v_{\mathrm{i}}$ across ∂P is unrestricted. One seeks a pair of functions $u = (u_{\mathrm{e}}, u_{\mathrm{i}}) \in H^1_{\text{per}}(P^c)\oplus H^1(P)$ such that

$$(Lk)^2\int_Q \left[\nabla u\cdot\nabla\bar{v} + i\eta\boldsymbol{\kappa}\cdot(u\nabla\bar{v} - \bar{v}\nabla u) + \eta^2 u\bar{v}\right]\mathrm{d}A - \eta^2\left(L\frac{\omega}{c}\right)^2\int_Q u\bar{v}\,\mathrm{d}s - i\eta^2 L\frac{\omega}{c}\int_{\partial P} b[u][\bar{v}]\mathrm{d}s = 0 \tag{2}$$

for all test functions $v \in H^1_{\text{per}}(P^c)\oplus H^1(P)$. We begin by fixing the frequency ω and the direction vector $\boldsymbol{\kappa}$ of the wave and formally expanding the field u and the wavenumber k in powers of η. It is also convenient to define the rescaled frequency ξ and rescaled coefficients ψ_m and δ_m of the field and square wavenumber:

$$\begin{array}{ll} u = u_0 + \eta u_1 + \eta^2 u_2 + \ldots, & u_m = i^m\psi_m, \\ k^2 = k_0^2 + \eta k_1^2 + \eta^2 k_2^2 + \ldots, & L^2 k_m^2 = i^m\delta_m, \quad \xi = L\frac{\omega}{c}. \end{array}$$

Inserting these expansions into (2) yields an infinite system for the coefficients:

$$\sum_{\ell=0}^{m}\delta_\ell\int_Q\left[(\nabla\psi_{m-\ell} + \boldsymbol{\kappa}\psi_{m-1-\ell})\cdot\nabla\bar{v} - (\boldsymbol{\kappa}\cdot\nabla\psi_{m-1-\ell} + \psi_{m-2-\ell})\bar{v}\right] + \xi^2\int_Q\psi_{m-2}\bar{v} + i\xi\int_{\partial P} b[\psi_{m-2}][\bar{v}] = 0 \tag{3}$$

for all $v \in H^1_{\text{per}}(P^c)\oplus H^1(P)$.

III. Solution of the Sequence of Cell Problems

The sequence of cell problems (3) can be solved inductively for all of the coefficients ψ_m and δ_m, and one must solve for them in a specific order. The problem for the fields ψ_m in each of the domains P and P^c is of Neumann type, with data coming from the current at order $m-2$. Thus solvability is subject to the condition of the Fredholm alternative. It is precisely this condition in P^c that determines the coefficients δ_m. The solvability condition in P can be satisfied through the stipulation of the free constant that is added to each field coefficient in P.

The first two equations ($m = 0, 1$) as well as the solvability condition for ψ_2 should be treated separately, as they establish the leading order of the fields, the current, and the dispersion relation. From this information, one also obtains the effective values of ϵ and μ for the homogenized medium as $\eta \to 0$.

1. Solving for ψ_0 and the leading-order current. Equation (3) for $m = 0$ is

$$\delta_0\int_Q\nabla\psi_0\cdot\nabla\bar{v} = 0 \qquad \forall v \in H^1_{\text{per}}(P^c)\oplus H^1(P). \tag{4}$$

This amounts to two decoupled homogeneous Neumann problems for ψ_0 in P^c and in P:

$$\nabla\cdot\nabla\psi_0 = 0 \text{ in } P \text{ and } P^c, \qquad \nabla\psi_0\cdot n = 0 \text{ on } \partial P.$$

Thus ψ_0 is constant in each of the regions P^c and P, but there is yet no relation between its values in the two regions. We will normalize ψ_0 to unity in P^c, and set ψ_0 equal to a yet undetermined complex constant γ_0 in P,

$$\psi_0 = 1 \text{ in } P^c, \quad \psi_0 = \gamma_0 \text{ in } P, \quad j_0 = \gamma_0 - 1 = \text{ leading-order current around } \partial P.$$

2. Solving for ψ_1, the corrector function. Equation (3) for $m = 1$ is

$$\delta_0 \int_Q (\nabla\psi_1 + \boldsymbol{\kappa}\psi_0)\cdot\nabla\bar{v} = 0 \qquad \forall v \in H^1_{\text{per}}(P^c) \oplus H^1(P). \tag{5}$$

The δ_1-term is absent because $\nabla\psi_0 = 0$. In strong form, this equation is a decoupled pair of inhomogeneous Neumann problems for ψ_1, namely,

$$\begin{aligned} &\nabla\cdot\nabla\psi_1 = 0 \text{ in } P^c, \quad \nabla\psi_1\cdot n = -\boldsymbol{\kappa}\cdot n \text{ on } \partial P_{\text{e}}, \quad \psi_1 \text{ is periodic on } \partial Q, \\ &\nabla\cdot\nabla\psi_1 = 0 \text{ in } P, \quad \nabla\psi_1\cdot n = -\gamma_0\boldsymbol{\kappa}\cdot n \text{ on } \partial P_{\text{i}}, \end{aligned}$$

in which ∂P_{e} denotes ∂P from the side of P^c and ∂P_{i} denotes ∂P from the side of P. In P^c, we normalize ψ_1 to mean zero, and, in P, we decompose ψ_1 into its mean-zero part plus a constant,

$$\begin{aligned} &\text{in } P^c: \quad \int_{P^c} \psi_1 = 0, \\ &\text{in } P: \quad \psi_1 = \gamma_0\psi_* + \gamma_1\,, \qquad \psi_* = -\boldsymbol{\kappa}\cdot\mathbf{y} + \frac{1}{|P|}\int_P \boldsymbol{\kappa}\cdot\mathbf{y}, \end{aligned}$$

where γ_1 is a constant to be determined and $|P|$ denotes the area of P.

3. Leading order of the dispersion relation. The equation for ψ_2 is

$$\delta_0 \int_Q \left[(\nabla\psi_2 + \boldsymbol{\kappa}\psi_1)\cdot\nabla\bar{v} - (\boldsymbol{\kappa}\cdot\nabla\psi_1 + \psi_0)\bar{v}\right] + \xi^2 \int_Q \psi_0\bar{v} + i\xi\int_{\partial P} b[\psi_0][\bar{v}] = 0 \tag{6}$$

for all $v \in H^1_{\text{per}}(P^c)\oplus H^1(P)$. The δ_2-term is absent because $\nabla\psi_0 = 0$, and the term $\delta_1\int_Q(\nabla\psi_1 + \boldsymbol{\kappa}\psi_0)\cdot\nabla\bar{v}$ vanishes because of equation (5). This equation is a pair of Neumann problems, coupled through the boundary current j_0. There are two solvability conditions, which yield a system of two equations for δ_0 and γ_0. The first is obtained by putting $v = 1$ in Q (so that $[v] = 0$), and the second is obtained by putting $v = 0$ in P^c and $v = 1$ in P (so $[v] = -1$):

$$-\delta_0 \int_{P^c} (\boldsymbol{\kappa}\cdot\nabla\psi_1 + \psi_0) + \xi^2 \int_Q \psi_0 = 0\,, \qquad \xi^2 \int_P \psi_0 - i\xi\int_{\partial P} b[\psi_0] = 0\,, \tag{7}$$

where we have used the fact that $\int_P(\boldsymbol{\kappa}\cdot\nabla\psi_1 + \psi_0) = 0$ by taking in (5) $v = \boldsymbol{\kappa}\cdot\mathbf{y}$ in P and $v = 0$ in P^c. Using $\psi_0 = \gamma_0$ in P and $[\psi_0] = 1 - \gamma_0$, and assuming $\xi \neq 0$, the latter equation gives

$$\gamma_0 = \frac{ib|\partial P|}{\xi|P| + ib|\partial P|}, \tag{8}$$

as long as the denominator does not vanish. Thus we have solved for the leading order value of the current $j_0 = \gamma_0 - 1$. Substituting this expression for γ_0 in the first equation of (7) yields

$$\delta_0 \int_{P^c} (\boldsymbol{\kappa}\cdot\nabla\psi_1 + 1) = \xi^2 \frac{\xi|P||P^c| + ib|\partial P|}{\xi|P| + ib|\partial P|}. \qquad \textbf{(dispersion relation)} \tag{9}$$

This is the dispersion relation for the homogenized medium relating wavevector to frequency.

One can show that the constant multiplying δ_0 is always nonzero. It is in fact equal to the reciprocal of the effective dielectric coefficient ϵ_* for the homogenized micro-resonator array, while the factor multiplying ξ^2 on the right is the frequency-dependent effective magnetic coefficient μ_* as derived in [5] (see equations (4.9, 4.11, 4.18, 4.19)). Thus we have recovered the limiting dispersion relation $\delta_0 = \xi^2\mu_*\epsilon_*$, or $k_0^2 = (\omega/c)^2\mu_*\epsilon_*$, as the cell size tends to zero. Observe that, if b has a small real part, then μ_* becomes large at a real frequency; this is the magnetic dipole resonance of the metamaterial, with moment directed out of the plane.

4. Solving for ψ_2 in P^c and P. This is possible now that we have imposed the solvability conditions ((8,9) or 7). Setting $v=0$ in P in (6) yields a Neumann problem for ψ_2 in P^c that is forced by the current j_0, and one has the freedom to take the mean of ψ_2 to vanish in P^c, that is, $\int_{P^c}\psi_2 = 0$. In P, let $\hat{\psi}_2$ be the mean-zero solution of the following modification of (6) with $v=0$ in P^c, in which ψ_1 is replaced by its mean-zero part $\gamma_0\psi_*$:

$$\delta_0\int_P \Big[(\nabla\hat{\psi}_2 + \boldsymbol{\kappa}(\gamma_0\psi_*))\cdot\nabla\bar{v} - (\boldsymbol{\kappa}\cdot\nabla(\gamma_0\psi_*) + \psi_0)\bar{v}\Big] + \xi^2\int_P \psi_0\bar{v} - i\xi\int_{\partial P} b[\psi_0]\bar{v} = 0. \tag{10}$$

By definition of ψ_*, one has $\int_P(\nabla\gamma_1\psi_* + \boldsymbol{\kappa}\gamma_1)\cdot\nabla\bar{v} = 0$, and by adding this to (10) one finds that equation (6) is satisfied if we substitute $\hat{\psi}_2 + \gamma_1\psi_*$ for ψ_2. Thus we obtain

$$\psi_2 = \hat{\psi}_2 + \gamma_1\psi_* + \gamma_2 \quad \text{and} \quad \int_P (\hat{\psi}_2 + \gamma_1\psi_*) = 0,$$

in which γ_2 is a constant to be determined.

5. Solving inductively for ψ_m and δ_m. At the m^{th} stage, one solves for δ_{m-2} and γ_{m-2} through the solvability conditions for ψ_m, then one solves for the field ψ_m in P^c and for $\hat{\psi}_2$ in P, such that

$$\begin{aligned} &\text{in } P^c: \quad \int_{P^c}\psi_m = 0,\\ &\text{in } P: \quad \psi_m = \hat{\psi}_m + \gamma_{m-1}\psi_* + \gamma_m \quad \text{with} \quad \int_P(\hat{\psi}_m + \gamma_{m-1}\psi_*) = 0. \end{aligned}$$

This is accomplished inductively for all $m > 2$, as follows. By putting $v = 1$ in Q in (3), we obtain a solvability condition for ψ_m that contains the as yet undetermined constants δ_{m-2} and γ_{m-2},

$$\underbrace{\delta_{m-2}}\int_{P^c}(\boldsymbol{\kappa}\cdot\nabla\psi_1 + 1) + \sum_{\ell=0}^{m-3}\delta_\ell\left(\int_{P^c}\boldsymbol{\kappa}\cdot\nabla\psi_{m-1-\ell} + \int_P \boldsymbol{\kappa}\cdot\nabla\hat{\psi}_{m-1-\ell}\right) - \xi^2\,\underbrace{\gamma_{m-2}}\,|P| = 0. \tag{11}$$

In simplifying this expression, we have used the mean values of the fields and the definition of ψ_*. Again, the δ_m-term is absent because $\nabla\psi_0 = 0$, and the term $\delta_{m-1}\int_Q(\nabla\psi_1 + \boldsymbol{\kappa}\psi_0)\cdot\nabla\bar{v}$ vanishes because of equation (5). The other solvability condition is obtained by putting $v = 0$ in P^c and $v = 1$ in P,

$$\underbrace{\gamma_{m-2}}\big(\xi^2|P| + i\xi b|\partial P|\big) = i\xi b\int_{\partial P}\Big(\psi_{m-2}|_{P^c} - (\hat{\psi}_{m-2} + \gamma_{m-3}\psi_*)|_P\Big) + \sum_{\ell=0}^{m-3}\delta_\ell\int_P \boldsymbol{\kappa}\cdot\nabla\hat{\psi}_{m-1-\ell}\,. \tag{12}$$

As long as $\xi^2|P| + i\xi b|\partial P| \neq 0$, one may solve for γ_{m-2} and substitute the result into (11) to obtain both δ_{m-2} and γ_{m-2} in terms of known fields and constants. Finally, from (3), one can solve for ψ_m in P^c ($v_{\text{i}}=0$) and $\hat{\psi}_m$ in P ($v_{\text{e}}=0$) by replacing ψ_{m-1} with its mean-zero part $\hat{\psi}_{m-1} + \gamma_{m-2}\psi_*$. The sum over ℓ in (3) can be taken to run from 0 to $m-2$ by virtue of (4) and (5), and therefore ψ_m and $\hat{\psi}_m$ are obtained in terms of known fields and constants.

IV. Convergence of the Series

To prove that the series converge for $\eta < R$ for some $R > 0$, we must prove that the coefficients are exponentially bounded. It will then follow from (2,3) that the series satisfy the Maxwell system. From equations (3,11,12), we obtain an infinite sequence of coupled inequalities for the coefficients δ_m and γ_m and the H^1 norms of the fields ψ_m in P^c and $\hat{\psi}_m$ in P. Denote

$$\bar{p}_m := \|\psi_m\|_{H^1(P^c)}, \quad \hat{p}_m := \|\hat{\psi}_m\|_{H^1(P)}.$$

We use crude constants K_i below for this short communication. They can be refined considerably and depend on P, b, and ξ, and certain of them are large when δ_0 or $\xi^2|P| + i\xi b|\partial P|$ is small. For $m \geq 2$,

$$
\begin{aligned}
&\bar{p}_m \le K_1\left(\bar{p}_{m-1}+\bar{p}_{m-2}+\hat{p}_{m-2}+|\gamma_{m-2}|+|\gamma_{m-3}|+\textstyle\sum_{\ell=1}^{m-2}|\delta_\ell|(\bar{p}_{m-\ell}+\bar{p}_{m-1-\ell}+\bar{p}_{m-2-\ell})\right),\\
&\hat{p}_m \le K_2\Big(\bar{p}_{m-2}+\hat{p}_{m-1}+\hat{p}_{m-2}+|\gamma_{m-2}|+|\gamma_{m-3}|+\\
&\qquad\qquad +\textstyle\sum_{\ell=1}^{m-2}|\delta_\ell|(\hat{p}_{m-\ell}+\hat{p}_{m-1-\ell}+\hat{p}_{m-2-\ell}+|\gamma_{m-1-\ell}|+|\gamma_{m-2-\ell}|+|\gamma_{m-3-\ell}|)\Big),\\
&|\gamma_{m-1}| \le K_3\left(\bar{p}_{m-1}+\hat{p}_{m-1}+|\gamma_{m-2}|+\textstyle\sum_{\ell=0}^{m-2}|\delta_\ell|\hat{p}_{m-\ell}\right),\\
&|\delta_{m-1}| \le K_4\left(|\gamma_{m-1}|+\textstyle\sum_{\ell=0}^{m-2}|\delta_\ell|(\bar{p}_{m-\ell}+\hat{p}_{m-\ell})\right).
\end{aligned}
\tag{13}
$$

Now identify $\bar{a}_m$ with $\bar{p}_m$, $\hat{a}_m$ with $\hat{p}_m$, c_m with $|\gamma_{m-1}|$, and d_m with $|\delta_{m-1}|$; put $\bar{a}_0=\bar{p}_0$, $\bar{a}_1=\bar{p}_1$, $\hat{a}_0=\hat{a}_1=c_0=d_0=0$, $c_1=|\gamma_0|$, and $d_1=|\delta_0|$; set all values to zero for $m<0$; and define $\bar{a}_m$, $\hat{a}_m$, c_m, and d_m for $m\ge 2$ through the recursion obtained by replacing inequality with equality in (13) (and shifting indices),

$$
\begin{aligned}
&\bar{a}_m = K_1\left(\bar{a}_{m-1}+\bar{a}_{m-2}+\hat{a}_{m-2}+c_{m-1}+c_{m-2}+\textstyle\sum_{\ell=2}^{m-1}d_\ell(\bar{a}_{m+1-\ell}+\bar{a}_{m-\ell}+\bar{a}_{m-1-\ell})\right),\\
&\hat{a}_m = K_2\Big(\bar{a}_{m-2}+\hat{a}_{m-1}+\hat{a}_{m-2}+c_{m-1}+c_{m-2}+\\
&\qquad\qquad +\textstyle\sum_{\ell=2}^{m-1}d_\ell(\hat{a}_{m+1-\ell}+\hat{a}_{m-\ell}+\hat{a}_{m-1-\ell}+c_{m+1-\ell}+c_{m-\ell}+c_{m-1-\ell})\Big),\\
&c_m = K_3\left(\bar{a}_{m-1}+\hat{a}_{m-1}+c_{m-1}+\textstyle\sum_{\ell=1}^{m-1}d_\ell\hat{a}_{m+1-\ell}\right),\\
&d_m = K_4\left(c_m+\textstyle\sum_{\ell=1}^{m-1}d_\ell(\bar{a}_{m+1-\ell}+\hat{a}_{m+1-\ell})\right).
\end{aligned}
\tag{14}
$$

Since the values of $\bar{p}_m$, *etc.*, coincide with those of $\bar{a}_m$, *etc.*, for $m=0,1$, we have $\bar{p}_m\le\bar{a}_m$, *etc.*, for all m and thus, to prove an exponential bound $\bar{p}_m,\hat{p}_m,|\gamma_m|,|\delta_m|\le CJ^m$ for some C and J, it is sufficient to prove such a bound on the sequences $(\bar{a}_m,\hat{a}_m,c_m,d_m)$. Define generating functions for the sequences $(\bar{a}_m,\hat{a}_m,c_m,d_m)_{m\ge1}$ through

$$
\bar{f}(z)=\textstyle\sum_{\ell=0}^{\infty}\bar{a}_{\ell+1}z^\ell,\quad \hat{f}(z)=\sum_{\ell=0}^{\infty}\hat{a}_{\ell+1}z^\ell,\quad g(z)=\sum_{\ell=0}^{\infty}c_{\ell+1}z^\ell,\quad h(z)=\sum_{\ell=0}^{\infty}d_{\ell+1}z^\ell.
$$

The recursive system (14) with the given initial values is equivalent to setting $\bar{F}=\hat{F}=G=H=0$ in the relations

$$
\begin{aligned}
\bar{F} &= -(\bar{f}-\bar{a}_1)+K_1\big(z\bar{f}+z(\bar{a}_0+z\bar{f})+z^2\hat{f}+(z+z^2)g+(h-d_1)(\bar{f}-\bar{a}_1)+\\
&\qquad\qquad\qquad\qquad +z(h-d_1)\bar{f}+z(h-d_1)(z\bar{f}+\bar{a}_0)\big),\\
\hat{F} &= -(\hat{f}-\hat{a}_1)+K_2\big((z+z^2)\hat{f}+z^2\bar{f}+(z+z^2)g+(h-d_1)(\hat{f}-\hat{a}_1)+\\
&\qquad\qquad +(z+z^2)(h-d_1)\hat{f}+(h-d_1)(g-c_1)+(z+z^2)(h-d_1)g\big),\\
G &= -(g-c_1)+K_3\big(z(\bar{f}+\hat{f}+g)+h(\hat{f}-\hat{a}_1)\big),\\
H &= -(h-d_1)+K_4\big(g-c_1+h(\bar{f}-\bar{a}_1+\hat{f}-\hat{a}_1)\big),
\end{aligned}
$$

in which $\bar{F}$, $\hat{F}$, G, and H are functions of the five variables $\bar{f}$, $\hat{f}$, g, h, and z. One can verify that, at the point $(\bar{f},\hat{f},g,h,z)=(\bar{a}_1,\hat{a}_1,c_1,d_1,0)$, we have $(\bar{F},\hat{F},G,H)=(0,0,0,0)$ and that the Jacobian matrix of $(\bar{F},\hat{F},G,H)$ with respect to $(\bar{f},\hat{f},g,h)$ at $z=0$ is

$$
\left.\frac{\partial(\bar{F},\hat{F},G,H)}{\partial(\bar{f},\hat{f},g,h)}\right|_{z=0}=
\begin{bmatrix}
-1+K_1(h-d_1) & 0 & 0 & \bar{f}-\bar{a}_1\\
0 & -1+K_2(h-d_1) & K_2(h-d_1) & K_2(g-c_1)\\
0 & K_3h & -1 & K_3(\hat{f}-\hat{a}_1)\\
K_4h & K_4h & K_4 & -1+K_4(\bar{f}-\bar{a}_1+\hat{f}-\hat{a}_1)
\end{bmatrix}.
$$

The determinant of this matrix at $(\bar{f},\hat{f},g,h)=(\bar{a}_1,\hat{a}_1,c_1,d_1)$ is equal to 1. By the implicit function theorem for real-analytic functions, there exist real-analytic functions $\bar{f}(z)$, $\hat{f}(z)$, $g(z)$, and $h(z)$, defined in an open interval about $z=0$, such that

$$
\begin{aligned}
&(\bar{f}(0),\hat{f}(0),g(0),h(0))=(\bar{a}_1,\hat{a}_1,c_1,d_1),\\
&X(\bar{f}(z),\hat{f}(z),g(z),h(z),z)=0\ \text{ for }\ X=\bar{F},\hat{F},G,H.
\end{aligned}
$$

These functions admit convergent power series representations within some radius $|z|<R>0$ and satisfy the same functional relations as the formal power series do, namely those that are equivalent to the recursive system (14) with the given initial conditions. This means that the Taylor coefficients of the functions $(\bar{f}(z), \hat{f}(z), g(z), h(z))$ are equal to the solution $(\bar{a}_m, \hat{a}_m, c_m, d_m)$ of the recursion. We conclude that these sequences are exponentially bounded,

$$\bar{a}_m, \hat{a}_m, c_m, d_m \ \leq \ CJ^m$$

for some positive numbers C and J. The numbers $(\bar{p}_m, \hat{p}_m, |\gamma_m|, |\delta_m|)$ are therefore also bounded by CJ^m.

V. Discussion

The availability of convergent power series for Bloch waves is a step toward a better practical understanding of metamaterials. The convergence places higher multipoles of the unit cell into a framework that elucidates their role in creating bulk properties of the composite, not only in the quasi-static limit, but also for structures for which the ratio of the period to the wavelength is small but not infinitesimal. The use of generating functions for proving convergence appears to be applicable to Bloch waves in very general infinite periodic metamaterials.

In order to be able to utilize a power series for the quantitative approximation of fields, one needs an estimate on its radius of convergence. This is a difficult problem, which requires analysis beyond the straightforward application of generating functions and the implicit function theorem. For periodic arrays of plasmonic inclusions [4], a lower bound for the radius of convergence was obtained by using fine properties of the Catalan sequence. The technique does not seem to be extensible to general problems, and the bound obtained is not optimal; thus new ideas are needed.

A challenging goal would be to compute the radius of convergence. In 1890, Hermann Schwarz [7] obtained power-series solutions for the problem

$$\Delta u + \lambda p(x) u = 0$$

in a bounded domain with $p(x) > 0$ and given Dirichlet boundary data, with λ as the expansion parameter. He showed that the radius of convergence was equal to the first Dirichlet eigenvalue of the problem. Proving the radius of convergence is facilitated by the fact that the coefficient fields of his series are positive. For Bloch fields in periodic high-contrast composites, the fields are by no means positive; in fact they oscillate in sign, as they represent higher-order multipoles. The lower bound obtained using the Catalan numbers is far from optimal because it was based on field bounds coming from Sobolev stability estimates. We ask the question: What is the analog of the first eigenvalue that Schwarz found; in other words, what phenomenon in the unit cell of our metamaterial characterizes the breakdown of convergence of the power-series solution?

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